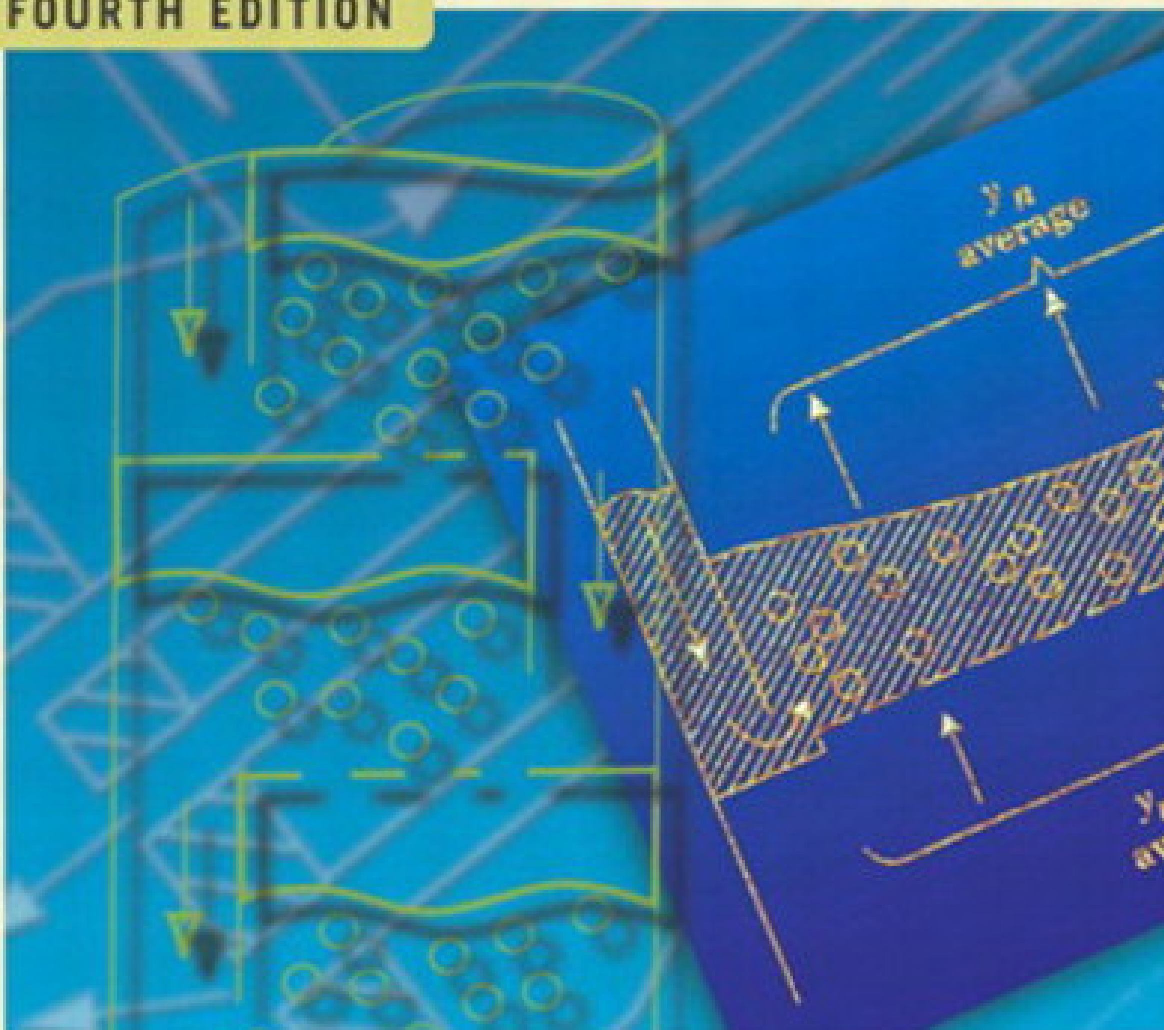
Transport Processes and Separation Process Principles

(INCLUDES UNIT OPERATIONS)

FOURTH EDITION



CHRISTIE JOHN GEANKOPLIS

Table of Contents

Chapter 5. Principles of Unsteady-State Heat Transfer

DERIVATION OF BASIC EQUATION

Introduction

In Chapter 4 we considered various heat-transfer systems in which the temperature at any given point and the heat flux were always constant over time, that is, in steady state. In the present chapter we will study processes in which the temperature at any given point in the system changes with time, that is, heat transfer is unsteady state or transient.

Before steady-state conditions can be reached in a process, some time must elapse after the heat-transfer process is initiated to allow the unsteady-state conditions to disappear. For example, in Section 4.2A we determined the heat flux through a wall at steady state. We did not consider the period during which one side of the wall was being heated up and the temperatures were increasing.

Unsteady-state heat transfer is important because of the large number of heating and cooling problems occurring industrially. In metallurgical processes it is necessary to predict cooling and heating rates for various geometries of metals in order to predict the time required to reach certain temperatures. In food processing, for example, the canning industry, perishable canned foods are heated by immersion in steam baths or chilled by immersion in cold water. In the paper industry wood logs are immersed in steam baths before processing. In most of these processes the material is suddenly immersed in a fluid of higher or lower temperature.

Derivation of Unsteady-State Conduction Equation

To derive the equation for unsteady-state conduction in one direction in a solid, we refer to Fig. 5.1-1. Heat is being conducted in the x direction in the cube Δx, Δy, Δz in size. For conduction in the x direction, we write

Equation 5.1-1.

$$q_x = -kA\frac{\partial T}{\partial x}$$

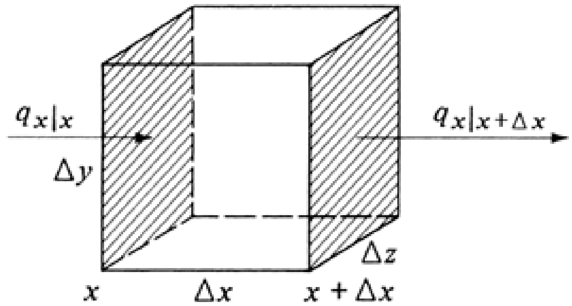


Figure 5.1-1. Unsteady-state conduction in one direction.

The term $\partial T/\partial x$ means the partial or derivative of T with respect to x, with the other variables, y, z, and time t, being held constant. Next, making a heat balance on the cube, we can write

Equation 5.1-2.

$$\text{rate of heat input} + \text{rate of generation} = \text{rate of heat output} + \text{rate of heat accumulation}$$

The rate of heat input to the cube is

Equation 5.1-3.

$$\text{rate of heat input} = q_{x|x} = -k(\Delta y\, \Delta z) \left.\frac{\partial T}{\partial x}\right|_x$$

Also,

Equation 5.1-4.

$$\text{rate of heat output} = q_{x|x+\Delta x} = -k(\Delta y\, \Delta z) \left.\frac{\partial T}{\partial x}\right|_{x+\Delta x}$$

The rate of accumulation of heat in the volume $\Delta x\, \Delta y\, \Delta z$ in time ∂t is

Equation 5.1-5.

$$\text{rate of heat accumulation} = (\Delta x\, \Delta y\, \Delta z)\rho c_p \frac{\partial T}{\partial t}$$

The rate of heat generation in volume $\Delta x\, \Delta y\, \Delta z$ is

Equation 5.1-6.

$$\text{rate of heat generation} = (\Delta x\, \Delta y\, \Delta z)\dot{q}$$

Substituting Eqs. (5.1-3)–(5.1-6) into (5.1-2) and dividing by $\Delta x\, \Delta y\, \Delta z$,

Equation 5.1-7.

$$\dot{q} + \frac{-k\left(\left.\frac{\partial T}{\partial x}\right|_x - \left.\frac{\partial T}{\partial x}\right|_{x+\Delta x}\right)}{\Delta x} = \rho c_p \frac{\partial T}{\partial t}$$

Letting Δx approach zero, we have the second partial of T with respect to x or $\partial^2 T/\partial x^2$ on the left side. Then, rearranging,

Equation 5.1-8.

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

where α is $k/\rho c_p$, thermal diffusivity. This derivation assumes constant k, ρ, and c_p. In SI units, α = m^2/s, T = K, t = *s*, k = W/m · K, ρ = kg/m^3, $\dot{q}$ = W/m^3, and c_p = J/kg · K. In English units, α = ft^2/h, T = °F, t = h, k = btu/h · ft·°F, ρ = lb$_m$/ft^3, $\dot{q}$ = btu/h · ft^3, and c_p = btu/lb$_m$ · °F.

For conduction in three dimensions, a similar derivation gives

Equation 5.1-9.

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\dot{q}}{\rho c_p}$$

In many cases, unsteady-state heat conduction is occurring but the rate of heat generation is zero. Then Eqs. (5.1-8) and (5.1-9) become

Equation 5.1-10.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Equation 5.1-11.

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

Equations (5.1-10) and (5.1-11) relate the temperature T with position x, y, and z and time t. The solutions of Eqs. (5.1-10) and (5.1-11) for certain specific cases as well as for the more general cases are considered in much of the remainder of this chapter.

SIMPLIFIED CASE FOR SYSTEMS WITH NEGLIGIBLE INTERNAL RESISTANCE

Basic Equation

We begin our treatment of transient heat conduction by analyzing a simplified case. In this situation we consider a solid which has a very high thermal conductivity or very low internal conductive resistance compared to the external surface resistance, where convection occurs from the external fluid to the surface of the solid. Since the internal resistance is very small, the temperature within the solid is essentially uniform at any given time.

An example would be a small, hot cube of steel at T_0 K at time t = 0, suddenly immersed in a large bath of cold water at T_∞ which is held constant with time. Assume that the heat-transfer coefficient h in W/m^2 · K is constant with time. Making a heat balance on the solid object for a small time interval of time dt s, the heat transfer from the bath to the object must equal the change in internal energy of the object:

Equation 5.2-1.

$$hA(T_\infty - T)dt = c_p \rho V\, dT$$

where A is the surface area of the object in m^2, T the average temperature of the object at time t in s, ρ the density of the object in kg/m^3, and V the volume in m^3. Rearranging the equation and integrating between the limits of $T = T_0$ when $t = 0$ and $T = T$ when $t = t$,

Equation 5.2-2.

$$\int_{T=T_0}^{T=T} \frac{dT}{T_\infty - T} = \frac{hA}{c_p \rho V} \int_{t=0}^{t=t} dt$$

Equation 5.2-3.

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-(hA/c_p \rho V)t}$$

This equation describes the time–temperature history of the solid object. The term $c_p \rho V$ is often called the *lumped thermal capacitance* of the system. This type of analysis is often called the *lumped capacity method* or *Newtonian heating or cooling method.*

Equation for Different Geometries

In using Eq. (5.2-3) the surface/volume ratio of the object must be known. The basic assumption of negligible internal resistance was made in the derivation. This assumption is reasonably accurate when

Equation 5.2-4.

$$N_{Bi} = \frac{hx_1}{k} < 0.1$$

where hx_1/k is called the Biot number N_{Bi}, which is dimensionless, and x_1 is a characteristic dimension of the body obtained from $x_1 = V/A$. The Biot number compares the relative values of internal conduction resistance and surface convective resistance to heat transfer.

For a sphere,

Equation 5.2-5.

$$x_1 = \frac{V}{A} = \frac{4\pi r^3/3}{4\pi r^2} = \frac{r}{3}$$

For a long cylinder,

Equation 5.2-6.

$$x_1 = \frac{V}{A} = \frac{\pi D^2 L/4}{\pi DL} = \frac{D}{4} = \frac{r}{2}$$

For a long square rod,

Equation 5.2-7.

$$x_1 = \frac{V}{A} = \frac{(2x)^2 L}{4(2x)L} = \frac{x}{2} \qquad (x = \tfrac{1}{2} \text{ thickness})$$

EXAMPLE 5.2-1. Cooling of a Steel Ball

A steel ball having a radius of 1.0 in. (25.4 mm) is at a uniform temperature of 800°F (699.9 K). It is suddenly plunged into a medium whose temperature is held constant at 250°F (394.3 K). Assuming a convective coefficient of h = 2.0 btu/h · ft^2 · °F (11.36 W/m^2 · K), calculate the temperature of the ball after 1 h (3600 s). The average physical properties are k = 25 btu/h · ft · °F (43.3 W/m · K), ρ = 490 lb$_m$/ft^3 (7849 kg/m^3), and c_p = 0.11 btu/lb$_m$ · °F (0.4606 kJ/kg · K). Use SI and English units.

Solution: For a sphere from Eq. (5.2-5),

$$x_1 = \frac{V}{A} = \frac{r}{3} = \frac{\frac{1}{12}}{3} = \tfrac{1}{36}\text{ ft}$$

$$= \frac{25.4}{1000 \times 3} = 8.47 \times 10^{-3}\text{ m}$$

From Eq. (5.2-4) for the Biot number,

$$N_{Bi} = \frac{hx_1}{k} = \frac{2(\frac{1}{36})}{25} = 0.00222$$

$$N_{Bi} = \frac{11.36(8.47 \times 10^{-3})}{43.3} = 0.00222$$

This value is <0.1; hence, the lumped capacity method can be used. Then,

$$\frac{hA}{c_p\rho V} = \frac{2}{0.11(490)(\frac{1}{36})} = 1.335\text{ h}^{-1}$$

$$\frac{hA}{c_p\rho V} = \frac{11.36}{(0.4606 \times 1000)(7849)(8.47 \times 10^{-3})} = 3.71 \times 10^{-4}\text{ s}^{-1}(1.335\text{ h}^{-1})$$

Substituting into Eq. (5.2-3) for t = 1.0 h and solving for T,

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{T - 250°\text{F}}{800 - 250} = e^{-(hA/c_p\rho V)t} = e^{-(1.335)(1.0)} \qquad T = 395°\text{F}$$

$$\frac{T - 394.3\text{ K}}{699.9 - 394.3} = e^{-(3.71\times10^{-4})(3600)} \qquad T = 474.9\text{ K}$$

Total Amount of Heat Transferred

The temperature of the solid at any time t can be calculated from Eq. (5.2-3). At any time t, the instantaneous rate of heat transfer $q(t)$ in W from the solid of negligible internal resistance can be calculated from

Equation 5.2-8.

$$q(t) = hA(T - T_\infty)$$

Substituting the instantaneous temperature T from Eq. (5.2-3) into Eq. (5.2-8),

Equation 5.2-9.

$$q(t) = hA(T_0 - T_\infty)e^{-(hA/c_p\rho V)t}$$

To determine the total amount of heat Q in W · s or J transferred from the solid from time $t = 0$ to $t = t$, we can integrate Eq. (5.2-9):

Equation 5.2-10.

$$Q = \int_{t=0}^{t=t} q(t)\, d(t) = \int_{t=0}^{t=t} hA(t_0 - T_\infty) e^{-(hA/c_p\rho V)t}\, dt$$

Equation 5.2-11.

$$Q = c_p \rho V (T_0 - T_\infty)[1 - e^{-(hA/c_p\rho V)t}]$$

EXAMPLE 5.2-2. Total Amount of Heat in Cooling

For the conditions in Example 5.2-1, calculate the total amount of heat removed up to time $t = 3600$ s.

Solution: From Example 5.2-1, $hA/c_p\rho V = 3.71 \times 10^{-4}\ s^{-1}$. Also, $V = 4\pi r^3/3 = 4(\pi)(0.0254)^3/3 = 6.864 \times 10^{-5}$ m^3. Substituting into Eq. (5.2-11),

$$\begin{aligned} Q &= (0.4606 \times 1000)(7849)(6.864 \times 10^{-5})(699.9 - 394.3) \\ &\quad \times [1 - e^{-(3.71\times10^{-4})(3600)}] \\ &= 5.589 \times 10^4 \text{ J} \end{aligned}$$

UNSTEADY-STATE HEAT CONDUCTION IN VARIOUS GEOMETRIES

Introduction and Analytical Methods

In Section 5.2 we considered a simplified case of negligible internal resistance where the object has a very high thermal conductivity. Now we will consider the more general situation where the internal resistance is not small, and hence the temperature is not constant in the solid. The first case that we shall consider is one where the surface convective resistance is negligible compared to the internal resistance. This could occur because of a very large heat-transfer coefficient at the surface or because of a relatively large conductive resistance in the object.

To illustrate an analytical method of solving this first case, we will derive the equation for unsteady-state conduction in the x direction only in a flat plate of thickness $2H$ as shown in Fig. 5.3-1. The initial profile of the temperature in the plate at $t = 0$ is uniform at $T = T_0$. At time $t = 0$, the ambient temperature is suddenly changed to T_1 and held there. Since there is no convection resistance, the temperature of the surface is also held constant at T_1. Since this is conduction in the x direction, Eq. (5.1-10) holds:

Equation 5.1-10.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

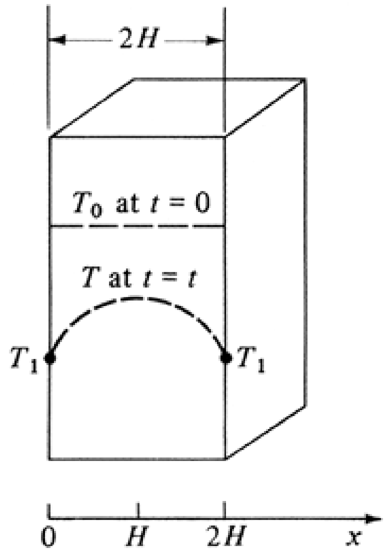


Figure 5.3-1. Unsteady-state conduction in a flat plate with negligible surface resistance.

The initial and boundary conditions are

Equation 5.3-1.

$$T = T_0, \qquad t = 0, \qquad x = x$$

$$T = T_1, \qquad t = t, \qquad x = 0$$

$$T = T_1, \qquad t = t, \qquad x = 2H$$

Generally, it is convenient to define a dimensionless temperature *Y* so that it varies between 0 and 1. Hence,

Equation 5.3-2.

$$Y = \frac{T_1 - T}{T_1 - T_0}$$

Substituting Eq. (5.3-2) into (5.1-10),

Equation 5.3-3.

$$\frac{\partial Y}{\partial t} = \alpha \frac{\partial^2 Y}{\partial x^2}$$

Redefining the boundary and initial conditions,

Equation 5.3-4.

$$Y = \frac{T_1 - T_0}{T_1 - T_0} = 1, \qquad t = 0, \qquad x = x$$

$$Y = \frac{T_1 - T_1}{T_1 - T_0} = 0, \qquad t = t, \qquad x = 0$$

$$Y = \frac{T_1 - T_1}{T_1 - T_0} = 0, \qquad t = t, \qquad x = 2H$$

A convenient procedure to use to solve Eq. (5.3-3) is the method of separation of variables, which leads to a product solution

Equation 5.3-5.

$$Y = e^{-a^2\alpha t}(A\cos ax + B\sin ax)$$

where A and B are constants and a is a parameter. Applying the boundary and initial conditions of Eq. (5.3-4) to solve for these constants in Eq. (5.3-5), the final solution is an infinite Fourier series (G1):

Equation 5.3-6.

$$\frac{T_1 - T_0}{T_1 - T_0} = \frac{4}{\pi}\left(\frac{1}{1}\exp\frac{-1^2\pi^2\alpha t}{4H^2}\sin\frac{1\pi x}{2H} + \frac{1}{3}\exp\frac{-3^2\pi^2\alpha t}{4H^2}\sin\frac{3\pi x}{2H} + \frac{1}{5}\exp\frac{-5^2\pi^2\alpha t}{4H^2}\sin\frac{5\pi x}{2H} + \cdots\right)$$

Hence from Eq. (5.3-6), the temperature T at any position x and time t can be determined. However, these types of equations are very time-consuming to use, and convenient charts have been prepared, which are discussed in Sections 5.3B, 5.3C, 5.3D, and 5.3E, where a surface resistance is present.

Unsteady-State Conduction in a Semi-infinite Solid

In Fig. 5.3-2 a semi-infinite solid is shown that extends to ∞ in the $+x$ direction. Heat conduction occurs only in the x direction. Originally, the temperature in the solid is uniform at T_0. At time $t = 0$, the solid is suddenly exposed to or immersed in a large mass of ambient fluid at temperature T_1, which is constant. The convection coefficient h in $W/m^2 \cdot K$ or $btu/h \cdot ft^2 \cdot °F$ is present and is constant; that is, a surface resistance is present. Hence, the temperature T_S at the surface is not the same as T_1.

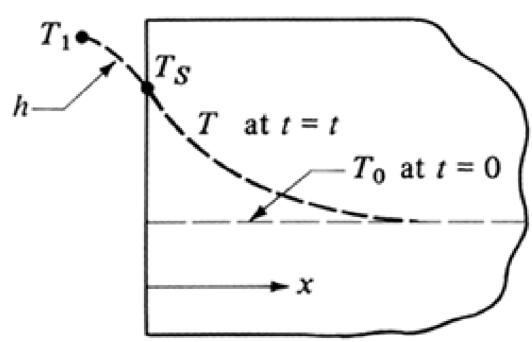


Figure 5.3-2. Unsteady-state conduction in a semi-infinite solid.

The solution of Eq. (5.1-10) for these conditions has been obtained (S1) and is

Equation 5.3-7.

$$\frac{T_1 - T_0}{T_1 - T_0} = 1 - Y = \operatorname{erfc}\frac{x}{2\sqrt{\alpha t}} - \exp\left[\frac{h\sqrt{\alpha t}}{k}\left(\frac{x}{\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h}{k}\sqrt{\alpha t}\right)$$

where x is the distance into the solid from the surface in m (SI units), t = time in s, $\alpha = k/\rho c_p$ in m^2/s. In English units, x = ft, t = h, and α = ft^2/h. The function erfc is (1 − erf), where erf is the error function and numerical values are tabulated in standard tables and texts (G1, P1, S1); Y is fraction of unaccomplished change $(T_1 - T)/(T_1 - T_0)$; and $1 - Y$ is fraction of change.

Figure 5.3-3, calculated using Eq. (5.3-7), is a convenient plot used for unsteady-state heat conduction into a semi-infinite solid with surface convection. If conduction into the solid is slow enough or h is very large, the top line with $h\sqrt{\alpha t}/k = \infty$ is used.

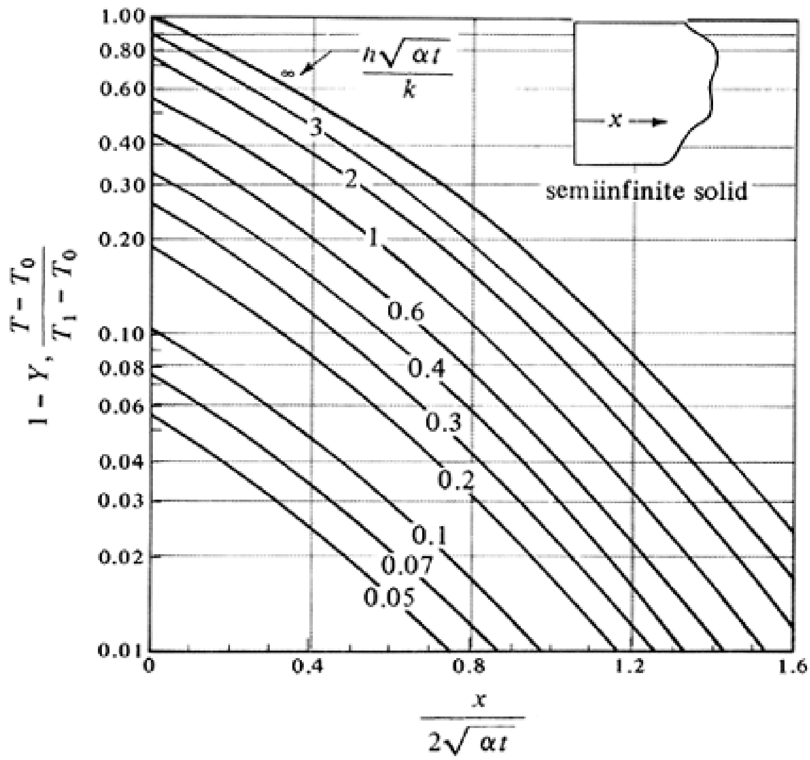


Figure 5.3-3. Unsteady-state heat conducted in a semi-infinite solid with surface convection. Calculated from Eq. (5.3-7)(S1).

EXAMPLE 5.3-1. Freezing Temperature in the Ground

The depth in the soil of the earth at which freezing temperatures penetrate is often of importance in agriculture and construction. On a certain fall day, the temperature in the earth is constant at 15.6°C (60°F) to a depth of several meters. A cold wave suddenly reduces the air temperature from 15.6 to −17.8°C (0°F). The convective coefficient above the soil is 11.36 W/m^2 · K (2 btu/h · ft^2 · °F). The soil properties can be assumed as $\alpha = 4.65 \times 10^{-7}$ m^2/s (0.018 ft^2/h) and k = 0.865 W/m · K (0.5 btu/h · ft · °F). Neglect any latent heat effects. Use SI and English units.

a. What is the surface temperature after 5 h?
b. To what depth in the soil will the freezing temperature of 0°C (32°F) penetrate in 5 h?

Solution: This is a case of unsteady-state conduction in a semi-infinite solid. For part (a), the value of x which is the distance from the surface is x = 0 m. Then the value of $x/2\sqrt{\alpha t}$ is calculated as follows for t = 5 h, $\alpha = 4.65 \times 10^{-7}$ m^2/s, k = 0.865 W/m · °C, and h = 11.36 W/m^2 · °C. Using SI and English units,

$$\frac{x}{2\sqrt{\alpha t}} = \frac{0}{2\sqrt{(4.65 \times 10^{-7})(5 \times 3600)}} \qquad \frac{x}{2\sqrt{\alpha t}} = \frac{0}{2\sqrt{0.018(5)}}$$

Also,

$$\frac{h\sqrt{\alpha t}}{k} = \frac{11.36\sqrt{(4.65 \times 10^{-7})(5 \times 3600)}}{0.865} = 1.2 \qquad \frac{h\sqrt{\alpha t}}{k} = \frac{2\sqrt{0.018(5)}}{0.5} = 1.2$$

Using Fig. 5.3-3, for $x/2\sqrt{\alpha t} = 0$ and $h\sqrt{\alpha t}/k = 1.2$, the value of $1 - Y = 0.63$ is read off the curve. Converting temperatures to K, $T_0 = 15.6°C + 273.2 = 288.8$ K (60°F) and $T_1 = -17.8°C + 273.2 = 255.4$ K (0°F). Then

$$1 - Y = \frac{T - T_0}{T_1 - T_0} = 0.63 = \frac{T - 288.8}{255.4 - 288.8}$$

Solving for T at the surface after 5 h,

$$T = 267.76 \text{ K} \quad \text{or} \quad -544°\text{C } (22.2°\text{F})$$

For part (b), $T = 273.2$ K or 0°C, and the distance x is unknown. Substituting the known values,

$$\frac{T - T_0}{T_1 - T_0} = \frac{273.2 - 288.8}{255.4 - 288.8} = 0.467$$

From Fig. 5.3-3 for $(T - T_0)/(T_1 - T_0) = 0.467$ and $h\sqrt{\alpha t}/k = 1.2$, a value of 0.16 is read off the curve for $x/2\sqrt{\alpha t}$. Hence,

$$\frac{x}{2\sqrt{\alpha t}} = \frac{x}{2\sqrt{(4.65 \times 10^{-7})(5 \times 3600)}} = 0.16 \qquad \frac{x}{2\sqrt{\alpha t}} = \frac{x}{2\sqrt{0.018(5)}} = 0.16$$

Solving for x, the distance the freezing temperature penetrates in 5 h,

$$x = 0.0293 \text{ m } (0.096 \text{ ft})$$

Unsteady-State Conduction in a Large Flat Plate

A geometry that often occurs in heat-conduction problems is a flat plate of thickness $2x_1$ in the x direction and having large or infinite dimensions in the y and z directions, as shown in Fig. 5.3-4. Heat is being conducted only from the two flat and parallel surfaces in the x direction. The original uniform temperature of the plate is T_0; at time $t = 0$, the solid is exposed to an environment at temperature T_1 and unsteady-state conduction occurs. A surface resistance is present.

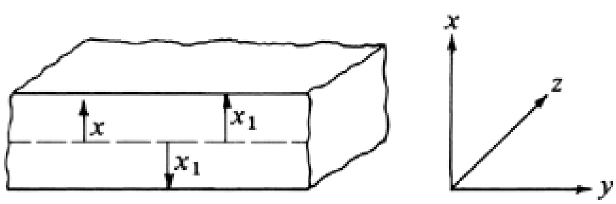


Figure 5.3-4. Unsteady-state conduction in a large flat plate.

The numerical results of this case are presented graphically in Figs. 5.3-5 and 5.3-6. Figure 5.3-5, from Gurney and Lurie (G2), is a convenient chart for determining the temperatures at any position in the plate and at any time t. The dimensionless parameters used in this and subsequent unsteady-state charts in this section are given in Table 5.3-1 (x is the distance from the center of the flat plate, cylinder, or sphere; x_1 is one-half the thickness of the flat plate, radius of cylinder, or radius of sphere; x = distance from the surface for a semi-infinite solid.)

Table 5.3-1. Dimensionless Parameters for Use in Unsteady-State Conduction Charts

$$Y = \frac{T_1 - T}{T_1 - T_0} \qquad m = \frac{k}{hx_1}$$

$$1 - Y = \frac{T - T_0}{T_1 - T_0} \qquad n = \frac{x}{x_1}$$

$$X = \frac{\alpha t}{x_1^2}$$

SI units: $\alpha = m^2/s$, $T = K$, $t = s$, $x = m$, $x_1 = m$, $k = W/m \cdot K$, $h = W/m^2 \cdot K$

English units: $\alpha = ft^2/h$, $T = °F$, $t = h$, $x = ft$, $x_1 = ft$, $k = btu/h \cdot ft \cdot °F$, $h = btu/h \cdot ft^2 \cdot °F$

Cgs units: $\alpha = cm^2/s$, $T = °C$, $t = s$, $x = cm$, $x_1 = cm$, $k = cal/s \cdot cm \cdot °C$, $h = cal/s \cdot cm^2 \cdot °C$

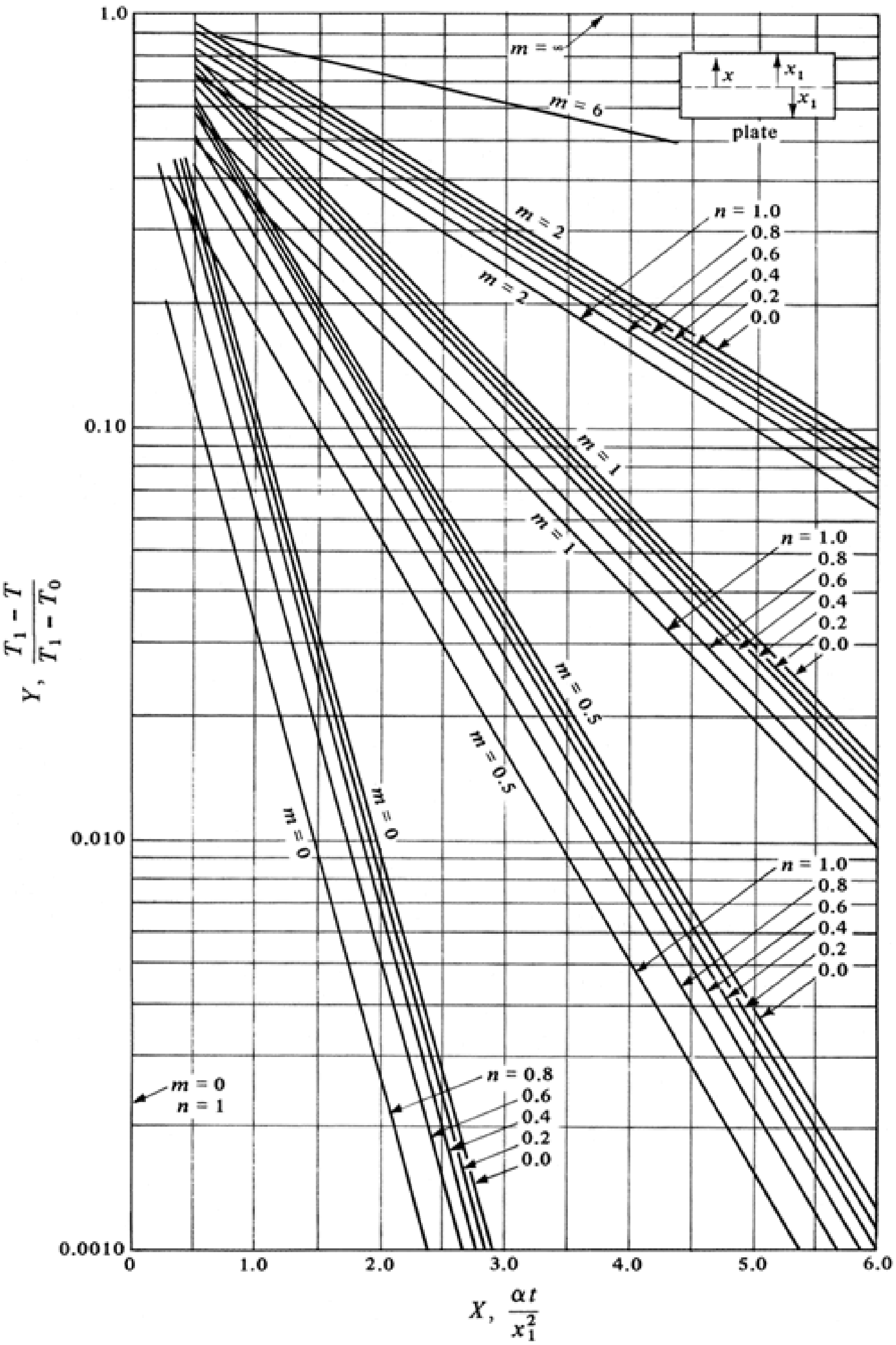


Figure 5.3-5. Unsteady-state heat conduction in a large flat plate. [From H. P. Gurney and J. Lurie, Ind. Eng. Chem., ***15,*** *1170 (1923).]*

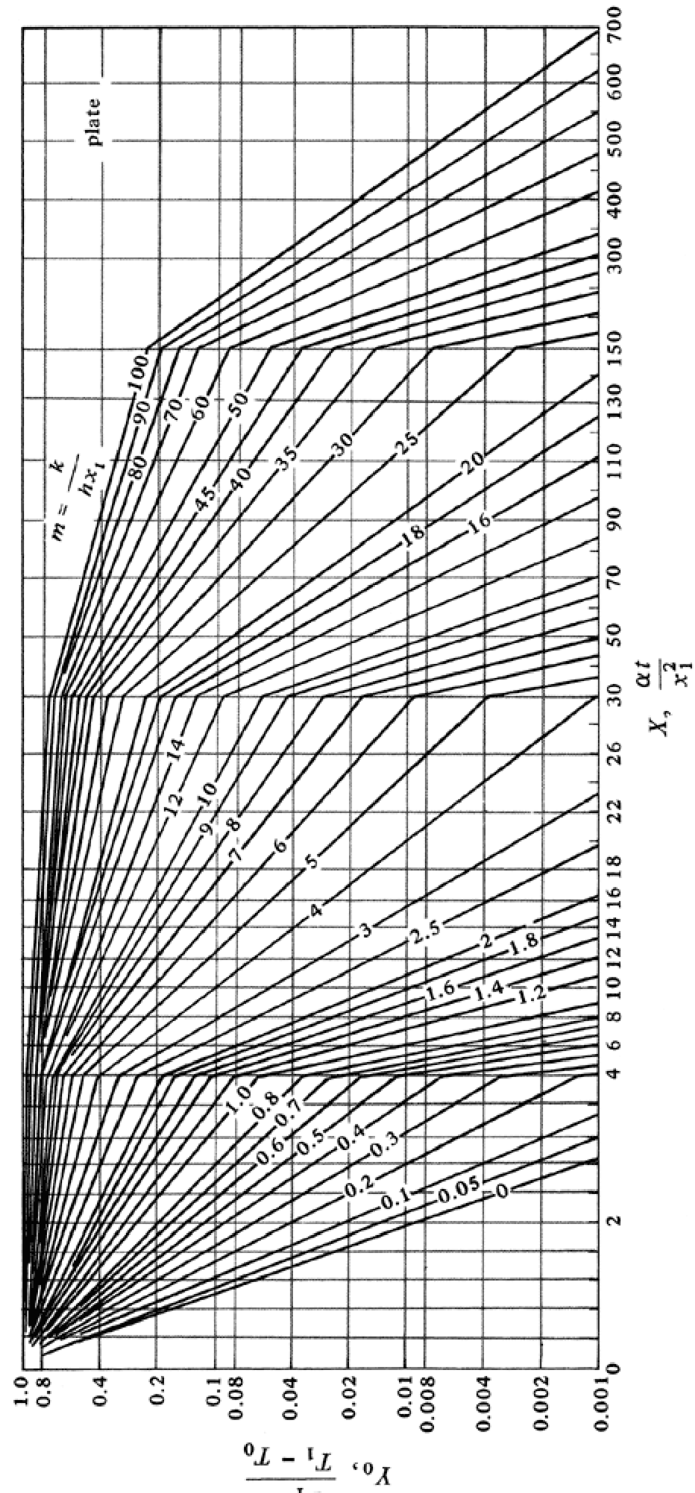


*Figure 5.3-6. Chart for determining temperature at the center of a large flat plate for unsteady-state heat conduction. [From H. P. Heisler, Trans. A.S.M.E., **69,** 227 (1947). With permission.]*

When $n = 0$, the position is at the center of the plate in Fig. 5.3-5. Often the temperature history at the center of the plate is quite important. A more accurate chart for determining only the center temperature is given in Fig. 5.3-6, the Heisler (H1) chart. Heisler (H1) has also prepared multiple charts for determining the temperatures at other positions.

EXAMPLE 5.3-2. Heat Conduction in a Slab of Butter

A rectangular slab of butter which is 46.2 mm thick at a temperature of 277.6 K (4.4°C) in a cooler is removed and placed in an environment at 297.1 K (23.9°C). The sides and bottom of the butter container can be considered to be insulated by the container side walls. The flat top surface of the butter is exposed to the environment. The convective coefficient is constant at 8.52 W/m^2 · K. Calculate the temperature in the butter at the surface, at 25.4 mm below the surface, and at 46.2 mm below the surface at the insulated bottom after 5 h of exposure.

Solution: The butter can be considered as a large, flat plate with conduction vertically in the x direction. Since heat is entering only at the top face and the bottom face is insulated, the 46.2 mm of butter is equivalent to a half plate with thickness x_1 = 46.2 mm. In a plate with two exposed surfaces, as in Fig. 5.3-4, the center at $x = 0$ acts as an insulated surface, and both halves are mirror images of each other.

The physical properties of butter from Appendix A.4 are k = 0.197 W/m · K, c_p = 2.30 kJ/kg · K = 2300 J/kg · K, and ρ = 998 kg/m^3. The thermal diffusivity is

$$\alpha = \frac{k}{\rho c_p} = \frac{0.197}{998(2300)} = 8.58 \times 10^{-8} \text{ m}^2/\text{s}$$

Also, x_1 = 46.2/1000 = 0.0462 m.

The parameters needed for use in Fig. 5.3-5 are

$$m = \frac{k}{hx_1} = \frac{0.197}{8.52(0.0462)} = 0.50$$

$$X = \frac{\alpha t}{x_1^2} = \frac{(8.58 \times 10^{-8})(5 \times 3600)}{(0.0462)^2} = 0.72$$

For the top surface, where $x = x_1$ = 0.0462 m,

$$n = \frac{x}{x_1} = \frac{0.0462}{0.0462} = 1.0$$

Then, using Fig. 5.3-5,

$$Y = 0.25 = \frac{T_1 - T}{T_1 - T_0} = \frac{297.1 - T}{297.1 - 277.6}$$

Solving, T = 292.2 K (19.0°C).

At the point 25.4 mm from the top surface, or 20.8 mm from the center, x = 0.0208 m, and

$$n = \frac{x}{x_1} = \frac{0.0208}{0.0462} = 0.45$$

From Fig. 5.3-5,

$$Y = 0.45 = \frac{T_1 - T}{T_1 - T_0} = \frac{297.1 - T}{297.1 - 277.6}$$

Solving, T = 288.3 K (15.1°C).

For the bottom point, or 0.0462 m from the top, $x = 0$ and

$$n = \frac{x}{x_1} = \frac{0}{x_1} = 0$$

Then, from Fig. 5.3-5,

$$Y = 0.50 = \frac{T_1 - T}{T_1 - T_0} = \frac{297.1 - T}{297.1 - 277.6}$$

Solving, T = 287.4 K (14.2°C). Alternatively, using Fig. 5.3-6, which is only for the center point, Y = 0.53 and T = 286.8 K (13.6°C).

Unsteady-State Conduction in a Long Cylinder

Here we consider unsteady-state conduction in a long cylinder where conduction occurs only in the radial direction. The cylinder is long so that either conduction at the ends can be neglected or the ends are insulated. Charts for this case are presented in Fig. 5.3-7 for determining the temperatures at any position and Fig. 5.3-8 for the center temperature only.

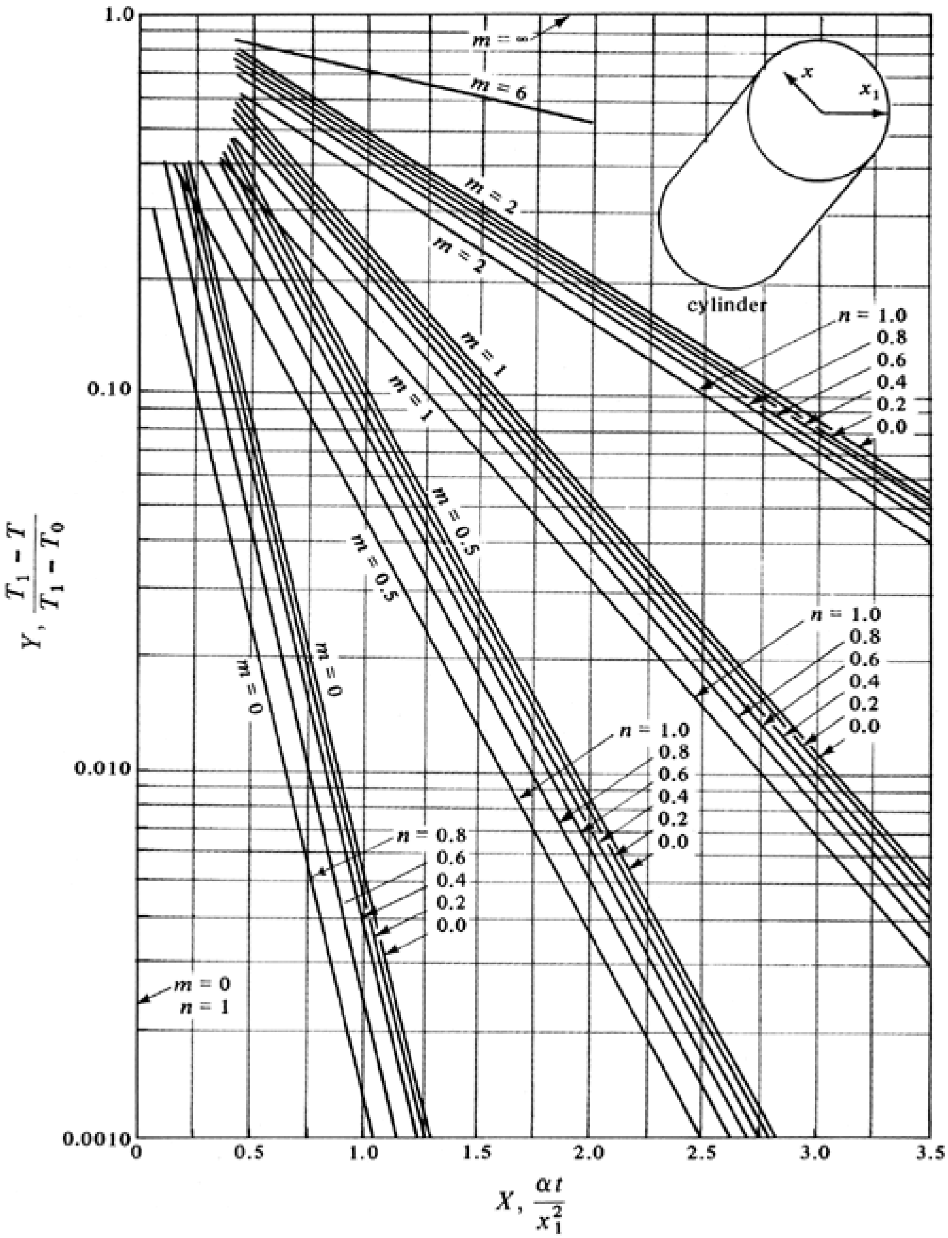


*Figure 5.3-7. Unsteady-state heat conduction in a long cylinder. [From H. P. Gurney and J. Lurie, Ind. Eng. Chem., **15,** 1170 (1923).]*

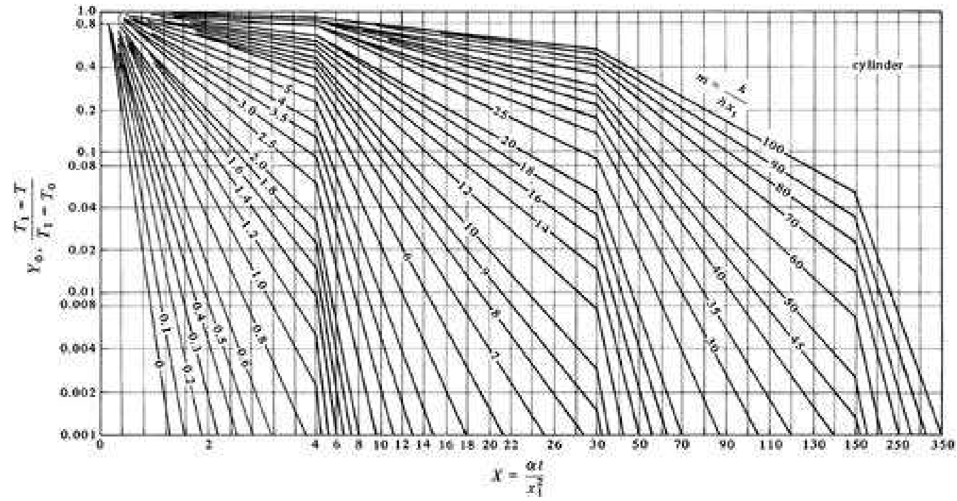


Figure 5.3-8. Chart for determining temperature at the center of a long cylinder for unsteady-state heat conduction. [From H. P. Heisler, Trans. A.S.M.E., ***69,*** *227 (1947). With permission.]*

EXAMPLE 5.3-3. Transient Heat Conduction in a Can of Pea Purée

A cylindrical can of pea purée (C2) has a diameter of 68.1 mm and a height of 101.6 mm and is initially at a uniform temperature of 29.4°C. The cans are stacked vertically in a retort, and steam at 115.6°C is admitted. For a heating time of 0.75 h at 115.6°C, calculate the temperature at the center of the can. Assume that the can is in the center of a vertical stack of cans and that it is insulated on its two ends by the other cans. The heat capacity of the metal wall of the can will be neglected. The heat-transfer coefficient of the steam is estimated as 4540 $W/m^2 \cdot K$. Physical properties of purée are k = 0.830 W/m · K and α = 2.007 × 10^{-7} m^2/s.

Solution: Since the can is insulated at the two ends, we can consider it as a long cylinder. The radius is x_1 = 0.0681/2 = 0.03405 m. For the center with x = 0,

$$n = \frac{x}{x_1} = \frac{0}{x_1} = 0$$

Also,

$$m = \frac{k}{hx_1} = \frac{0.830}{4540(0.03405)} = 0.00537$$

$$X = \frac{\alpha t}{x_1^2} = \frac{(2.007 \times 10^{-7})(0.75 \times 3600)}{(0.03405)^2} = 0.468$$

Using Fig. 5.3-8 from Heisler for the center temperature,

$$Y = 0.13 = \frac{T_1 - T}{T_1 - T_0} = \frac{115.6 - T}{115.6 - 29.4}$$

Solving, T = 104.4°C.

Unsteady-State Conduction in a Sphere

Figure 5.3-9 shows a chart by Gurney and Lurie for determining the temperatures at any position in a sphere. Figure 5.3-10 is a chart by Heisler for determining only the center temperature in a sphere.

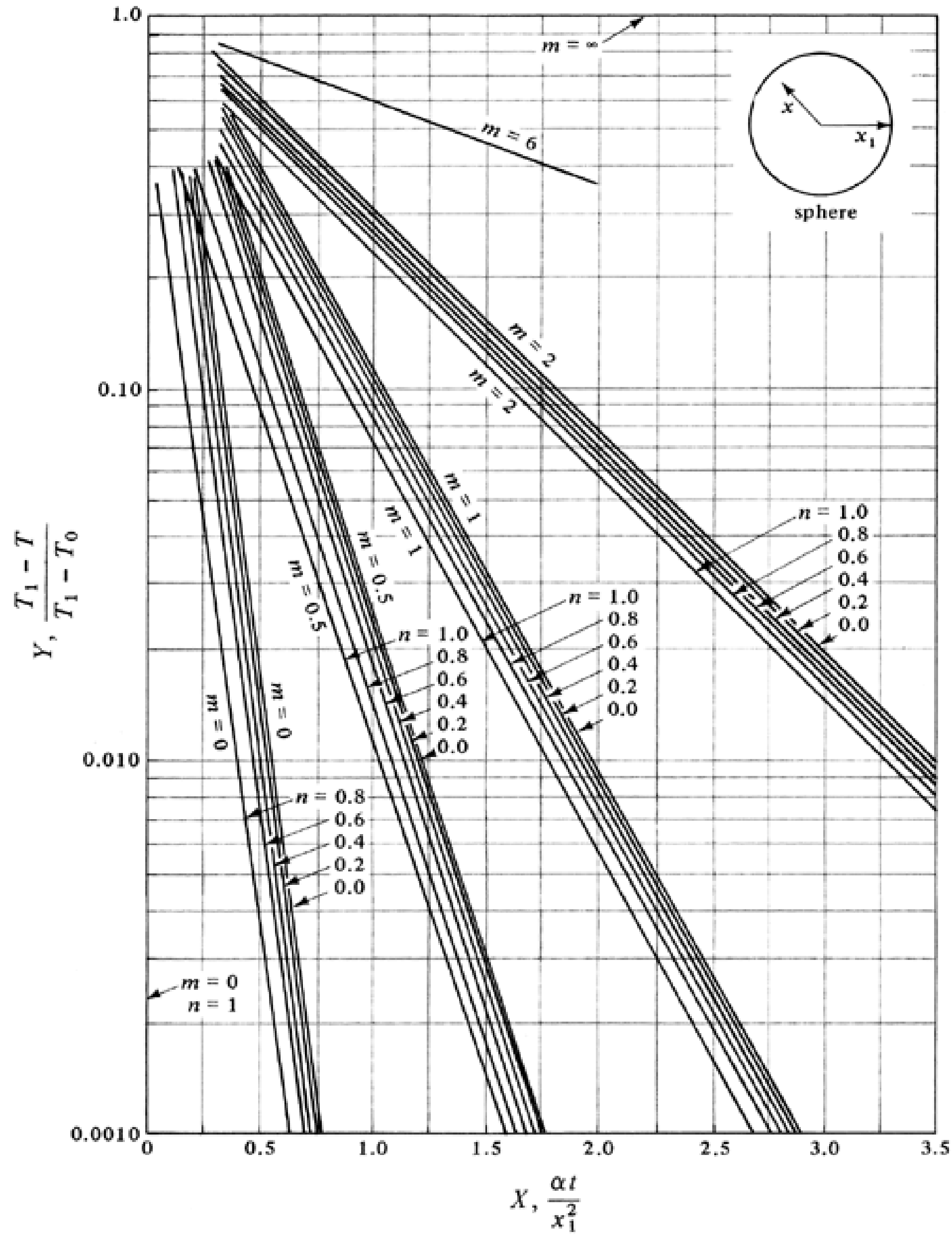


*Figure 5.3-9. Unsteady-state heat conduction in a sphere. [From H. P. Gurney and J. Lurie, Ind. Eng. Chem., **15,** 1170 (1923).]*

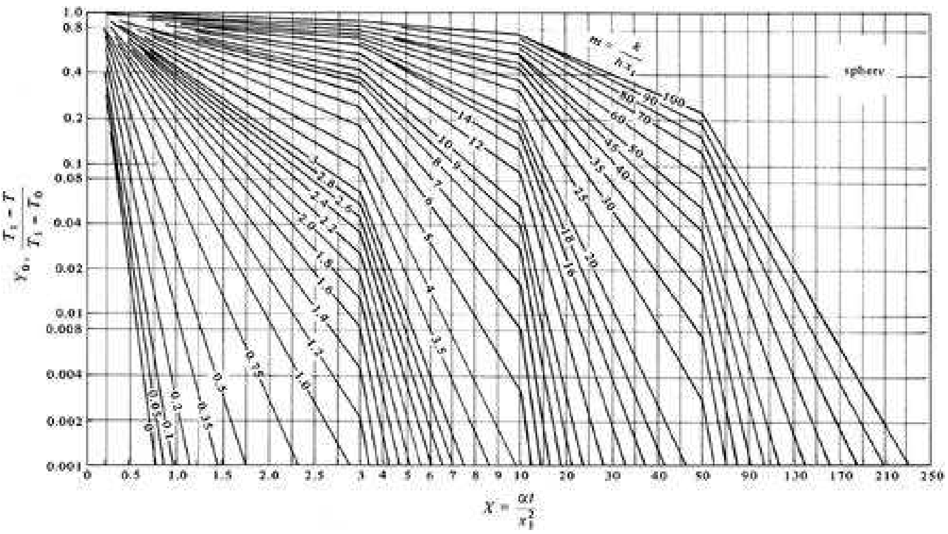


Figure 5.3-10. Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction [From H. P. Heisler, Trans. A.S.M.E., ***69,*** *227 (1947). With Permission.]*

Unsteady-State Conduction in Two- and Three-Dimensional Systems

The heat-conduction problems considered so far have been limited to one dimension. However, many practical problems involve simultaneous unsteady-state conduction in two and three directions. We shall illustrate how to combine one-dimensional solutions to yield solutions for several-dimensional systems.

Newman (N1) used the principle of superposition and showed mathematically how to combine the solutions for one-dimensional heat conduction in the x, the y, and the z direction into an overall solution for simultaneous conduction in all three directions. For example, a rectangular block with dimensions $2x_1$, $2y_1$, and $2z_1$ is shown in Fig. 5.3-11. For the Y value in the x direction, as before,

Equation 5.3-8.

$$Y_x = \frac{T_1 - T_x}{T_1 - T_0}$$

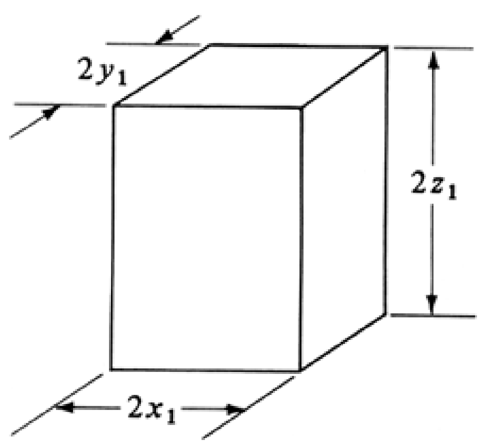


Figure 5.3-11. Unsteady-state conduction in three directions in a rectangular block.

where T_x is the temperature at time t and position x distance from the center line, as before. Also, $n = x/x_1$, $m = k/hx_1$, and $X_x = \alpha t/x_1^2$, as before. Then for the y direction,

Equation 5.3-9.

$$Y_y = \frac{T_1 - T_y}{T_1 - T_0}$$

and $n = y/y_1$, $m = k/hy_1$, and $X_y = \alpha t/y_1^2$. Similarly, for the z direction,

Equation 5.3-10.

$$Y_z = \frac{T_1 - T_z}{T_1 - T_0}$$

Then, for simultaneous transfer in all three directions,

Equation 5.3-11.

$$Y_{x,y,z} = (Y_x)(Y_y)(Y_z) = \frac{T_1 - T_{x,y,z}}{T_1 - T_0}$$

where $T_{x,y,z}$ is the temperature at the point *x, y, z* from the center of the rectangular block. The value of Y_x for the two parallel faces is obtained from Figs. 5.3-5 and 5.3-6 for conduction in a flat plate. The values of Y_y and Y_z are similarly obtained from the same charts.

For a short cylinder with radius x_1 and length $2y_1$, the following procedure is followed. First Y_x for the radical conduction is obtained from the figures for a long cylinder. Then Y_y for conduction between two parallel planes is obtained from Fig. 5.3-5 or 5.3-6 for conduction in a flat plate. Then,

Equation 5.3-12.

$$Y_{x,y} = (Y_x)(Y_y) = \frac{T_1 - T_{x,y}}{T_1 - T_0}$$

EXAMPLE 5.3-4. Two-Dimensional Conduction in a Short Cylinder

Repeat Example 5.3-3 for transient conduction in a can of pea purée but assume that conduction also occurs from the two flat ends.

Solution: The can, which has a diameter of 68.1 mm and a height of 101.6 mm, is shown in Fig. 5.3-12. The given values from Example 5.3-3 are x_1 = 0.03405 m, y_1 = 0.1016/2 = 0.0508 m, k = 0.830 W/m · K, α = 2.007 × 10^{-7} m^2/s, h = 4540 W/m^2 · K, and t = 0.75(3600) = 2700 s.

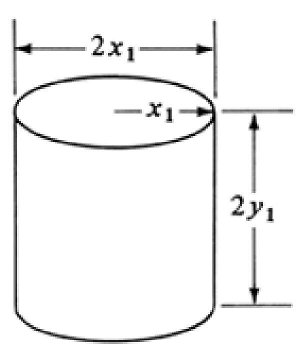


Figure 5.3-12. Two dimensional conduction in a short cylinder in Example 5.3-4.

For conduction in the x (radial) direction as calculated previously,

$$n = \frac{x}{x_1} = \frac{0}{x_1} = 0, \qquad m = \frac{k}{hx_1} = \frac{0.830}{4540(0.03405)} = 0.00537$$

$$X = \frac{\alpha t}{x_1^2} = \frac{(2.007 \times 10^{-7})2700}{(0.03405)^2} = 0.468$$

From Fig. 5.3-8 for the center temperature,

$$Y_x = 0.13$$

For conduction in the *y* (axial) direction for the center temperature,

$$n = \frac{y}{y_1} = \frac{0}{0.0508} = 0$$

$$m = \frac{k}{hy_1} = \frac{0.830}{4540(0.0508)} = 0.00360$$

$$X = \frac{\alpha t}{y_1^2} = \frac{(2.007 \times 10^{-7})2700}{(0.0508)^2} = 0.210$$

Using Fig. 5.3-6 for the center of a large plate (two parallel opposed planes),

$$Y_y = 0.80$$

Substituting into Eq. (5.3-12),

$$Y_{x,y} = (Y_x)(Y_y) = 0.13(0.80) = 0.104$$

Then,

$$\frac{T_1 - T_{x,y}}{T_1 - T_0} = \frac{115.6 - T_{x,y}}{115.6 - 29.4} = 0.104$$

$$T_{x,y} = 106.6°\text{C}$$

This compares with 104.4°C obtained in Example 5.3-3 for only radial conduction.

Charts for Average Temperature in a Plate, Cylinder, and Sphere with Negligible Surface Resistance

If the surface resistance is negligible, the curves given in Fig. 5.3-13 will give the total fraction of unaccomplished change, *E*, for slabs, cylinders, or spheres for unsteady-state conduction. The value of *E* is

Equation 5.3-13.

$$E = \frac{T_1 - T_{av}}{T_1 - T_0}$$

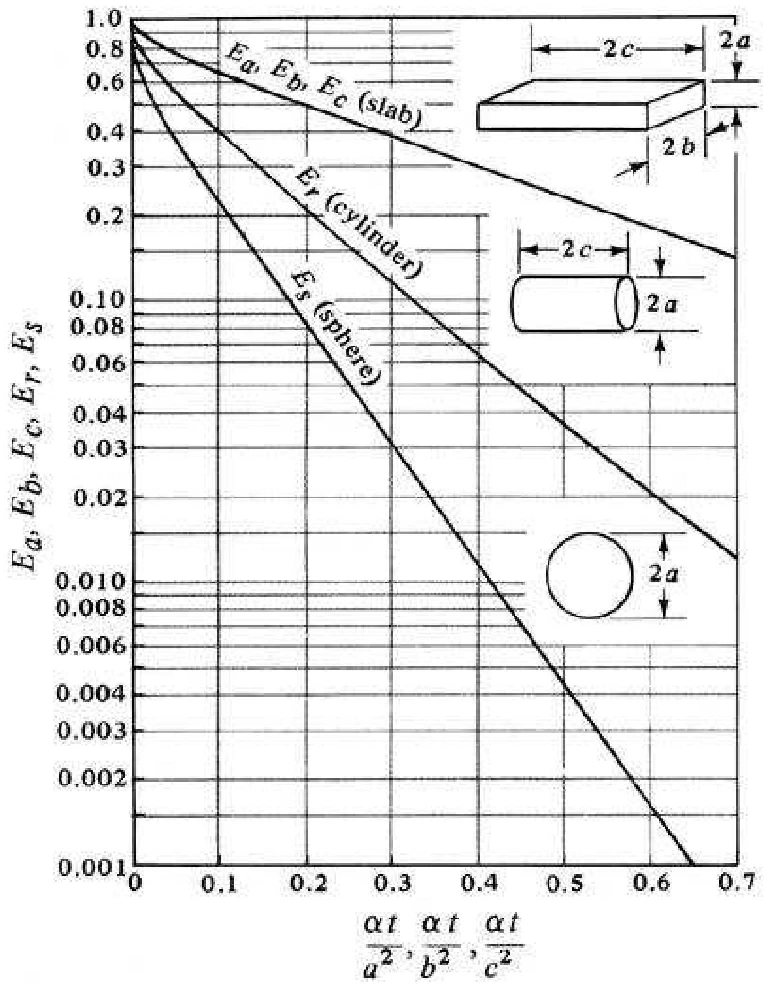


Figure 5.3-13. Unsteady-state conduction and average temperatures for negligible surface resistance. (From R. E. Treybal, Mass Transfer Operations, 2nd ed. New York: McGraw-Hill Book Company, 1968. With permission.)

where T_0 is the original uniform temperature, T_1 is the temperature of the environment to which the solid is suddenly subjected, and T_{av} is the average temperature of the solid after t hours.

The values of E_a, E_b, and E_c are each used for conduction between a pair of parallel faces, as in a plate. For example, for conduction in the a and b directions in a rectangular bar,

Equation 5.3-14.

$$E = E_a E_b$$

For conduction from all three sets of faces,

Equation 5.3-15.

$$E = E_a E_b E_c$$

For conduction in a short cylinder $2c$ long and radius a,

Equation 5.3-16.

$$E = E_c E_r$$

NUMERICAL FINITE-DIFFERENCE METHODS FOR UNSTEADY-STATE CONDUCTION

Unsteady-State Conduction in a Slab

Introduction

As discussed in previous sections of this chapter, the partial differential equations for unsteady-state conduction in various simple geometries can be solved analytically if the boundary conditions are constant at $T = T_1$ with time. Also, in the solutions the initial profile of the temperature at $t = 0$ is uniform at $T = T_0$. The unsteady-state charts used also have these same boundary conditions and initial condition. However, when the boundary conditions are not constant with time and/or the initial conditions are not constant with position, numerical methods must be used.

Numerical calculation methods for unsteady-state heat conduction are similar to numerical methods for steady state discussed in Section 4.15. The solid is subdivided into sections or slabs of equal length and a fictitious node is placed at the center of each section. Then a heat balance is made for each node. This method differs from the steady-state method in that we have heat accumulation in a node for unsteady-state conduction. The methods are well suited for a spreadsheet calculation.

Equations for a slab

The unsteady-state equation for conduction in the x direction in a slab is

Equation 5.1-10.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

This can be set up for a numerical solution by expressing each partial derivative as an actual finite difference in ΔT, Δt, and Δx. However, an alternative method will be used to derive the final result by making a heat balance. Figure 5.4-1 shows a slab centered at position n, represented by the shaded area. The slab has a width of Δx m and a cross-sectional area of A m^2. The node at position n having a temperature of T_n is placed at the center of the shaded section; this node represents the total mass and heat capacity of the section or slab. Each node is imagined to be connected to the adjacent node by a fictitious, small conducting rod. (See Fig. 4.15-3 for an example.)

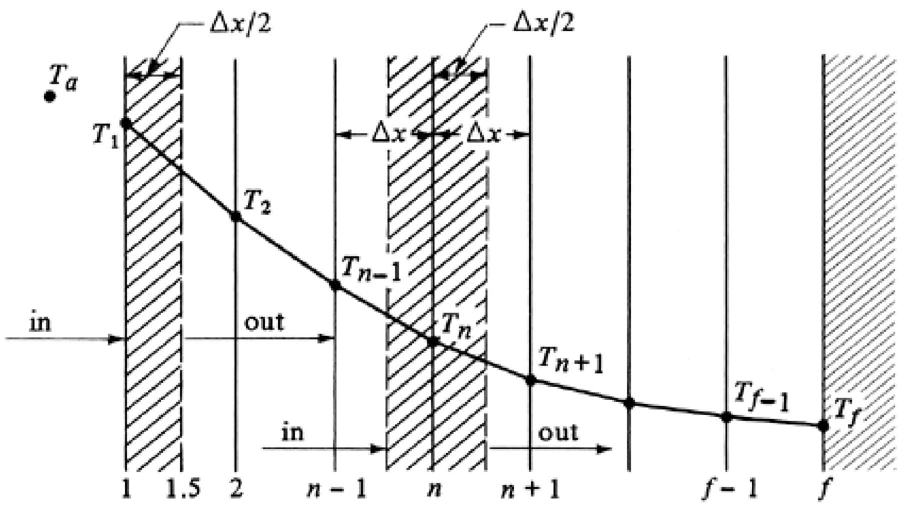


Figure 5.4-1. Unsteady-state conduction in a slab.

The figure shows the temperature profile at a given instant of time t s. Making a heat balance on this node or slab, the rate of heat in – the rate of heat out = the rate of heat accumulation in Δt s:

Equation 5.4-1.

$$\frac{kA}{\Delta x}({}_tT_{n-1} - {}_tT_n) - \frac{kA}{\Delta x}({}_tT_n - {}_tT_{n+1}) = \frac{(A\,\Delta x)\rho c_p}{\Delta t}({}_{t+\Delta t}T_n - {}_tT_n)$$

where ${}_tT_n$ is the temperature at point n at time t and ${}_{t+\Delta t}T_n$ is the temperature at point n at time t + 1 Δt later. Rearranging and solving for ${}_{t+\Delta t}T_n$,

Equation 5.4-2.

$${}_{t+\Delta t}T_n = \frac{1}{M}[{}_tT_{n+1} + (M-2){}_tT_n + {}_tT_{n-1}]$$

where

Equation 5.4-3.

$$M = \frac{(\Delta x)^2}{\alpha\,\Delta t}$$

Note that in Eq. (5.4-2) the temperature ${}_{t+\Delta t}T_n$ at position or node n and at a new time $t + \Delta t$ is calculated from the three points which are known at time t, the starting time. This is called the *explicit method,* because the temperature at a new time can be calculated explicitly from the temperatures at the previous time. In this method the calculation proceeds directly from one time increment to the next until the final temperature distribution is calculated at the desired final time. Of course, the temperature distribution at the initial time and the boundary conditions must be known.

Once the value of Δx has been selected, then from Eq. (5.4-3) a value of M or the time increment Δt may be picked. For a given value of M, smaller values of Δx mean smaller values of Δt. The value of M must be as follows:

Equation 5.4-4.

$$M \geq 2$$

If M is less than 2, the second law of thermodynamics is violated. It also can be shown that for stability and convergence of the finite-difference solution, M must be ≥ 2.

Stability means the errors in the solution do not grow exponentially as the solution proceeds but damp out. Convergence means that the solution of the difference equation approaches the exact solution of the partial differential equation as Δt and Δx go to zero with M fixed. Using smaller sizes of Δt and Δx increases the accuracy in general but greatly increases the number of calculations required. Hence, a digital computer is often ideally suited for this type of calculation using a spreadsheet.

Simplified Schmidt method for a slab

If the value of $M = 2$, then a great simplification of Eq. (5.4-2) occurs, giving the Schmidt method:

Equation 5.4-5.

$${}_{t+\Delta t}T_n = \frac{{}_tT_{n-1} + {}_tT_{n+1}}{2}$$

This means that when a time 1 Δt has elapsed, the new temperature at a given point n at $t + \Delta t$ is the arithmetic average of the temperatures at the two adjacent nodes $n + 1$ and $n - 1$ at the original time t.

Boundary Conditions for Numerical Method for a Slab

Convection at the boundary

For the case where there is a finite convective resistance at the boundary and the temperature of the environment or fluid outside is suddenly changed to T_a, we can derive the following for a slab. Referring to Fig. 5.4-1, we make a heat balance on the outside half-element. The rate of heat in by convection − the rate of heat out by conduction = the rate of heat accumulations in Δt s:

Equation 5.4-6.

$$hA({}_tT_a - {}_tT_1) - \frac{kA}{\Delta x}({}_tT_1 - {}_tT_2) = \frac{(A\,\Delta x/2)\rho c_p}{\Delta t}({}_{t+\Delta t}T_{1.25} - {}_tT_{1.25})$$

where ${}_tT_{1.25}$ is the temperature at the midpoint of the 0.5 Δx outside slab. As an approximation, the temperature T_1 at the surface can be used to replace that of $T_{1.25}$. Rearranging,

Equation 5.4-7.

$${}_{t+\Delta t}T_1 = \frac{1}{M}[2N{}_tT_a + [M - (2N + 2)]{}_tT_1 + 2{}_tT_2]$$

where

Equation 5.4-8.

$$N = \frac{h\,\Delta x}{k}$$

Note that the value of M must be such that

Equation 5.4-9.

$$M \geq 2N + 2$$

Insulated boundary condition

In the case for the boundary condition where the rear face is insulated, a heat balance is made on the rear $\frac{1}{2}\Delta x$ slab just as on the front $\frac{1}{2}\Delta x$ slab in Fig. 5.4-1. The resulting equation is the same as Eqs. (5.4-6) and (5.4-7), but $h = 0$ or $N = 0$ and ${}_tT_{f-1} = {}_tT_{f+1}$ because of symmetry.

Equation 5.4-10.

$${}_{t+\Delta t}T_f = \frac{1}{M}[(M - 2){}_tT_f + 2{}_tT_{f-1}]$$

Alternative convective condition

To use the equations above for a given problem, the same values of M, Δx, and Δt must be used. If N gets too large, so that M may be inconveniently too large, another form of Eq. (5.4-7) can be derived. By neglecting the heat accumulation in the front half-slab in Eq. (5.4-6),

Equation 5.4-11.

$$_{t+\Delta t}T_1 = \frac{N}{N+1}\,{}_{t+\Delta t}T_a + \frac{1}{N+1}\,{}_{t+\Delta t}T_2$$

Here the value of *M* is not restricted by the *N* value. This approximation works fairly well when a large number of increments in Δ*x* are used so that the amount of heat neglected is small compared to the total.

Procedures for use of initial boundary temperature

When the temperature of the environment outside is suddenly changed to T_a, the following procedures should be used.

1. When *M* = 2 and a hand calculation of a limited number of increments is used, a special procedure should be used in Eqs. (5.4-5) and (5.4-7) or (5.4-11). For the first time increment, one should use an average value for ${}_1T_a$ of $(T_a + {}_0T_1)/2$, where ${}_0T_1$ is the initial temperature at point 1. For all succeeding Δ*t* values, the value of T_a should be used (D1, K1). This special procedure for determining the value of T_a to use for the first time increment increases the accuracy of the numerical method, especially after a few time intervals. If T_a varies with time *t*, a new value can be used for each Δ*t* interval.
2. When *M* = 2 and many time increments are used with a digital computer, this special procedure is not needed, and the same value of T_a is used for all time increments.
3. When *M* = 3 or more and a hand calculation of a limited number of increments or a digital-computer calculation of many increments is used, only one value of T_a is used for all time increments. Note that when *M* = 3 or more, many more calculations are needed compared to the case for *M* = 2. The most accurate results are obtained when *M* = 4, which is the preferred method, with slightly less accurate results for *M* = 3 (D1, K1, K2).

EXAMPLE 5.4-1. Unsteady-State Conduction and the Schmidt Numerical Method

A slab of material 1.00 m thick is at a uniform temperature of 100°C. The front surface is suddenly exposed to a constant environmental temperature of 0°C. The convective resistance is zero ($h = \infty$). The back surface of the slab is insulated. The thermal diffusivity $\alpha = 2.00 \times 10^{-5}$ m^2/s. Using five slices each 0.20 m thick and the Schmidt numerical method with *M* = 2.0, calculate the temperature profile at *t* = 6000 s. Use the special procedure for the first time increment.

Solution: Figure 5.4-2 shows the temperature profile at *t* = 0 and the environmental temperature of $T_\alpha = 0$°C with five slices used. For the Schmidt method, *M* = 2. Substituting into Eq. (5.4-3) with $\alpha = 2.00 \times 10^{-5}$ and Δ*x* = 0.20 and solving for Δ*t*,

Equation 5.4-3.

$$M = 2 = \frac{(\Delta x)^2}{\alpha\,\Delta t} = \frac{(0.20)^2}{(2.00\times 10^{-5})\Delta t} \qquad \Delta t = 1000\text{ s}$$

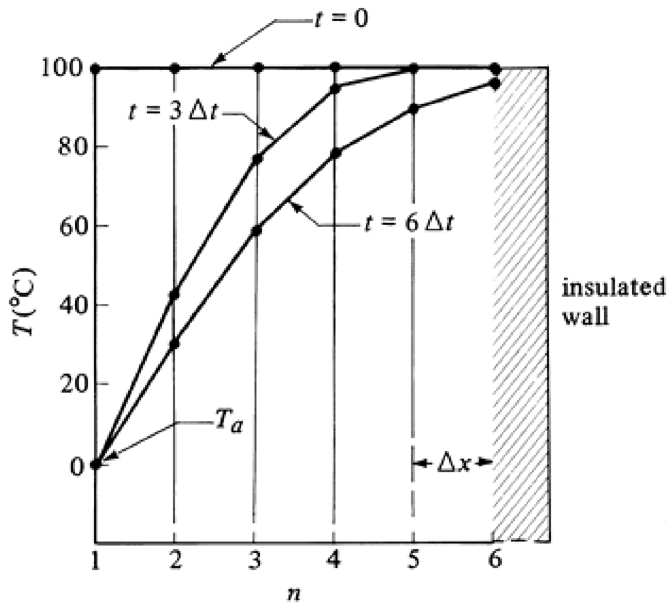


Figure 5.4-2. Temperature for numerical method, Example 5.4-1.

This means that (6000 s)/(1000 s/increment), or six time increments, must be used to reach 6000 s.

For the front surface, where $n = 1$, the temperature ${}_1T_a$ to use for the first Δt time increment, as stated previously, is

Equation 5.4-12.

$${}_1T_a = \frac{T_a + {}_0T_1}{2} = {}_1T_1 \qquad n = 1$$

where ${}_0T_1$ is the initial temperature at point 1. For the remaining time increments,

Equation 5.4-13.

$$T_1 = T_a \qquad n = 1$$

To calculate the temperatures for all time increments for the slabs $n = 2$ to 5, using Eq. (5.4-5),

Equation 5.4-14.

$${}_{t+\Delta t}T_n = \frac{{}_tT_{n-1} + {}_tT_{n+1}}{2} \qquad n = 2, 3, 4, 5$$

For the insulated end for all time increments at $n = 6$, substituting $M = 2$ and $f = 6$ into Eq. (5.4-10),

Equation 5.4-15.

$${}_{t+\Delta t}T_6 = \frac{(2-2){}_tT_6 + 2{}_tT_5}{2} = {}_tT_5$$

For the first time increment of $t + \Delta t$, and calculating the temperature at $n = 1$ by Eq. (5.4-12),

$${}_{t+\Delta t}T_1 = \frac{T_a + {}_0T_1}{2} = \frac{0 + 100}{2} = 50°\text{C} = {}_1T_a$$

For $n = 2$, using Eq. (5.4-14),

$$_{t+\Delta t}T_2 = \frac{_tT_1 + {_tT_3}}{2} = \frac{50 + 100}{2} = 75$$

Continuing for *n* = 3, 4, 5, we have

$$_{t+\Delta t}T_3 = \frac{_tT_2 + {_tT_4}}{2} = \frac{100 + 100}{2} = 100$$

$$_{t+\Delta t}T_4 = \frac{_tT_3 + {_tT_5}}{2} = \frac{100 + 100}{2} = 100$$

$$_{t+\Delta t}T_5 = \frac{_tT_4 + {_tT_6}}{2} = \frac{100 + 100}{2} = 100$$

For *n* = 6, using Eq. (5.4-15),

$$_{t+\Delta t}T_6 = {_tT_5} = 100$$

For 2 Δ*t*, using Eq. (5.4-13) for *n* = 1, and continuing for *n* = 2 to 6, using Eqs. (5.4-14) and (5.4-15),

$$_{t+2\Delta t}T_1 = T_a = 0$$

$$_{t+2\Delta t}T_2 = \frac{_{t+\Delta t}T_1 + {_{t+\Delta t}T_3}}{2} = \frac{0 + 100}{2} = 50$$

$$_{t+2\Delta t}T_3 = \frac{_{t+\Delta t}T_2 + {_{t+\Delta t}T_4}}{2} = \frac{75 + 100}{2} = 87.5$$

$$_{t+2\Delta t}T_4 = \frac{_{t+\Delta t}T_3 + {_{t+\Delta t}T_5}}{2} = \frac{100 + 100}{2} = 100$$

$$_{t+2\Delta t}T_5 = \frac{_{t+\Delta t}T_4 + {_{t+\Delta t}T_6}}{2} = \frac{100 + 100}{2} = 100$$

$$_{t+2\Delta t}T_6 = {_{t+\Delta t}T_5} = 100$$

For 3 Δ*t*,

$$_{t+3\Delta t}T_1 = 0$$

$$_{t+3\Delta t}T_2 = \frac{0 + 87.5}{2} = 43.75$$

$$_{t+3\Delta t}T_3 = \frac{50 + 100}{2} = 75$$

$$_{t+3\Delta t}T_4 = \frac{87.5 + 100}{2} = 93.75$$

$$_{t+3\Delta t}T_5 = \frac{100 + 100}{2} = 100$$

$$_{t+3\Delta t}T_6 = 100$$

For 4 Δt,

$$_{t+4\Delta t}T_1 = 0$$

$$_{t+4\Delta t}T_2 = \frac{0 + 75}{2} = 37.5$$

$$_{t+4\Delta t}T_3 = \frac{43.75 + 93.75}{2} = 68.75$$

$$_{t+4\Delta t}T_4 = \frac{75 + 100}{2} = 87.5$$

$$_{t+4\Delta t}T_5 = \frac{93.75 + 100}{2} = 96.88$$

$$_{t+4\Delta t}T_6 = 100$$

For 5 Δt,

$$_{t+5\Delta t}T_1 = 0$$

$$_{t+5\Delta t}T_2 = \frac{0 + 68.75}{2} = 34.38$$

$$_{t+5\Delta t}T_3 = \frac{37.5 + 87.5}{2} = 62.50$$

$$_{t+5\Delta t}T_4 = \frac{68.75 + 96.88}{2} = 82.81$$

$$_{t+5\Delta t}T_5 = \frac{87.5 + 100}{2} = 93.75$$

$$_{t+5\Delta t}T_6 = 96.88$$

For 6 Δt (final time),

$$_{t+6\Delta t}T_1 = 0$$

$$_{t+6\Delta t}T_2 = \frac{0 + 62.5}{2} = 31.25$$

$$_{t+6\Delta t}T_3 = \frac{34.38 + 82.81}{2} = 58.59$$

$$_{t+6\Delta t}T_4 = \frac{62.50 + 93.75}{2} = 78.13$$

$$_{t+6\Delta t}T_5 = \frac{82.81 + 96.88}{2} = 89.84$$

$$_{t+6\Delta t}T_6 = 93.75$$

The temperature profiles for 3 Δt increments and the final time of 6 Δt increments are plotted in Fig. 5.4-2. This example shows how a hand calculation can be done. To increase the accuracy, more slab increments and more time increments are required. This, then, is ideally suited for computation using a spreadsheet with a computer.

EXAMPLE 5.4-2. Unsteady-State Conduction Using the Digital Computer

Repeat Example 5.4-1 using the digital computer. Use $\Delta x = 0.05$ m. Write the spreadsheet program and compare the final temperatures with Example 5.4-1. Use the explicit method of Schmidt for $M = 2$. Although not needed for many time increments using the digital computer, use the special procedure for the value of $_1T_a$ for the first time increment. Thus a direct comparison can be made with Example 5.4-1 of the effect of the number of increments on the results.

Solution: The number of slabs to use is 1.00 m/(0.05 m/slab) or 20 slabs. Substituting into Eq. (5.4-3) with $\alpha = 2.00 \times 10^{-5}$ m^2/s, $\Delta x = 0.05$ m, and $M = 2$, and solving for Δt,

$$M = 2 = \frac{(\Delta x)^2}{\alpha \, \Delta t} = \frac{(0.05)^2}{(2.00 \times 10^{-5})(\Delta t)}$$

$$\Delta t = 62.5 \text{ s}$$

Hence, (6000/62.5) = 96 time increments to be used. The value of n goes from $n = 1$ to 21.

The equations to use to calculate the temperatures are again Eqs. (5.4-12)– (5.4-15). However, the only differences are that in Eq. (5.4-14) n goes from 2 to 20, and in Eq. (5.4-15) $n = 21$, so that $_{t+\Delta t}T_{21} = {_t}T_{20}$.

The spreadsheet for these equations is easily written and is left up to the reader. The results are tabulated in Table 5.4-1 for comparison with Example 5.4-1, where only five slices were used. The table shows that the results for five slices are reasonably close to those for 20 slices, with values in both cases deviating by 2% or less from each other.

Table 5.4-1. Comparison of Results for Examples 5.4-1 and 5.4-2

	Results Using $\Delta x = 0.20$ m	Results Using $\Delta x = 0.05$ m

Distance from Front Face		Temperature		Temperature
m	n	°C	n	°C
0	1	0.0	1	0.0
0.20	2	31.25	5	31.65
0.40	3	58.59	9	58.47
0.60	4	78.13	13	77.55
0.80	5	89.84	17	88.41
1.00	6	93.75	21	91.87

As a rule-of-thumb guide for hand calculations, using a minimum of five slices and at least 8–10 time increments should give sufficient accuracy for most purposes. Only when very high accuracy is desired or several cases are to be solved is it desirable to solve the problem using a spreadsheet calculation with a computer.

EXAMPLE 5.4-3. Unsteady-State Conduction with Convective Boundary Condition

Use the same conditions as Example 5.4-1, but a convective coefficient of h = 25.0 W/m^2 · K is now present at the surface. The thermal conductivity k = 10.0 W/m · K.

Solution: Equations (5.4-7) and (5.4-8) can be used for convection at the surface. From Eq. (5.4-8), $N = h\Delta x/k$ = 25.0(0.20)/10.0 = 0.50. Then $2N + 2$ = 2(0.50) + 2 = 3.0. However, by Eq. (5.4-9), the value of M must be equal to or greater than $2N + 2$. This means that a value of M = 2 cannot be used. We will select the preferred method where M = 4.0. [Another, less accurate alternative is to use Eq. (5.4-11) for convection, and then the value of M is not restricted by the N value.]

Substituting into Eq. (5.4-3) and solving for Δt,

$$M = 4 = \frac{(\Delta x)^2}{\alpha \Delta t} = \frac{(0.20)^2}{(2.00 \times 10^{-5})(\Delta t)} \qquad \Delta t = 500 \text{ s}$$

Hence, 6000/500 = 12 time increments must be used.

For the first Δt time increment and for all time increments, the value of the environmental temperature T_a to use is T_a = 0°C since $M > 3$. For convection at the node or point n = 1 we use Eq. (5.4-7), where M = 4 and N = 0.50:

Equation 5.4-16.

$$\begin{aligned} {}_{t+\Delta t}T_1 &= \tfrac{1}{4}[2(0.5)T_a + [4 - (2 \times 0.5 + 2)]_tT_1 + 2_tT_2] \\ &= 0.25T_a + 0.25_tT_1 + 0.50_tT_2 \qquad n = 1 \end{aligned}$$

For n = 2, 3, 4, 5, we use Eq. (5.4-2),

Equation 5.4-17.

$$\begin{aligned} {}_{t+\Delta t}T_n &= \tfrac{1}{4}[{}_tT_{n+1} + (4 - 2)_tT_n + {}_tT_{n-1}] \\ &= 0.25_tT_{n+1} + 0.50_tT_n + 0.25_tT_{n-1} \qquad n = 2, 3, 4, 5 \end{aligned}$$

For n = 6 (insulated boundary), we use Eq. (5.4-10) and f = 6.

Equation 5.4-18.

$$\begin{aligned} {}_{t+\Delta t}T_6 &= \tfrac{1}{4}[(4 - 2)_tT_6 + 2_tT_5] \\ &= 0.50_tT_6 + 0.50_tT_5 \qquad n = 6 \end{aligned}$$

For 1 Δt, for the first time increment of $t + \Delta t$, $T_a = 0$. Using Eq. (5.4-16) to calculate the temperature at node 1,

$$_{t+\Delta t}T_1 = 0.25(0) + 0.25(100) + 0.50(100) = 75.0$$

For n = 2, 3, 4, 5, using Eq. (5.4-17),

$$_{t+\Delta t}T_2 = 0.25\,_tT_3 + 0.50\,_tT_2 + 0.25\,_tT_1$$

$$= 0.25(100) + 0.50(100) + 0.25(100) = 100.0$$

Also, in a similar calculation, T_3, T_4, and T_5 = 100.0. For n = 6, using Eq. (5.4-18), T_6 = 100.0.

For 2 Δt, $T_a = 0$. Using Eq. (5.4-16),

$$_{t+2\Delta t}T_1 = 0.25(0) + 0.25(75.0) + 0.50(100) = 68.75$$

Using Eq. (5.4-17) for n = 2, 3, 4, 5,

$$_{t+2\Delta t}T_2 = 0.25(100) + 0.50(100) + 0.25(75.00) = 93.75$$

$$_{t+2\Delta t}T_3 = 0.25(100) + 0.50(100) + 0.25(100) = 100.0$$

Also, T_4 and T_5 = 100.0. For n = 6, using Eq. (5.4-18), T_6 = 100.0.

For 3 Δt, $T_a = 0$. Using Eq. (5.4-16),

$$_{t+3\Delta t}T_1 = 0.25(0) + 0.25(68.75) + 0.50(93.75) = 64.07$$

Using Eq. (5.4-17) for n = 2, 3, 4, 5,

$$_{t+3\Delta t}T_2 = 0.25(100) + 0.50(93.75) + 0.25(68.75) = 89.07$$

$$_{t+3\Delta t}T_3 = 0.25(100) + 0.50(100) + 0.25(93.75) = 98.44$$

$$_{t+3\Delta t}T_4 = 0.25(100) + 0.50(100) + 0.25(100) = 100.0$$

Also, T_5 = 100.0 and T_6 = 100.0.

In a similar manner the calculations can be continued for the remaining time until a total of 12 Δt increments have been used.

Other Numerical Methods for Unsteady-State Conduction

Unsteady-state conduction in a cylinder

In deriving the numerical equations for unsteady-state conduction in a flat slab, the cross-sectional area was constant throughout. In a cylinder it changes radially. To derive the equation for a cylinder, Fig. 5.4-3 is used, where the cylinder is divided into concentric hollow cylinders whose walls are Δx m thick. Assuming a cylinder 1 m long and making a heat balance on the slab at point n, the rate of heat in − rate of heat out = rate of heat accumulation:

Equation 5.4-19.

$$\frac{k[2\pi(n + 1/2)\,\Delta x]}{\Delta x}(_tT_{n+1} - {}_tT_n) - k\,\frac{[2\pi(n - 1/2)\,\Delta x]}{\Delta x}(_tT_n - {}_tT_{n-1}) = \frac{2\pi n(\Delta x)^2 \rho c_p}{\Delta t}(_{t+\Delta t}T_n - {}_tT_n)$$

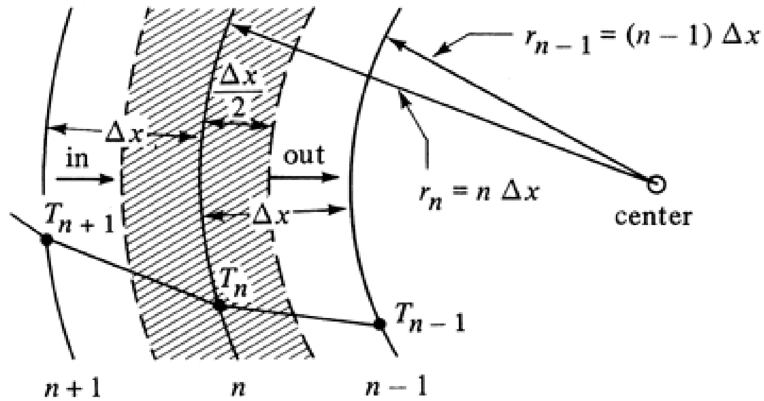


Figure 5.4-3. Unsteady-state conduction in a cylinder.

Rearranging, the final equation is

Equation 5.4-20.

$$_{t+\Delta t}T_n = \frac{1}{M}\left[\frac{2n+1}{2n}\,{}_tT_{n+1} + (M-2)\,{}_tT_n + \frac{2n-1}{2n}\,{}_tT_{n-1}\right]$$

where $M = (\Delta x)^2/(\alpha\,\Delta t)$ as before. Also, at the center where $n = 0$,

Equation 5.4-21.

$$_{t+\Delta t}T_0 = \frac{4}{M}\,{}_tT_1 + \frac{M-4}{M}\,{}_tT_0$$

To use Equations (5.4-20) and (5.4-21),

Equation 5.4-22.

$$M \ge 4$$

Equations for convection at the outer surface of the cylinder have been derived (D1). If the heat capacity of the outer half-slab is neglected,

Equation 5.4-23.

$$_{t+\Delta t}T_n = \frac{nN}{\dfrac{2n-1}{2} + nN}\,{}_{t+\Delta t}T_a + \frac{(2n-1)/2}{\dfrac{2n-1}{2} + nN}\,{}_{t+\Delta t}T_{n-1}$$

where T_n is the temperature at the surface and T_{n-1} the temperature at a position in the solid 1 Δx below the surface.

Equations for numerical methods for two-dimensional unsteady-state conduction have been derived and are available in a number of references (D1, K2).

Unsteady-state conduction and implicit numerical method

In some practical problems the restrictions imposed on the value $M \ge 2$ by stability requirements may prove inconvenient.Also, to minimize the stability problems, implicit methods using different finite-difference formulas have been developed.An important one of these formulas is the Crank–Nicolson method, which will be considered here.

In deriving Eqs. (5.4-1) and (5.4-2), the rate at which heat entered the slab in Fig. 5.4-1 was taken to be the rate at time t:

Equation 5.4-24.

$$\text{Rate of heat in at } t = \frac{kA}{\Delta A}({}_tT_{n-1} - {}_tT_n)$$

It was then assumed that this rate could be used during the whole interval from t to $t + \Delta t$. This is an approximation, however since the rate changes during this Δt interval. A better value would be the average value of the rate at t and at $t + \Delta t$, or

Equation 5.4-25.

$$\text{average rate of heat in} = \frac{kA}{\Delta x}\left[\frac{({}_tT_{n-1} - {}_tT_n) + ({}_{t+\Delta t}T_{n-1} - {}_{t+\Delta t}T_n)}{2}\right]$$

For the heat leaving, a similar type of average is used. The final equation is

Equation 5.4-26.

$$_{t+\Delta t}T_{n+1} - (2M + 2)_{t+\Delta t}T_n + {}_{t+\Delta t}T_{n-1} = -{}_tT_{n+1} + (2 - 2M)_tT_n - {}_tT_{n-1}$$

This means that now a new value of ${}_{t+\Delta t}T_n$ cannot be calculated only from values at time t, as in Eq. (5.4-2), but that all the new values of T at $t + \Delta t$ at all points must be calculated simultaneously. To do this, an equation similar to Eq. (5.4-26) is written for each of the internal points. Each of these equations and the boundary equations are linear algebraic equations. These then can be solved simultaneously by the standard methods, such as the Gauss–Seidel iteration technique, matrix inversion technique, and so on (G1, K1).

An important advantage of Eq. (5.4-26) is that the stability and convergence criteria are satisfied for all positive values of M. This means that M can have values less than 2.0. A disadvantage of the implicit method is the larger number of calculations needed for each time step. Explicit methods are simpler to use, but because of stability considerations, especially in complex situations, implicit methods are often preferred.

CHILLING AND FREEZING OF FOOD AND BIOLOGICAL MATERIALS

Introduction

Unlike many inorganic and organic materials which are relatively stable, food and other biological materials decay and deteriorate more or less rapidly with time at room temperature. This spoilage is due to a number of factors. Tissues of foods such as fruits and vegetables continue to undergo metabolic respiration after harvesting, and ripen and eventually spoil. Enzymes of the dead tissues of meats and fish remain active and induce oxidation and other deteriorating effects. Microorganisms attack all types of foods by decomposing the foods so that spoilage occurs; chemical reactions also occur, such as the oxidation of fats.

At low temperatures the growth rate of microorganisms will be slowed if the temperature is below that which is optimum for growth. Enzyme activity and chemical reaction rates are also reduced at low temperatures. The rates of most chemical and biological reactions in storage of chilled or frozen foods and biological materials are reduced by factors of $\frac{1}{2}$ to $\frac{1}{3}$ for each 10 K (10°C) drop in temperature.

Water plays an important part in these rates of deterioration, and it is present to a substantial percentage in most biological materials. To reach a temperature low enough for most of these rates to approximately cease, most of the water must be frozen. Materials such as food do not freeze at 0°C (32°F), as pure water does, but at a range of temperatures below 0°C. However, because of some of the physical effects of ice crystals and other effects, such as concentrating of solutions, chilling of biological materials is often used for preservation instead of freezing.

Chilling of materials involves removing the sensible heat and heat of metabolism and reducing the temperature, usually to a range of 4.4°C (40°F) to just above freezing. Essentially no latent heat of freezing is involved. The materials can be stored for a week or so up to a few months, depending on the product stored and the gaseous atmosphere. Each material has its optimum chill storage temperature.

In the freezing of food and biological materials, the temperature is reduced so that most of the water is frozen to ice. Depending on the final storage temperature, down to as low as −30°C, the materials can be stored for up to a year or so. Often in the production of frozen foods, they are first treated by blanching or scalding to destroy enzymes.

Chilling of Food and Biological Materials

In the chilling of food and biological materials, the temperature of the materials is reduced to the desired chill storage temperature, which can be about −1.1°C (30°F) to 4.4°C (40°F). For example, after slaughter, beef has a temperature of 37.8°C (100°F) to 40°C (104°F), and it is often cooled to about 4.4°C (40°F). Milk from cows must be chilled quickly to temperatures just above freezing. Some fish fillets at the time of packing are at a temperature of 7.2°C (45°F) to 10°C (50°F) and are chilled to close to 0°C.

These rates of chilling or cooling are governed by the laws of unsteady-state heat conduction discussed in Sections 5.1 to 5.4. The heat is removed by convection at the surface of the material and by unsteady-state conduction in the material. The fluid outside the foodstuff or biological materials is used to remove this heat; in many cases it is air. The air has previously been cooled by refrigeration to −1.1°C to +4.4°C, depending on the material and other conditions. The convective heat-transfer coefficients, which usually include radiation effects, can also be predicted by the methods in Chapter 4; for air the coefficient varies from about 8.5 to 40 $W/m^2 \cdot K$ (1.5 to 7 $btu/h \cdot ft^2 \cdot °F$), depending primarily on air velocity.

In some cases the fluid used for chilling is a liquid flowing over the surface, and the values of h can vary from about 280 to 1700 $W/m^2 \cdot K$ (50–300 $btu/h \cdot ft^2 \cdot °F$). In other cases, a contact or plate cooler is used, where chilled plates are in direct contact with the material. Then the temperature of the surface of the material is usually assumed to be equal or close to that of the contact plates. Contact freezers are used for freezing biological materials.

Where the food is packaged in boxes or the material is tightly covered by a film of plastic, this additional resistance must be considered. One method for doing this is to add the resistance of the package covering to that of the convective film:

Equation 5.5-1.

$$R_T = R_P + R_C$$

where R_P is the resistance of covering, R_C the resistance of the outside convective film, and R_T the total resistance. Then, for each resistance,

Equation 5.5-2.

$$R_C = \frac{1}{h_c A}$$

Equation 5.5-3.

$$R_P = \frac{\Delta x}{kA}$$

Equation 5.5-4.

$$R_T = \frac{1}{hA}$$

where h_c is the convective gas or liquid coefficient, *A* is the area, Δx is the thickness of the covering, *k* is the thermal conductivity of the covering, and *h* is the overall coefficient. The overall coefficient *h* is the one to use in the unsteady-state charts. This assumes a negligible heat capacity of the covering, which is usually the case. Also, it assumes that the covering closely touches the food material so there is no resistance between the covering and the food.

The major sources of error in using the unsteady-state charts are inadequate data on the density, heat capacity, and thermal conductivity of the foods and the prediction of the convective coefficient. Food materials are irregular anisotropic substances, whose physical properties are often difficult to evaluate. Also, if evaporation of water occurs on chilling, latent heat losses can affect the accuracy of the results.

EXAMPLE 5.5-1. Chilling Dressed Beef

Hodgson (H2) gives physical properties of beef carcasses during chilling of $\rho = 1073$ kg/m^3, $c_p = 3.48$ kJ/kg · K, and $k = 0.498$ W/m · K. A large slab of beef 0.203 m thick and initially at a uniform temperature of 37.8°C is to be cooled so that the center temperature is 10°C. Chilled air at 1.7°C (assumed constant) with an $h = 39.7$ W/m^2 · K is used. Calculate the time needed.

***Solution*:** The thermal diffusivity α is

$$\alpha = \frac{k}{\rho c_p} = \frac{0.498}{(1073)(3.48 \times 1000)} = 1.334 \times 10^{-7}\ \text{m}^2/\text{s}$$

Then, for the half-thickness x_1 of the slab,

$$x_1 = \frac{0.203}{2} = 0.1015\ \text{m}$$

For the center of the slab,

$$n = \frac{x}{x_1} = \frac{0}{x_1} = 0$$

Also,

$$m = \frac{k}{hx_1} = \frac{0.498}{(39.7)(0.1015)} = 0.123$$

$$T_1 = 1.7°\text{C} + 237.2 = 274.9\ \text{K} \qquad T_0 = 37.8 + 273.2 = 311.0\ \text{K}$$

$$T = 10 + 273.2 = 283.2\ \text{K}$$

$$Y = \frac{T_1 - T}{T_1 - T_0} = \frac{274.9 - 283.2}{274.9 - 311.0} = 0.230$$

Using Fig. 5.3-6 for the center of a large flat plate,

$$X = 0.90 = \frac{\alpha t}{x_1^2} = \frac{(1.334 \times 10^{-7})(t)}{(0.1015)^2}$$

Solving, $t = 6.95 \times 10^4$ s (19.3 h).

Freezing of Food and Biological Materials

Introduction

In the freezing of food and other biological materials, the removal of sensible heat in chilling occurs first and then the removal of the latent heat of freezing. The latent heat of freezing water of 335 kJ/kg (144 btu/lb_m) is a substantial portion of the total heat removed on freezing. Other slight effects, such as the heats of solution of salts and so on, may be present but are quite small. Actually, when materials such as meats are frozen to −29°C, only about 90% of the water is frozen to ice, with the rest thought to be bound water (B1).

Riedel (R1) gives enthalpy–temperature–composition charts for the freezing of many different foods. These charts show that freezing does not occur at a given temperature but extends over a range of several degrees. As a consequence, there is no one freezing point with a single latent heat of freezing.

Since the latent heat of freezing is present in the unsteady-state process of freezing, the standard unsteady-state conduction equations and charts given in this chapter cannot be used for prediction of freezing times. A full analytical solution of the rate of freezing of food and biological materials is very difficult because of the variation of physical properties with temperature, the amount of freezing varying with temperature, and other factors. An approximate solution by Plank is often used.

Approximate solution of Plank for freezing

Plank (P2) has derived an approximate solution for the time of freezing which is often sufficient for engineering purposes. The assumptions in the derivation are as follows. Initially, all the food is at the freezing temperature but is unfrozen. The thermal conductivity of the frozen part is constant. All the material freezes at the freezing point, with a constant latent heat. The heat transfer by conduction in the frozen layer occurs slowly enough that it is under pseudo-steady-state conditions.

In Fig. 5.5-1 a slab of thickness a m is cooled from both sides by convection. At a given time t s, a thickness of x m of frozen layer has formed on both sides. The temperature of the environment is constant at T_1 K and the freezing temperature is constant at T_f. An unfrozen layer in the center at T_f is present.

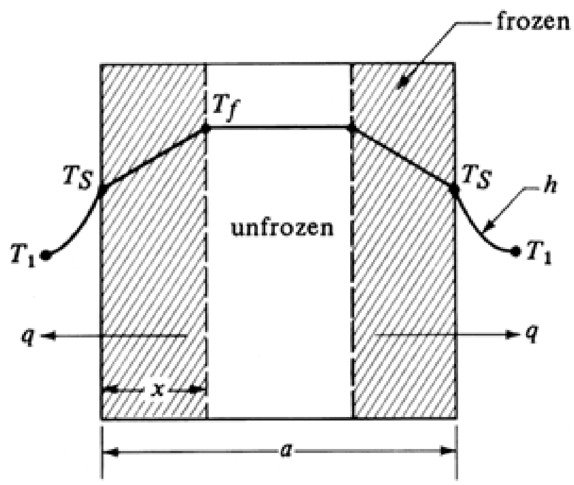


Figure 5.5-1. Temperature profile during freezing.

The heat leaving at time t is q W. Since we are at pseudo-steady state, at time t, the heat leaving by convection on the outside is

Equation 5.5-5.

$$q = hA(T_S - T_1)$$

where A is the surface area. Also, the heat being conducted through the frozen layer of x thickness at steady state is

Equation 5.5-6.

$$q = \frac{kA}{x}(T_f - T_S)$$

where k is the thermal conductivity of the frozen material. In a given time dt s, a layer dx thick of material freezes. Then multiplying A times dx times ρ gives the kg mass frozen. Multiplying this by the latent heat λ in J/kg and dividing by dt,

Equation 5.5-7.

$$q = \frac{A\,dx\,\rho\lambda}{dt} = A\rho\lambda\frac{dx}{dt}$$

where ρ is the density of the unfrozen material.

Next, to eliminate T_S from Eqs. (5.5-5) and (5.5-6), Eq. (5.5-5) is solved for T_S and substituted into Eq. (5.5-6), giving

Equation 5.5-8.

$$q = \frac{(T_f - T_1)A}{x/k + 1/h}$$

Equating Eq. (5.5-8) to (5.5-7),

Equation 5.5-9.

$$\frac{(T_f - T_1)A}{x/k + 1/h} = A\rho\lambda\frac{dx}{dt}$$

Rearranging and integrating between $t = 0$ and $x = 0$, to $t = t$ and $x = a/2$,

Equation 5.5-10.

$$(T_f - T_1)\int_0^t dt = \lambda\rho\int_0^{a/2}\left(\frac{x}{k} + \frac{1}{h}\right)dx$$

Integrating and solving for t,

Equation 5.5-11.

$$t = \frac{\lambda\rho}{T_f - T_1}\left(\frac{a}{2h} + \frac{a^2}{8k}\right)$$

To generalize the equation for other shapes,

Equation 5.5-12.

$$t = \frac{\lambda\rho}{T_f - T_1}\left(\frac{Pa}{h} + \frac{Ra^2}{k}\right)$$

where a is the thickness of an infinite slab (as in Fig. 5.5-1), diameter of a sphere, diameter of a long cylinder, or smallest dimension of a rectangular block or brick. Also,

$$P = \tfrac{1}{2} \text{ for infinite slab, } \tfrac{1}{6} \text{ for sphere, } \tfrac{1}{4} \text{ for infinite cylinder}$$

$$R = \tfrac{1}{8} \text{ for infinite slab, } \tfrac{1}{24} \text{ for sphere, } \tfrac{1}{16} \text{ for infinite cylinder}$$

For a rectangular brick having dimensions a by $\beta_1 a$ by $\beta_2 a$, where a is the shortest side, Ede (B1) has prepared a chart to determine the values of P and R to be used to calculate t in Eq. (5.5-12). Equation (5.5-11) can also be used for calculation of thawing times by replacing the k of the frozen material by the k of the thawed material.

EXAMPLE 5.5-2. Freezing of Meat

Slabs of meat 0.0635 m thick are to be frozen in an air-blast freezer at 244.3 K (−28.9°C). The meat is initially at the freezing temperature of 270.4 K (−2.8°C). The meat contains 75% moisture. The heat-transfer coefficient is h = 17.0 W/m^2 · K. The physical properties are ρ = 1057 kg/m^3 for the unfrozen meat and k = 1.038 W/m · K for the frozen meat. Calculate the freezing time.

***Solution*:** Since the latent heat of fusion of water to ice is 335 kJ/kg (144 btu/lb$_m$), for meat with 75% water,

$$\lambda = 0.75(335) = 251.2 \text{ kJ/kg}$$

The other given variables are a = 0.0635 m, T_f = 270.4 K, T_1 = 244.3 K, ρ = 1057 kg/m^3, h = 17.0 W/m^2 · K, k = 1.038 W/m · K. Substituting into Eq. (5.5-11),

$$t = \frac{\lambda\rho}{T_f - T_1}\left(\frac{a}{2h} + \frac{a^2}{8k}\right) = \frac{(2.512 \times 10^5)1057}{270.4 - 244.3}\left[\frac{0.0635}{2(17.0)} + \frac{(0.0635)^2}{8(1.038)}\right]$$

$$= 2.395 \times 10^4 \text{ s } (6.65 \text{ h})$$

Other methods for calculating freezing times

Neumann (C1, C2) has derived a complicated equation for freezing in a slab. He assumes the following conditions. The surface temperature is the same as the environment, that is, no surface resistance. The temperature of freezing is constant. This method suffers from the limitation that a convection coefficient cannot be used at the surface, since it assumes no surface resistance. However, the method does include the effect of cooling from an original temperature, which may be above the freezing point.

Plank's equation does not make provision for an original temperature, which may be above the freezing point. An approximate method for calculating the additional time necessary to cool from temperature T_0 down to the freezing point T_f is as follows. Using the unsteady-state charts calculate the time for the average temperature in the material to reach T_f, assuming that no freezing occurs and using the physical properties of the unfrozen material. If there is no surface resistance, Fig. 5.3-13 can be used directly for this. If a resistance is present, the temperature at several points in the material will have to be obtained from the unsteady-state charts and the average temperature calculated from these point temperatures. This may be partially trial and error, since the time is unknown and must be assumed. If the average temperature calculated is not at the freezing point, a new time must be assumed. This is an approximate method since some material will actually freeze.

DIFFERENTIAL EQUATION OF ENERGY CHANGE

Introduction

In Sections 3.6 and 3.7 we derived a differential equation of continuity and a differential equation of momentum transfer for a pure fluid. These equations were derived because overall mass, energy, and momentum balances made on a finite volume in the earlier parts of Chapter 2 did not tell us what goes on inside a control volume. In the overall balances performed, a new balance was made for each new system studied. However, it is often easier to start with the differential equations of continuity and momentum transfer in general form and then simplify the equations by discarding unneeded terms for each specific problem.

In Chapter 4 on steady-state heat transfer and Chapter 5 on unsteady-state heat transfer, new overall energy balances were made on a finite control volume for each new situation. To progress further in our study of heat or energy transfer in flow and nonflow systems, we must use a differential volume to investigate in greater detail what goes on inside this volume. The balance will be made on a single phase and the boundary conditions at the phase boundary will be used for integration.

In the next section we derive a general differential equation of energy change: the conservation-of-energy equation. Then this equation is modified for certain special cases that occur frequently. Finally, applications of the uses of these equations are given. Cases for both steady-state and unsteady-state energy transfer are studied using this conservation-of-energy equation, which is perfectly general and holds for steady- or unsteady-state conditions.

Derivation of Differential Equation of Energy Change

As in the derivation of the differential equation of momentum transfer, we write a balance on an element of volume of size Δx, Δy, Δz which is stationary. We then write the law of conservation of energy, which is really the first law of thermodynamics for the fluid in this volume element at any time. The following is the same as Eq. (2.7-7) for a control volume given in Section 2.7.

Equation 5.6-1.

$$\begin{pmatrix}\text{rate of}\\ \text{energy in}\end{pmatrix} - \begin{pmatrix}\text{rate of}\\ \text{energy out}\end{pmatrix} - \begin{pmatrix}\text{rate of external}\\ \text{work done by}\\ \text{system on surroundings}\end{pmatrix} = \begin{pmatrix}\text{rate of}\\ \text{accumulation of}\\ \text{energy}\end{pmatrix}$$

As in momentum transfer, the transfer of energy into and out of the volume element is by convection and molecular transport or conduction. There are two kinds of energy being transferred. The first is internal energy U in J/kg (btu/lb$_m$) or any other set of units. This is the energy associated with random translational and internal motions of the molecules plus molecular interactions. The second is kinetic energy $\rho v^2/2$, which is the energy associated with the bulk fluid motion, where v is the local fluid velocity, m/s (ft/s). Hence, the total energy per unit volume is $(\rho U + \rho v^2/2)$. The rate of accumulation of energy in the volume element in m^3 (ft^3) is then

Equation 5.6-2.

$$\Delta x\ \Delta y\ \Delta z \frac{\partial}{\partial t}\left(\rho U + \frac{\rho v^2}{2}\right)$$

The total energy entering by convection in the x direction at x minus that leaving at $x + \Delta x$ is

Equation 5.6-3.

$$\Delta y\ \Delta z\left[v_x\left(\rho U + \frac{\rho v^2}{2}\right)\right]_x - \Delta y\ \Delta z\left[v_x\left(\rho U + \frac{\rho v^2}{2}\right)\right]_{x+\Delta x}$$

Similar equations can be written for the y and z directions using velocities v_y and v_z, respectively.

The net rate of energy entering the element by conduction in the x direction is

Equation 5.6-4.

$$\Delta y\ \Delta z[(q_x)_x - (q_x)_{x+\Delta x}]$$

Similar equations can be written for the y and z directions, where q_x, q_y, and q_z are the components of the heat flux vector q, which is in W/m^2 (btu/s · ft^2) or any other convenient set of units.

The net work done by the system on its surroundings is the sum of the following three parts for the x direction. For the net work done against the gravitational force,

Equation 5.6-5.

$$-\rho\ \Delta x\ \Delta y\ \Delta z(v_x g_x)$$

where g_x is gravitational force. The net work done against the static pressure p is

Equation 5.6-6.

$$\Delta y\ \Delta z[(pv_x)_{x+\Delta x} - (pv_x)_x]$$

where p is N/m^2 (lb$_f$/ft^2) or any other convenient set of units. For the net work against the viscous forces,

Equation 5.6-7.

$$\Delta y\, \Delta z[(\tau_{xx}v_x + \tau_{xy}v_y + \tau_{xz}v_z)_{x+\Delta x} - (\tau_{xx}v_x + \tau_{xy}v_y + \tau_{xz}v_z)_x]$$

In Section 3.7 these viscous forces are discussed in more detail.

Writing equations similar to (5.6-3)–(5.6-7) in all three directions; substituting these equations and Eq. (5.6-2) into (5.6-1); dividing by Δx, Δy, and Δz; and letting Δx, Δy, and Δz approach zero, we obtain

Equation 5.6-8.

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho U + \frac{\rho v^2}{2}\right) = &-\left[\frac{\partial}{\partial x} v_x\left(\rho U + \frac{\rho v^2}{2}\right) + \frac{\partial}{\partial y} v_y\left(\rho U + \frac{\rho v^2}{2}\right) + \frac{\partial}{\partial z} v_z\left(\rho U + \frac{\rho v^2}{2}\right)\right]\\ &- \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) + \rho(v_x g_x + v_y g_y + v_z g_z)\\ &- \left[\frac{\partial}{\partial x}(p v_x) + \frac{\partial}{\partial y}(p v_y) + \frac{\partial}{\partial z}(p v_z)\right]\\ &- \left[\frac{\partial}{\partial x}(\tau_{xx}v_x + \tau_{xy}v_y + \tau_{xz}v_z) + \frac{\partial}{\partial z}(\tau_{yx}v_x + \tau_{yy}v_y + \tau_{yz}v_z)\right.\\ &\quad\left. + \frac{\partial}{\partial z}(\tau_{zx}v_x + \tau_{zy}v_y + \tau_{zz}v_z)\right]\end{aligned}$$

For further details of this derivation, see (B2).

Equation (5.6-8) is the final equation of energy change relative to a stationary point. However, it is not in a convenient form. We first combine Eq. (5.6-8) with the equation of continuity, Eq. (3.6-23), with the equation of motion, Eq. (3.7-13), and express the internal energy in terms of fluid temperature *T* and heat capacity. Then writing the resultant equation for a Newtonian fluid with constant thermal conductivity *k*, we obtain

Equation 5.6-9.

$$\rho c_v \frac{DT}{Dt} = k\nabla^2 T - T\left(\frac{\partial p}{\partial T}\right)_\rho (\nabla \cdot v) + \mu\phi$$

This equation utilizes Fourier's second law in three directions, where

Equation 5.6-10.

$$k\nabla^2 T = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

The viscous-dissipation term $\mu\varphi$ is generally negligible except where extremely large velocity gradients exist. It will be omitted in the discussions to follow. Equation (5.6-9) is the equation of energy change for a Newtonian fluid with constant *k* in terms of the fluid temperature *T*.

Special Cases of the Equation of Energy Change

The following special forms of Eq. (5.6-9) for a Newtonian fluid with constant thermal conductivity are commonly encountered. First, Eq. (5.6-9) will be written in rectangular coordinates without the $\mu\varphi$ term:

Equation 5.6-11.

$$\rho c_v\left(\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)-T\left(\frac{\partial p}{\partial T}\right)_\rho\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)$$

Fluid at constant pressure

The equations below can be used for constant-density fluids as well as for constant pressure.

Equation 5.6-12.

$$\rho c_p\frac{DT}{Dt}=k\nabla^2 T$$

In rectangular coordinates,

Equation 5.6-13.

$$\rho c_p\left(\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)$$

In cylindrical coordinates,

Equation 5.6-14.

$$\rho c_p\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right)$$

In spherical coordinates,

Equation 5.6-15.

$$\rho c_p\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)=k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 \mathrm{T}}{\partial\phi^2}\right]$$

For definitions of cylindrical and spherical coordinates, see Section 3.6. If the velocity v is zero, DT/Dt becomes $\partial T/\partial t$.

Fluid at constant density

Equation 5.6-16.

$$\rho c_p\frac{DT}{Dt}=k\nabla^2 T$$

Note that this is identical to Eq. (5.6-12) for constant pressure.

Solid

Here we consider ρ is constant and $v = 0$.

Equation 5.6-17.

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T$$

This is often referred to as *Fourier's second law* of heat conduction. This also holds for a fluid with zero velocity at constant pressure.

Heat generation

If there is heat generation in the fluid by electrical or chemical means, then $\dot{q}$ can be added to the right side of Eq. (5.6-17).

Equation 5.6-18.

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T + \dot{q}$$

where $\dot{q}$ is the rate of heat generation in W/m^3 (btu/h · ft^3) or other suitable units. Viscous dissipation is also a heat source, but its inclusion greatly complicates problem solving because the equations for energy and motion are then coupled.

Other coordinate systems

Fourier's second law of unsteady-state heat conduction can be written as follows.

For rectangular coordinates,

Equation 5.6-19.

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\nabla^2 T = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

where $\alpha = k/\rho c_p$, thermal diffusivity in m^2/s (ft^2/h).

For cylindrical coordinates,

Equation 5.6-20.

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

For spherical coordinates,

Equation 5.6-21.

$$\frac{\partial T}{\partial t} = \alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right]$$

Uses of Equation of Energy Change

In Section 3.8 we used the differential equations of continuity and motion to set up fluid-flow problems. We did this by discarding the terms that are zero or near zero and using the remaining equations to solve for the velocity and pressure distributions. This was done instead of making new mass and momentum balances for each new situation. In a similar manner, to solve problems of heat transfer, the differential equations of continuity, motion, and energy will be used, with the unneeded terms being discarded. Several examples will be given to illustrate the general methods used.

EXAMPLE 5.6-1. Temperature Profile with Heat Generation

A solid cylinder in which heat generation is occurring uniformly as $\dot{q}$ W/m^3 is insulated on the ends. The temperature of the surface of the cylinder is held constant at T_w K. The radius of the cylinder is $r = R$ m. Heat flows only in the radial direction. Derive the equation for the temperature profile at steady state if the solid has a constant thermal conductivity.

Solution: Equation (5.6-20) will be used for cylindrical coordinates. The term $\dot{q}/\rho c_p$ for generation will be added to the right side, giving

Equation 5.6-22.

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\dot{q}}{\rho c_p}$$

For steady state $\partial T/\partial t = 0$. Also, for conduction only in the radial direction, $\partial^2 T/\partial z^2 = 0$ and $\partial^2 T/\partial \theta^2 = 0$. This gives the following differential equation:

Equation 5.6-23.

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = -\frac{\dot{q}}{k}$$

This can be rewritten as

Equation 5.6-24.

$$r\frac{d^2T}{dr^2} + \frac{dT}{dr} = -\frac{\dot{q}r}{k}$$

Note that Eq. (5.6-24) can be rewritten as follows:

Equation 5.6-25.

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\dot{q}r}{k}$$

Integrating Eq. (5.6-25) once,

Equation 5.6-26.

$$r\frac{dT}{dr} = -\frac{\dot{q}r^2}{2k} + K_1$$

where K_1 is a constant. Integrating again,

Equation 5.6-27.

$$T = -\frac{\dot{q}r^2}{4k} + K_1 \ln r + K_2$$

where K_2 is a constant. The boundary conditions are when $r = 0$, $dT/dr = 0$ (by symmetry), and when $r = R$, $T = T_w$. The final equation is

Equation 5.6-28.

$$T = \frac{\dot{q}(R^2 - r^2)}{4k} + T_w$$

This is the same as Eq. (4.3-29), which was obtained by another method.

EXAMPLE 5.6-2. Laminar Flow and Heat Transfer Using Equation of Energy Change

Using the differential equation of energy change, derive the partial differential equation and boundary conditions needed for the case of laminar flow of a constant-density fluid in a horizontal tube which is being heated. The fluid is flowing at a constant velocity v_z. At the wall of the pipe where the radius $r = r_0$, the heat flux is constant at q_0. The process is at steady state and it is assumed at $z = 0$ at the inlet that the velocity profile is established. Constant physical properties will be assumed.

Solution: From Example 3.8-3, the equation of continuity gives $\partial v_z/\partial z = 0$. Solution of the equation of motion for steady state using cylindrical coordinates gives the parabolic velocity profile:

Equation 5.6-29.

$$v_z = v_{z\,\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right]$$

Since the fluid has a constant density, Eq. (5.6-14) in cylindrical coordinates will be used for the equation of energy change. For this case $v_r = 0$ and $v_\theta = 0$. Since this will be symmetrical, $\partial T/\partial\theta$ and $\partial^2 T/\partial\theta^2$ will be zero. For steady state, $\partial T/\partial t = 0$. Hence, Eq. (5.6-14) reduces to

Equation 5.6-30.

$$v_z \frac{\partial T}{\partial z} = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right)$$

Usually conduction in the z direction ($\partial^2 T/\partial z^2$ term) is small compared to the convective term $v_z \partial T/\partial z$ and can be dropped. Finally, substituting Eq. (5.6-29) into (5.6-30), we obtain

Equation 5.6-31.

$$v_{z\,\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right]\frac{\partial T}{\partial z} = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)$$

The boundary conditions are

$$\text{At } z = 0, \quad T = T_0 \quad (\text{all } r)$$

$$\text{At } r = 0, \quad T = \text{finite}$$

$$\text{At } r = r_0, \quad q_0 = -k\frac{\partial T}{\partial r} \quad (\text{constant})$$

For details on the actual solution of this equation, see Siegel et al. (S2).

BOUNDARY-LAYER FLOW AND TURBULENCE IN HEAT TRANSFER

Laminar Flow and Boundary-Layer Theory in Heat Transfer

In Section 3.10C an exact solution was obtained for the velocity profile for isothermal laminar flow past a flat plate. The solution of Blasius can be extended to include the convective heat-transfer problem for the same geometry and laminar flow. In Fig. 5.7-1 the thermal boundary layer is shown. The temperature of the fluid approaching the plate is T_∞ and that of the plate is T_S at the surface.

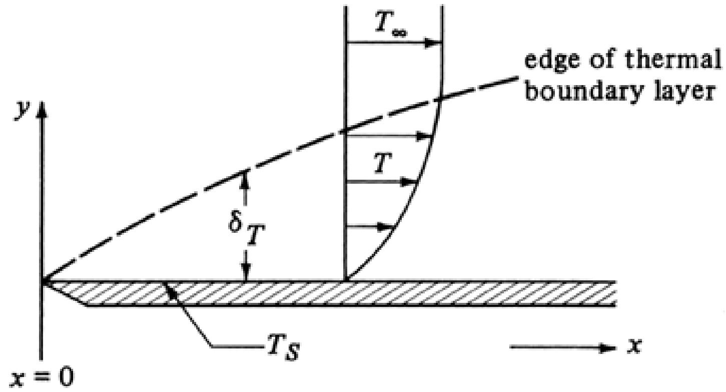


Figure 5.7-1. Laminar flow of fluid past a flat plate and thermal boundary layer.

We start by writing the differential energy balance, Eq. (5.6-13):

Equation 5.7-1.

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

If the flow is in the *x* and *y* directions, $v_z = 0$. At steady state, $\partial T/\partial t = 0$. Conduction is neglected in the *x* and *z* directions, so $\partial^2 T/\partial x^2 = \partial^2 T/\partial z^2 = 0$. Conduction occurs in the *y* direction. The result is

Equation 5.7-2.

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2}$$

The simplified momentum-balance equation used in the velocity boundary-layer derivation is very similar:

Equation 3.10-5.

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{\mu}{\rho} \frac{\partial^2 v_x}{\partial y^2}$$

The continuity equation used previously is

Equation 3.10-3.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Equations (3.10-5) and (3.10-3) were used by Blasius for solving the case for laminar boundary-layer flow. The boundary conditions used were

Equation 5.7-3.

$$\frac{v_x}{v_\infty} = \frac{v_y}{v_\infty} = 0 \quad \text{at } y = 0$$

$$\frac{v_x}{v_\infty} = 1 \qquad \text{at } y = \infty$$

$$\frac{v_x}{v_\infty} = 1 \qquad \text{at } x = 0$$

The similarity between Eqs. (3.10-5) and (5.7-2) is obvious. Hence, the Blasius solution can be applied if $k/\rho c_p = \mu/\rho$. This means the Prandtl number $c_p\mu/k = 1$. Also, the boundary conditions must be the same. This is done by replacing the temperature T in Eq. (5.7-2) by the dimensionless variable $(T - T_S)/(T_\infty - T_S)$. The boundary conditions become

Equation 5.7-4.

$$\frac{v_x}{v_\infty} = \frac{v_y}{v_\infty} = \frac{T - T_S}{T_\infty - T_S} = 0 \quad \text{at } y = 0$$

$$\frac{v_x}{v_\infty} = \frac{T - T_S}{T_\infty - T_S} = 1 \qquad \text{at } y = \infty$$

$$\frac{v_x}{v_\infty} = \frac{T - T_S}{T_\infty - T_S} = 1 \qquad \text{at } x = 0$$

We see that the equations and boundary conditions are identical for the temperature profile and the velocity profile. Hence, for any point x, y in the flow system, the dimensionless velocity variables v_x/v_∞ and $(T - T_S)/(T_\infty - T_S)$ are equal. The velocity-profile solution is the same as the temperature-profile solution.

This means that the transfer of momentum and heat are directly analogous, and the boundary-layer thickness δ for the velocity profile (hydrodynamic boundary layer) and the thermal boundary-layer thickness δ_T are equal. This is important for gases, where the Prandtl numbers are close to 1.

By combining Eqs. (3.10-7) and (3.10-8), the velocity gradient at the surface is

Equation 5.7-5.

$$\left(\frac{\partial v_x}{\partial y}\right)_{y=0} = 0.332\frac{v_\infty}{x} N_{\text{Re},x}^{1/2}$$

where $N_{\text{Re},x} = xv_\infty\rho/\mu$. Also,

Equation 5.7-6.

$$\frac{v_x}{v_\infty} = \frac{T - T_S}{T_\infty - T_S}$$

Combining Eqs. (5.7-5) and (5.7-6),

Equation 5.7-7.

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = (T_\infty - T_S)\left(\frac{0.332}{x} N_{\text{Re},x}^{1/2}\right)$$

The convective equation can be related to the Fourier equation by the following, where q_y is in J/s or W (btu/h):

Equation 5.7-8.

$$\frac{q_y}{A} = h_x(T_S - T_\infty) = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

Combining Eqs. (5.7-7) and (5.7-8),

Equation 5.7-9.

$$\frac{h_x x}{k} = N_{\text{Nu},x} = 0.332 N_{\text{Re},x}^{1/2}$$

where $N_{\text{Nu},x}$ is the dimensionless Nusselt number and h_x is the local heat-transfer coefficient at point x on the plate.

Pohlhausen (K1) was able to show that the relation between the hydrodynamic and thermal boundary layers for fluids with Prandtl number >0.6 gives approximately

Equation 5.7-10.

$$\frac{\delta}{\delta_T} = N_{\text{Pr}}^{1/3}$$

As a result, the equation for the local heat-transfer coefficient is

Equation 5.7-11.

$$h_x = 0.332 \frac{k}{x} N_{\text{Re},x}^{1/2} N_{\text{Pr}}^{1/3}$$

Also,

Equation 5.7-12.

$$\frac{h_x x}{k} = N_{\text{Nu},x} = 0.332 N_{\text{Re},x}^{1/2} N_{\text{Pr}}^{1/3}$$

The equation for the mean heat-transfer coefficient h from $x = 0$ to $x = L$ for a plate of width b and area bL is

Equation 5.7-13.

$$h = \frac{b}{A}\int_0^L h_x\, dx$$

$$= \frac{1}{L} 0.332k\left(\frac{\rho v_\infty}{\mu}\right)^{1/2} N_{\text{Pr}}^{1/3} \int_0^L \frac{dx}{x^{1/2}}$$

Integrating,

Equation 5.7-14.

$$h = 0.644 \frac{k}{L} N_{Re,L}^{1/2} N_{Pr}^{1/3}$$

Equation 5.7-15.

$$\frac{hL}{k} = N_{Nu} = 0.644 N_{Re,L}^{1/2} N_{Pr}^{1/3}$$

As pointed out previously, this laminar boundary layer on smooth plates holds up to a Reynolds number of about 5 × 10^5. In using the results above, the fluid properties are usually evaluated at the film temperature $T_f = (T_S + T_\infty)/2$.

Approximate Integral Analysis of the Thermal Boundary Layer

As discussed in the analysis of the hydrodynamic boundary layer, the Blasius solution is accurate but limited in its scope. Other, more complex systems cannot be solved by this method. The approximate integral analysis used by von Kármán to calculate the hydrodynamic boundary layer was covered in Section 3.10. This approach can be used to analyze the thermal boundary layer.

This method will be outlined briefly. First, a control volume, as previously given in Fig. 3.10-5, is used to derive the final energy integral expression:

Equation 5.7-16.

$$\frac{k}{\rho c_p}\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{d}{dx}\int_0^{\delta_T} v_x (T_\infty - T)\, dy$$

This equation is analogous to Eq. (3.10-48) combined with Eq. (3.10-51) for the momentum analysis, giving

Equation 5.7-17.

$$\frac{\mu}{\rho}\left(\frac{\partial v_x}{\partial y}\right)_{y=0} = \frac{d}{dx}\int_0^{\delta} v_x (v_\infty - v_x)\, dy$$

Equation (5.7-16) can be solved if both a velocity profile and temperature profile are known. The assumed velocity profile used is Eq. (3.10-50):

Equation 3.10-50.

$$\frac{v_x}{v_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

The same form of temperature profile is assumed:

Equation 5.7-18.

$$\frac{T - T_S}{T_\infty - T_S} = \frac{3}{2}\frac{y}{\delta_T} - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3$$

Substituting Eqs. (3.10-50) and (5.7-18) into the integral expression and solving,

Equation 5.7-19.

$$N_{\mathrm{Nu},x} = 0.36 N_{\mathrm{Re},x}^{1/2} N_{\mathrm{Pr}}^{1/3}$$

This is only about 8% greater than the exact result in Eq. (5.7-11), which indicates that this approximate integral method can be used with confidence in cases where exact solutions cannot be obtained.

In a similar fashion, the integral momentum analysis method used for the turbulent hydrodynamic boundary layer in Section 3.10 can be used for the thermal boundary layer in turbulent flow. Again, the Blasius $\frac{1}{7}$-power law is used for the temperature distribution. These give results that are quite similar to the experimental equations given in Section 4.6.

Prandtl Mixing Length and Eddy Thermal Diffusivity

Eddy momentum diffusivity in turbulent flow

In Section 3.10F the total shear stress $\bar{\tau}^t_{yx}$ for turbulent flow was written as follows when the molecular and turbulent contributions are summed together:

Equation 5.7-20.

$$\bar{\tau}^t_{yx} = -\rho\left(\frac{\mu}{\rho} + \varepsilon_t\right)\frac{d\bar{v}_x}{dy}$$

The molecular momentum diffusivity μ/ρ in m^2/s is a function only of the fluid molecular properties. However, the turbulent momentum eddy diffusivity ε_t depends on the fluid motion. In Eq. (3.10-29) we related ε_t to the Prandtl mixing length L as follows:

Equation 3.10-29.

$$\varepsilon_t = L^2\left|\frac{d\bar{v}_x}{dy}\right|$$

Prandtl mixing length and eddy thermal diffusivity

We can derive the eddy thermal diffusivity α_t for turbulent heat transfer in a similar manner, as follows. Eddies or clumps of fluid are transported a distance L in the y direction. At this point L the clump of fluid differs in mean velocity from the adjacent fluid by the velocity v'_x, which is the fluctuating velocity component discussed in Section 3.10F. Energy is also transported the distance L with a velocity v'_y in the y direction together with the mass being transported. The instantaneous temperature of the fluid is $T = T' + \bar{T}$ where $\bar{T}$ is the mean value and T' the deviation from the mean value. This fluctuating T' is similar to the fluctuating velocity v'_x. The mixing length is small enough that the temperature difference can be written as

Equation 5.7-21.

$$T' = L\frac{d\bar{T}}{dy}$$

The rate of energy transported per unit area is q_y/A and is equal to the mass flux in the y direction times the heat capacity times the temperature difference:

Equation 5.7-22.

$$\frac{q_y}{A} = \frac{-v'_y \rho c_p L \ d\overline{T}}{dy}$$

In Section 3.10F we assumed $v'_x \cong v'_y$ and that

Equation 5.7-23.

$$v'_x = v'_y = L\left|\frac{d\bar{v}_x}{dy}\right|$$

Substituting Eq. (5.7-23) into (5.7-22),

Equation 5.7-24.

$$\frac{q_y}{A} = -\rho c_p L^2 \left|\frac{d\bar{v}_x}{dy}\right| \frac{d\overline{T}}{dy}$$

According to Eq. (3.10-29) the term $L^2|d\bar{v}_x/dy|$ is the momentum eddy diffusivity ε_t. When this term is in the turbulent heat-transfer equation (5.7-24), it is called α_t, eddy thermal diffusivity. Then Eq. (5.7-24) becomes

Equation 5.7-25.

$$\frac{q_y}{A} = -\rho c_p \alpha_t \frac{d\overline{T}}{dy}$$

Combining this with the Fourier equation written in terms of the molecular thermal diffusivity α,

Equation 5.7-26.

$$\frac{q_y}{A} = -\rho c_p(\alpha + \alpha_t) \frac{d\overline{T}}{dy}$$

Similarities among momentum, heat, and mass transport

Equation (5.7-26) is similar to Eq. (5.7-20) for total momentum transport. The eddy thermal diffusivity α_t and the eddy momentum diffusivity ε_t have been assumed equal in the derivations. Experimental data show that this equality is only approximate. An eddy mass diffusivity for mass transfer has been defined in a similar manner using the Prandtl mixing length theory and is assumed equal to α_t and ε_t.

PROBLEMS

5.2-1. ***Temperature Response in Cooling a Wire.*** A small copper wire with a diameter of 0.792 mm and initially at 366.5 K is suddenly immersed in a liquid held constant at 311 K. The convection coefficient h = 85.2 W/m^2 · K. The physical properties can be assumed constant and are k = 374 W/m · K, c_p = 0.389 kJ/kg · K, and ρ = 8890 kg/m^3.

a. Determine the time in seconds for the average temperature of the wire to drop to 338.8 K (one-half the initial temperature difference).

b. Do the same but for $h = 11.36$ W/m$^2 \cdot$ K.

c. For part (b), calculate the total amount of heat removed for a wire 1.0 m long.

A1: **Ans.** (a) $t = 5.66$ s

5.2-2. ***Quenching Lead Shot in a Bath.*** Lead shot having an average diameter of 5.1 mm is at an initial temperature of 204.4°C. To quench the shot it is added to a quenching oil bath held at 32.2°C and falls to the bottom. The time of fall is 15 s. Assuming an average convection coefficient of $h = 199$ W/m$^2 \cdot$ K, what will be the temperature of the shot after the fall? For lead, $\rho = 11\ 370$ kg/m^3 and $c_p = 0.138$ kJ/kg · K.

5.2-3. ***Unsteady-State Heating of a Stirred Tank.*** A vessel is filled with 0.0283 m^3 of water initially at 288.8 K. The vessel, which is well stirred, is suddenly immersed in a steam bath held at 377.6 K. The overall heat-transfer coefficient U between the steam and water is 1136 W/m$^2 \cdot$ K and the area is 0.372 m^2. Neglecting the heat capacity of the walls and agitator, calculate the time in hours to heat the water to 338.7 K. [*Hint:* Since the water is well stirred, its temperature is uniform. Show that Eq. (5.2-3) holds by starting with Eq. (5.2-1).]

5.3-1. ***Temperature in a Refractory Lining.*** A combustion chamber has a 2-in.-thick refractory lining to protect the outer shell. To predict the thermal stresses at start-up, the temperature 0.2 in. below the surface is needed 1 min after start-up. The following data are available. The initial temperature $T_0 = 100$°F, the hot gas temperature $T_1 = 3000$°F, $h = 40$ btu/h · ft^2 · °F, $k = 0.6$ btu/h · ft · °F, and $\alpha = 0.020$ ft^2/h. Calculate the temperature at a 0.2-in. depth and a 0.6-in. depth. Use Fig. 5.3-3 and justify its use by seeing if the lining acts as a semi-infinite solid during this 1-min period.

A4: **Ans.** For $x = 0.2$ in., $(T - T_0)/(T_1 - T_0) = 0.28$ and $T = 912$°F (489°C); for $x = 0.6$ in., $(T - T_0)/(T_1 - T_0) = 0.02$ and $T = 158$°F (70°C)

5.3-2. ***Freezing Temperature in the Soil.*** The average temperature of the soil to a considerable depth is approximately 277.6 K (40°F) during a winter day. If the outside air temperature suddenly drops to 255.4 K (0°F) and stays there, how long will it take for a pipe 3.05 m (10 ft) below the surface to reach 273.2 K (32°F)? The convective coefficient is $h = 8.52$ W/m$^2 \cdot$ K (1.5 btu/h · ft^2 · °F). The soil physical properties can be taken as 5.16×10^{-7} m^2/s (0.02 ft^2/h) for the thermal diffusivity and 1.384 W/m · K (0.8 btu/h · ft · °F) for the thermal conductivity. (*Note:* The solution is trial and error, since the unknown time appears twice in the graph for a semi-infinite solid.)

5.3-3. ***Cooling a Slab of Aluminum.*** A large piece of aluminum that can be considered a semi-infinite solid initially has a uniform temperature of 505.4 K. The surface is suddenly exposed to an environment at 338.8 K with a surface convection coefficient of 455 W/m$^2 \cdot$ K. Calculate the time in hours for the temperature to reach 388.8 K at a depth of 25.4 mm. The average physical properties are $\alpha = 0.340$ m^2/h and $k = 208$ W/m · K.

5.3-4. ***Transient Heating of a Concrete Wall.*** A wall made of concrete 0.305 m thick is insulated on the rear side. The wall at a uniform temperature of 10°C (283.2 K) is exposed on the front side to a gas at 843°C (1116.2 K). The convection coefficient is 28.4 W/m^2 · K, the thermal diffusivity is 1.74 × 10^{-3} m^2/h, and the thermal conductivity is 0.935 W/m · K.

a. Calculate the time for the temperature at the insulated face to reach 232°C (505.2 K).

b. Calculate the temperature at a point 0.152 m below the surface at this same time.

A7: **Ans.** (a) $\alpha t/x_1^2 = 0.25$,t =13.4 h

5.3-5. ***Cooking a Slab of Meat.*** A slab of meat 25.4 mm thick originally at a uniform temperature of 10°C is to be cooked from both sides until the center reaches 121°C in an oven at 177°C. The convection coefficient can be assumed constant at 25.6 W/m^2 · K. Neglect any latent heat changes and calculate the time required. The thermal conductivity is 0.69 W/m · K and the thermal diffusivity 5.85 × 10^{-4} m^2/h. Use the Heisler chart.

A8: **Ans.** 0.80 h (2880 s)

5.3-6. ***Unsteady-State Conduction in a Brick Wall.*** A flat brick wall 1.0 ft thick is the lining on one side of a furnace. If the wall is at a uniform temperature of 100°F and one side is suddenly exposed to a gas at 1100°F, calculate the time for the furnace wall at a point 0.5 ft from the surface to reach 500°F. The rear side of the wall is insulated. The convection coefficient is 2.6 btu/h · ft^2 · °F and the physical properties of the brick are k = 0.65 btu/h · ft · °F and α = 0.02 ft^2/h.

5.3-7. ***Cooling a Steel Rod.*** A long steel rod 0.305 m in diameter is initially at a temperature of 588 K. It is immersed in an oil bath maintained at 311 K. The surface convective coefficient is 125 W/m^2 · K. Calculate the temperature at the center of the rod after 1 h. The average physical properties of the steel are k = 38 W/m · K and α = 0.0381 m^2/h.

A10: **Ans.** T = 391 K

5.3-8. ***Effect of Size on Heat Processing Meat.*** An autoclave held at 121.1°C is being used to process sausage meat 101.6 mm in diameter and 0.61 m long which is originally at 21.1°C. After 2 h the temperature at the center is 98.9°C. If the diameter is increased to 139.7 mm, how long will it take for the center to reach 98.9°C? The heat-transfer coefficient to the surface is h = 1100 W/m^2 · K, which is very large, so the surface resistance can be considered negligible. (Show this.) Neglect the heat transfer from the ends of the cylinder. The thermal conductivity k = 0.485 W/m · K.

A11: **Ans.** 3.78 h

5.3-9. ***Temperature of Oranges on Trees During Freezing Weather.*** In orange-growing areas, the freezing of the oranges on the trees during cold nights is of serious economic concern. If the oranges are initially at a temperature of 21.1°C, calculate the center temperature of the orange if exposed to air at −3.9°C for 6 h. The oranges are 102 mm in diameter and the convective coefficient is estimated as 11.4 $W/m^2 \cdot K$. The thermal conductivity *k* is 0.431 W/m · K and α is 4.65 × 10^{-4} m^2/h. Neglect any latent heat effects.

A12: **Ans.** $(T_1 - T)/(T_1 - T_0) = 0.05$, $T = -2.65$°C

5.3-10. ***Hardening a Steel Sphere.*** To harden a steel sphere having a diameter of 50.8 mm, it is heated to 1033 K and then dunked into a large water bath at 300 K. Determine the time for the center of the sphere to reach 366.5 K. The surface coefficient can be assumed as 710 $W/m^2 \cdot K$, k = 45 W/m · K, and α = 0.0325 m^2/h.

5.3-11. ***Unsteady-State Conduction in a Short Cylinder.*** An aluminum cylinder is initially heated to a uniform temperature of 204.4°C. Then it is plunged into a large bath held at 93.3°C, where h = 568 $W/m^2 \cdot K$. The cylinder has a diameter of 50.8 mm and is 101.6 mm long. Calculate the center temperature after 60 s. The physical properties are α = 9.44 × 10^{-5} m^2/s and k = 207.7 W/m · K.

5.3-12. ***Conduction in Three Dimensions in a Rectangular Block.*** A rectangular steel block 0.305 m by 0.457 m by 0.61 m is initially at 315.6°C. It is suddenly immersed in an environment at 93.3°C. Determine the temperature at the center of the block after 1 h. The surface convection coefficient is 34 $W/m^2 \cdot K$. The physical properties are k = 38 W/m · K and α = 0.0379 m^2/h.

5.4-1. ***Schmidt Numerical Method for Unsteady-State Conduction.*** A material in the form of an infinite plate 0.762 m thick is at an initial uniform temperature of 366.53 K. The rear face of the plate is insulated. The front face is suddenly exposed to a temperature of 533.2 K. The convective resistance at this face can be assumed as zero. Calculate the temperature profile after 0.875 h using the Schmidt numerical method with M = 2 and slabs 0.1524 m thick. The thermal diffusivity is 0.0929 m^2/h.

A16: **Ans.** Δt = 0.125 h, seven time increments needed

5.4-2. ***Unsteady-State Conduction with Nonuniform Initial Temperature Profile.*** Use the same conditions as in Problem 5.4-1 but with the following change. The initial temperature profile is not uniform but is 366.53 K at the front face and 422.1 K at the rear face with a linear variation between the two faces.

5.4-3. ***Unsteady-State Conduction Using the Digital Computer.*** Repeat Problem 5.4-2 but use the computer and a spreadsheet. Use slabs 0.03048 m thick and M = 2.0. Calculate the temperature profile after 0.875 h.

5.4-4. ***Chilling Meat Using Numerical Methods.*** A slab of beef 45.7 mm thick and initially at a uniform temperature of 283 K is being chilled by a surface contact cooler at 274.7 K on the front face. The rear face of the meat is insulated. Assume that the convection resistance at the front surface is zero. Using five slices and M = 2, calculate the temperature profile after 0.54 h. The thermal diffusivity is 4.64 × 10^{-4} m^2/h.

A19: **Ans.** Δt = 0.090 h, six time increments

5.4-5. ***Cooling Beef with Convective Resistance.*** A large slab of beef is 45.7 mm thick and is at an initial uniform temperature of 37.78°C. It is being chilled at the front surface in a chilled air blast at −1.11°C with a convective heat-transfer coefficient of $h = 38.0$ W/m^2 · K. The rear face of the meat is insulated. The thermal conductivity of the beef is $k = 0.498$ W/m · K and $\alpha = 4.64 \times 10^{-4}$ m^2/h. Using a numerical method with five slices and $M = 4.0$, calculate the temperature profile after 0.27 h. [*Hint:* Since there is a convective resistance, the value of N must be calculated. Also, Eq. (5.4-7) should be used.]

A20: **Ans.** 17.16°C (n = 1), 28.22°C (2), 34.48°C (3), 37.00°C (4), 37.67°C (5), 37.77°C (6)

5.4-6. ***Cooling Beef Using the Digital Computer.*** Repeat Problem 5.4-5 using the digital computer. Use 20 slices and $M = 4.0$. Use a spreadsheet calculation.

5.4-7. ***Convection and Unsteady-State Conduction.*** For the conditions of Example 5.4-3, continue the calculations for a total of 12 time increments. Plot the temperature profile.

5.4-8. ***Alternative Convective Boundary Condition for Numerical Method.*** Repeat Example 5.4-3 but instead use the alternative boundary condition, Eq. (5.4-11). Also, use $M = 4$. Calculate the profile for the full 12 time increments.

5.4-9. ***Numerical Method for Semi-infinite Solid and Convection.*** A semi-infinite solid initially at a uniform temperature of 200°C is cooled at its surface by convection. The cooling fluid at a constant temperature of 100°C has a convective coefficient of $h = 250$ W/m^2 · K. The physical properties of the solid are $k = 20$ W/m · K and $\alpha = 4 \times 10^{-5}$ m^2/s. Using a numerical method with $\Delta x = 0.040$ m and $M = 4.0$, calculate the temperature profile after 50 s total time.

A24: **Ans.** $T_1 = 157.72$, $T_2 = 181.84$, $T_3 = 194.44$, $T_4 = 198.93$, $T_5 = 199.90$°C

5.5-1. ***Chilling Slab of Beef.*** Repeat Example 5.5-1, where the slab of beef is cooled to 10°C at the center, but use air of 0°C at a lower value of $h = 22.7$ W/m^2 · K.

A25: **Ans.** $(T_1 - T)/(T_1 - T_0) = 0.265$, $X = 0.92$, $t = 19.74$ h

5.5-2. ***Chilling Fish Fillets.*** Codfish fillets originally at 10°C are packed to a thickness of 102 mm. Ice is packed on both sides of the fillets and wet-strength paper separates the ice and fillets. The surface temperature of the fish can be assumed as essentially 0°C. Calculate the time for the center of the fillets to reach 2.22°C and the temperature at this time at a distance of 25.4 mm from the surface. Also, plot temperature versus position for the slab. The physical properties are (B1) $k = 0.571$ W/m · K, $\rho = 1052$ kg/m^3, and $c_p = 4.02$ kJ/kg · K.

5.5-3. ***Average Temperature in Chilling Fish.*** Fish fillets having the same physical properties given in Problem 5.5-2 are originally at 10°C. They are packed to a thickness of 102 mm with ice on each side. Assuming that the surface temperature of the fillets is 0°C, calculate the time for the average temperature to reach 1.39°C. (*Note:* This is a case where the surface resistance is zero. Can Fig. 5.3-13 be used for this case?)

5.5-4. ***Time to Freeze a Slab of Meat.*** Repeat Example 5.5-2 using the same conditions except that a plate or contact freezer is used where the surface coefficient can be assumed as $h = 142\ W/m^2 \cdot K$.

A28: **Ans.** $t = 2.00$ h

5.5-5. ***Freezing a Cylinder of Meat.*** A package of meat containing 75% moisture and in the form of a long cylinder 5 in. in diameter is to be frozen in an air-blast freezer at −25°F. The meat is initially at the freezing temperature of 27°F. The heat-transfer coefficient is $h = 3.5\ btu/h \cdot ft^2 \cdot °F$. The physical properties are $\rho = 64\ lb_m/ft^3$ for the unfrozen meat and $k = 0.60\ btu/h \cdot ft \cdot °F$ for the frozen meat. Calculate the freezing time.

5.6-1. ***Heat Generation Using Equation of Energy Change.*** A plane wall with uniform internal heat generation of $\dot{q}\ W/m^3$ is insulated at four surfaces, with heat conduction only in the x direction. The wall has a thickness of $2L$ m. The temperature at one wall at $x = +L$ and at the other wall at $x = -L$ is held constant at T_w K. Using the differential equation of energy change, Eq. (5.6-18), derive the equation for the final temperature profile.

A30: **Ans.**

$$T = \frac{\dot{q}(L^2 - x^2)}{2k} + T_w$$

5.6-2. ***Heat Transfer in a Solid Using Equation of Energy Change.*** A solid of thickness L is at a uniform temperature of T_0 K. Suddenly the front surface temperature of the solid at $z = 0$ m is raised to T_1 at $t = 0$ and held there, and at $z = L$ at the rear to T_2 and held constant. Heat transfer occurs only in the z direction. For constant physical properties and using the differential equation of energy change, do as follows:

a. Derive the partial differential equation and the boundary conditions (B.C.) for unsteady-state energy transfer.
b. Do the same for steady state and integrate the final equation.

A31: **Ans.** (a) $\partial T/\partial t = \alpha\, \partial^2 T/\partial z^2$; B.C.(1): $t = 0$, $z = z$, $T = T_0$; B.C.(2): $t = t$, $z = 0$, $T = T_1$; B.C.(3): $t = t$, $z = L$, $T = T_2$; (b) $T = (T_2 - T_1)z/L + T_1$

5.6-3. ***Radial Temperature Profile Using the Equation of Energy Change.*** Radial heat transfer is occurring by conduction through a long, hollow cylinder of length L with the ends insulated.

a. What is the final differential equation for steady-state conduction? Start with Fourier's second law in cylindrical coordinates, Eq. (5.6-20).
b. Solve the equation for the temperature profile from part (a) for the boundary conditions given as follows: $T = T_i$ for $r = r_i$, $T = T_o$ for $r = r_o$.
c. Using part (b), derive an expression for the heat flow q in W.

A32: **Ans.**

$$T = T_i - \frac{T_i - T_o}{\ln(r_o/r_i)} \ln \frac{r}{r_i}$$

5.6-4. ***Heat Conduction in a Sphere.*** Radial energy flow is occurring in a hollow sphere with an inside radius of r_i and an outside radius of r_o. At steady state the inside surface temperature is constant at T_i and constant at T_o on the outside surface.

a. Using the differential equation of energy change, solve the equation for the temperature profile.

b. Using part (a), derive an expression for the heat flow in W.

5.6-5. ***Variable Heat Generation and Equation of Energy Change.*** A plane wall is insulated so that conduction occurs only in the x direction. The boundary conditions which apply at steady state are $T = T_0$ at $x = 0$ and $T = T_L$ at $x = L$. Internal heat generation per unit volume is occurring and varies as $\dot{q} = \dot{q}_0 e^{-\beta x/L}$, where $\dot{q}_0$ and ß are constants. Solve the general differential equation of energy change for the temperature profile.

5.7-1. ***Thermal and Hydrodynamic Boundary Layer Thicknesses.*** Air at 294.3 K and 101.3 kPa with a free stream velocity of 12.2 m/s is flowing parallel to a smooth, flat plate held at a surface temperature of 383 K. Do the following:

a. At the critical $N_{Re,L} = 5 \times 10^5$, calculate the critical length $x = L$ of the plate, the thickness δ of the hydrodynamic boundary layer, and the thickness δ_T of the thermal boundary layer. Note that the Prandtl number is not 1.0.

b. Calculate the average heat-transfer coefficient over the plate covered by the laminar boundary layer.

5.7-2. ***Boundary-Layer Thicknesses and Heat Transfer.*** Air at 37.8°C and 1 atm abs flows at a velocity of 3.05 m/s parallel to a flat plate held at 93.3°C. The plate is 1 m wide. Calculate the following at a position 0.61 m from the leading edge:

a. The thermal boundary-layer thickness δ_T and the hydrodynamic boundary-layer thickness δ.

b. Total heat transfer from the plate.

REFERENCES

Bibliography

[ch05biblio01entry01] (B1) N Blakebrough,. *Biochemical and Biological Engineering Science*, Vol. 2. New York: Academic Press, Inc., 1968.

[ch05biblio01entry02] (B2) R. B., Bird, W. E., Stewart, and E. N. Lightfoot, *Transport Phenomena*. New York: John Wiley & Sons, Inc., 1960.

[ch05biblio01entry03] (C1) H. S., Carslaw, and J. C. Jaeger, *Conduction of Heat in Solids*. Oxford: Clarendon Press, 1959.

[ch05biblio01entry04] (C2) S. E. Charm, *The Fundamentals of Food Engineering*, 2nd ed. Westport, Conn.: Avi Publishing Co., Inc., 1971.

[ch05biblio01entry05] (D1) G. M. Dusinberre, *Heat Transfer Calculations by Finite Differences*. Scranton, Pa.: International Textbook Co., Inc., 1961.

[ch05biblio01entry06] (G1) C. J. Geankoplis, *Mass Transport Phenomena*. Columbus, Ohio: Ohio State University Bookstores, 1972.

[ch05biblio01entry07] (G2) H. P., Gurney, and J. Lurie, *Ind. Eng. Chem.*, **15**, 1170 (1923).

[ch05biblio01entry08] (H1) H. P. Heisler, *Trans. A.S.M.E.*, **69**, 227 (1947).

[ch05biblio01entry09] (H2) T. Hodgson, *Fd. Inds. S. Afr.*, **16**, 41 (1964); *Int. Inst. Refrig. Annexe*, **1966**, 633 (1966).

[ch05biblio01entry10] (K1) F. Kreith, *Principles of Heat Transfer*, 2nd ed. Scranton, Pa.: International Textbook Company, 1965.

[ch05biblio01entry11] (K2) F., Kreith, and W. Z. Black, *Basic Heat Transfer*. New York: Harper & Row, Publishers, 1980.

[ch05biblio01entry12] (N1) A. H. Newman, *Ind. Eng. Chem.*, **28**, 545 (1936).

[ch05biblio01entry13] (P1) R. H., Perry, and C. H. Chilton, *Chemical Engineers' Handbook*, 5th ed. New York: McGraw-Hill Book Company, 1973.

[ch05biblio01entry14] (P2) R. Z. Plank, *Ges. Kalteind.*, **20**, 109 (1913); *Z. Ges. Kalteind. Bieh. Reih*, **10** (3), 1 (1941).

[ch05biblio01entry15] (R1) L. Riedel, *Kaltetchnik*, **8**, 374 (1956); **9**, 38 (1957); **11**, 41 (1959); **12**, 4 (1960).

[ch05biblio01entry16] (S1) P. J. Schneider, *Conduction Heat Transfer*. Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1955.

[ch05biblio01entry17] (S2) R., Siegel, E. M., Sparrow, and T. M. Hallman, *Appl. Sci. Res.*, **A7**, 386 (1958).