Heat Conduction Equation

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Content

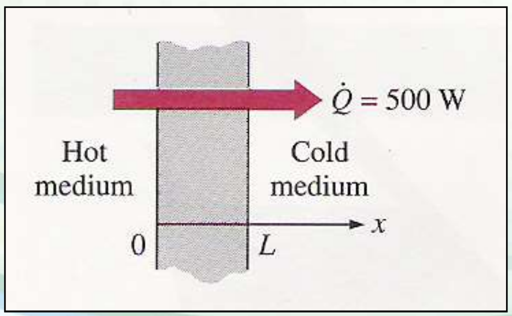
- Multidimensionality and time dependence of heat transfer.
- The differential equation of heat conduction in various coordinate systems.
- Thermal conditions on surfaces.
- One-dimensional heat conduction problems.
- One-dimensional heat conduction in solids that involve heat generation.
- Evaluation of heat conduction in solids with temperature-dependent thermal conductivity.

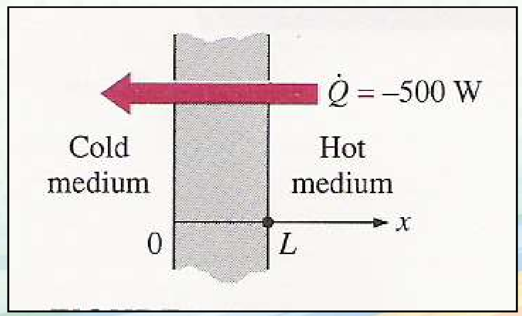
Introduction

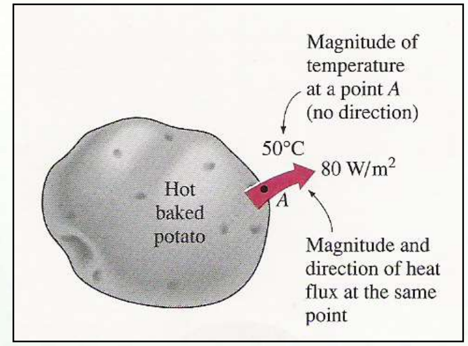
- Heat transfer has a direction as well as a magnitude

→ q is a vector

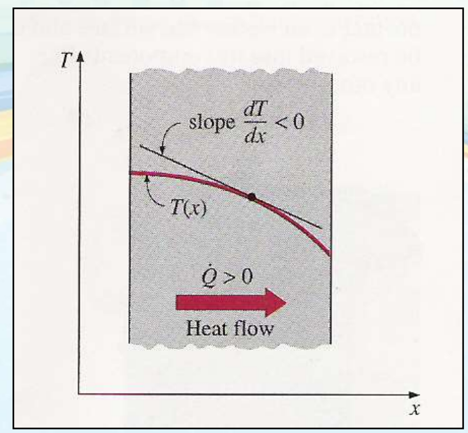
- Sign convention for heat transfer:







- Conduction is proportional to the temperature gradient.



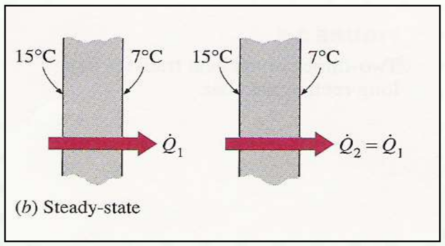
Introduction

- Classification of conduction problems:

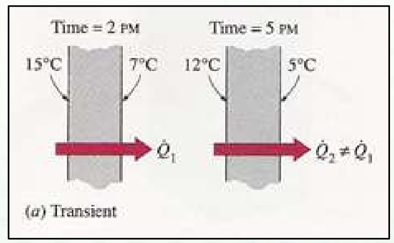
  in general $T = T(x, y, z, t)$

⇒ Steady state versus transient conduction

$T = T(x)$



(*b*) Steady-state

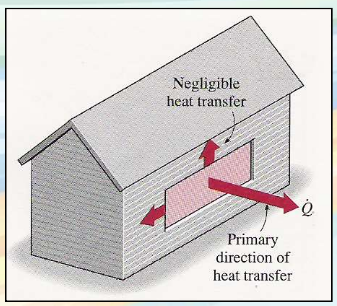


(*a*) Transient

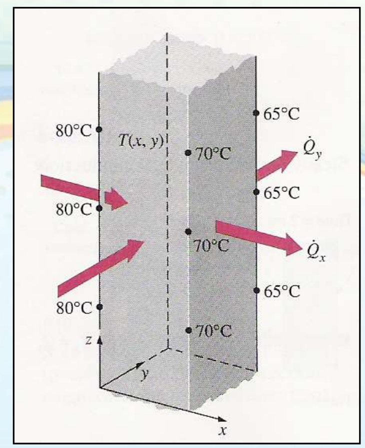
$T = T(x, t)$

⇒ One-dimensional versus two dimensional conduction

$T = T(x)$

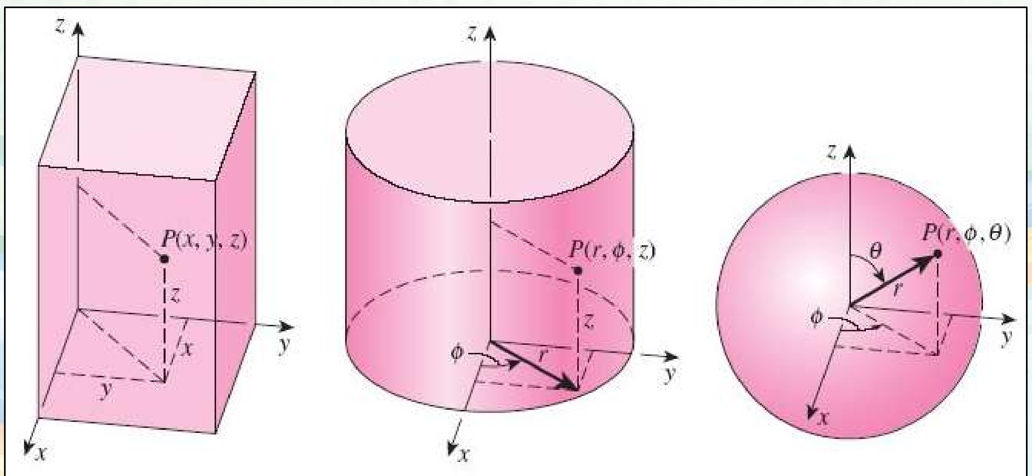


$T = T(x, y)$

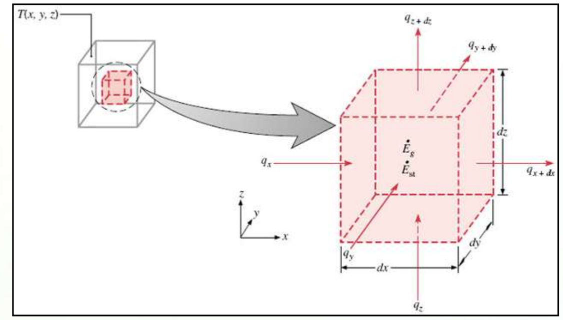


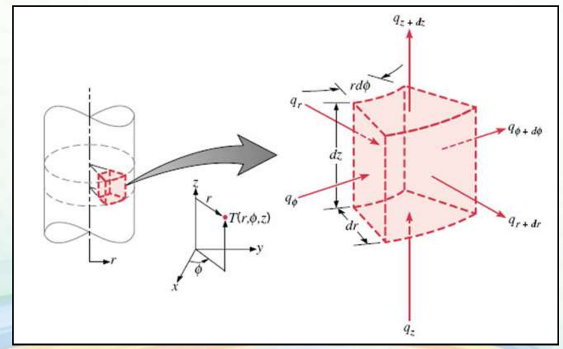
Introduction

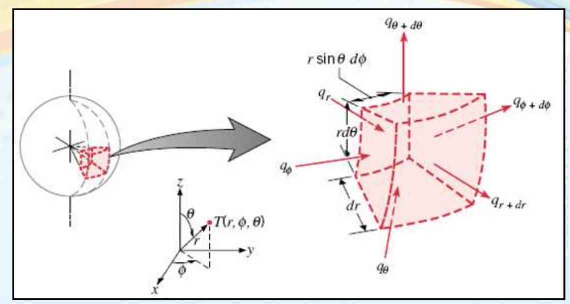
- Three-dimensional conduction cases
  - ⇒ Rectangular: T = T(x, y, z)
  - ⇒ Cylindrical: T = T(r, φ, z)
  - ⇒ Spherical: T = T(r, φ ,θ)



(a) Rectangular coordinates (b) Cylindrical coordinates (c) Spherical coordinates







Fourier's Law

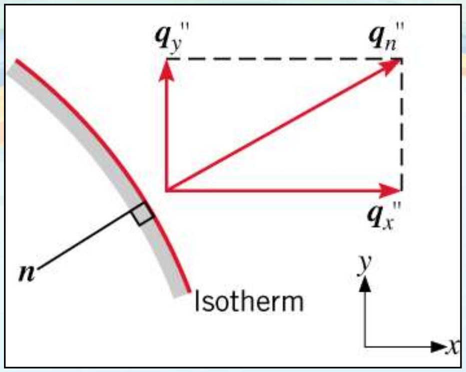
- A rate equation that allows determination of the conduction heat flux from knowledge of the temperature distribution in a medium

- Its most general (vector) form for multidimensional conduction is:

$$\left(k \equiv -\overrightarrow{q''}/\vec{\nabla}T\right) \qquad \overrightarrow{q''} = -k\vec{\nabla}T$$

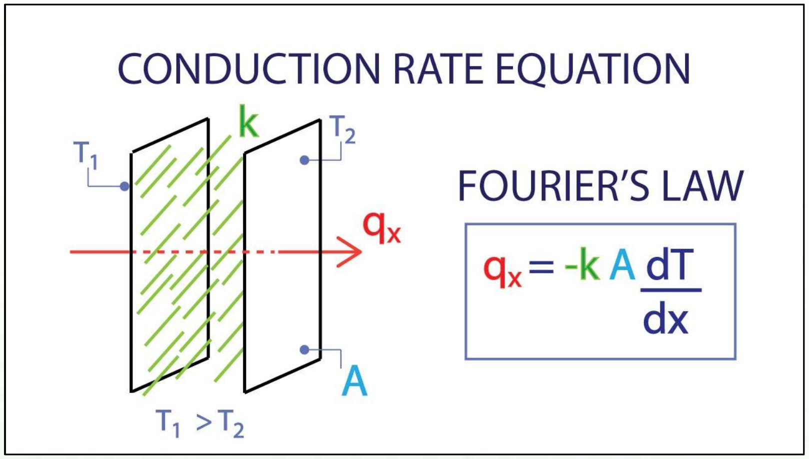
Implications:

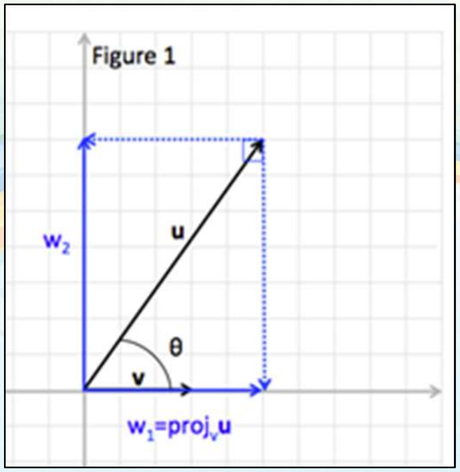
- Heat transfer is in the direction of decreasing temperature (basis for minus sign).



Fourier's Law

- Fourier's Law serves to define the thermal conductivity of the medium
- Direction of heat transfer is perpendicular to lines of constant temperature (isotherms).
- Heat flux vector may be resolved into orthogonal components.





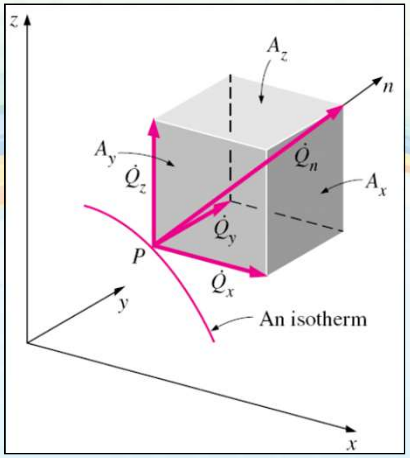
General Relation for Fourier's Law

- In rectangular coordinates, the heat conduction vector can be expressed in terms of its components as

$$\vec{\dot{Q}}_n = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k}$$

- which can be determined from Fourier's law as

$$\begin{cases} \dot{Q}_x = -kA_x\dfrac{\partial T}{\partial x} = -kdydz\dfrac{\partial T}{\partial x} \\ \dot{Q}_y = -kA_y\dfrac{\partial T}{\partial y} = -kdxdz\dfrac{\partial T}{\partial y} \\ \dot{Q}_z = -kA_z\dfrac{\partial T}{\partial z} = -kdxdy\dfrac{\partial T}{\partial z} \end{cases}$$



Heat Generation

- Examples of heat generation:
  1. Electrical energy being converted to heat at a rate of I^2R,
  2. Fuel elements of nuclear reactors,
  3. Exothermic chemical reactions.
- Heat generation is a volumetric phenomenon.
- The rate of heat generation units: W/m^3 or $Btu/h \cdot ft^3$.
- The rate of heat generation in a medium may vary with time as well as position within the medium.
- The total rate of heat generation in a medium of volume V can be determined from

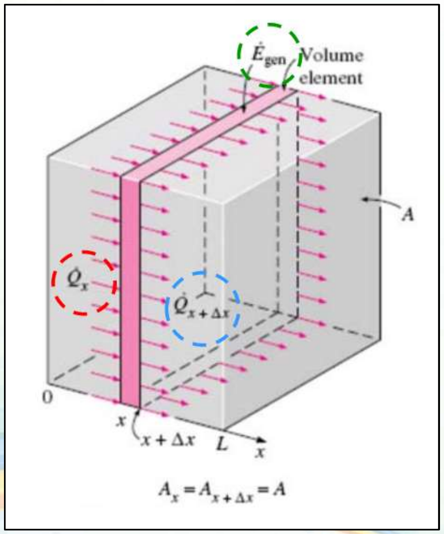
$$\dot{E}_{gen} = \int_V \dot{e}_{gen}\, dV \quad (W)$$

One-Dimensional Heat Conduction Equation - Plane Wall

- Applying conservation of energy to a differential control volume through which energy transfer is exclusively by conduction.

(Rate of heat conduction at x) - (Rate of heat conduction at x+Dx) + (Rate of heat generation inside the element) = (Rate of change of the energy content of the element)

$$\dot{Q}_x \quad -\dot{Q}_{x+\Delta x} \quad +\dot{E}_{gen,element} \quad = \frac{\Delta E_{element}}{\Delta t}$$



- The change in the energy content and the rate of heat generation can be expressed as

$$\begin{cases} \Delta E_{element} = E_{t+\Delta t} - E_t = mc(T_{t+\Delta t} - T_t) = \rho c A \Delta x (T_{t+\Delta t} - T_t) \\ \dot{E}_{gen,element} = \dot{e}_{gen} V_{element} = \dot{e}_{gen} A \Delta x \end{cases}$$

- Substituting these two equations above, we get

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{e}_{gen} A\Delta x = \rho c A\Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

- Dividing by A.Δx, taking the limit as Δx ⟶ 0 and Δt ⟶ 0 and from Fourier's law $\dot{Q}_x = -kA\frac{\partial T}{\partial x}$

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{e}_{gen} = \rho c\frac{\partial T}{\partial t}$$

solution provides the temperature distribution in a stationary medium.

- The area *A* is constant for a plane wall → the one dimensional transient heat conduction equation in a plane wall is

✓ Variable conductivity:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{e}_{gen} = \rho c\frac{\partial T}{\partial t}$$

✓ Constant conductivity:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \; ; \; \alpha = \frac{k}{\rho c}$$

One-Dimensional Heat Conduction Equation - Plane Wall

- The one-dimensional conduction equation may be reduces to the following forms under special conditions

1) Steady-state:

$$\frac{d^2T}{dx^2} + \frac{\dot{e}_{gen}}{k} = 0$$

2) Transient, no heat generation:

$$\frac{\partial^2 T}{dx^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

3) Steady-state, no heat generation:

$$\frac{d^2T}{dx^2} = 0$$

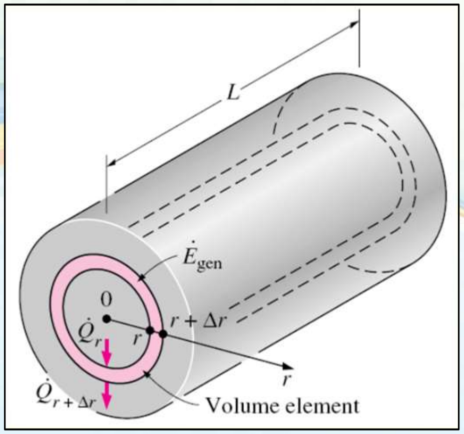
One-Dimensional Heat Conduction Equation - Long Cylinder

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r\end{pmatrix} - \begin{pmatrix}\text{Rate of}\\ \text{heat}\\ \text{conduction}\\ \text{at } r+\Delta r\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

$$\dot{Q}_r \quad - \dot{Q}_{r+\Delta r} \quad + \dot{E}_{gen,element} \quad = \frac{\Delta E_{element}}{\Delta t}$$

- The change in the energy content and the rate of heat generation can be expressed as:

$$\begin{cases}\Delta E_{element} = E_{t+\Delta t} - E_t = mc(T_{t+\Delta t} - T_t) = \rho c A \Delta r (T_{t+\Delta t} - T_t) \\ \dot{E}_{gen,element} = \dot{e}_{gen} V_{element} = \dot{e}_{gen} A \Delta r\end{cases}$$



One-Dimensional Heat Conduction Equation - Long Cylinder

- Substituting into the main equation, we get

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{e}_{gen} A\Delta r = \rho c A\Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

- Dividing by $A.\Delta r$, taking the limit as $\Delta r \rightarrow 0$ and $\Delta t \rightarrow 0$, and from Fourier's law:

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t}$$

- Noting that the area varies with the independent variable r according to $A=2\pi rL$, the one dimensional transient heat conduction equation in a long cylinder becomes

Variable conductivity:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\dot{e}_{gen} = \rho c\frac{\partial T}{\partial t}$$

Constant conductivity:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

- The one-dimensional conduction equation may be reduces to the following forms under special conditions

1) Steady-state:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{e}_{gen}}{k}=0$$

2) Transient, no heat generation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

3) Steady-state, no heat generation:

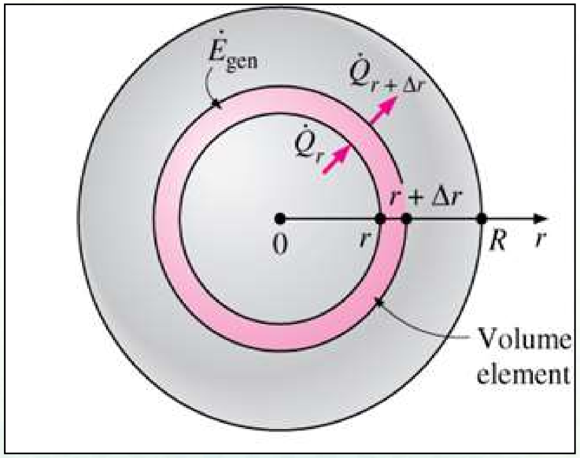
$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

One-Dimensional Heat Conduction Equation - Sphere

- Same approach used for the cylinder with:

  $A = 4\pi r^2$ instead of

  $A = 2\pi rL$



- Variable conductivity:

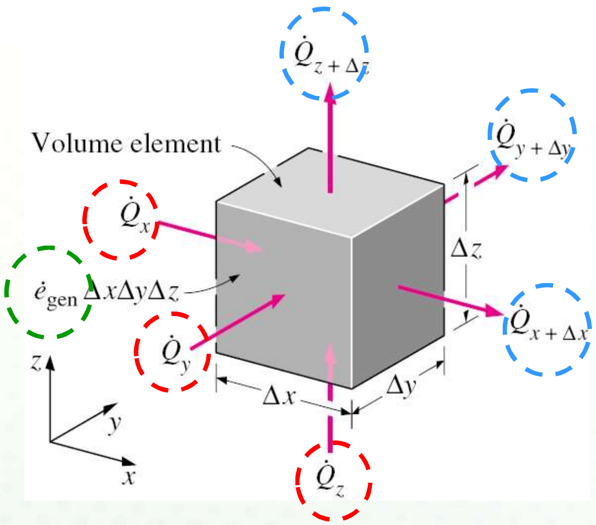
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k\frac{\partial T}{\partial r}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

- Constant conductivity:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial t}\right)+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

General Heat Conduction Equation

- Cartesian Coordinates:



(Rate of heat conduction at *x*, *y*, and *z*) - (Rate of heat Conduction at *x+Dx, y+Dy, and z+Dz*) + (Rate of heat Generation inside the element) = (Rate of change of the energy content of the element)

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z \quad -\dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} \quad +E_{gen,element} \quad = \frac{\Delta E_{element}}{\Delta t}$$

- Repeating the mathematical approach used for the one-dimensional heat conduction the three-dimensional heat conduction equation is determined to be

Two-dimensional

- Constant conductivity:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Three-dimensional

1) Steady-state: (Poisson equation)

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{e}_{gen}}{k}=0$$

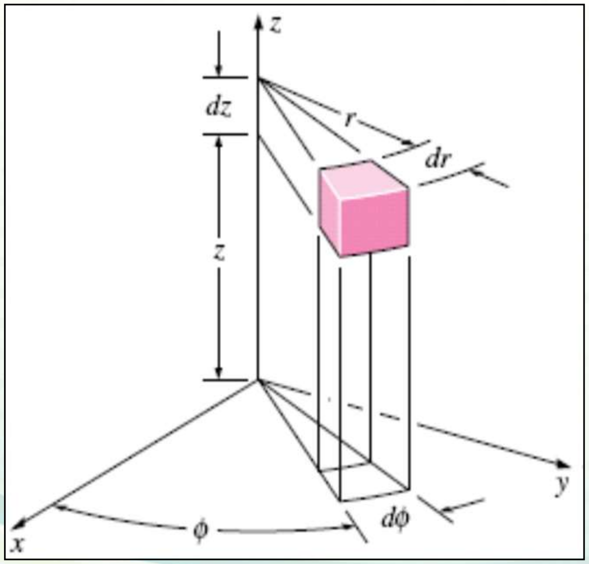
2) Transient, no heat generation: (Diffusion equation)

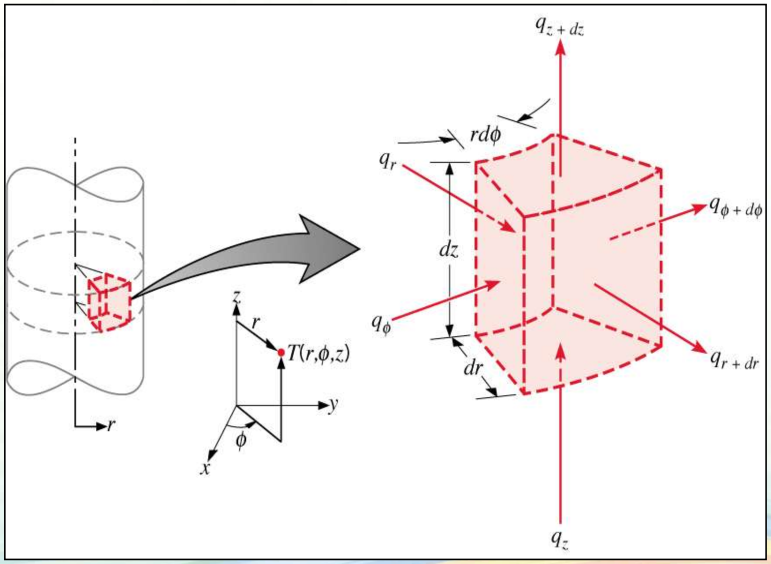
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

3) Steady-state, no heat generation: (Laplace equation)

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0$$

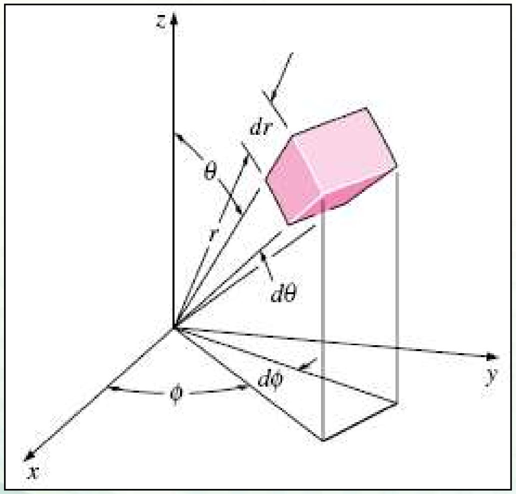
Cylindrical Coordinates

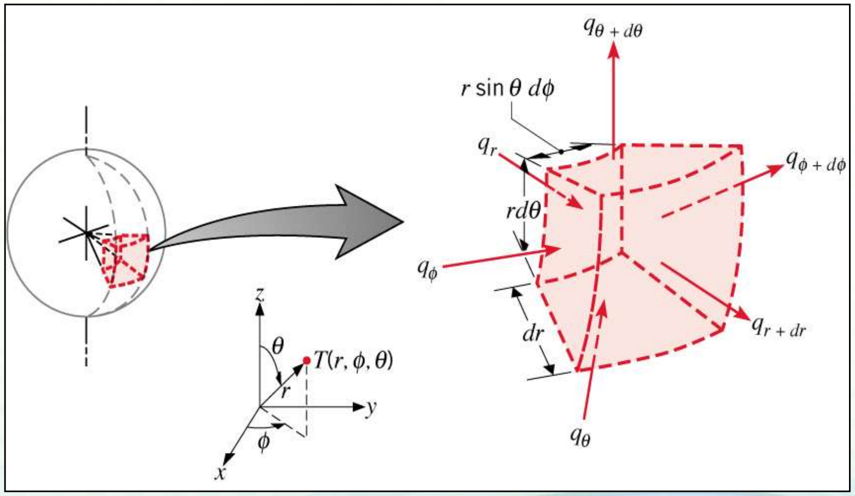




$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial T}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{e}_{gen}=\ \rho c\frac{\partial T}{\partial t}$$

Spherical Coordinates





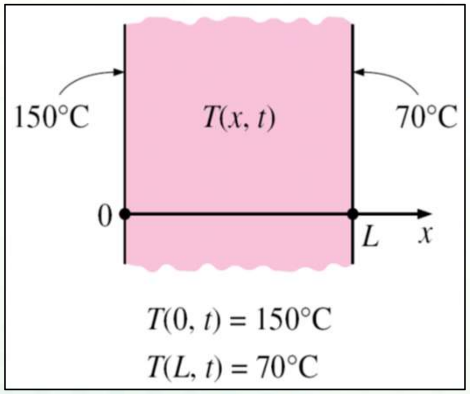
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta^2}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta\frac{\partial T}{\partial\theta}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

Boundary and Initial Conditions

1. Specified Temperature Boundary Condition
2. Specified Heat Flux Boundary Condition
3. Convection Boundary Condition
4. Radiation Boundary Condition
5. Interface Boundary Conditions
6. Generalized Boundary Conditions

Specified Temperature Boundary Condition

- For one-dimensional heat transfer through a plane wall of thickness L, for example, the specified temperature boundary conditions can be expressed as



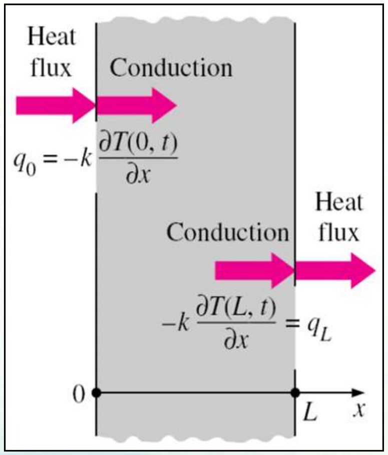
$$T(0, t) = T_1$$
$$T(L, t) = T_2$$

- The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.

Specified Heat Flux Boundary Condition

- The heat flux in the positive *x*-direction anywhere in the medium, including the boundaries, can be expressed by *Fourier's law* of heat conduction as

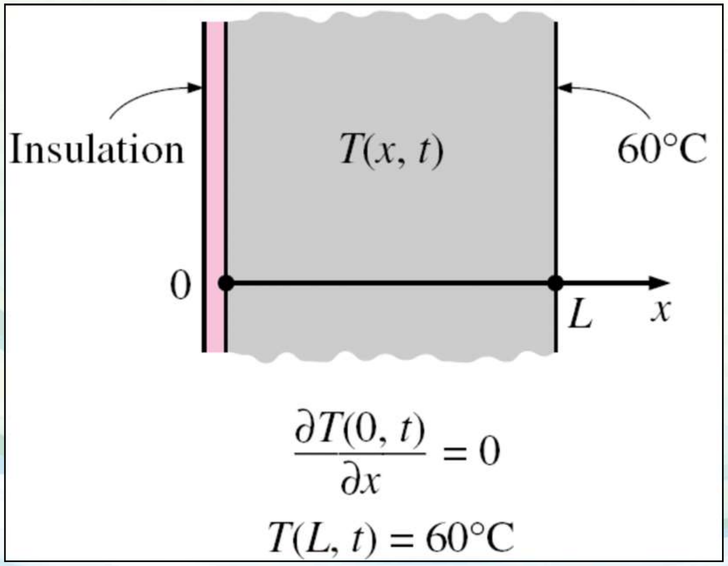
$$\dot{q} = -k\frac{dT}{dx} = \left(\text{Heat flux in the positive x-direction}\right)$$



- The sign of the specified heat flux is determined by inspection: *positive* if the heat flux is in the positive direction of the coordinate axis, and *negative* if it is in the opposite direction.

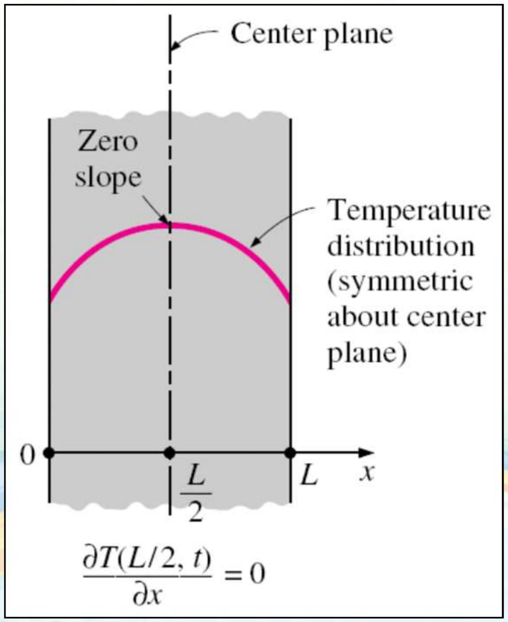
Two Special Cases

Insulated boundary



$$k\frac{\partial T(0,t)}{\partial x} = 0 \quad \text{or} \quad \frac{\partial T(0,t)}{\partial x} = 0$$

Thermal symmetry



$$\frac{\partial T\left({}^{L}/_{2},t\right)}{\partial x} = 0$$

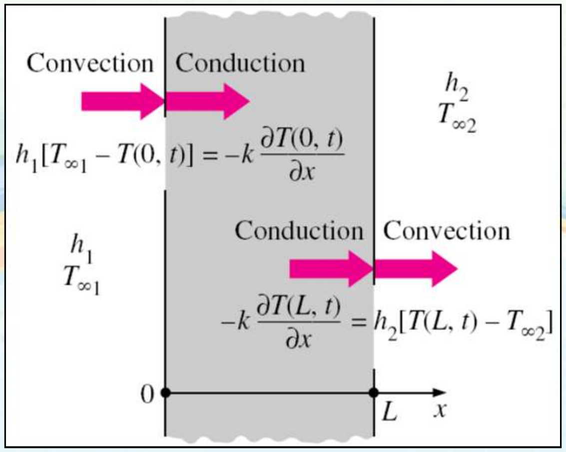
Convection Boundary Condition

$$\begin{pmatrix} \text{Heat conduction} \\ \text{at the surface in a} \\ \text{selected direction} \end{pmatrix} = \begin{pmatrix} \text{Heat convection} \\ \text{at the surface in} \\ \text{the same direction} \end{pmatrix}$$

$$-k\frac{\partial T(0,t)}{\partial x} = h_1[T_{\infty 1} - T(0,t)]$$

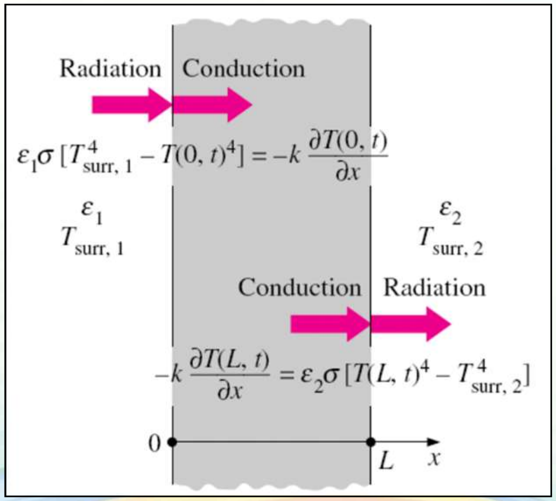
and

$$-k\frac{\partial T(L,t)}{\partial x} = h_2[T(L,t) - T_{\infty 2}]$$



Radiation Boundary Condition

(Heat conduction at the surface in a selected direction) = (Radiation exchange at the surface in the same direction)



$$-k\frac{\partial T(0,t)}{\partial x} = \varepsilon_1\sigma\left[T^4_{surr,1} - T(0,t)^4\right]$$

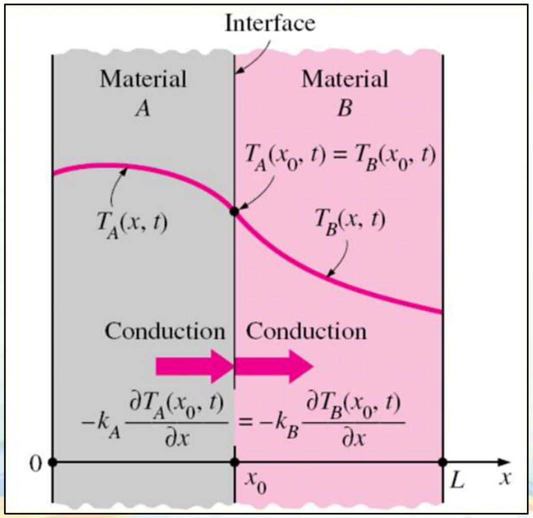
and

$$-k\frac{\partial T(T,t)}{\partial x} = \varepsilon_2\sigma\left[T(L,t)^4 - T^4_{surr,2}\right]$$

Interface Boundary Conditions

- At the interface the requirements are:

1. Two bodies in contact must have the *same temperature* at the area of contact,
2. An interface (which is a surface) cannot store any energy, and thus the *heat flux* on the two sides of an interface *must be the same.*



$$T_A(x_0, t) = T_B(x_0, t)$$

and

$$-k_A \frac{\partial T_A(x_0, t)}{\partial x} = -k_B \frac{\partial T_B(x_0, t)}{\partial x}$$

Generalized Boundary Conditions

- In general a surface may involve convection, radiation, and specified heat flux simultaneously. The boundary condition in such cases is again obtained from a surface energy balance, expressed as

$$\left(\begin{array}{c}\text{Heat transfer to}\\ \text{the surface in all}\\ \text{modes}\end{array}\right) = \left(\begin{array}{c}\text{Heat transfer from}\\ \text{the surface in all}\\ \text{modes}\end{array}\right)$$

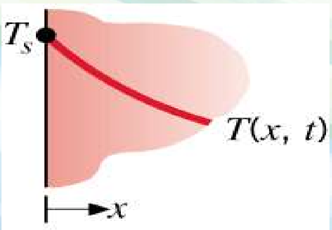
Boundary and Initial Conditions: summary

- For transient conduction, heat equation is first order in time, requiring specification of an initial temperature distribution:

$$T\left(x,t\right)_{t=0} = T\left(x,0\right)$$

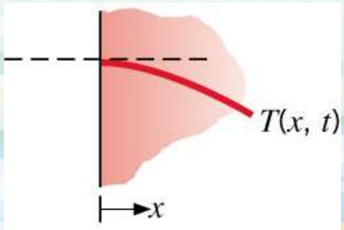
- Since heat equation is second order in space, two boundary conditions must be specified. Some common cases:

Constant Surface Temperature



$T(0,t) = T_S$

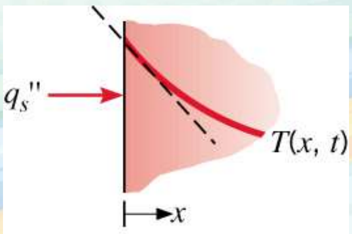
Insulated Surface



$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

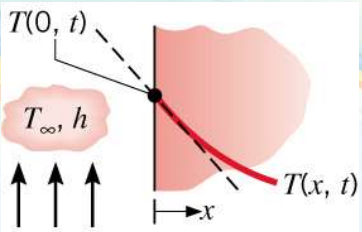
Constant Heat Flux:

Applied Flux



$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q_S''$$

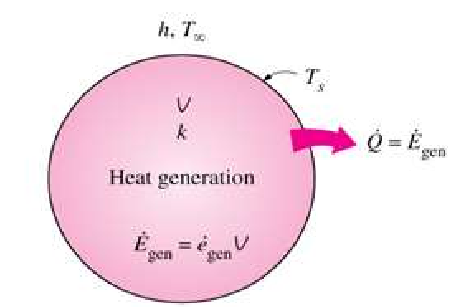
Convection



$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0,t)]$$

Heat Generation in Solids -The Surface Temperature

(Rate of heat transfer from the solid) = (Rate of energy generation within the solid)



At steady conditions, the entire heat generated in a solid must leave the solid through its outer surface.

For *uniform* heat generation within the medium

$$\dot{Q} = \dot{e}_{gen}V \qquad (W)$$

The heat transfer rate by convection can also be expressed from Newton's law of cooling as

$$\dot{Q} = hA_s(T_s - T_\infty) \qquad (W)$$

$$T_s = T_\infty + \frac{\dot{e}_{gen}V}{hA_s}$$

Heat Generation in Solids -The Surface Temperature

- For a large *plane wall* of thickness 2L ($A_s = A_{wall}$ and $V = 2LA_{wall}$)

$$T_{s,plane\ wall} = T_\infty + \frac{\dot{e}_{gen} L}{h}$$

- For a long solid *cylinder* of radius r_0 ($A_s = 2\pi\, r_0 L$ and $V = \pi\, r_0^2 L$)

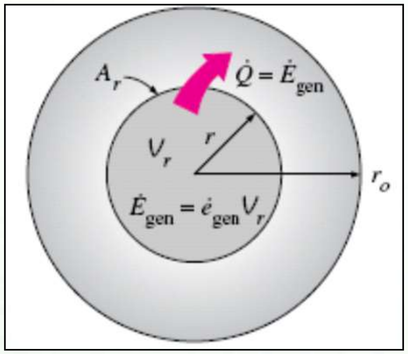
$$T_{s,cylinder} = T_\infty + \frac{\dot{e}_{gen} r_0}{2h}$$

- For a solid *sphere* of radius r_0 ($A_s = 4\, \pi\, r_0^2$ and $V = {}^4/_3\, \pi\, r_0^3$)

$$T_{s,spher} = T_\infty + \frac{\dot{e}_{gen} r_0}{3h}$$

Heat Generation in Solids -The maximum Temperature in a Cylinder (the Centerline)

- The *heat generated* within an inner cylinder must be equal to the *heat conducted* through its outer surface.



$$-kA_r \frac{dT}{dr} = \dot{e}_{gen} V_r$$

- Substituting these expressions into the above equation and separating the variables, we get

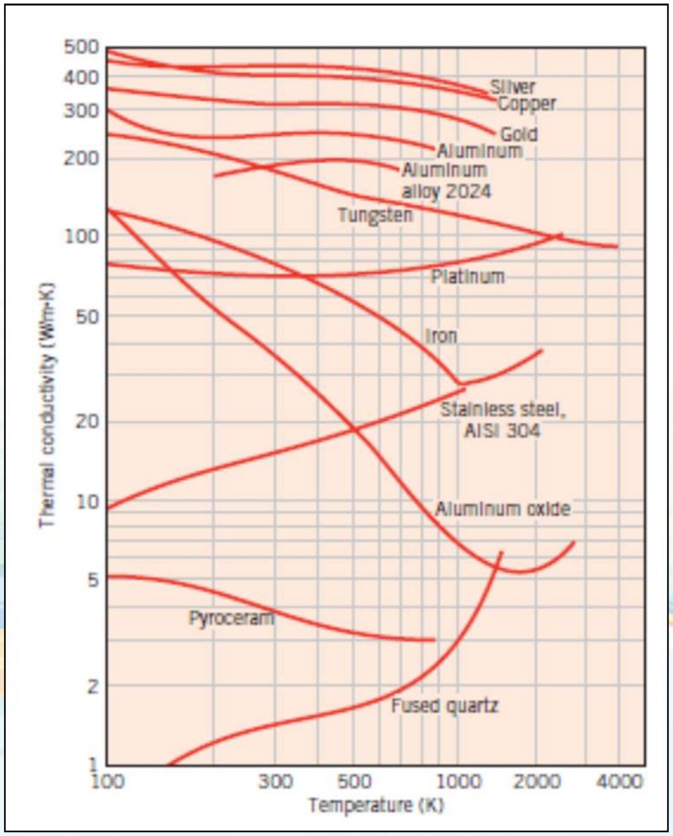
$$-k(2\pi r L)\frac{dT}{dr} = \dot{e}_{gen}\left(\pi r^2 L\right) \rightarrow dT = -\frac{\dot{e}_{gen}}{2k} r dr$$

- Integrating from r =0 where T(0) =T_0 to r=r_o

$$\Delta T_{max,cylinder} = T_0 - T_s = \frac{\dot{e}_{gen} r_0^2}{4k}$$

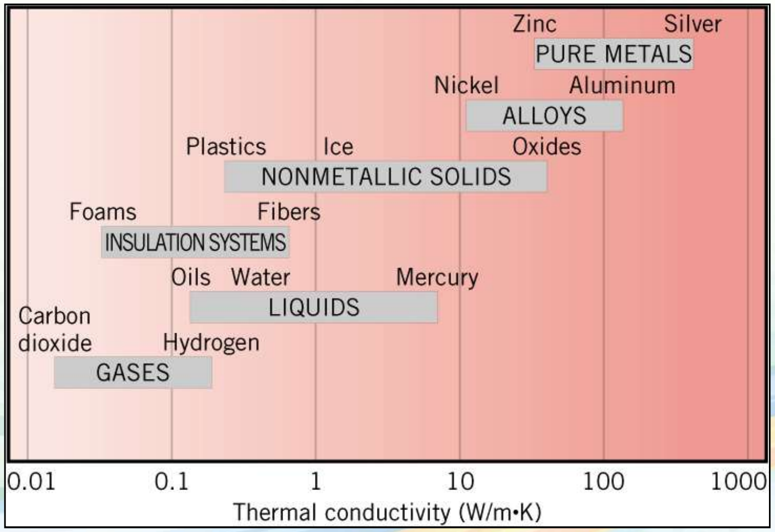
Variable Thermal Conductivity, k(T)

- The thermal conductivity of a material, in general, varies with temperature.
- An average value for the thermal conductivity is commonly used when the variation is mild.
- This is also common practice for other temperature-dependent properties such as the density and specific heat.



Thermophysical Properties

- Thermal Conductivity: A measure of a material's ability to transfer thermal energy by conduction.



- Thermal Diffusivity: A measure of a material's ability to respond to changes in its thermal environment.

Variable Thermal Conductivity for 1-D Cases

- When the variation of thermal conductivity with temperature $k(T)$ is known, the average value of the thermal conductivity in the temperature range between T_1 and T_2 can be determined from

$$k_{ave} = \frac{\int_{T_1}^{T_2} k(T)dT}{T_2 - T_1}$$

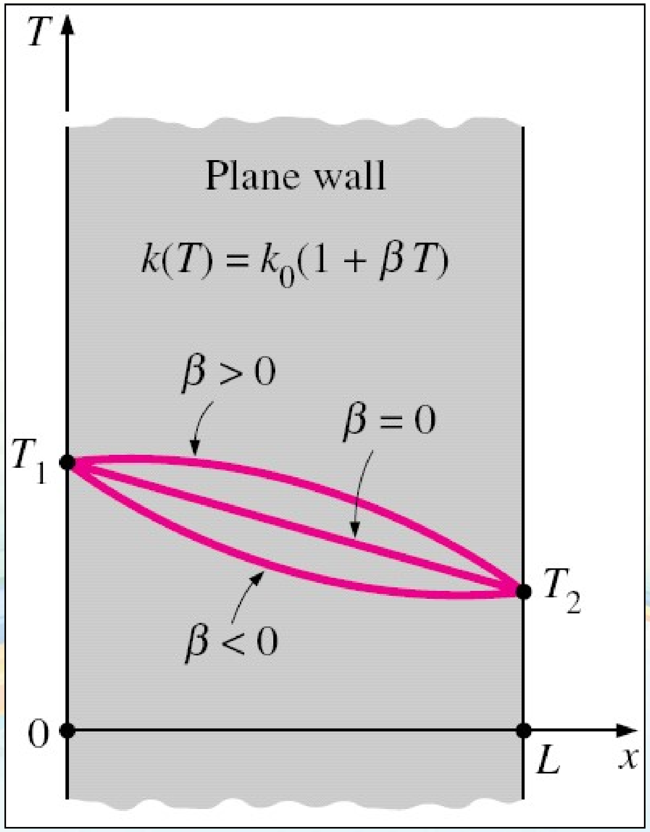
- The variation in thermal conductivity of a material which can often be approximated as a linear function and expressed as

$$k(T) = k_0(1 + \beta T)$$

$$k_{avg} \frac{\int_{T_1}^{T_2} k_0(1 + \beta T)dT}{T_2 - T_1} = k_0\left(1 + \beta \frac{T_2 + T_1}{2}\right) = \mathrm{k}(T_{avg})$$

Variable Thermal Conductivity

- For a plane wall the temperature varies linearly during steady one-dimensional heat conduction when the thermal conductivity is constant.
- This is no longer the case when the thermal conductivity changes with temperature (even linearly).



Methodology of a Conduction Analysis

- Consider possible micro- or nanoscale effects in problems involving very small physical dimensions or very rapid changes in heat or cooling rates.
- Solve appropriate form of heat equation to obtain the temperature distribution.
- Knowing the temperature distribution, apply Fourier's Law to obtain the heat flux at any time, location and direction of interest.