Heat Conduction Equation

Steady-State Conduction without Thermal Energy Generation

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Content

- One-Dimensional, Steady-State Conduction without Thermal Energy Generation

1. Plane Wall
2. Tube and cylinders
3. Spherical shell

Methodology of a Conduction Analysis

1. Specify appropriate form of the heat equation.

2. Solve for the temperature distribution.

3. Apply Fourier's law to determine the heat flux.

- Simplest Case: One-Dimensional, Steady-State, and Conduction with No Thermal Energy Generation.

- Common Geometries:

  1. The Plane Wall: Described in rectangular (x) coordinate. Area perpendicular to direction of heat transfer is constant (independent of x).

  2. The Tube Wall: Radial conduction through tube wall.

  3. The Spherical Shell: Radial conduction through shell wall.

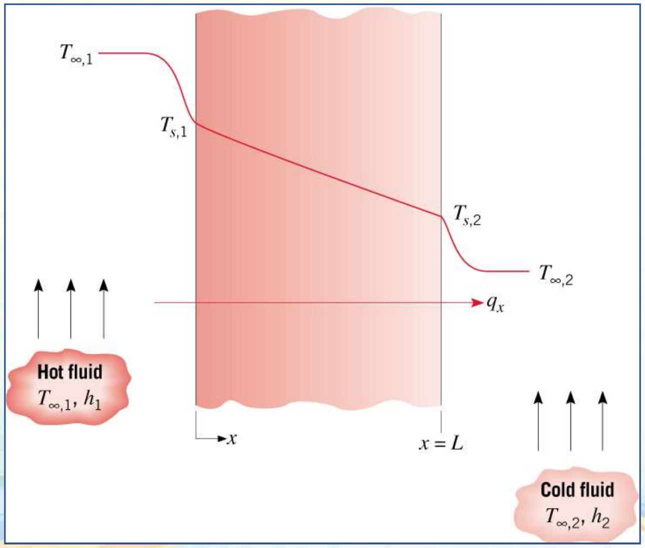
The Plane Wall

- Consider a plane wall between two fluids of different temperature:
- Heat Equation:

For steady-state conditions with no distributed source or sink of energy within the wall

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

$$T(x) = C_1 x + C_2$$



- Implications:

the heat flux is constant, independent of x

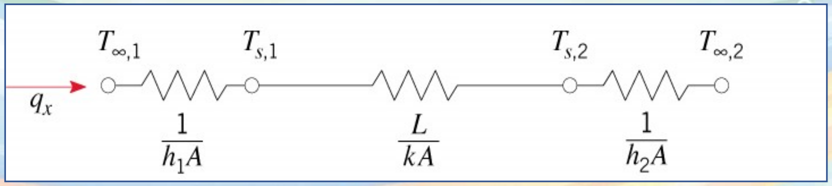
- ✓ Boundary Conditions: $T(0) = T_{s,1}$ and $T(L) = T_{s,2}$
- ✓ Temperature Distribution for Constant *k* : $T(x) = \left(T_{s,2} - T_{s,1}\right)\frac{x}{L} + T_{s,1}$

The Plane Wall

- Heat Flux and Heat Rate:

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) \quad \rightarrow \quad q''_x = \frac{q_x}{A} = \frac{k}{L}\left(T_{s,1} - T_{s,2}\right)$$

- Thermal Resistances and Thermal Circuits:
  - Conduction in a plane wall: $$R_{t,cond} \equiv \frac{\left(T_{s,1} - T_{s,2}\right)}{q_x} = \frac{L}{kA}$$
  - Convection: $$R_{t,conv} \equiv \frac{\left(T_s - T_\infty\right)}{q_x} = \frac{1}{hA}$$
  - Thermal circuit for plane wall with adjoining fluids:



$$q_x = \frac{\left(T_{\infty,1} - T_{s,1}\right)}{1/h_1A} = \frac{\left(T_{s,1} - T_{s,2}\right)}{L/kA} = \frac{\left(T_{s,2} - T_{\infty,2}\right)}{1/h_2A}$$

$$R_{tot} = \frac{1}{h_1A} + \frac{L}{kA} + \frac{1}{h_2A} \quad \rightarrow \quad q_x = \frac{\left(T_{\infty,1} - T_{\infty,2}\right)}{R_{tot}}$$

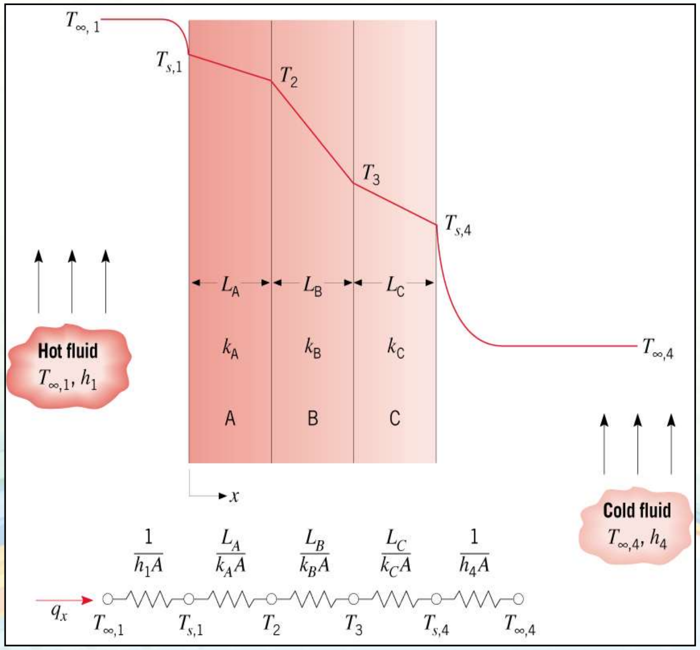
The Plane Wall

- Parallel composite wall with negligible contact resistance:

$$Heat\ flow = \frac{thermal\ potential\ difference}{thermal\ resistance}$$

$$q_x = \frac{(T_{\infty,1} - T_{\infty,4})}{\sum R_{tot}}$$

$$q_x = \frac{(T_{\infty,1} - T_{s,1})}{1/h_1 A} = \frac{(T_{s,1} - T_2)}{L_A/k_A A} = \frac{(T_2 - T_3)}{L_B/k_B A} = ----\ -$$



$$q_x = \frac{(T_{\infty,1} - T_{\infty,4})}{[(1/h_1 A) + (L_A/k_A A) + (L_B/k_B A) + (L_C/k_C A) + (1/h_4 A)]}$$

The Plane Wall

Overall Heat Transfer Coefficient (U) :

- A modified form of Newton's Law of Cooling to encompass multiple resistances to heat transfer.

$$q_x = UA\Delta T$$

Where

$$U = \frac{1}{R_{tot}A} = \frac{1}{[(1/h_1) + (L_A/k_A) + (L_B/k_B) + (L_C/k_C) + (1/h_4)]}$$

$$R_{tot} = \sum R_t = \frac{\Delta T}{q} = \frac{1}{UA}$$

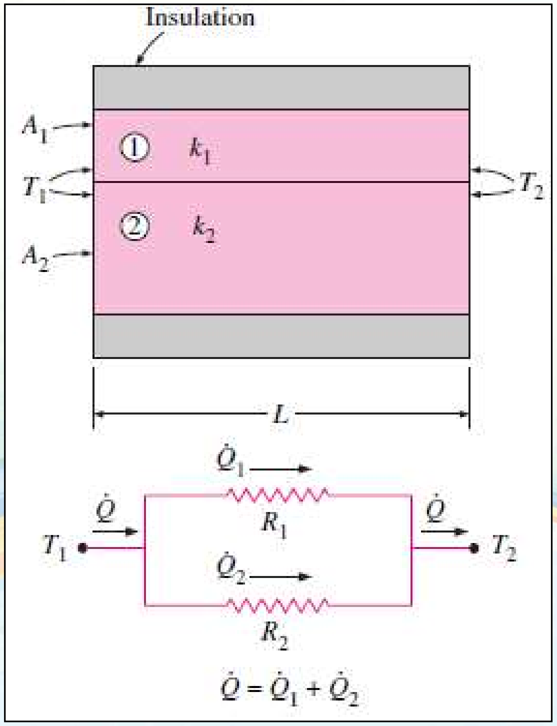
The Plane Wall

Series composite wall:

$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} = (T_1 - T_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\dot{Q} = \frac{T_1 - T_2}{R_{total}}$$

$$\frac{1}{R_{total}} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \rightarrow R_{total} = \frac{R_1 R_2}{R_1 + R_2}$$



The Plane Wall

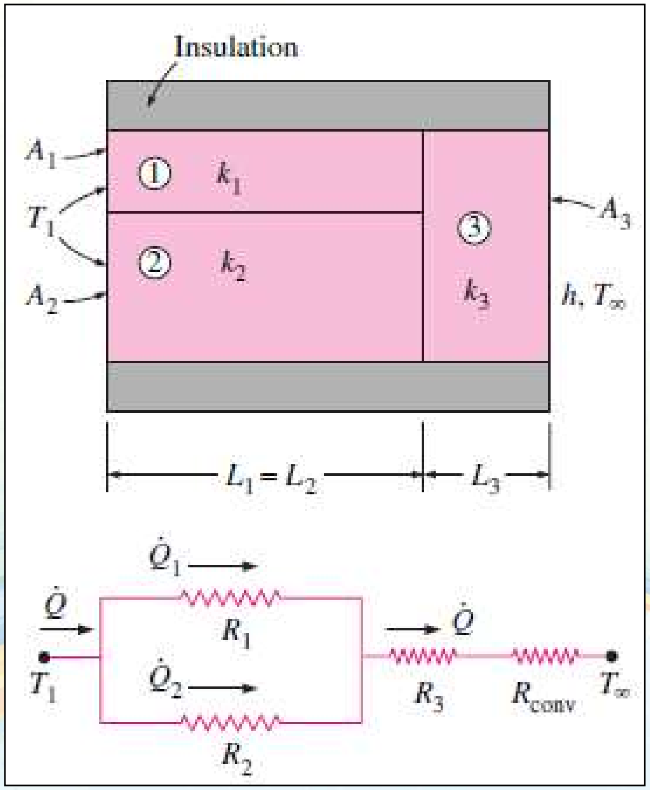
Series – Parallel Composite Wall:

$$\dot{Q} = \frac{T_1 - T_2}{R_{total}}$$

$$R_{total} = R_{12} + R_3 + R_{conv}$$

$$R_{total} = \frac{R_1 R_2}{R_1 + R_2} + R_3 + R_{conv}$$

$$R_1 = \frac{L_1}{k_1 A_1}, R_2 = \frac{L_2}{k_2 A_2}, R_3 = \frac{L_3}{k_3 A_3}, R_{conv} = \frac{1}{h A_3}$$



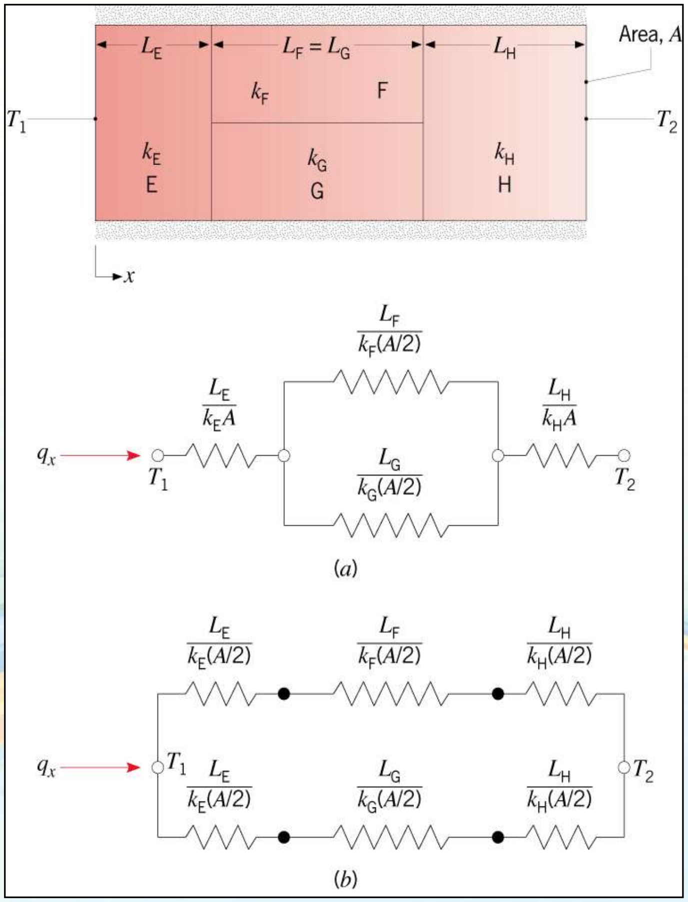
The Plane Wall

Series – Parallel Composite Wall:

- Note departure from one-dimensional conditions for

$$k_F \neq k_G$$

- Circuits based on assumption of isothermal surfaces normal to x direction or adiabatic surfaces parallel to x direction provide approximations for q_x .



The Plane Wall

Series – Parallel Resistances :

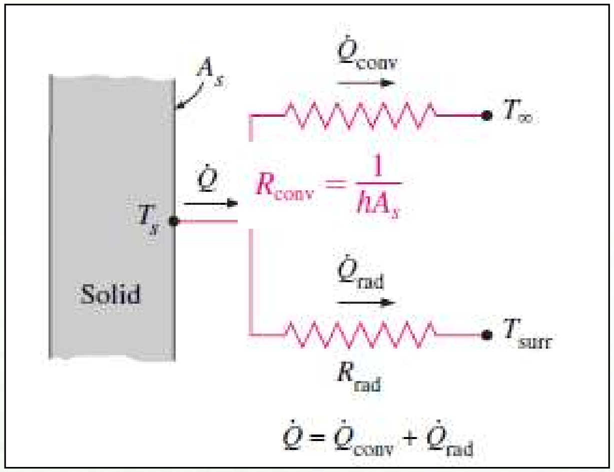
$$\dot{Q} = \dot{Q}_{conv} + \dot{Q}_{rad}$$

$$R_{rad} = \frac{1}{h_{rad} A_s} \quad (K/W)$$

$$h_{rad} = \frac{\dot{Q}_{rad}}{A_s(T_s - T_{surr})}$$

$$h_{rad} = \varepsilon\sigma\left(T_s^2 + T_{surr}^2\right)(T_s + T_{surr})$$

$$\dot{Q}_{rad} = \varepsilon\sigma A_s(T_s^4 - T_{surr}^4) = h_{rad} A_s (T_s - T_{surr}) = \frac{(T_s - T_{surr})}{R_{rad}}$$



- Thermal Resistance for Unit Surface Area:

$$R''_{t,cond} = \frac{L}{k} \qquad R''_{t,conv} = \frac{1}{h} \qquad R''_{t,rad} = \frac{1}{h_{rad}}$$

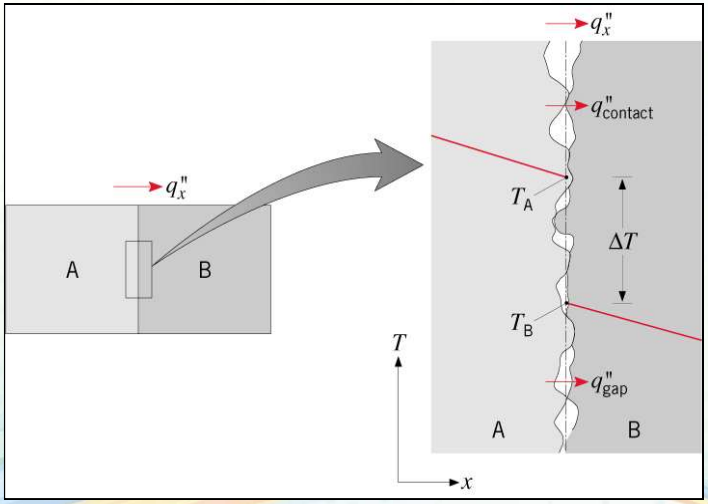
Units: $R_t \leftrightarrow$ W/K

$R''_t \leftrightarrow m^2 \cdot K/W$

The Plane Wall

Contact Resistance:

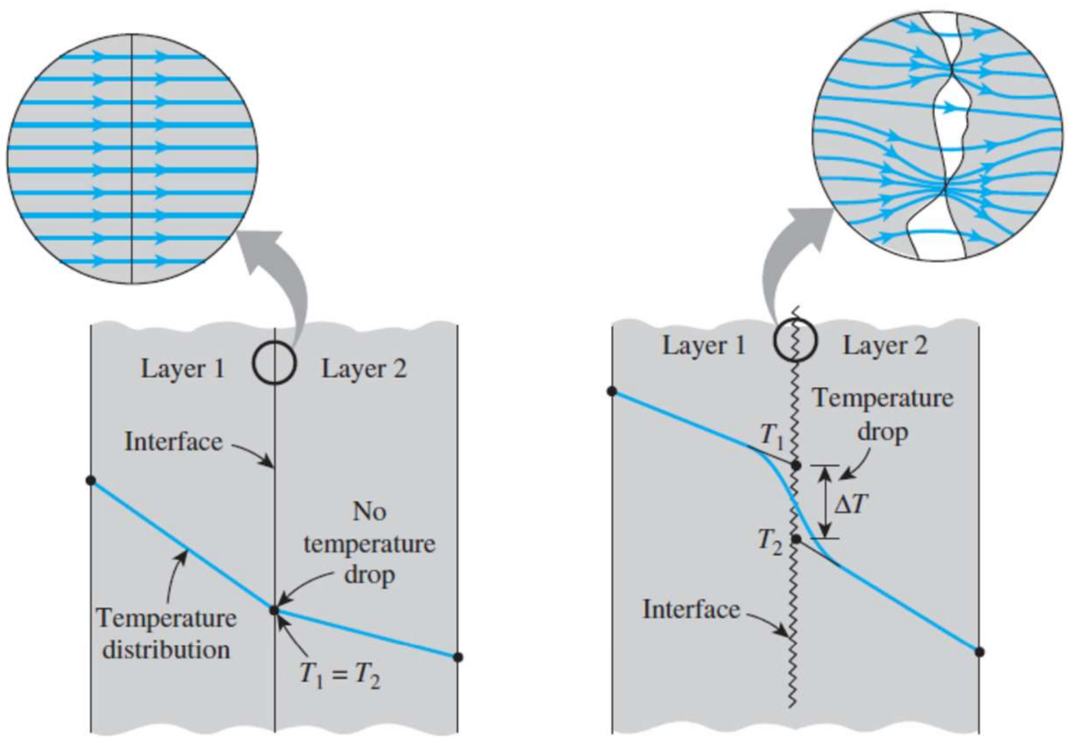
- In composite systems, the temperature drop across the interface between materials may be appreciable.
- This temperature change is attributed to what is known as the thermal contact resistance



$$R''_{t,c} = \frac{T_A - T_B}{q''_x}$$

$$R_{t,c} = \frac{R''_{t,c}}{A_c}$$

Ideal and Actual Contact



(*a*) Ideal (perfect) thermal contact

(*b*) Actual (imperfect) thermal contact

Contact Resistance:

TABLE 3.2 Thermal resistance of representative solid/solid interfaces

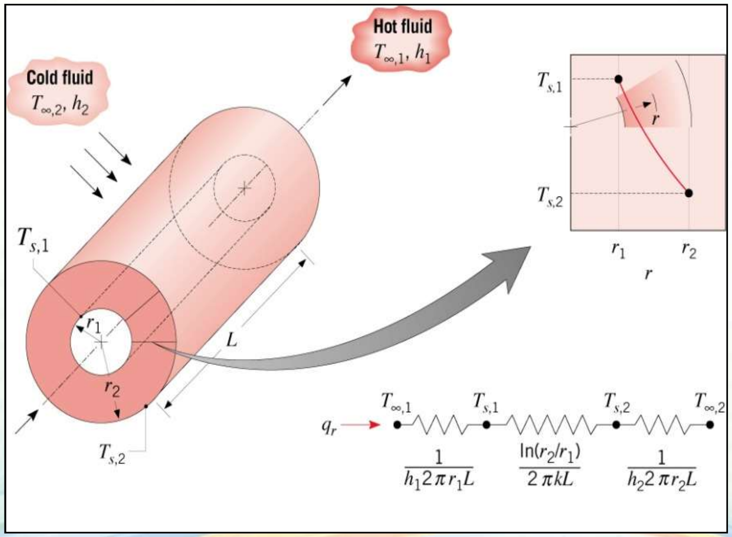
Interface	$R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)	Source
Silicon chip/lapped aluminum in air (27–500 kN/m^2)	0.3–0.6	[2]
Aluminum/aluminum with indium foil filler (~100 kN/m^2)	~0.07	[1, 3]
Stainless/stainless with indium foil filler (~3500 kN/m^2)	~0.04	[1, 3]
Aluminum/aluminum with metallic (Pb) coating	0.01–0.1	[4]
Aluminum/aluminum with Dow Corning 340 grease (~100 kN/m^2)	~0.07	[1, 3]
Stainless/stainless with Dow Corning 340 grease (~3500 kN/m^2)	~0.04	[1, 3]
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9	[5]
Brass/brass with 15-μm tin solder	0.025–0.14	[6]

Cylindrical tube and shell

- Heat Equation:

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

$$T(r) = C_1 ln(r) + C_2$$



- Given that

$$T(r_1) = T_{s,1} \quad and \quad T(r_2) = T_{s,2}$$

- Temperature Distribution for Constant k :

$$T(r) = \frac{T_{s,1} - T_{s,2}}{ln(r_1/r_2)} ln\left(\frac{r}{r_2}\right) + T_{s,2}$$

Cylindrical tube and shell

- Heat Flux and Heat Rate:

$$q_r = -kA_r \frac{dT}{dr} \quad \rightarrow \quad q_r = -2\pi krL \frac{dT}{dr}$$

$$\rightarrow \quad q_r = \frac{2\pi Lk\left(T_{s,1} - T_{s,2}\right)}{ln(r_2/r_1)}$$

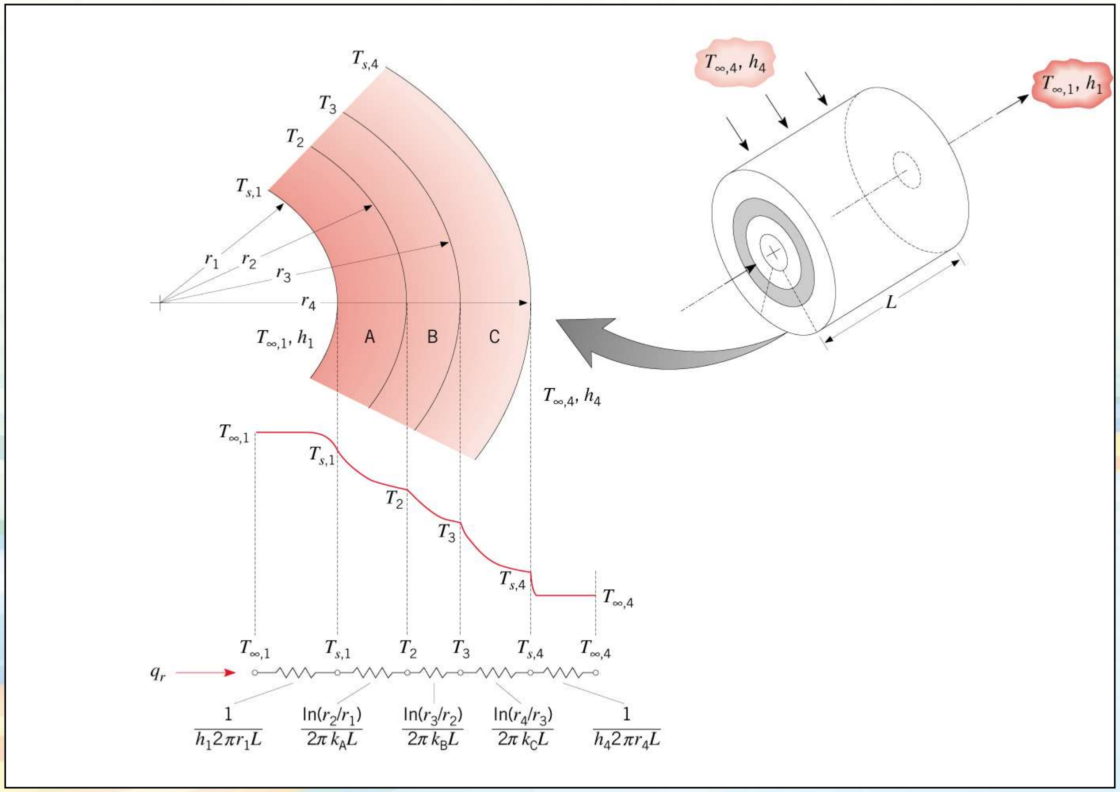
- Conduction Resistance:

$$R_{t,cond} = \frac{ln(r_2/r_1)}{2\pi Lk} \quad \text{K/W}$$

- Why is it inappropriate to base the thermal resistance on a unit surface area?

Cylindrical tube and shell

- Composite Wall with Negligible Contact Resistance



Cylindrical tube and shell

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{tot}} = UA(T_{\infty,1} - T_{\infty,4})$$

$$\Rightarrow \quad q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{2\pi r_1 L h_1} + \frac{ln(r_2/r_1)}{2\pi k_A L} + \frac{ln(r_3/r_2)}{2\pi k_B L} + \frac{ln(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}}$$

- If U is defined in terms of the inside area, $A_1 = 2\pi r_1 L$,

$$U_1 = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A} ln\left(\frac{r_2}{r_1}\right) + \frac{r_1}{k_B} ln\left(\frac{r_3}{r_2}\right) + \frac{r_1}{k_C} ln\left(\frac{r_4}{r_3}\right) + \frac{r_1}{r_4}\frac{1}{h_1}}$$

- This definition is arbitrary, and the overall coefficient may also be defined in terms of A_4 or any of the intermediate areas.

$$U_1 A_1 = U_2 A_2 = U_3 A_3 = U_4 A_4 = \left(\sum R_t\right)^{-1}$$

Cylindrical tube and shell

$$R_{total} = R_{conv,1} + R_{cyl,1} + R_{cyl,2} + R_{cyl,3} + R_{conv,2}$$

$$= \frac{1}{h_1 A_1} + \frac{ln(r_2/r_1)}{2\pi L k_1} + \frac{ln(r_3/r_2)}{2\pi L k_2} + \frac{ln(r_4/r_3)}{2\pi L k_3} + \frac{1}{h_4 A_4}$$

$$\dot{Q} = \frac{T_{\infty 1} - T_2}{R_{conv,1} + R_{cyl,1}} = \frac{T_{\infty 1} - T_2}{\dfrac{1}{h_1(2\pi r_1 L)} + \dfrac{ln(r_2/r_1)}{2\pi L k_1}}$$

$$\dot{Q} = \frac{T_2 - T_{\infty 2}}{R_2 + R_3 + R_{conv,2}}$$

$$= \frac{T_2 - T_{\infty 2}}{\dfrac{ln(r_3/r_2)}{2\pi L k_2} + \dfrac{ln(r_4/r_3)}{2\pi L k_3} + \dfrac{1}{h_0(2\pi r_4 L)}}$$

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{R_{conv,1}}$$

$$= \frac{T_{\infty 1} - T_2}{R_{conv,1} + R_1}$$

$$= \frac{T_1 - T_3}{R_1 + R_2}$$

$$= \frac{T_2 - T_3}{R_2}$$

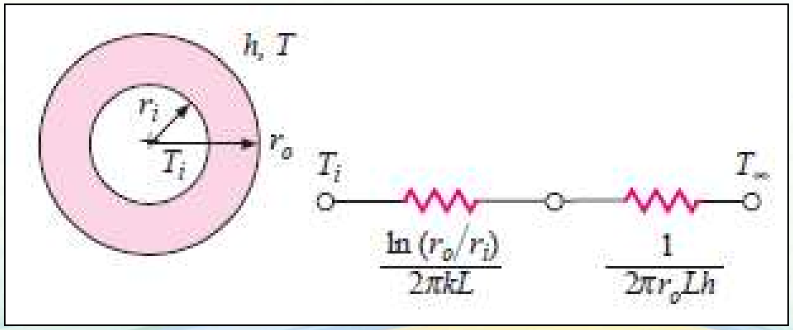
$$= \frac{T_2 - T_{\infty 2}}{R_2 + R_{conv,2}}$$

$$= \cdots$$

Critical Thickness of Insulation

- The inner temperature of the insulation is fixed at T_i
- The outer surface is exposed to a convection environment at T_∞

$$q = \frac{2\pi L(T_i - T_\infty)}{\frac{ln(r_0/r_i)}{k} + \frac{1}{r_0 h}}$$



- The maximum condition is

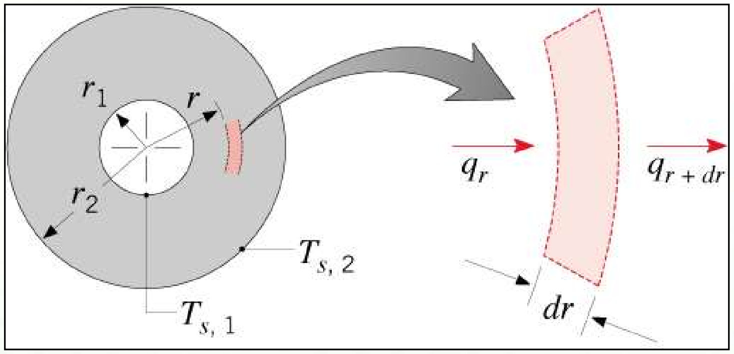
$$\frac{dq}{dr_0} = 0 = \frac{-2\pi L(T_i - T_\infty)\left(\frac{1}{kr_0} - \frac{1}{hr_0^2}\right)}{\left[\frac{ln(r_0/r_i)}{k} + \frac{1}{r_0 h}\right]^2}$$

$$\Rightarrow \quad r_0 = \frac{k}{h}$$

Spherical Shell

- Heat Equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) = 0$$



- Temperature Distribution for Constant k :

$$T(r) = T_{s,1} - \left(T_{s,1} - T_{s,2}\right)\frac{1-(r_1/r)}{1-(r_1/r_2)}$$

But

$$q_r = -kA\frac{dT}{dr} = -k\left(4\pi r^2\right)\frac{dT}{dr}$$

Spherical Shell

- for steady-state conditions with no heat generation and no heat loss from the sides.
- q_r is a constant, independent of r,

$$\frac{q_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{T_{s,1}}^{T_{s,2}} k(T)dT$$

$\Rightarrow$

$$q_r = \frac{4\pi k\left(T_{s,1} - T_{s,2}\right)}{(1/r_1) - (1/r_2)}$$

and

$$R_{t,cond} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

- For Composite Shell:

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{tot}} = UA\left(T_{\infty,1} - T_{\infty,4}\right)$$

and

$$U = \frac{1}{R_{tot}A}$$