Heat Conduction Equation

Steady-State Conduction with Thermal Energy Generation

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Implications of Energy Generation

- Involves a local (volumetric) source of thermal energy due to conversion from another form of energy in a conducting medium.

- The source may be uniformly distributed, as in the conversion from electrical to thermal energy (Ohmic heating):

Volumetric generation rate

$$\dot{q} \equiv \frac{\dot{E}_g}{V} = \frac{I^2 R_e}{V} \quad (W/m^3)$$

or it may be non-uniformly distributed, as in the absorption of radiation passing through a semi-transparent medium.

- Generation affects the temperature distribution in the medium and causes the heat rate to vary with location, thereby precluding inclusion of the medium in a thermal circuit.

The Plane Wall

- Consider one-dimensional, steady-state conduction in a plane wall of constant k, uniform generation, and asymmetric surface conditions:
- Heat Equation:

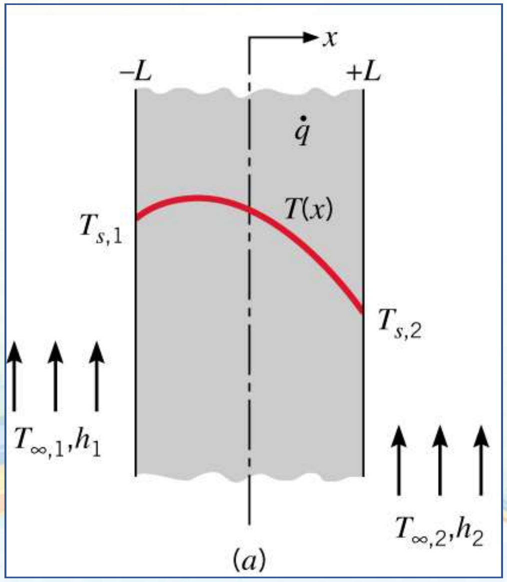
$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = 0 \quad \Rightarrow \quad \frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

- General Solution: $T = \frac{-\dot{q}}{2k}x^2 + C_1x + C_2$
- Boundary conditions

$$T(-L) = T_{s,1} \quad and \quad T(L) = T_{s,2}$$

$$C_1 = \frac{T_{s,2}-T_{s,1}}{2L} \quad and \quad C_2 = \frac{\dot{q}}{2k}L^2 + \frac{T_{s,1}+T_{s,2}}{2}$$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1-\frac{x^2}{L^2}\right) + \frac{T_{s,2}-T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1}+T_{s,2}}{2}$$



Asymmetrical boundary conditions

The Plane Wall

- Is the heat flux q'' independent of x?

Answer is required

Symmetric Surface Conditions **or** One Surface Insulated:

- Boundary conditions

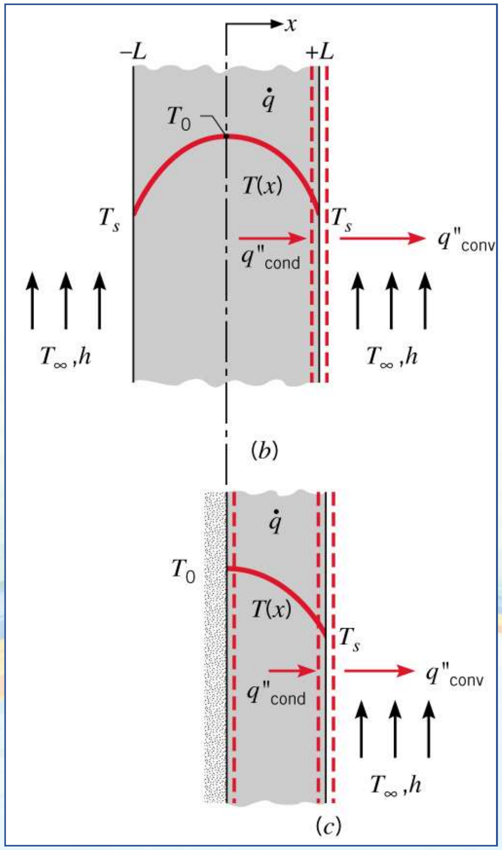
$$at\ x = \pm L \quad T_{s,1} = T_{s,2} \equiv T_s$$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

- At the mid-plane, x =0.0
- The maximum temperature exists $(dT/dx)_{x=0} = 0$

$$T(0) \equiv T_0 = \frac{\dot{q}L^2}{2k} + T_s$$

Symmetrical boundary conditions



Adiabatic surface at midplane

The Plane Wall

- Hence, the temperature distribution, may be expressed as

$$T_0 - T_s = \frac{\dot{q}L^2}{2k}$$

$$\frac{T(x) - T_0}{T_s - T_0} = \left(\frac{x}{L}\right)^2$$

If the temperature of an adjoining fluid T_∞ is known and not T_s

- Surface energy balance →

$$-k\frac{dT}{dx}\bigg|_{x=L} = h(T_s - T_\infty) \quad \Rightarrow \quad T_s = T_\infty + \frac{\dot{q}L}{h}$$

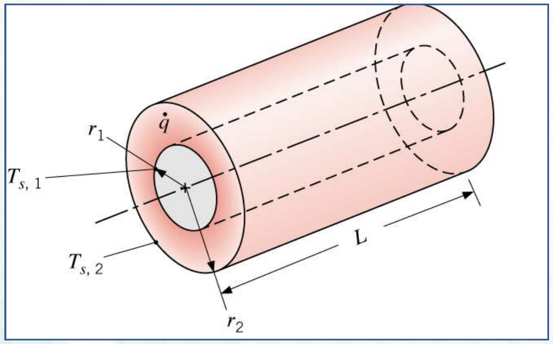
- Overall energy balance on the wall →

$$\dot{E}_g = \dot{E}_{out} \quad \Rightarrow \quad \dot{q}L = h(T_s - T_\infty)$$

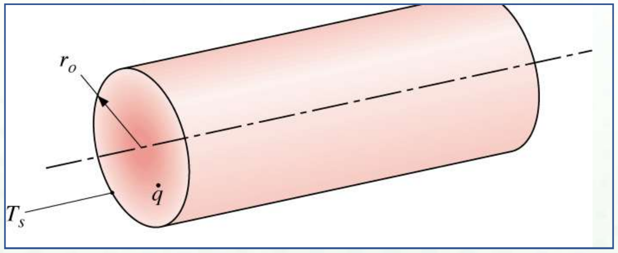
- How do we determine the heat rate at *x* = *L*?
  Merge equations

Cylindrical Systems

Cylindrical (Tube) Wall



Solid Cylinder (Circular Rod)



Heat Equations: $\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$

- Separating variables and assuming uniform generation and constant k

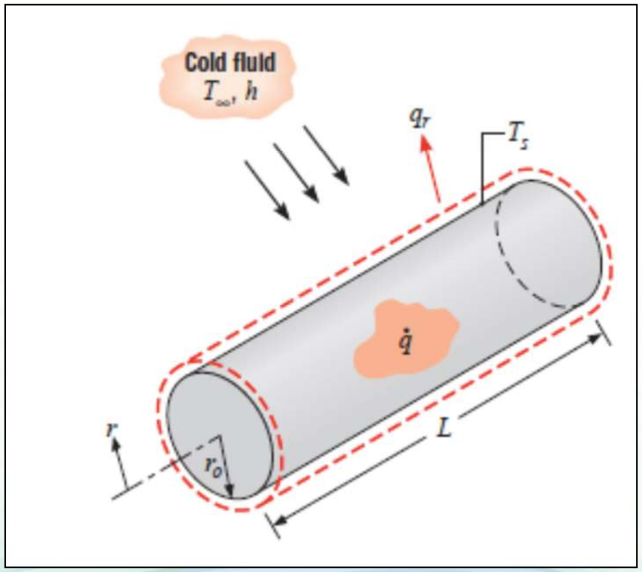
$$r\frac{dT}{dr}=-\frac{\dot{q}}{2k}r^2+C_1 \quad \rightarrow \quad T(r)=-\frac{\dot{q}}{4k}r^2+C_1\ln r+C_2$$

Cylindrical Systems (Solid)

- Boundary conditions

$$\left.\frac{dT}{dr}\right|_{r=0} = 0 \quad and \quad T(r_0) = T_s$$

$C_1 = 0$ and $C_2 = T_s + \frac{\dot{q}}{4k} r_o^2$



- The temperature distribution

$$T(r) = \frac{\dot{q} r_o^2}{4k}\left(1 - \frac{r^2}{r_0^2}\right) + T_s$$

$$at\ r = 0 \quad T(0) = T_0 \quad \rightarrow \quad T_0 = \frac{\dot{q} R^2}{4k} + T_s$$

- The temperature distribution in non-dimensional form

$$\frac{T(r) - T_s}{T_0 - T_s} = 1 - \left(\frac{r}{r_0}\right)^2$$

Cylindrical Systems

Surface Temperature

Overall energy balance:

$$\dot{E}_g = \dot{E}_{out}$$

$$\dot{q}\left(\pi r_0^2 L\right) = h(2\pi r_0 L)(T_s - T_\infty)$$

surface energy balance:

$$-k\left.\frac{dT}{dr}\right|_{r=r_0} = h(T_s - T_\infty)$$

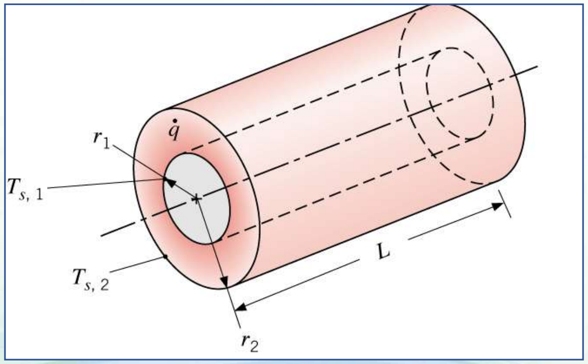
$$T_s = T_\infty + \frac{\dot{q} r_0}{2h}$$

- A summary of temperature distributions is provided in Appendix C for plane, cylindrical and spherical walls, as well as for solid cylinders and spheres. Note how boundary conditions are specified and how they are used to obtain surface temperatures.

Cylindrical Systems (Hollow)

- For a hollow cylinder with uniformly distributed heat sources the appropriate boundary conditions would be

$$T = T_i \quad at\ r = r_i\ (inside\ surface)$$
$$T = T_0 \quad at\ r = r_0\ (outside\ surface)$$



- The general solution is still

$$T(r) = -\frac{\dot{q}}{4k}r^2 + C_1 \ln r + C_2$$

$$T - T_0 = \frac{\dot{q}}{4k}(r_0^2 - r^2) + C_1 \ln\frac{r}{r_0}$$

where

$$C_1 = \frac{T_i - T_0 + \dot{q}(r_i^2 - r_0^2)/4k}{\ln(r_i/r_0)}$$

- For a hollow cylinder with uniformly distributed heat sources and the external surface is insulated the the appropriate boundary conditions would be:

$$\left.\frac{dT}{dr}\right|_{r=ro} = 0 \quad and \quad T(r_0) = T_s$$

$$C_1 = \frac{\dot{q}}{2k} r_o^2$$

$$C_2 = T_s + \frac{\dot{q}}{4k} r_o^2 - \frac{\dot{q}}{2k} r_o^2 ln r_o$$

$$T(r) = T_s + \frac{\dot{q}}{4k}\left(r_o^2 - r^2\right) - \frac{\dot{q}}{2k} r_o^2 ln\frac{r_o}{r}$$

- For a hollow cylinder with uniformly distributed heat sources and the internal surface is insulated the the appropriate boundary conditions would be:

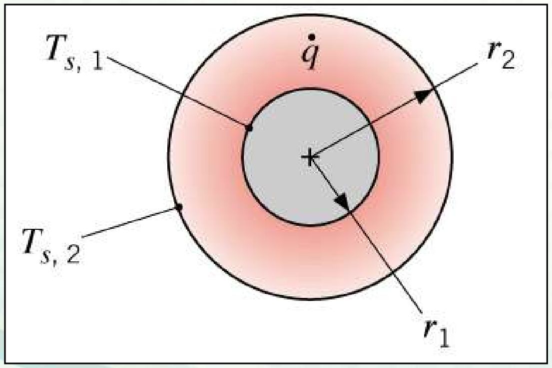
$$\left.\frac{dT}{dr}\right|_{r=ri} = 0 \quad and \quad T(r_0) = T_s$$

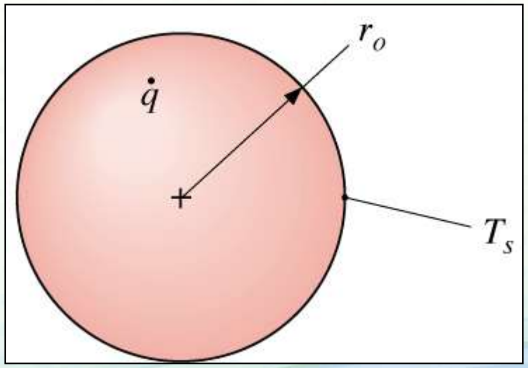
$$T(r) = T_s + \frac{\dot{q}}{4k}\left(r_o^2 - r^2\right) + \frac{\dot{q}}{2k} r_i^2 ln\frac{r}{r_o}$$

Spheres

Spherical Wall (Shell)

Solid Sphere





- Heat Equations:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

Spheres

- **Temperature Distribution**

$$T = -\frac{qr^2}{6k} + \frac{C_1}{r} + C_2$$

- B.C 1:

$$\left.\frac{dT}{dr}\right|_{r=0} = 0 \rightarrow C_1 = 0$$

- B.C 2:

$$T(r_0) = T_s \rightarrow C_2 = T_s + \frac{\dot{q}r_0^2}{6k}$$

$$T(r) = \frac{\dot{q}r_0^2}{6k}\left(1 - \frac{r^2}{r_0^2}\right) + T_s$$

Spheres

Surface Temperature

- Overall energy balance:

$$-\dot{E}_{out} = \dot{E}_g = 0 \quad \Rightarrow \quad -k\left.\frac{dT}{dr}\right|_{r=r_0} = h(T_s - T_\infty) \quad \Rightarrow \quad \rightarrow T_s = T_\infty + \frac{\dot{q}r_i^3}{3hr_o^2}$$

- Or from a surface energy balance:

$$\dot{E}_{in} - \dot{E}_{out} = 0 \qquad \rightarrow q_{cond}\,(r_0) = q_{conv}$$

$$\rightarrow T_s = T_\infty + \frac{\dot{q}r_i^3}{3hr_o^2}$$