CHEM 0915341

Summer Semester 21/22

CHAPTER 3
Continuity Equation

INTRODUCTION

- Continuity equation represents that the product of cross-sectional area of the pipe and the fluid speed at any point along the pipe is always constant.
- This product is equal to the volume flow per second or simply the flow rate.
- The continuity equation is given as:

$$R = Av = constant$$

where:

- ✓ R is the volume flow rate
- ✓ A is the flow area
- ✓ *v* is the flow velocity

BERNOULLI'S PRINCIPLE

- Bernoulli's principle formulated by Daniel Bernoulli states that as the speed of a moving fluid increases (liquid or gas), the pressure within the fluid decreases.

- The total mechanical energy of the moving fluid comprising the gravitational potential energy of elevation, the energy associated with the fluid pressure and the kinetic energy of the fluid motion, remains constant.

- Bernoulli's equation formula is a relation between pressure, kinetic energy, and gravitational potential energy of a fluid in a container.

- The formula for Bernoulli's principle is given as follows:

$$p + \frac{1}{2}\rho v^2 + \rho g h = constant$$

- Where *p* is the pressure exerted by the fluid, ν is the velocity of the fluid, *ρ* is the density of the fluid and *h* is the height of the container.

ASSUMPTION OF CONTINUITY EQUATION

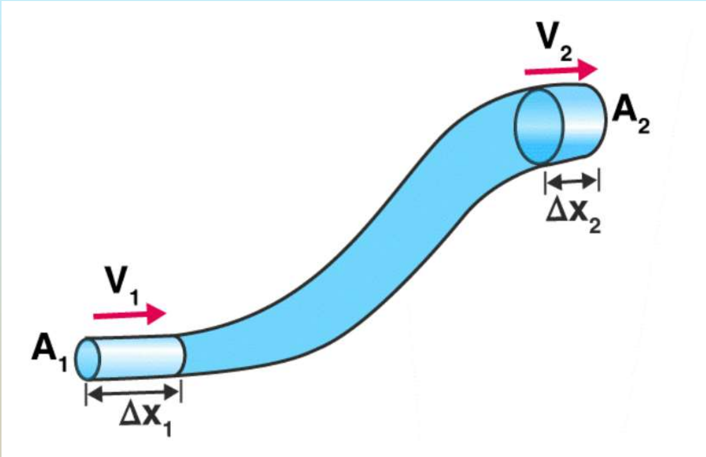
Following are the assumptions of continuity equation:

- The tube is having a single entry and single exit
- The fluid flowing in the tube is non-viscous
- The flow is incompressible
- The fluid flow is steady

DERIVATION

- Consider the following diagram:
- Now, consider the fluid flows for a short interval of time in the tube.
- So, assume that short interval of time as Δt.
- In this time, the fluid will cover a distance of Δx_1 with a velocity v_1 at the lower end of the pipe.
- At this time, the distance covered by the fluid will be:

$$\Delta x_1 = v_1 \Delta t$$



- Now, at the lower end of the pipe, the volume of the fluid that will flow into the pipe will be:

$$V = A_1 \Delta x_1 = A_1 v_1 \Delta t$$

- It is known that mass (m) = Density (ρ) × Volume (V). So, the mass of the fluid in Δx_1 region will be:

$$\Delta m_1 = Density \times Volume$$

$$\rightarrow \Delta m_1 = \rho_1 A_1 v_1 \Delta t \qquad Eq.1$$

- Now, the mass flux has to be calculated at the lower end.
- Mass flux is simply defined as the mass of the fluid per unit time passing through any cross-sectional area.

- For the lower end with cross-sectional area A_1, mass flux will be:

$$\frac{\Delta m_1}{\Delta t} = \rho_1 A_1 v_1 \qquad Eq. 2$$

- Similarly, the mass flux at the upper end will be:

$$\frac{\Delta m_2}{\Delta t} = \rho_2 A_2 v_2 \qquad Eq. 2$$

- Here, v_2 is the velocity of the fluid through the upper end of the pipe i.e. through Δx_2 , in Δt time and A_2, is the cross-sectional area of the upper end.

- In this, the density of the fluid between the lower end of the pipe and the upper end of the pipe remains the same with time as the flow is steady.
- So, the mass flux at the lower end of the pipe is equal to the mass flux at the upper end of the pipe i.e. Equation 2 = Equation 3.
- Thus:

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \qquad Eq. 4$$

- This can be written in a more general form as:

$$\rho A v = constant$$

- The equation proves the law of conservation of mass in fluid dynamics. Also, if the fluid is incompressible, the density will remain constant for steady flow.

- So, $\rho_1 = \rho_2$
- Thus, Equation 4 can be now written as:

$$A_1 v_1 = A_2 v_2$$

- This equation can be written in general form as:

$$Av = constant$$

- Now, if R is the volume flow rate, the above equation can be expressed as:

$$R = Av = constant$$

- This was the derivation of continuity equation.

CONTINUITY EQUATION IN CYLINDRICAL COORDINATES

- Following is the continuity equation in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial r \rho u}{\partial r} + \frac{1}{r}\frac{\partial \rho v}{\partial \theta} + \frac{\partial \rho \omega}{\partial z} = 0$$

- Steady Flow Continuity Equation

  Following is the continuity equation in cylindrical coordinates:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho \omega) = 0$$

TYPES OF TIME DERIVATIVES AND VECTOR NOTATION

1. Partial time derivative

- Various types of time derivatives are used in the derivations to follow.
- The most common type of derivative is the partial time derivative.
- For example, suppose that we are interested in the mass concentration or density ρ in kg/m^3 in a flowing stream as a function of position x, y, z and time *t*.
- The partial time derivative of ρ *is* $\frac{\partial \rho}{\partial t}$.
- This is the local change of density with time at a fixed point x, y, and z.

2. Total time derivative

- Suppose that we want to measure the density in the stream while we are moving about in the stream with velocities in the x, y, and z directions of $\frac{dx}{dt}, \frac{dy}{dt}, and\ \frac{dz}{dt}$, respectively.

- The total derivative $\frac{d\rho}{dt}$ is:

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt}$$

- This means that the density is a function of *t* and of the velocity components $\frac{dx}{dt}, \frac{dy}{dt}, and\ \frac{dz}{dt}$ at which the observer is moving.

3. Substantial time derivative

- Another useful type of time derivative is obtained if the observer floats along with the velocity *v* of the flowing stream and notes the change in density with respect to time.
- This is called the derivative that follows the motion, or the substantial time derivative, $\frac{D\rho}{Dt}$.

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial t} + (v . \nabla \rho)$$

where v_x, v_y, and v_z are the velocity components of the stream velocity *v*, which is a vector.

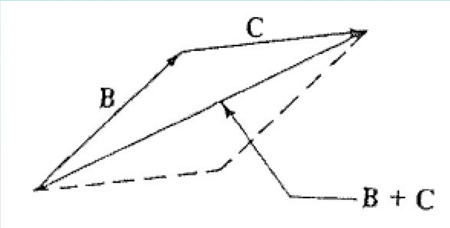
- This substantial derivative is applied to both scalar and vector variables.

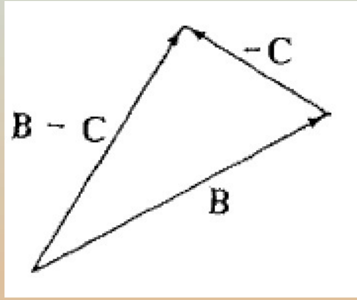
4- Scalars

- The physical properties encountered in momentum, heat, and mass transfer can be placed in several categories: scalars, vectors, and tensors.
- Scalars are quantities such as concentration, temperature, length, volume, time, and energy.
- They have magnitude but no direction and are considered to be zero-order tensors.
- The common mathematical algebraic laws hold for the algebra of scalars.
- For example, $bc = cd, b(cd) = (bc)d$, and so on.

5. Vectors

- Velocity, force, momentum, and acceleration are considered vectors since they have magnitude and direction.
- They are regarded as first-order tensors and are written in boldface letters in this text, such as v for velocity.
- The addition of the two vectors $B + C$ by parallelogram construction and the subtraction of two vectors $B - C$ is shown.
- The vector B is represented by its three projections B_x, B_y, and B_z on the x, y, and z axes and





$$B = iB_x + jB_y + kB_z$$

where *i*, *j*, and *k* are unit vectors along the axes x, y, and z, respectively.

6. Differential operations with scalars and vectors

- The gradient or "grad" of a scalar field is

$$\nabla\rho = i\frac{\partial\rho}{\partial x} + j\frac{\partial\rho}{\partial y} + k\frac{\partial\rho}{\partial z}$$

where ρ is a scalar such as density.

- The divergence or "div" of a vector v is

$$(\nabla.v) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

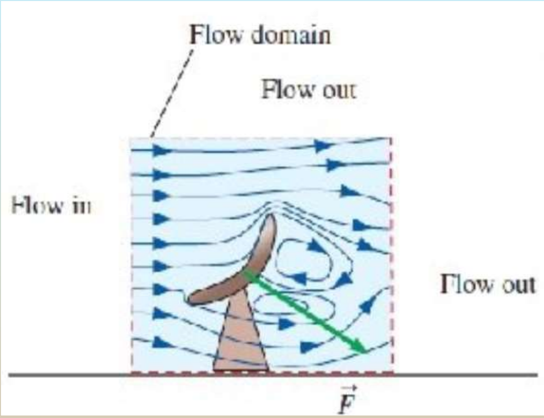
where *v* is a function of v_x, v_y, and v_z.

- The Laplacian of a scalar field is

$$\nabla^2\rho = \frac{\partial^2\rho}{\partial x^2} + \frac{\partial^2\rho}{\partial y^2} + \frac{\partial^2\rho}{\partial z^2}$$

WHY DIFFERENTIAL ANALYSIS?

- Integral analysis allows us to compute overall (global) flow behavior without concern for the detailed flow inside a device.
- Integral analysis requires careful integration at a system boundaries (velocity profile at exits must be given or assumed)
- Differential analysis is required when we need to know the detailed flow behavior at points inside a system (velocity profiles are computed directly).



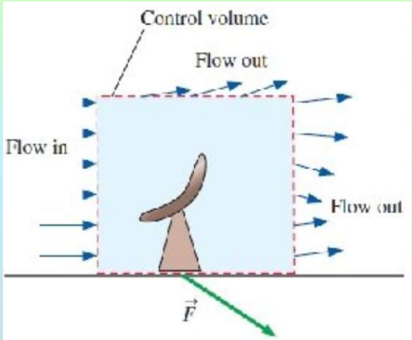
- **The control volume technique**

  It is useful when we are interested in the overall features of a flow, such as mass flow rate into and out of the control volume or net forces applied to bodies.

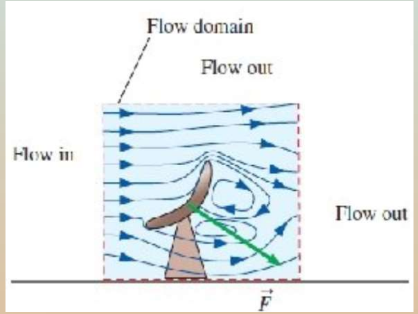
- **Differential analysis**

  Involves application of differential equations of fluid motion to any and every point in the flow field over a region called the flow domain.

- **Boundary conditions** for the variables must be specified at all boundaries of the flow domain, including inlets, outlets, and walls.
- If the flow is unsteady, we must march our solution along in time as the flow field changes.



In control volume analysis, the interior of the control volume is treated like a black box



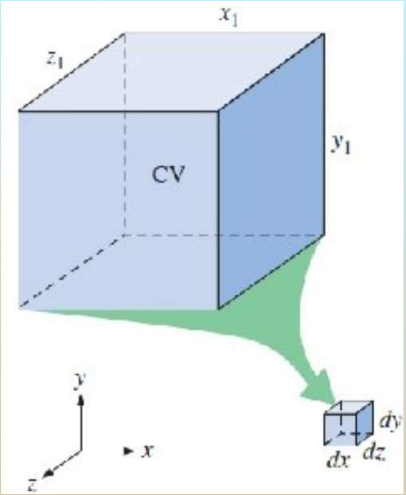
in differential analysis, all the details of the flow are solved at every point within the flow domain

CONSERVATION OF MASS—THE CONTINUITY EQUATION

- The net rate of change of mass within the control volume is equal to the rate at which mass flows into the control volume minus the rate at which mass flows out of the control volume.
- Conservation of mass for a CV:

(rate of mass accumulation) = (rate of mass in) (rate of mass out)

$$\int_{CV} \frac{\partial \rho}{\partial t} dV = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$

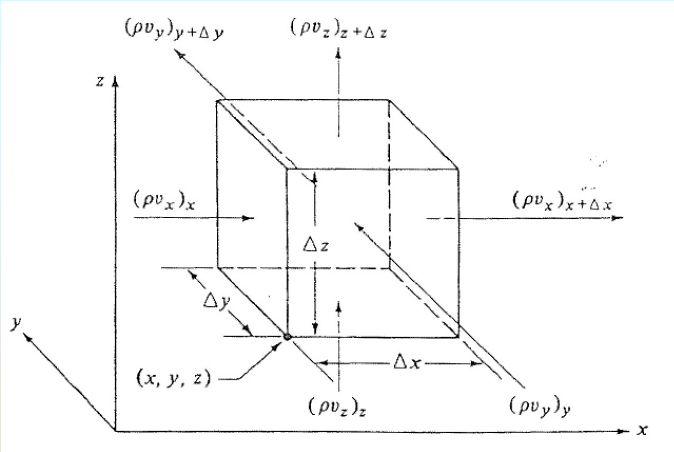


To derive a differential conservation equation, we imagine shrinking a control volume to infinitesimal size

- In the x direction the rate of mass entering the face at x having an area of $\Delta y \, \Delta z$ m^2 is

$(\rho v_x)_x \Delta y \Delta z$ kg/s and that leaving at x + Δx is $(\rho v_x)_{x+\Delta x} \Delta y \Delta z$.

The term (ρv_x) is a mass flux in kg/s.m^2



- Mass entering and that leaving in the y and the z directions are also shown in the figure.

- The rate of mass accumulation in the volume $\Delta x \, \Delta y \, \Delta z$ is

$$Rate\ of\ mass\ accumulation = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

- Substituting all these expressions into the mass balance equation and dividing both sides by $\Delta x \, \Delta y \, \Delta z$,

$$\frac{[(\rho v_x)_x - (\rho v_x)_{x+\Delta x}]}{\Delta x} + \frac{\left[(\rho v_y)_y - (\rho v_y)_{y+\Delta y}\right]}{\Delta y} + \frac{[(\rho v_z)_z - (\rho v_z)_{z+\Delta z}]}{\Delta z} = \frac{\partial \rho}{\partial t}$$

- Taking the limit as Δx, Δy, and Δ z approach zero, we obtain the equation of continuity or conservation of mass for a pure fluid:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial (\rho v_x)_x}{\partial x} + \frac{\partial (\rho v_y)_y}{\partial y} + \frac{\partial (\rho v_z)_z}{\partial z}\right] = -(\nabla . \rho v)$$

- The vector notation on the right side comes from the fact that v is a vector.
- This equation tells us how density ρ changes with time at a fixed point resulting from the changes in the mass velocity vector ρ v.

We can convert the last equation into another form by carrying out the actual partial differentiation:

$$\frac{\partial \rho}{\partial t} = -\rho\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] - \left[v_x\frac{\partial \rho}{\partial x} + v_y\frac{\partial \rho}{\partial y} + v_z\frac{\partial \rho}{\partial z}\right]$$

Rearranging

$$\frac{\partial \rho}{\partial t} + \left[v_x\frac{\partial \rho}{\partial x} + v_y\frac{\partial \rho}{\partial y} + v_z\frac{\partial \rho}{\partial z}\right] = -\rho\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right]$$

The left-hand side of this equation is the same as the substantial derivative in slide 13. Hence, this equation becomes:

$$\frac{D\rho}{Dt} = -\rho\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] = -\rho(\nabla . v)$$

EQUATION OF CONTINUITY FOR CONSTANT DENSITY

- Often in engineering with liquids that are relatively incompressible, the density ρ is essentially constant.
- Then ρ remains constant for a fluid element as it moves along a path following the fluid motion, or $\frac{D\rho}{Dt} = 0$
- Hence, the continuity equation becomes for a fluid of constant density at steady or unsteady state

$$(\nabla . v) = \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right] = 0$$

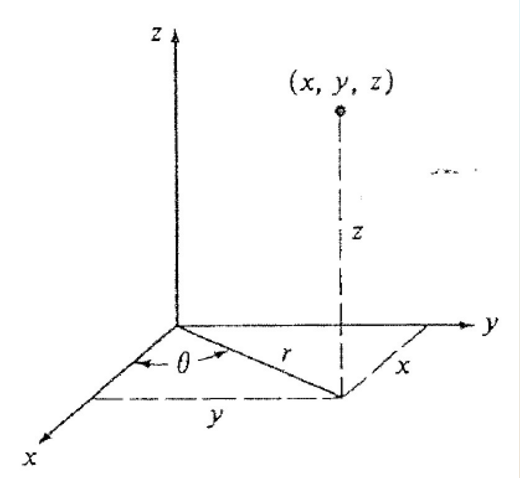
- At steady state, $\frac{\partial \rho}{\partial t} = 0$

CONTINUITY EQUATION IN CYLINDRICAL AND SPHERICAL COORDINATES

- The coordinate system as related to rectangular coordinates is shown in the figure.
- The relations between rectangular x, y, z and cylindrical r, θ , z coordinates are:

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad \mathrm{z} = \mathrm{z}$$

$$r = +\sqrt{x^2 + y^2} \qquad and \quad \theta = \tan^{-1}\frac{y}{x}$$



Using these relations, the equation of continuity in cylindrical coordinates is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

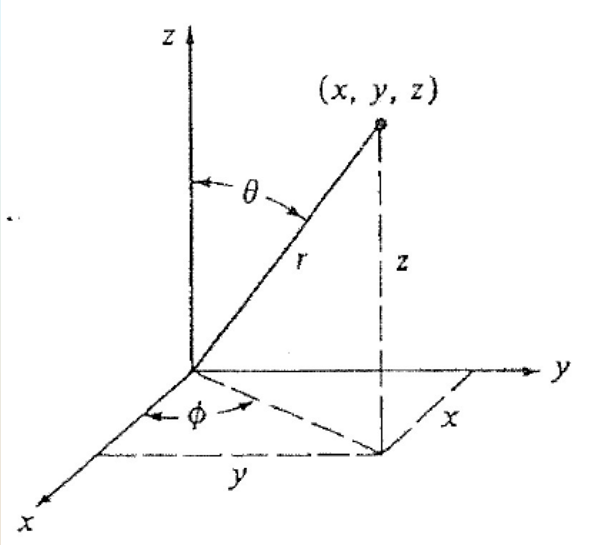
Continuity equation in cylindrical and spherical coordinates

For spherical coordinates the variables r, θ, and ϕ are related to x, y, z by the following as shown in figure.

$$x = r\,sin(\theta)\,cos(\phi) \qquad y = r\,sin(\theta)\,sin(\phi) \quad \mathrm{z} = \mathrm{z}\,cos(\theta)$$

$$r = +\sqrt{x^2+y^2+z^2} \qquad and \quad \theta = tan^{-1}\frac{\sqrt{x^2+y^2}}{x}$$

$$\phi = tan^{-1}\frac{y}{x}$$



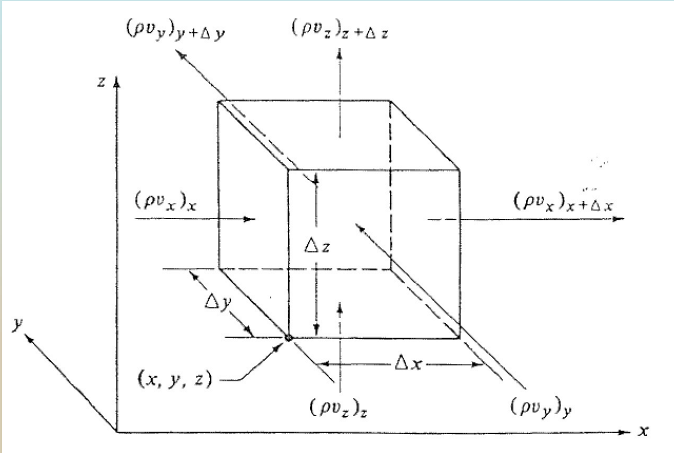
Using these relations, the equation of continuity in cylindrical coordinates is

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial\left(\rho r^2 v_r\right)}{\partial r} + \frac{1}{rsin(\theta)}\frac{\partial\left(\rho v_\theta sin(\theta)\right)}{\partial\theta} + \frac{1}{rsin(\theta)}\frac{\partial(\rho v_z)}{\partial\phi} = 0$$

DERIVATION OF EQUATIONS OF MOMENTUM TRANSFER

- The equation of motion is really the equation for the conservation-of-momentum which we can write as:

(rate of momentum in) - (rate of momentum out) + (sum of forces acting on system) = (rate of momentum accumulated)



- We will make a balance on an element.
- First we shall consider only the x component of each term in the momentum equation.
- The y and z components can be described in an analogous manner.

- The rate at which the x component of momentum enters the face at x in the x direction by convection is $(\rho v_x v_x)_x \Delta y \Delta z$, and the rate at which it leaves at x + Δx is $(\rho v_x v_x)_{x+\Delta x} \Delta y \Delta z$.
- The quantity $(\rho v_x)$is the concentration in momentum/m^3 or (kg·m/s)/m^3, and it is multiplied by v_x to give the momentum flux as momentum/s· m^2
- The x component of momentum entering the face at y is $\left(\rho v_y v_x\right)_y \Delta x \Delta z$, and leaving at y + Δy is $\left(\rho v_y v_x\right)_{y+\Delta y} \Delta x \Delta z$.
- For the face at z we have $(\rho v_z v_x)_z \Delta x \Delta y$, and leaving at z + Δz is$(\rho v_z v_x)_{z+\Delta z} \Delta x \Delta y$.

- The net convective x momentum flow into the volume element $\Delta x\Delta y\Delta z$ is

$$[(\rho v_x v_x)_x-(\rho v_x v_x)_{x+\Delta x}]\Delta y\Delta z+[(\rho v_y v_x)_y-(\rho v_y v_x)_{y+\Delta y}]\Delta x\Delta z+[(\rho v_z v_x)_z-(\rho v_z v_x)_{z+\Delta z}]\Delta x\Delta y$$

- Momentum flows in and out of the volume element by the mechanisms of convection or bulk flow as given by the above equation and also by molecular transfer.
- The rate at which the x component of momentum enters the face at x by molecular transfer is $(\tau_{xx})_x\Delta y\Delta z$, and the rate at which it leaves the surface at x + Δx is $(\tau_{xx})_{x\ \ \Delta x}\Delta y\Delta z$
- The same thing can be done for faces y and z.

- The net x component of momentum by molecular transfer is

$$[(\tau_{xx})_x - (\tau_{xx})_{x+\Delta x}]\Delta y \Delta z$$

$$+[(\tau_{yx})_y - (\tau_{yx})_{y+\Delta y}]\Delta x \Delta z + [(\tau_{zx})_z - (\tau_{zx})_{z+\Delta z}]\Delta x \Delta y$$

- These molecular fluxes of momentum may be considered as shear stresses and normal stresses
- Hence, τ_{yx} is the x direction shear stress on the y face and τ_{zx} the shear stress on the z face. Also, τ_{xx} is the normal stress on the x face.
- The net fluid pressure force acting on the element in the x direction is the difference between the force acting at x and x + Δx.

$$(p_x - p_{x+\Delta x})\Delta y \Delta z$$

- The gravitational force g_x acting on a unit mass in the x direction is multiplied by the mass of the element to give

$$\rho g_x \Delta x \Delta y \Delta z$$

- The rate of accumulation of x momentum in the element is

$$\Delta x \Delta y \Delta z \frac{\partial(\rho v_x)}{\partial x}$$

- By substitution in the conservation-of-momentum equation, then x component of the differential equation of motion:

$$\frac{\partial(\rho v_x)}{\partial t} = \left[\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right] - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x$$

- The y and z components of the differential equation of motion are, respectively

$$\frac{\partial(\rho v_y)}{\partial t} = \left[\frac{\partial(\rho v_x v_y)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_z v_y)}{\partial z}\right] - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) - \frac{\partial p}{\partial y} + \rho g_y$$

$$\frac{\partial(\rho v_z)}{\partial t} = \left[\frac{\partial(\rho v_x v_z)}{\partial x} + \frac{\partial(\rho v_y v_z)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z}\right] - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) - \frac{\partial p}{\partial z} + \rho g_z$$

- Using the x component of the differential equation of motion and

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho v_x)_x}{\partial x} + \frac{\partial(\rho v_y)_y}{\partial y} + \frac{\partial(\rho v_z)_z}{\partial z}\right] = -(\nabla . \rho v)$$

we obtain the equation of motion for the x component:

$$\rho\left[\frac{\partial(v_x)}{\partial t} + v_x \frac{\partial(v_x)}{\partial x} + v_y \frac{\partial(v_x)}{\partial y} + v_z \frac{\partial(v_x)}{\partial z}\right] = -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} + \rho g_x$$

- The equation of motion for y and z are:

$$\rho\left[\frac{\partial(v_y)}{\partial t}+v_x\frac{\partial(v_y)}{\partial x}+v_y\frac{\partial(v_y)}{\partial y}+v_z\frac{\partial(v_y)}{\partial z}\right]=-\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)-\frac{\partial p}{\partial y}+\rho g_y$$

$$\rho\left[\frac{\partial(v_z)}{\partial t}+v_x\frac{\partial(v_z)}{\partial x}+v_y\frac{\partial(v_z)}{\partial y}+v_z\frac{\partial(v_z)}{\partial z}\right]=-\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right)-\frac{\partial p}{\partial z}+\rho g_z$$

- Adding vectorially, we obtain an equation of motion for a pure fluid:

$$\rho\frac{Dv}{Dt}=-(\nabla.\tau)-\nabla p+\rho g$$

EQUATIONS OF MOTION FOR NEWTONIAN FLUIDS WITH VARYING DENSITY AND VISCOSITY

- In order to use the previous equations to determine velocity distributions, expressions
- must be used for the various stresses in terms of velocity gradients and fluid properties.
- For Newtonian fluids the expressions for the stresses have been related to the velocity gradients and the fluid viscosity μ.

1- Shear-stress components for Newtonian fluids in rectangular coordinates

$$\tau_{xx} = -2\mu\frac{\partial v_x}{\partial x} + \frac{2}{3}\mu(\nabla . v)$$

$$\tau_{yy} = -2\mu\frac{\partial v_y}{\partial y} + \frac{2}{3}\mu(\nabla . v)$$

$$\tau_{zz} = -2\mu\frac{\partial v_z}{\partial z} + \frac{2}{3}\mu(\nabla . v)$$

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)$$

$$\tau_{zx} = \tau_{xz} = -\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)$$

$$(\nabla . v) = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

2. Shear-stress components for Newtonian fluids in cylindrical coordinates

$$\tau_{\mathrm{rr}} = -\mu\left[2\frac{\partial v_r}{\partial r} - \frac{2}{3}(\nabla . v)\right]$$

$$\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \frac{2}{3}(\nabla . v)\right]$$

$$\tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z} - \frac{2}{3}(\nabla . v)\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial (v_\theta / r)}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]$$

$$\tau_{zr} = \tau_{\mathrm{rz}} = -\mu\left[\frac{\partial v_\theta}{\partial r} + \frac{\partial v_r}{\partial z}\right]$$

$$(\nabla . v) = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

3. Shear-stress components for Newtonian fluids in spherical coordinates

$$\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r} - \frac{2}{3}(\nabla . v)\right] \quad \text{and} \quad \tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \frac{2}{3}(\nabla . v)\right]$$

$$\tau_{\phi\phi} = -\mu\left[2\left(\frac{1}{r\,sin(\theta)}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\,cot(\theta)}{r}\right) - \frac{2}{3}(\nabla . v)\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial(v_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu\left[\frac{sin(\theta)}{r}\frac{\partial(v_\theta/sin(\theta))}{\partial \theta} + \frac{1}{r\,sin(\theta)}\frac{\partial v_\phi}{\partial \phi}\right]$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu\left[\frac{1}{r\,sin(\theta)}\frac{\partial v_r}{\partial \phi} + r\frac{\partial\left(v_\phi/r\right)}{\partial r}\right]$$

$$(\nabla . v) = \frac{1}{r^2}\frac{\partial\left(r^2 v_r\right)}{\partial r} + \left[\frac{1}{rsin(\theta)}\frac{\partial\left(v_\theta sin(\theta)\right)}{\partial \theta} + \frac{1}{r\,sin(\theta)}\frac{\partial v_\phi}{\partial \phi}\right]$$

4. Equation of Motion for Newtonian fluids with varying density and viscosity

$$\rho \frac{Dv_x}{Dt} = \frac{\partial}{\partial x}\left[2\mu \frac{\partial v_x}{\partial x} - \frac{2}{3}\mu(\nabla . v)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right] - \frac{\partial p}{\partial x} + \rho g_x$$

EQUATIONS OF MOTION FOR NEWTONIAN FLUIDS WITH CONSTANT DENSITY AND VISCOSITY

- The equations above are seldom used in their complete forms.
- When the density ρ and the viscosity μ are constant where $(\nabla . v) = 0$, the equations are simplified and we obtain the equations of motion for Newtonian fluids.
- These equations are also called the Navier-Stokes equations

1. Equation of motion in rectangular coordinates

$$\rho\left[\frac{\partial(v_x)}{\partial t}+v_x\frac{\partial(v_x)}{\partial x}+v_y\frac{\partial(v_x)}{\partial y}+v_z\frac{\partial(v_x)}{\partial z}\right]=\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)-\frac{\partial p}{\partial x}+\rho g_x$$

$$\rho\left[\frac{\partial(v_y)}{\partial t}+v_x\frac{\partial(v_y)}{\partial x}+v_y\frac{\partial(v_y)}{\partial y}+v_z\frac{\partial(v_y)}{\partial z}\right]=\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)-\frac{\partial p}{\partial y}+\rho g_y$$

$$\rho\left[\frac{\partial(v_z)}{\partial t}+v_x\frac{\partial(v_z)}{\partial x}+v_y\frac{\partial(v_z)}{\partial y}+v_z\frac{\partial(v_z)}{\partial z}\right]=\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)-\frac{\partial p}{\partial x}+\rho g_z$$

Combining the three equations for the three components, we obtain

$$\rho\frac{Dv}{Dt}=-\nabla p+\rho g+\mu\nabla^2 v$$

2. Equation of motion in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r}$$

$$+ \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$+ \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z}$$

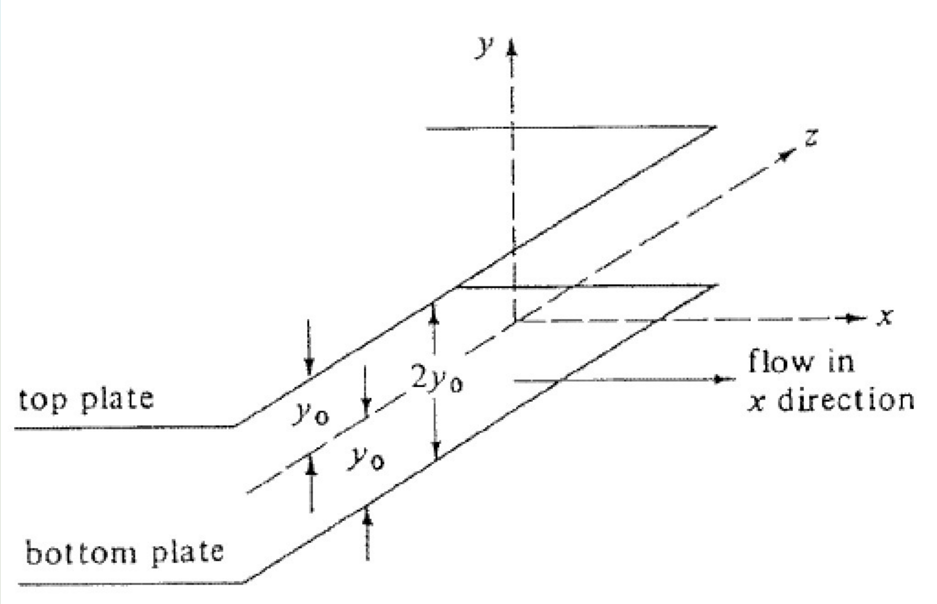
$$+ \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

USE OF DIFFERENTIAL EQUATIONS OF CONTINUITY AND MOTION

- To apply these equations to any viscous-flow problem.
- For a given specific problem, the terms that are zero or near zero are simply discarded and the remaining equations used in the solution to solve for the velocity, density, and pressure distributions.
- It is necessary to know the initial conditions and the boundary conditions to solve the equations

Differential Equations of Continuity and Motion for Flow between Parallel Plates

- Fluid with constant viscosity which is flowing between two flat and parallel plates
- The velocities v_y and v_z are then zero. The plates are a distance $2y_o$ apart
- The continuity equation becomes



$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

since v_y and v_z are constants, then

$$\frac{\partial v_x}{\partial x} = 0$$

- The Navier-Stokes equations becomes

$$\rho\left[\frac{\partial(v_x)}{\partial t}+v_x\frac{\partial(v_x)}{\partial x}+v_y\frac{\partial(v_x)}{\partial y}+v_z\frac{\partial(v_x)}{\partial z}\right]=\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)-\frac{\partial p}{\partial x}+\rho g_x$$

- The following terms equal zero
  - ✓ $\frac{\partial(v_x)}{\partial t}$ steady state
  - ✓ v_y and v_z
  - ✓ $\frac{\partial v_x}{\partial x}=0$ also $\frac{\partial^2 v_x}{\partial x^2}$
  - ✓ $\frac{\partial(v_x)}{\partial z}$ since there is no change of v_x with z. Also, $\frac{\partial^2 v_x}{\partial z^2}=0$
  - ✓ $g_x=0$ for the present case of a horizontal pipe
- This will simplify the general equation to: $\frac{\partial p}{\partial x}=\mu\frac{\partial^2 v_x}{\partial y^2}$

$\frac{\partial p}{\partial x}$ is a constant in this problem since *Vx* is not a function of *x.*

Then, $\frac{\partial p}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}$ becomes an ordinary differential equation

$$\frac{d^2 v_x}{dy^2} = \frac{1}{\mu} \frac{\mathrm{dp}}{\mathrm{dx}} = constant$$

The conditions given in this question are:

$$\frac{\partial v_x}{\partial y} = 0 \quad at\ y = 0$$

$$v_x = 0\ at\ y = y_0$$

Solving the nonhomogeneous 2nd order differential equation ($\frac{d^2 v_x}{dy^2} = \frac{1}{\mu} \frac{\mathrm{dp}}{\mathrm{dx}}$) will give

$$v_g = C_1 + C_2\, y$$

$$v_N = \frac{1}{2\mu} \frac{dp}{dx} y^2$$

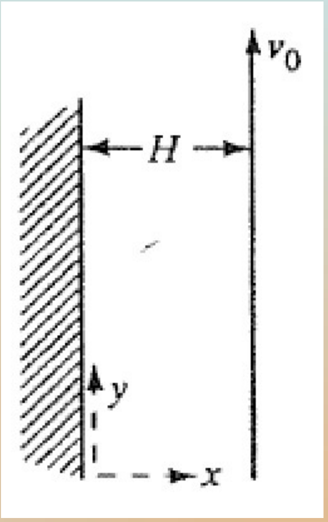
Apply the conditions, the final solution will be $\boldsymbol{v_x = \frac{1}{2\mu} \frac{dp}{dx} \left(y^2 - y_0^2\right)}$

The maximum velocity occurs when y = 0, $\boldsymbol{v_{max} = \frac{1}{2\mu} \frac{dp}{dx} \left(-y_0^2\right)}$

Laminar Flow Between Vertical Plates with One Plate Moving

A Newtonian fluid is confined between two parallel and vertical plates as shown in the Fig. The surface on the left is stationary and the other is moving vertically at a constant velocity Vo. Assuming that the flow is laminar, solve for the velocity profile

Assumption:



- ✓ At steady state, $\frac{\partial(v_y)}{\partial t} = 0$.
- ✓ The velocities v_y and v_z are zeros.
- ✓ $\frac{\partial(v_y)}{\partial y} = 0$ from the continuity equation
- ✓ $\frac{\partial(v_y)}{\partial z} = 0$, and
- ✓ $\rho g_y = -\rho g$

Navier-Stokes equation for the y coordinate

$$\rho\left[\frac{\partial(v_y)}{\partial t}+v_x\frac{\partial(v_y)}{\partial x}+v_y\frac{\partial(v_y)}{\partial y}+v_z\frac{\partial(v_y)}{\partial z}\right]=\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)-\frac{\partial p}{\partial y}+\rho g_y$$

Substitution of the assumption into Navier equation:

$$\mu\frac{d^2 v_y}{dx^2}-\frac{\mathrm{d}p}{\mathrm{d}y}-\rho g_y=0$$

The pressure gradient $\frac{\mathrm{d}p}{\mathrm{d}y}$ is constant

The general solution for this equation is

$$v_y-\frac{x^2}{2\mu}\left(\frac{dy}{dx}+\rho g\right)=C_1 x+C_2$$

The boundary conditions are:

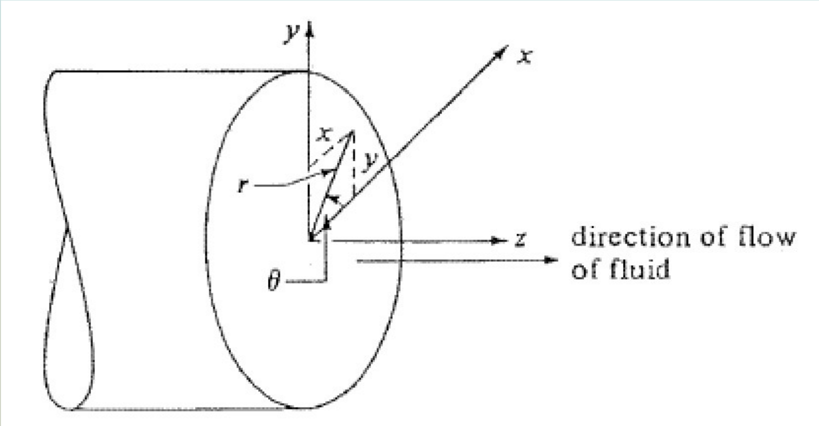
At $x = 0,\ v_y = 0$

At $x = H,\ v_y = v_0$

$$\mathbf{v_y - \frac{1}{2\mu}\left(\frac{dy}{dx} + \rho g\right)\left(Hx - x^2\right) + v_0 \frac{x}{H}}$$

Laminar Flow in a Circular Tube

Derive the equation for steady-state viscous flow in a horizontal tube of radius r_o, where the fluid is far from the tube inlet. The fluid is incompressible and μ is a constant. The flow is driven in one direction by a constant pressure gradient.



The cylindrical coordinates equation can be used for the z component and the terms that are zero discarded.

$$\rho\left[\frac{\partial(v_z)}{\partial t}+v_r\frac{\partial(v_z)}{\partial r}+\frac{v_\theta}{r}\frac{\partial(v_z)}{\partial\theta}+v_z\frac{\partial(v_z)}{\partial z}\right]$$
$$=\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(v_z)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]-\frac{\partial p}{\partial x}+\rho g_z$$

As before:

$\frac{\partial(v_z)}{\partial t}=0,\ \frac{\partial v_z}{\partial\theta}=0,\ \frac{\partial v_z}{\partial z}=0$

also $\frac{\partial^2 v_z}{\partial\theta^2}=0,\quad v_r=0$

The boundary conditions at $r=0,\ \frac{\partial v_z}{\partial r}=0$

Also, at $r=r_0,\ v_z=0$

Substitute all of these terms in the general momentum equation for cylindrical coordinates:

$$\frac{1}{\mu}\frac{\mathrm{dp}}{\mathrm{dz}} = \text{constant} = \frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{d(v_z)}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d(v_z)}{dr}\right)$$

Rearrange the equation:

$$\frac{d^2 v_z}{dr^2} + \frac{1}{r}\frac{d(v_z)}{dr} = \frac{1}{\mu}\frac{\mathrm{dp}}{\mathrm{dz}} = \text{constant}$$

This is a 2nd order non-homogeneous ordinary differential equation of Eulers form

$$r^2\frac{d^2 v_z}{dr^2} + r\frac{d(v_z)}{dr} = \frac{r^2}{\mu}\frac{\mathrm{dp}}{\mathrm{dz}}$$

The general solution of this equation has a form of:

$$v_{zG} = v_{zH} + v_{zN}$$

For v_{zH}, solve this homogeneous equation $r^2\frac{d^2 v_z}{dr^2} + r\frac{d(v_z)}{dr} = 0$ by assuming the solution has a form of

$$v_{zH} = v^m$$

The root m is repeated and is $m_1 = m_2 = 0$; the $v_{zH} = C_1 + C_2 ln(r)$

For v_{zN}, since the non-homogeneous term has the form of $constant\ \times r^2$, then the form of v_{zN} will be:

$$v_{zN} = A + Br + Dr^2$$

Substitute in the differential equation and equate the two sides of the equation, one will get:

$$A = B = 0$$

$$D = \frac{1}{4\mu}\frac{dP}{dz}$$

$$\therefore v_{zH} = \frac{1}{4\mu}\frac{dP}{dz}r^2$$

$$v_{zG} = v_{zH} + v_{zN} = C_1 + C_2 ln(r) + \frac{1}{4\mu}\frac{dP}{dz}r^2$$

Apply the two conditions to this general solution, we get:

$C_2 = 0$ and $C_1 = -\frac{1}{4\mu}\frac{dP}{dz}r_0^2$

$$\therefore v_z = \frac{1}{4\mu}\frac{dP}{dz}\left(r^2 - r_0^2\right)$$

The maximum velocity will be in the center of the tube where $r = 0$

$$\therefore v_{z\,max} = -\frac{1}{4\mu}\frac{dP}{dz}r_0^2$$

Converting v_z to the maximum velocity:

$$\therefore v_z = v_{z\,max}\left(1 - \frac{r^2}{r_0^2}\right)$$

The pressure drop $p_1 - p_2$ can be obtained by integrating $v_z = \frac{1}{4\mu}\frac{dP}{dz}\left(r^2 - r_0^2\right)$ w.r.t. z, Integrating to obtain the pressure drop from $z = o\ for\ p = p_1\ to\ z = L\ for\ p = p_2$

and using the average velocity $v_{z\,avg} = -\frac{r_0^2}{8\mu}\frac{dP}{dz}$:

$$p_1 - p_2 = -\frac{8\mu v_{z\,avg}L}{r_0^2} = \frac{32\mu v_{z\,avg}L}{D^2}$$

Where $D = 2r_0$