Mass Balance

Mass balance for flow system

- For flow problem, apply mass conservation over all the system or part of it.

$$\text{input} = \text{output} + \text{accumulation}$$

- In case of flow problems, we usually deal with flow rates and steady state condition,

$$\text{rate of input} = \text{rate of output (steady state)}$$

Look to the following flow system

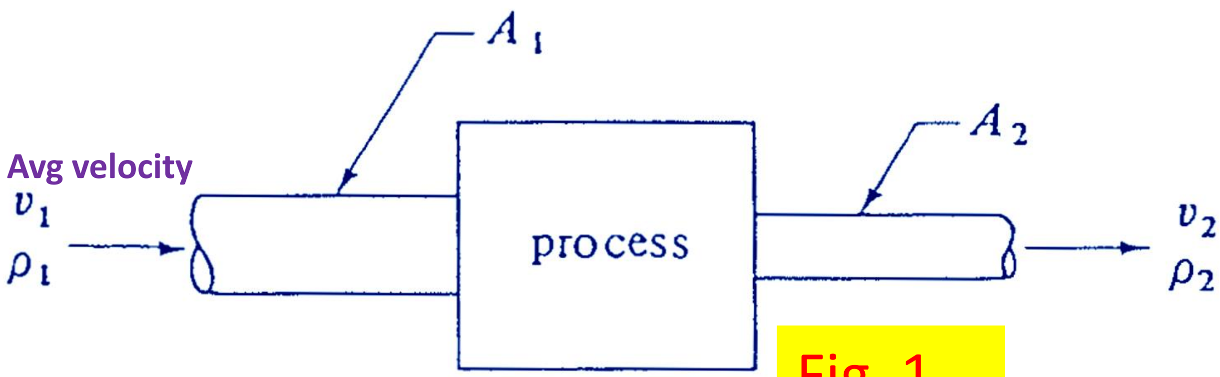


Fig. 1

Mass Balance:

$$m = \rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

Check units

Note:
Mass velocity or mass flux G
$G = \rho\, v$ with units kg/s m^2

Control Volume

Control volume for flow through a conduit.

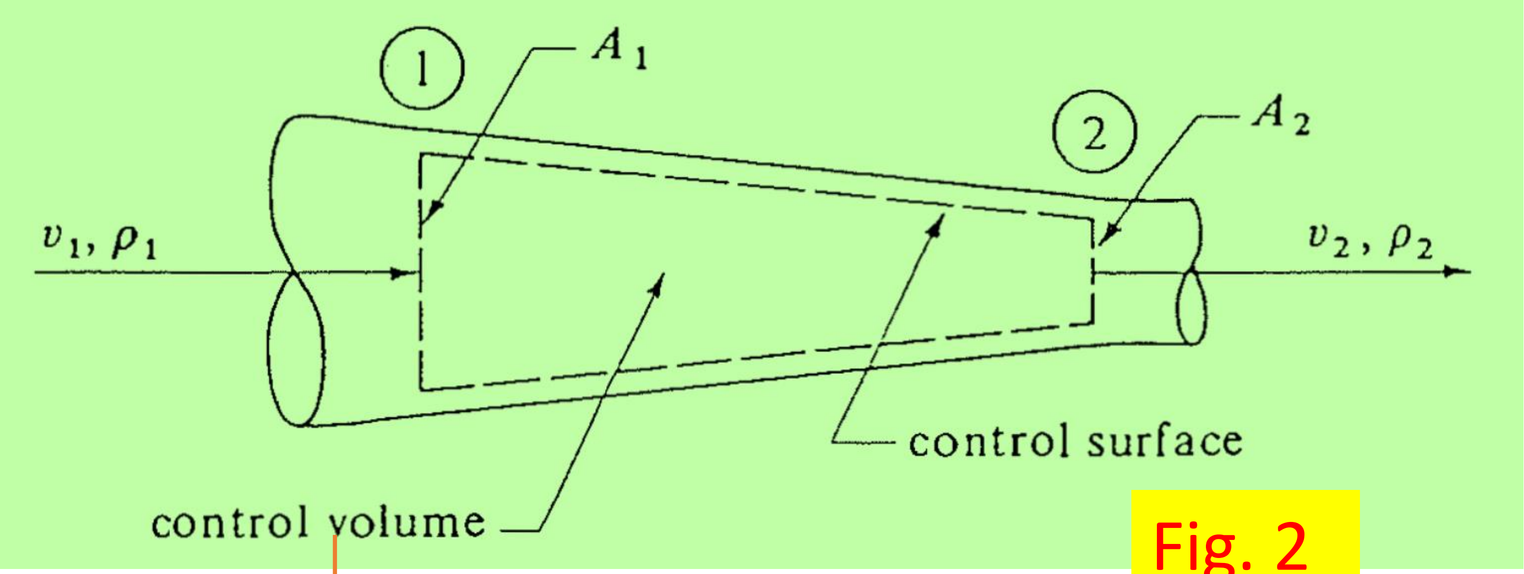


Fig. 2

a region fixed in space through which the fluid flows.

Note:

In most problems the control surface is taken as the wall of the conduit.

Mass balance Eq. over a control volume

In deriving the general equation for the overall balance of the property mass, the law of conservation of mass may be stated as follows for a control volume where no mass is being generated.

$$\begin{pmatrix}\text{rate of mass output}\\ \text{from control volume}\end{pmatrix} - \begin{pmatrix}\text{rate of mass input}\\ \text{from control volume}\end{pmatrix}$$

$$+ \begin{pmatrix}\text{rate of mass accumulation}\\ \text{in control volume}\end{pmatrix} = 0 \ (\text{rate of mass generation})$$

……………………………(1)

Assume a general control volume as shown below, and focus on a differential element dA.

Flow through a differential area dA on a control surface

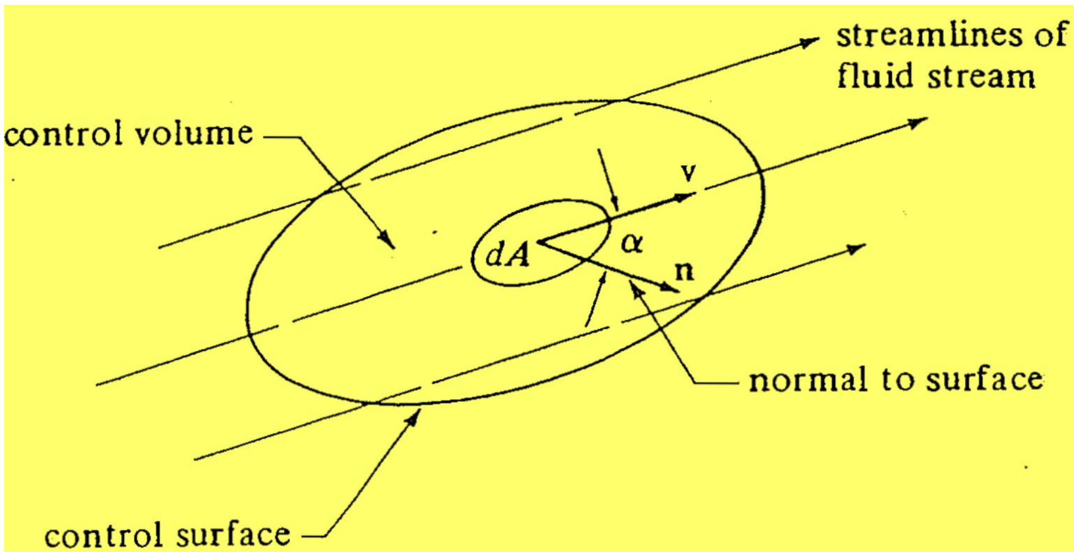


Fig. 3

the rate of mass efflux from this element $= (\rho v)(dA \cos \alpha)$, where $(dA \cos \alpha)$ is the area dA projected in a direction normal to the velocity vector **v**, α is the angle between the velocity vector **v** and the outward-directed unit normal vector **n** to dA

From vector algebra we recognize that $(\rho v)(dA \cos \alpha)$ is the scalar or dot product $\rho(\mathbf{v} \cdot \mathbf{n})\,dA$. If we now integrate this quantity over the entire control surface A we have the net outflow of mass across the control surface, or the net mass efflux in kg/s from the entire control volume V:

$\rho \upsilon$ = mass velocity G 'flux'

Scalar or dot product

(v .n) = ||v || ||n|| cosα
= υ . 1 cosα = υ cosα

$$\begin{pmatrix}\text{net mass efflux}\\ \text{from control volume}\end{pmatrix} = \iint_A v\rho \cos \alpha \, dA = \iint_A \rho(\mathbf{v} \cdot \mathbf{n})\, dA$$

..............................(2)

Rate of accumulation

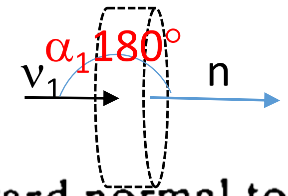
$$\begin{pmatrix}\text{rate of mass accumulation}\\ \text{in control volume}\end{pmatrix} = \frac{\partial}{\partial t} \iiint_V \rho \, dV = \frac{dM}{dt}$$

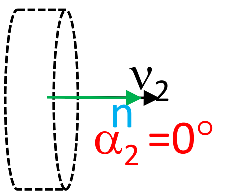
..............................(3)

Substituting Eq.[s] (2) and (3) into (1) leads to general form of overall mass balance.

$$\iint_A \rho(\mathbf{v}\cdot\mathbf{n})\, dA + \frac{\partial}{\partial t}\iiint_V \rho\, dV = 0 \qquad \text{..............................(4)}$$

- <u>Apply this eq. to Fig. 2</u>





where all the flow inward is normal to A_1 and outward normal to A_2

When the velocity v_2 leaving is normal to A_2, the angle α_2 between the normal to the control surface and the direction of the velocity is 0°

and $\cos\alpha_2 = 1.0$. Where v_1 is directed inward, $\alpha_1 > \pi/2$ α_1 is 180° ($\cos\alpha_1 = -1.0$)

Since α_2 is 0° and α_1 is 180°, using Eq.(4)

$$\iint_A v\rho\cos\alpha\, dA = \iint_{A_2} v\rho\cos\alpha_2\, dA + \iint_{A_1} v\rho\cos\alpha_1\, dA \qquad \text{..............................(5)}$$

$$= v_2\rho_2 A_2 - v_1\rho_1 A_1$$

For steady state, $dM/dt = 0$ in Eq. (3) And (4) becomes

$$m = \rho_1 v_1 A_1 = \rho_2 v_2 A_2$$

Notes: material balance can be done over species or components

$$m_{i2} - m_{i1} + \frac{dM_i}{dt} = R_i \quad \text{.....................(5a)}$$

out

in

accu mulat ion

gener ation

i means component i in multicomponent system

Average Velocity to Use in Overall Mass Balance

If the velocity is not constant but varies across the surface area, an average or bulk velocity is defined by

$$v_{av} = \frac{1}{A} \iint_A v \, dA$$

for a surface over which v is normal to A and the density ρ is assumed constant.

Example 1

For the case of imcompressible flow (ρ is constant) through a circular pipe of radius R, the velocity profile is parabolic for laminar flow as follows:

$$v = v_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

where v_{max} is the maximum velocity at the center where $r = 0$ and v is the velocity at a radial distance r from the center. Derive an expression for the average or bulk velocity v_{av} to use in the overall mass-balance equation.

Solution

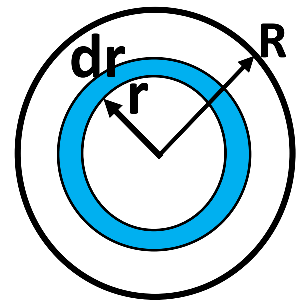
The average velocity is represented by

$$v_{av} = \frac{1}{A} \iint_A v \, dA$$

In Cartesian coordinates dA is $dx\ dy$.

However, using polar coordinates which are more appropriate for a pipe, $dA = r\ dr\ d\theta$,

where θ is the angle in polar coordinates. Substituting $dA = r\ dr\ d\theta$, and $A = \pi R^2$ and υ in above equation and integrating



$$v_{av} = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R v_{max} \left[1 - \left(\frac{r}{R} \right)^2 \right] r\ dr\ d\theta$$

$$= \frac{v_{max}}{\pi R^4} \int_0^{2\pi} \int_0^R (R^2 - r^2) r\ dr\ d\theta$$

$$= \frac{v_{max}}{\pi R^4} (2\pi - 0) \left(\frac{R^4}{2} - \frac{R^4}{4} \right)$$

$$v_{av} = \frac{v_{max}}{2}$$

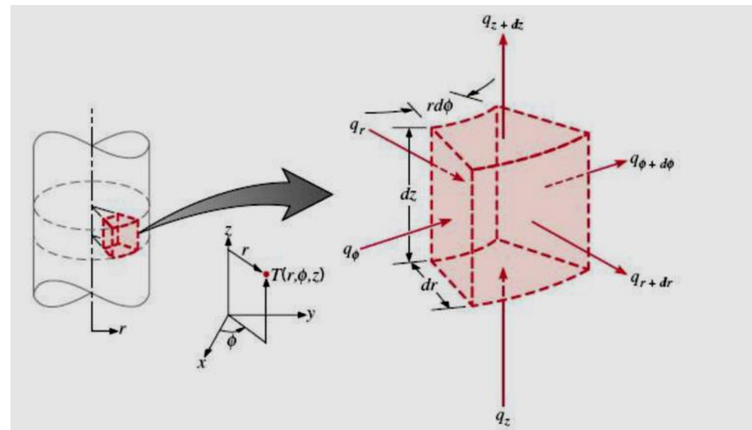


FIGURE 2.12 Differential control volume, $dr \cdot r\,d\phi \cdot dz$, for conduction analysis in cylindrical coordinates (r, ϕ, z).

OVERALL ENERGY BALANCE

- Apply the principle of energy conservation to the control volume in the same manner as mass conservation. Begin with 1st law of Thermodynamics

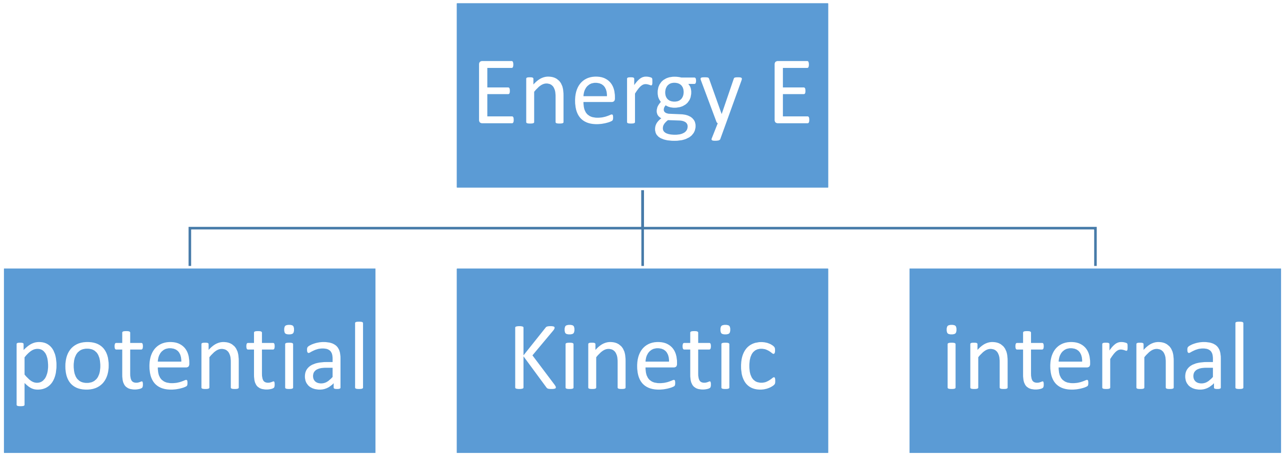
$$\Delta E = Q - W$$

where E is the total energy per unit mass of fluid,
Q is the heat *absorbed* per unit mass of fluid,
and W is the work of all kinds done per unit mass of fluid *upon* the surroundings.

Derivation of Overall Energy-Balance Equation

$$\text{rate of entity output} - \text{rate of entity input} + \text{rate of entity accumulation} = 0 \quad \text{.............................(6)}$$

The energy E present within a system can be classified in three ways.



The total energy of the fluid per unit mass is then

$$E = U + \frac{v^2}{2} + zg \qquad \text{(SI)}$$

$$\begin{pmatrix}\text{rate of energy accumulation} \\ \text{in control volume}\end{pmatrix} = \frac{\partial}{\partial t} \iiint_V \left(U + \frac{v^2}{2} + zg\right) \rho \, dV \qquad \text{.............................(7)}$$

Notes:

The mass added or removed from the system carries internal, kinetic, and potential energy.

In addition, energy is transferred when mass flows into and out of the control volume.

pressure–volume work per unit mass fluid is pV.

$H = U + pV$

the total energy carried with a unit mass is $(H + v^2/2 + zg)$

For a small area dA on the control surface the rate of energy efflux is

$$(H + v^2/2 + zg)(\rho v)(dA \cos \alpha)$$

$$\begin{pmatrix}\text{net energy efflux}\\ \text{from control volume}\end{pmatrix} = \iint_A \left(H + \frac{v^2}{2} + zg\right)(\rho v)\cos\alpha \, dA$$

..............................(8)

To obtain the overall energy balance, we substitute Eqs. **(8) And (7) into (6)** and equate the resulting equation to $q - \dot{W}_S$

$$\iint_A \left(H + \frac{v^2}{2} + zg\right)(\rho v)\cos\alpha \, dA + \frac{\partial}{\partial t}\iiint_V \left(U + \frac{v^2}{2} + zg\right)\rho \, dV = q - \dot{W}_S$$

..............................(9)

Overall Energy Balance for Steady-State Flow System

Consider the following **figure 4** with the following assumptions:

- Steady-state system.
- Single inlet and outlet.
- Ignore inlet a and exit height z, density ρ and enthalpy H variations. Eq. 9 becomes
- 1-d flow cross the boundary.

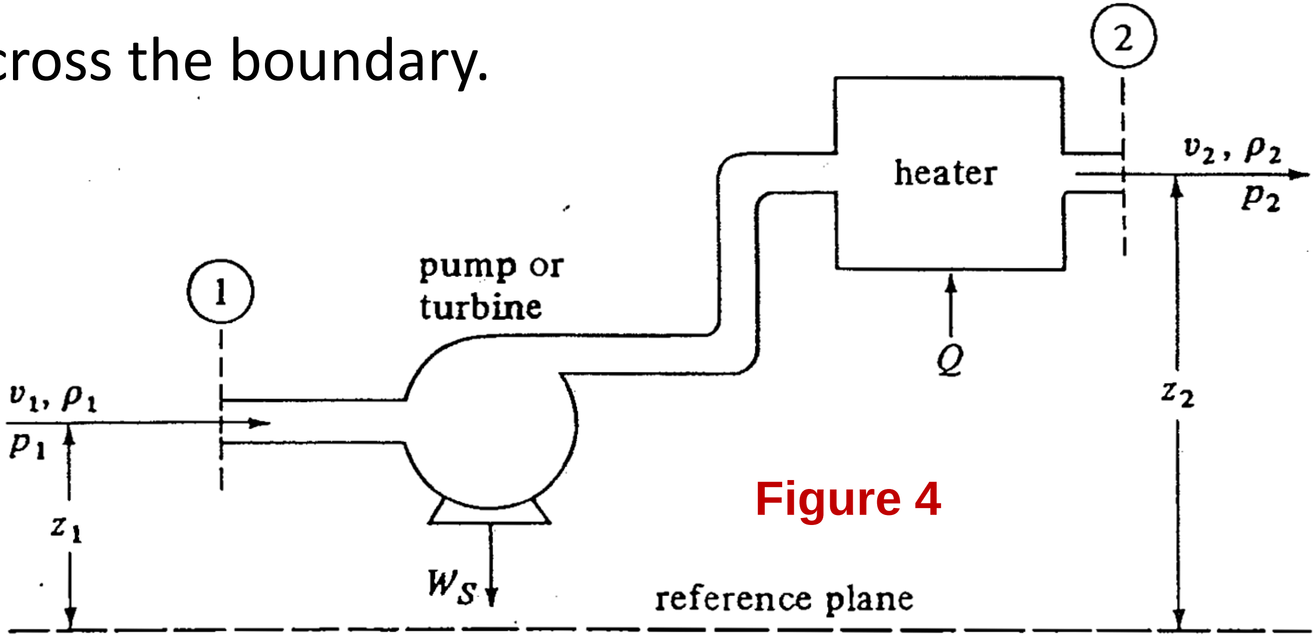


Figure 4

To determine the overall energy balance for the common system shown in figure 4, we can consider the following: since the angle between the velocity vector and unit normal vector is $\alpha = 0$ and since the accumulation term is 0 at steady state, Eq. 9 can be reduced to

$$\iint_A \left(H + \frac{v^2}{2} + zg\right)(\rho v) \cos \alpha \, dA + \frac{\partial}{\partial t} \iiint_V \left(U + \frac{v^2}{2} + zg\right) \rho \, dV = q - \dot{W}_S$$

……………………………(9)

$$H_2 m_2 - H_1 m_1 + \frac{m_2 (v_2^3)_{av}}{2v_{2\,av}} - \frac{m_1 (v_1^3)_{av}}{2v_{1\,av}} + g m_2 z_2 - g m_1 z_1 = q - \dot{W}_S$$

..............................(9)

For steady state, $m_1 = \rho_1 v_{1\,av} A_1 = m_2 = m$.

Dividing through by m so that the equation is on a unit mass basis,

$$H_2 - H_1 + \frac{1}{2}\left[\frac{(v_2^3)_{av}}{v_{2\,av}} - \frac{(v_1^3)_{av}}{v_{1\,av}}\right] + g(z_2 - z_1) = Q - W_S$$

..............................(10)

The term $(v^3)_{av}/(2v_{av})$ can be replaced by $v_{av}^2/2\alpha$.

where α is the kinetic-energy velocity correction factor and is equal to $v_{av}^3/(v^3)_{av}$

Eq. 10 can be rewritten as

$$H_2 - H_1 + \frac{1}{2\alpha}(v_{2\,av}^2 - v_{1\,av}^2) + g(z_2 - z_1) = Q - W_S$$

..............................(10)

Note: Kinetic energy term

$$\text{kinetic energy} = \iint_A \left(\frac{v^2}{2}\right)(\rho v) \cos \alpha \, dA \quad \text{.........................(11)}$$

Assume ρ = cons. And set $\cos \alpha = 1$.

Then multiplying the numerator and denominator by $v_{av} A$ noting that $m = \rho v_{av} A$, Eq. (11) becomes

$$\frac{\rho}{2} \iint_A (v^3) \, dA = \frac{\rho v_{av} A}{2 v_{av} A} \iint_A (v^3) \, dA = \frac{m}{2 v_{av}} \frac{1}{A} \iint_A (v^3) \, dA \quad \text{.........................(12)}$$

Dividing through by m so that Eq. (12) becomes per unit mass

$$\left(\frac{1}{2 v_{av}}\right) \frac{1}{A} \iint_A (v^3) \, dA = \frac{(v^3)_{av}}{2 v_{av}} = \frac{v^2{}_{av}}{2\alpha} \quad \text{.........................(13)}$$

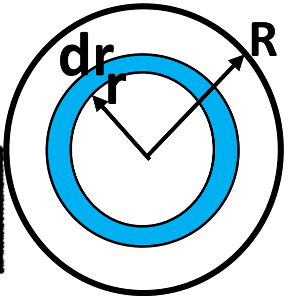
where α is defined as $$\alpha = \frac{v_{av}^3}{(v^3)_{av}}$$

and $(v^3)_{av}$ is defined as follows:

$$(v^3)_{av} = \frac{1}{A} \iint_A (v^3)\, dA \qquad \text{..........................(14)}$$

The local velocity v varies across the cross-sectional area of a pipe. To evaluate $(v^3)_{av}$ and, hence, the value of α, we must have an equation relating v as a function of position in the cross-sectional area.

For laminar regime



- Combine $v_{av} = \frac{v_{max}}{2}$ and $v = v_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$

$$v = 2v_{av}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

- Substitute in eq. 14 and noting that $A = \pi R^2$ and $dA = r\,dr\,d\theta$

$$(v^3)_{av} = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R \left[2v_{av}\left(1 - \frac{r^2}{R^2}\right)\right]^3 r\,dr\,d\theta$$

$$= \frac{(2\pi)2^3 v_{av}^3}{\pi R^2}\int_0^R \frac{(R^2 - r^2)^3}{R^6} r\,dr = \frac{16 v_{av}^3}{R^8}\int_0^R (R^2 - r^2)^3 r\,dr$$

- *On integration and rearrangement*

$$(v^3)_{av} = \frac{16v_{av}^3}{R^8}\int_0^R (R^6 - 3r^2R^4 + 3r^4R^2 - r^6)r\,dr$$

$$= \frac{16v_{av}^3}{R^8}\left(\frac{R^8}{2} - \frac{3}{4}R^8 + \frac{1}{2}R^8 - \frac{1}{8}R^8\right)$$

$$= 2v_{av}^3$$

$\therefore$ **α becomes**

$$\alpha = \frac{v_{av}^3}{(v^3)_{av}} = \frac{v_{av}^3}{2v_{av}^3} = 0.50$$

Hence, for laminar flow the value of α to use in the kinetic-energy term of Eq. (10) is 0.5

Note: for turbulent flow α is taken 1 for details see the text.

Example

Initially, a tank contains 500 kg of salt solution containing 10% salt. At point (1) in the control volume in Fig. 4, a stream enters at a constant flow rate of 10 kg/h containing 20% salt. A stream leaves at point (2) at a constant rate of 5 kg/h. The tank is well stirred. Derive an equation relating the weight fraction w_A of the salt in the tank at any time t in hours.

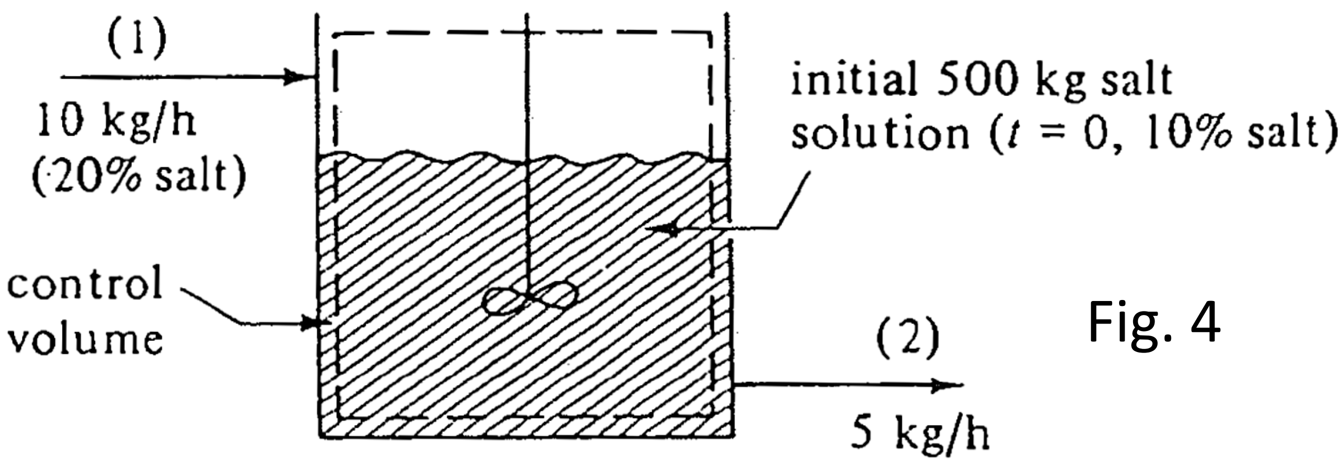


Fig. 4

Solution

$$\iint_A v\rho \cos \alpha \, dA = m_2 - m_1 = 5 - 10 = -5 \text{ kg solution/h}$$

$$\frac{\partial}{\partial t} \iiint_V \rho \, dV = \frac{dM}{dt}$$

- Hence the total mass bal. eq. becomes:

$$-5 + \frac{dM}{dt} = 0$$

$$\int_{M=500}^{M} dM = 5 \int_{t=0}^{t} dt$$

$$M = 5t + 500$$ **…………………(a)**

- Now, make component bal., bal over salt

$$\iint_A v\rho \cos\alpha \, dA = (5)w_A - 10(0.20) = 5w_A - 2 \text{ kg salt/h}$$

$$\frac{\partial}{\partial t}\iiint_V \rho \, dV = \frac{d}{dt}(Mw_A) = \frac{M\,dw_A}{dt} + w_A\frac{dM}{dt} \text{ kg salt/h}$$

- Hence salt bal. eq. becomes {component Bal. Eq.}

$$5w_A - 2 + M\frac{dw_A}{dt} + w_A\frac{dM}{dt} = 0 \qquad \text{.....................(b)}$$

- Substitute M from eq.(a) in eq. (b)

$$5w_A - 2 + (500 + 5t)\frac{dw_A}{dt} + w_A\frac{d(500+5t)}{dt} = 0$$

- On rearrangement and integration

$$5w_A - 2 + (500 + 5t)\frac{dw_A}{dt} + 5w_A = 0$$

$$\int_{w_A=0.10}^{w_A} \frac{dw_A}{2 - 10w_A} = \int_{t=0}^{t} \frac{dt}{500 + 5t}$$

$$-\frac{1}{10}\ln\left(\frac{2 - 10w_A}{1}\right) = \frac{1}{5}\ln\left(\frac{500 + 5t}{500}\right)$$

$$w_A = -0.1\left(\frac{100}{100 + t}\right)^2 + 0.20$$

Comments

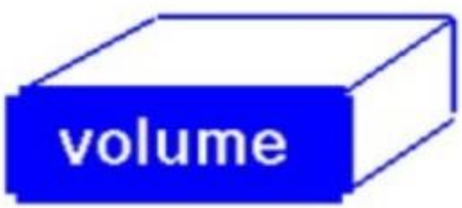
- ✓Overall or macroscopic material (or energy or momentum) gives an idea about the system from outside the enclosure.
- ✓Therefore, overall balances (material; energy; momentum) do not tell us the details of what happens inside the system.
- ✓The previous discussion is useful for the next step "momentum balance and shell momentum balance'.
- ✓ The shell momentum balance will be made in order to obtain the details of what happens inside the system.

Scalars and Vectors

A scalar quantity has only magnitude.
A vector quantity has both magnitude and direction.

Scalar Quantities

length, area, volume
speed
mass, density
pressure
temperature
energy, entropy
work, power



Vector Quantities

displacement
velocity
acceleration
momentum
force
lift , drag , thrust
weight