Momentum Balance

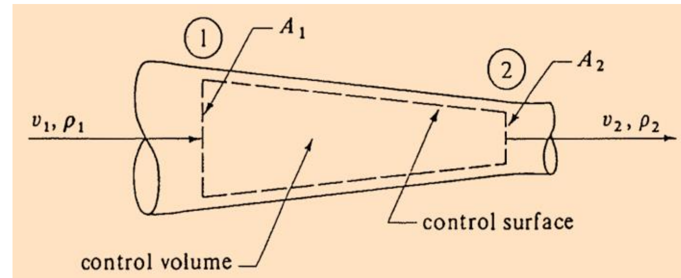
Overall momentum balance &Integral momentum equation

Application: flow system in one direction &

OVERALL MOMENTUM BALANCE

General Equation

Derivation is similar to mass balance.
But momentum is a vector quantity.



The total linear momentum vector **P** of the total mass *M* of a moving fluid having a velocity of **v** is

$$\mathbf{P} = M\mathbf{v} \qquad \text{..............................(1)}$$

The term $M\mathbf{v}$ is the momentum of this moving mass *M* enclosed at a particular instant in the control volume shown Above.

The units of $M\mathbf{v}$ are kg · m/s in the SI system.

How can we develop the integral momentum equation?

Starting with Newton's second law we will develop the integral momentum-balance equation for linear momentum

Newton's law may be stated: The time rate of change of momentum of a system is equal to the summation of all forces acting on the system and takes place in the direction of the net force.

$$\sum \mathbf{F} = \frac{d\mathbf{P}}{dt} \quad \text{.............................(2)}$$

where **F** is force. In the SI system **F** is in newtons (N) and 1 N = 1 kg · m/s^2. Note that in the SI system g_c is not needed, but it is needed in the English system.

The equation for the conservation of momentum with respect to a control volume can be written as follows:

$$\begin{pmatrix}\text{sum of forces acting}\\ \text{on control volume}\end{pmatrix} = \begin{pmatrix}\text{rate of momentum}\\ \text{out of control volume}\end{pmatrix} - \begin{pmatrix}\text{rate of momentum}\\ \text{into control volume}\end{pmatrix} + \begin{pmatrix}\text{rate of accumulation of momentum}\\ \text{in control volume}\end{pmatrix}$$

..............................(3)

Notes

This is in the same form as the general mass-balance equation with the sum of the forces as the generation rate term.

Hence, momentum is not conserved, since it is generated by external forces on the system.

If external forces are absent, momentum is conserved.

To Evaluate each term in Eq. (3)

- Using the same method as that being used in general mass balance.
- For a small element of area dA on the control surface, we write

$$\text{rate of momentum efflux} = \mathbf{v}(\rho v)(dA \cos \alpha) \qquad \text{.............................(4)}$$

- Note that the rate of mass efflux is $(\rho v)(dA \cos \alpha)$. $(dA \cos \alpha)$ is the area dA projected in a direction normal to the velocity vector $\mathbf{v}$
  α is the angle between the velocity vector $\mathbf{v}$ and the outward-directed-normal vector $\mathbf{n}$.
- Re-write eq. (4) as

$$\mathbf{v}(\rho v)(dA \cos \alpha) = \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n})\, dA \qquad \text{.............................(5)}$$

Integrating over the entire control surface A,

$$\begin{pmatrix}\text{net momentum efflux}\\ \text{from control volume}\end{pmatrix} = \iint_A \mathbf{v}(\rho v)\cos \alpha \, dA = \iint_A \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n})\, dA \qquad \text{.............................(6)}$$

- ***Note***

  The net efflux represents the first two terms on the right-hand side of Eq. (3)

The rate of accumulation term

$$\begin{pmatrix}\text{rate of accumulation of momentum} \\ \text{in control volume}\end{pmatrix} = \frac{\partial}{\partial t}\iiint_V \rho \mathbf{v}\, dV$$

...........................(7)

Substituting eq. (2) , (6) and (7) into (3)

$$\sum \mathbf{F} = \iint_A \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})\, dA + \frac{\partial}{\partial t}\iiint_V \rho\mathbf{v}\, dV$$ ....(8)

$\sum$ F in general may have a component in any direction, and the F is the force the surroundings exert on the control-volume fluid.

Since eq.(8) is a vector eq., we may write the component scalar eq.s in x, y, and z directions.

$$\sum F_x = \iint_A v_x \rho v \cos\alpha\, dA + \frac{\partial}{\partial t}\iiint_V \rho v_x\, dV$$ ....(9)

$$\sum F_y = \iint_A v_y \rho v \cos \alpha \, dA + \frac{\partial}{\partial t} \iiint_V \rho v_y \, dV \qquad \text{.........................(10)}$$

$$\sum F_z = \iint_A v_z \rho v \cos \alpha \, dA + \frac{\partial}{\partial t} \iiint_V \rho v_z \, dV \qquad \text{.........................(11)}$$

The force term $\sum F_x$ is composed of the sum of several forces.

1. *Body force.* The body force F_{xg} is the x-directed force caused by gravity acting on the total mass M in the control volume. This force, F_{xg}, is Mg_x. It is zero if the x direction is horizontal.
2. *Pressure force.* The force F_{xp} is the x-directed force caused by the pressure forces acting on the surface of the fluid system. When the control surface cuts through the fluid, the pressure is taken to be directed inward and perpendicular to the surface. In some cases part of the control surface may be a solid, and this wall is included inside the control surface. Then there is a contribution to F_{xp} from the pressure on the outside of this wall, which is typically atmospheric pressure. If gage pressure is used, the integral of the constant external pressure over the entire outer surface can be automatically ignored.

3. *Friction force.* When the fluid is flowing, an x-directed shear or friction force F_{xs} is present, which is exerted on the fluid by a solid wall when the control surface cuts between the fluid and the solid wall. In some or many cases this frictional force may be negligible compared to the other forces and is neglected.
4. *Solid surface force.* In cases where the control surface cuts through a solid, there is present force R_x, which is the x component of the resultant of the forces acting on the control volume at these points. This occurs in typical cases when the control volume includes a section of pipe and the fluid it contains. This is the force exerted by the solid surface on the fluid.

Newton's second Law of Motion

- Statement:

  Sum of all forces acting on the control volume must equal the net rate at which momentum leaves the control volume (outflow – inflow)*.

- There are three kinds of forces acting on the Boundary Layer:
  1. Body forces which are proportional to the volume.
  2. Surface forces which are proportional to area.
  3. Solid surface forces(R_x) which exerted by the solid surface on the fluid.

* Assuming no accumulation term i.e. steady state.

Types of Fluid Forces

Total Body Force includes:

1. Gravitational Force F_{xg}
2. Centrifugal force
3. Magnetic and/or electric forces

We designate the x and y components of of this force per unit volume of fluid as X and Y, respectively.

Surface Forces F_s includes:

1. Fluid static pressure F_{xp}
2. Friction forces or Viscous stresses F_{xs}

At any point in the B.L., the viscous force (a force per unit area) may be resolved into two perpendicular components, which include a normal stress σ_{ii} and a shear stress τ_{ij}

- Based on the previous discussion the Sum of the forces, $\sum F_x$, in eq. (11) is

$$\sum F_x = F_{xg} + F_{xp} + F_{xs} + R_x \quad \text{.........................(12)}$$

Body Force | Pressure Force | friction Force | Solid surface Force

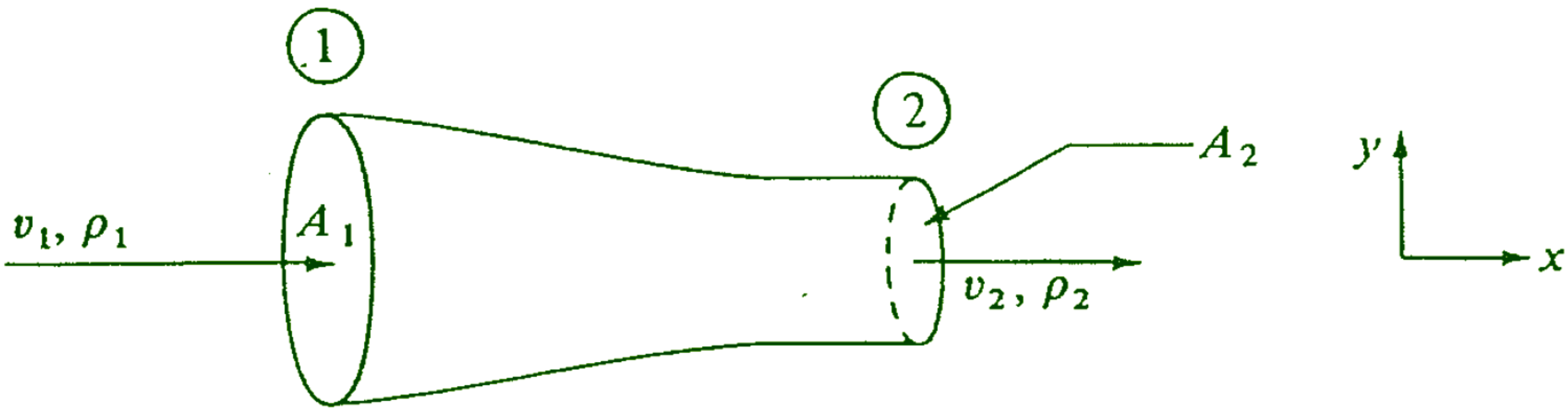
Similar equations can be written for the y and z directions.

- Eq. (9) becomes

$$\sum F_x = F_{xg} + F_{xp} + F_{xs} + R_x = \iint_A v_x \rho v \cos\alpha \, dA + \frac{\partial}{\partial t} \iiint_V \rho v_x \, dV \quad \text{.........................(13)}$$

Overall Momentum Balance in Flow System in One Direction

A quite common application of the overall momentum-balance equation is the case of a section of a conduit with its axis in the x direction. The fluid will be assumed to be flowing at steady state



Flow through a horizontal nozzle in the x direction only.

Momentum eq. (13), for x direction $v = v_x$ **is**

$$\sum F_x = F_{xg} + F_{xp} + F_{xs} + R_x = \iint_A v_x \, \rho v_x \cos\alpha \, dA \qquad \text{..........................(14)}$$

Integrating with $\cos\alpha = \pm 1.0$ and $\rho A = m/v_{av}$

$$v_{av} = \frac{1}{A}\iint_A v\, dA$$

$$F_{xg} + F_{xp} + F_{xs} + R_x = m\frac{(v_{x2}^2)_{av}}{v_{x2\,av}} - m\frac{(v_{x1}^2)_{av}}{v_{x1\,av}} \quad \text{..........................(15)}$$

where if the velocity is not constant and varies across the surface area,

$$(v_x^2)_{av} = \frac{1}{A}\iint_A v_x^2\, dA \quad \text{..........................(16)}$$

Note 1

$$\beta = \frac{(v_{av})^2}{(v^2)_{av}}$$

The ratio $(v_x^2)_{av}/v_{x\,av}$ is replaced by $v_{x\,av}/\beta$, where β, which is the momentum velocity correction factor, has a value of 0.95 to 0.99 for turbulent flow and $\frac{3}{4}$ for laminar flow. For most applications in turbulent flow, $(v_x)^2_{av}/v_{x\,av}$ is replaced by $v_{x\,av}$, the average bulk velocity. Note that the subscript x on v_x and F_x can be dropped since $v_x = v$ and $F_x = F$ for one directional flow.

Note 2

The term F_{xp}, which is the force caused by the pressures acting on the surface of the control volume, is $F_{xp} = p_1 A_1 - p_2 A_2$ ....(17)

- Ignore friction force, $F_{xs} = 0$
- Body force, $F_{xg} = 0$ since gravity is acting only in the y-direction.
- Substituting F_{xp} into Eq. (15) and replacing $(v_x^2)_{av}/v_{x\,av}$ by v/β (where $v_{x\,av} = v$), setting $\beta = 1.0$
- Eq. (15) becomes

$$R_x = mv_2 - mv_1 + p_2 A_2 - p_1 A_1$$

..........................(18)

Example 1

The momentum velocity correction factor β is defined as follows for flow in one direction where the subscript x is dropped.

$$\frac{(v^2)_{av}}{v_{av}} = \frac{v_{av}}{\beta}$$

$$\beta = \frac{(v_{av})^2}{(v^2)_{av}}$$

Determine β for laminar flow in a tube.

Assume laminar flow through a pipe with the following velocity distribution

$$v = 2v_{av}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Solution

- Using $A = \pi R^2$ and $dA = r\,dr\,d\theta$ and substituting into eq. (16) or {drop x for 1-D}

$$(v_x^2)_{av} = \frac{1}{A}\iint_A v_x^2\,dA$$

$$(v^2)_{av} = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R \left[2v_{av}\left(1 - \frac{r^2}{R^2}\right)\right]^2 r\,dr\,d\theta$$

$$= \frac{(2\pi)2^2 v_{av}^2}{\pi R^2}\int_0^R \frac{(R^2 - r^2)^2}{R^4}\,r\,dr$$

- On integration and rearrangement

$$(v^2)_{av} = \frac{8v_{av}^2}{R^6}\left(\frac{R^6}{2} - \frac{R^6}{2} + \frac{R^6}{6}\right) = \tfrac{4}{3}v_{av}^2$$

- Substituting the previous result in the definition of β

$$\beta = \frac{(v_{av})^2}{(v^2)_{av}}$$

$$\therefore \quad \beta = \frac{3}{4}$$

Example 2

Water is flowing at a rate of 0.03154 m^3/s through a horizontal nozzle shown in Fig. A1 and discharges to the atmosphere at point 2. The nozzle is attached at the upstream end at point 1 and frictional forces are considered negligible. The upstream ID is 0.0635 m and the downstream 0.0286 m. Calculate the resultant force on the nozzle. The density of the water is 1000 kg/m^3.

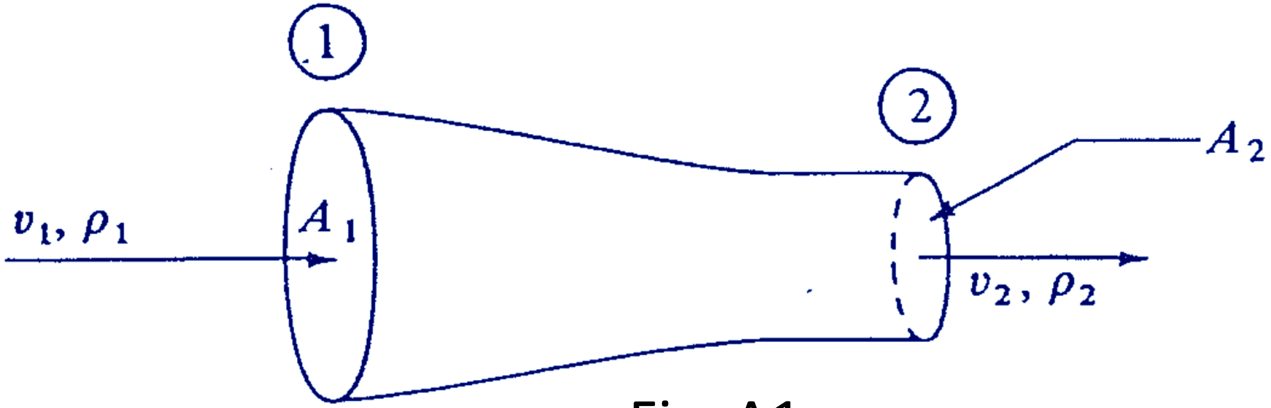


Fig. A1

Solution

To evaluate the upstream pressure p_1 we use the mechanical-energy balance equation assuming no frictional losses and turbulent flow

$$\frac{v_1^2}{2} + \frac{p_1}{\rho} = \frac{v_2^2}{2} + \frac{p_2}{\rho}$$

Setting $p_2 = 0$ gage pressure, $\rho = 1000\ \mathrm{kg/m^3}$, $v_1 = 9.96$ m/s, $v_2 = 49.1$ m/s, and solving for p_1,

$$p_1 = \frac{(1000)(49.1^2 - 9.96^2)}{2} = 1.156 \times 10^6\ \mathrm{N/m^2} \qquad \text{(gage pressure)}$$

For the x direction, the momentum balance equation

$$R_x = mv_2 - mv_1 + p_2 A_2 - p_1 A_1$$

See eq. 18 given before.

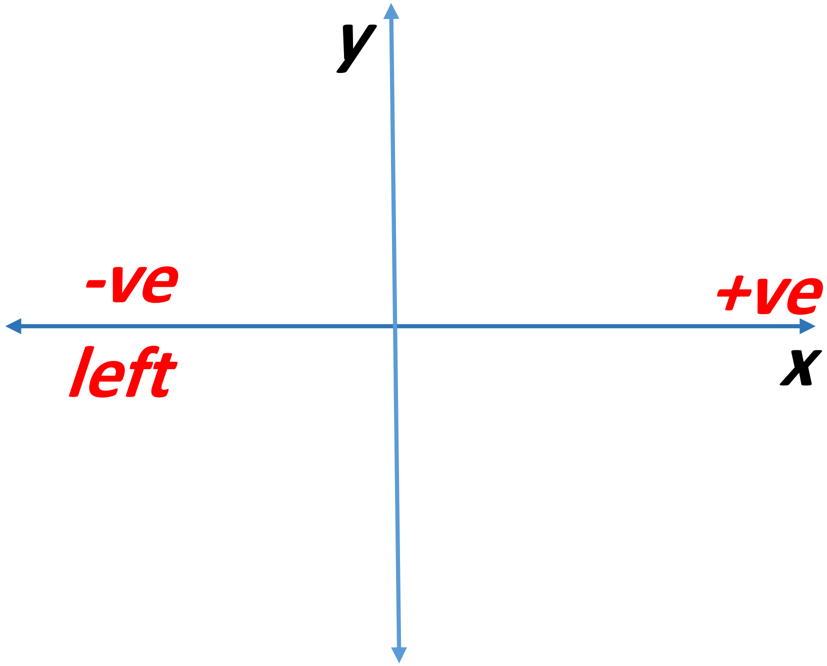
Substituting the known values and solving for R_x,

$$R_x = 31.54(49.10 - 9.96) + 0 - (1.156 \times 10^6)(3.167 \times 10^{-3})$$

$$= -2427\ \mathrm{N}(-546\ \mathrm{lb_f})$$

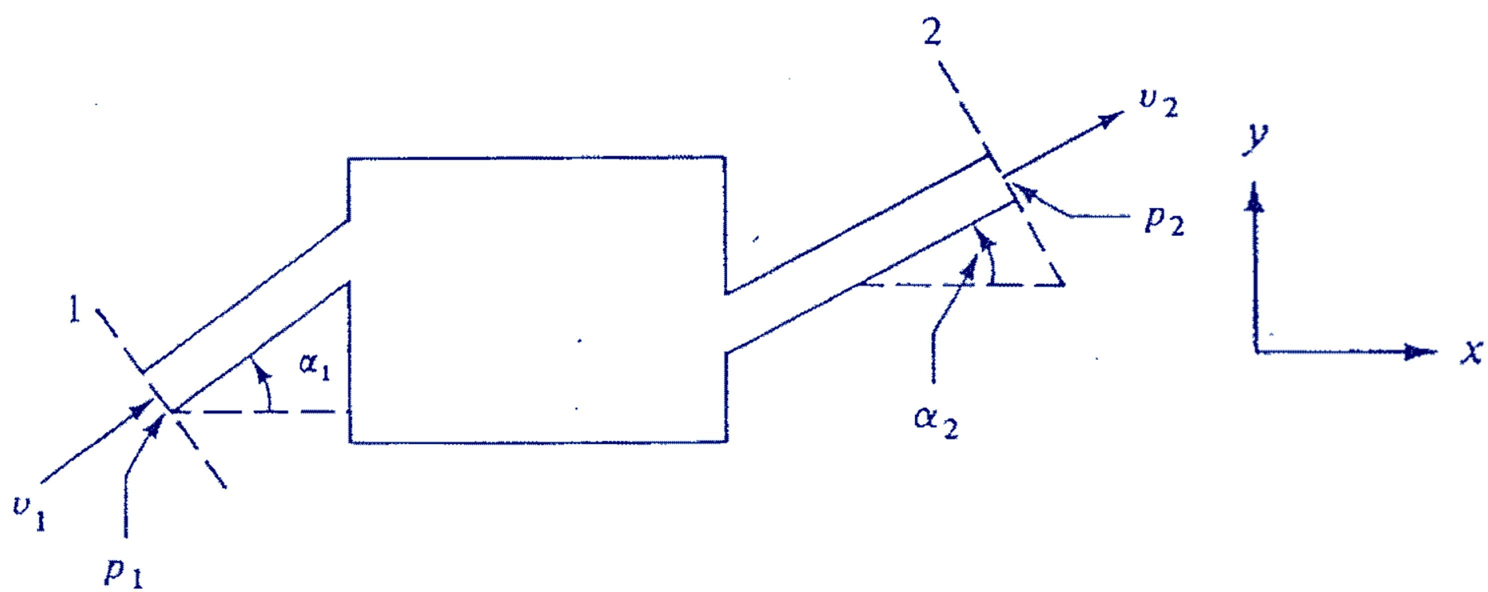
Note:

Since the force is negative, it is acting in the negative x direction or to the left. This is the force of the nozzle on the fluid. The force of the fluid on the solid is $-R_x$ or $+2427$ N.



Overall Momentum Balance in Two Directions

Another application of the overall momentum balance is given in the following Fig. for a flow system with fluid entering a conduit at point 1 inclined at an angle of α_1 relative to the horizontal *x* direction and leaving a conduit at point 2 at an angle α_2.



- Assumed steady state flowing and ignore the effect frictional force F_{xs} .
- Apply overall momentum bal. eq. in x-direction (no accumulation term).

$$F_{xg} + F_{xp} + R_x = \iint_A \upsilon_x \rho \upsilon \cos \alpha \, dA$$ ....(19)

- Integrating the surface (area) integral,

$$F_{xg} + F_{xp} + R_x = m \frac{(\upsilon_2^2)_{av}}{\upsilon_{2\,av}} \cos \alpha_2 - m \frac{(\upsilon_1^2)_{av}}{\upsilon_{1\,av}} \cos \alpha_1$$ ....(20)

- The term $(\upsilon^2)_{av} / \upsilon_{av}$ can again be replaced by υ_{av}/β with β being set at 1.0
- The term F_{xp} is (as given in the previous section)

$$F_{xp} = p_1 A_1 \cos \alpha_1 - p_2 A_2 \cos \alpha_2$$ ....(21)

- Then Eq. (20) becomes as follows after solving for R_x :

$$R_x = mv_2 \cos\alpha_2 - mv_1 \cos\alpha_1 + p_2 A_2 \cos\alpha_2 - p_1 A_1 \cos\alpha_1$$

.........................(21)

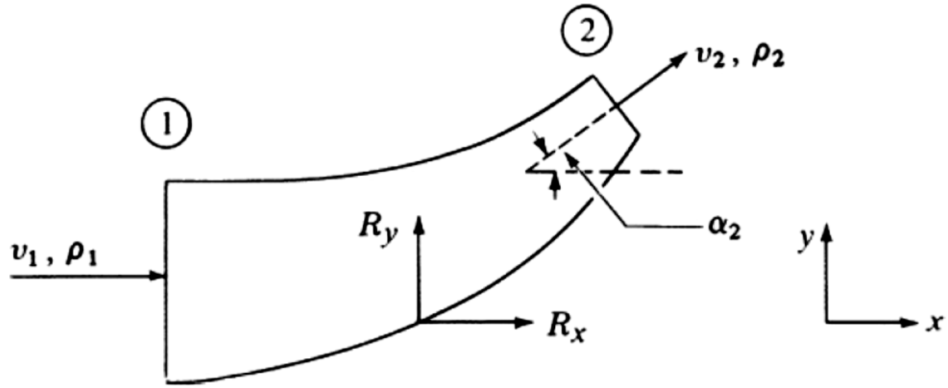
- Note: the term $F_{xg} = 0$ in this case (x-direction).
- For R_y the body force F_{yg} is in the negative y direction and $F_{yg} = - m_t g$, where m_t is the total mass fluid in the control volume. Replacing cosα by sinα, the equation for the y direction becomes

$$R_y = mv_2 \sin\alpha_2 - mv_1 \sin\alpha_1 + p_2 A_2 \sin\alpha_2 - p_1 A_1 \sin\alpha_1 + m_t g$$

.........................(22)

Example

Fluid is flowing at steady state through a reducing pipe bend, as shown in the attached Figure. Turbulent flow will be assumed with frictional forces negligible. The volumetric flow rate of the liquid and the pressure P_2 at point 2 are known as are the pipe diameters at both ends. Derive the equations to calculate the forces on the bend. Assume that the density ρ is constant.



Solution

The velocities v_1 and v_2 can be obtained from the volumetric flow rate and the areas. Also, $m = \rho_1 v_1 A_1 = \rho_2 v_2 A_2$. As in the previous example the mechanical-energy-balance equation is used to obtain the upstream pressure, p_1 .

$$\frac{v_1^2}{2} + \frac{p_1}{\rho} = \frac{v_2^2}{2} + \frac{p_2}{\rho}$$

For the x direction Eq. (21) is used for the momentum balance. Since $\alpha_1 = 0°$, $\cos \alpha_1 = 1.0$. Equation (21) becomes:

$$R_x = mv_2 \cos \alpha_2 - mv_1 \cos \alpha_1 + p_2 A_2 \cos \alpha_2 - p_1 A_1 \cos \alpha_1$$

$$R_x = mv_2 \cos \alpha_2 - mv_1 + p_2 A_2 \cos \alpha_2 - p_1 A_1$$

……………………….(21)

- For the *y* direction the momentum balance Eq. (22) is used where $\sin \alpha_1 = 0$.
- $$\therefore R_y = mv_2 \sin \alpha_2 - mv_1 \sin \alpha_1 + p_2 A_2 \sin \alpha_2 - p_1 A_1 \sin \alpha_1 + m_t g$$

$$R_y = mv_2 \sin \alpha_2 + p_2 A_2 \sin \alpha_2 + m_t g \quad \text{.....................(22)}$$

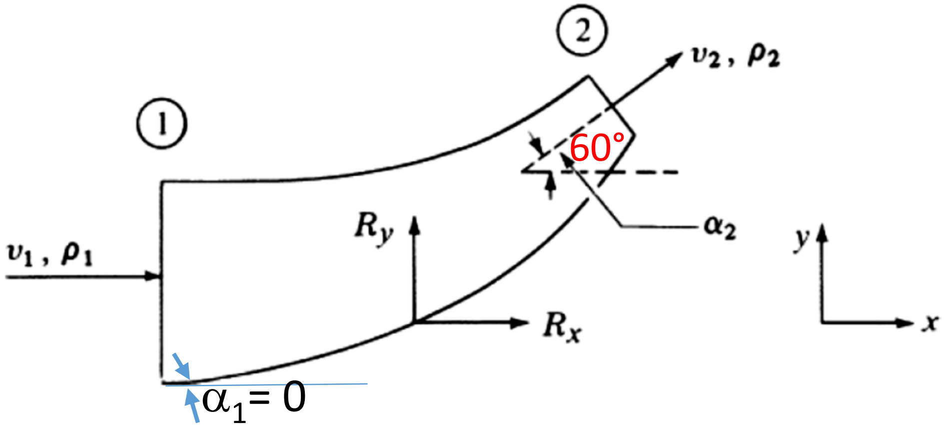
- The magnitude of the resultant force of the bend acting on the control volume fluid is

$$|\mathbf{R}| = \sqrt{R_x^2 + R_y^2}$$

- The angle this makes with the vertical is $\theta = \arctan(R_x / R_y)$. The gravity force F_{yg} is often small compared to the other terms in Eq. (22) and is neglected.

Problem

Water is flowing at steady state and 363 K at a rate of 0.0566 m^3/s through a 60° reducing bend ($\alpha_2 = 60°$) in the below Figure. The inlet pipe diameter is 0.1016 m and the outlet 0.0762 m. The friction loss in the pipe bend can be estimated as $v_2^2/5$. Neglect gravity forces. The exit pressure $p_2 = 111.5\ kN/m^2$ gage. Calculate the forces on the bend in newtons.



Solution

- From App. (A.2-3) density of water at 363 K

  $\rho = 962\ kg/m^3$

- From continuity eq. [$m = \rho\ v\ A$ and $Q = v\ A$] obtain velocities $v_1 = 0.0566/((\pi/4)(o.1016)^{\wedge}2) = 6.982$ m/s

  Repeat, for $v_2 = 12.41$ m/s

- Apply Mechanical energy eq. (text 2.7-28) to obtain p_1

$$\frac{1}{2\alpha}(v_{2\,av}^2 - v_{1\,av}^2) + g(z_2 - z_1) + \frac{p_2 - p_1}{\rho} + \sum F + W_S = 0$$

- For Turbulent flow Re > 2100 take $\alpha = 1.0$, cancel Potential and W_s terms. ΣF (Friction loss) = $v_2^2/5$. Then substitute numerical values and find p_1. *The result is* $p_1 = 1.9176 \times 10^5$

- Apply Momentum balance equations "equations (21) and (22). m = ρ_1 v_1 A_1 = 54.45 kg/s

$$R_x = mv_2 \cos \alpha_2 - mv_1 \cos \alpha_1 + p_2 A_2 \cos \alpha_2 - p_1 A_1 \cos \alpha_1$$

- After substitution the numerical values

R_x = -1343.7 N 'force on fluid'

- For y direction (Neglect gravity force)

$$R_y = mv_2 \sin \alpha_2 - mv_1 \sin \alpha_1 + p_2 A_2 \sin \alpha_2 - p_1 A_1 \sin \alpha_1 + \cancel{m_t g}$$

- After substitution,

R_y = +1026 N 'force on fluid'

- Finally, -R_x = +1343.7 N 'force on bend'

-R_y = -1026 N 'force on bend'

- The magnitude of the resultant force of the bend acting on the control volume fluid is

$$|\mathbf{R}| = \sqrt{R_x^2 + R_y^2}$$

$$|R| = 1691\ \text{N}$$

Check Angles