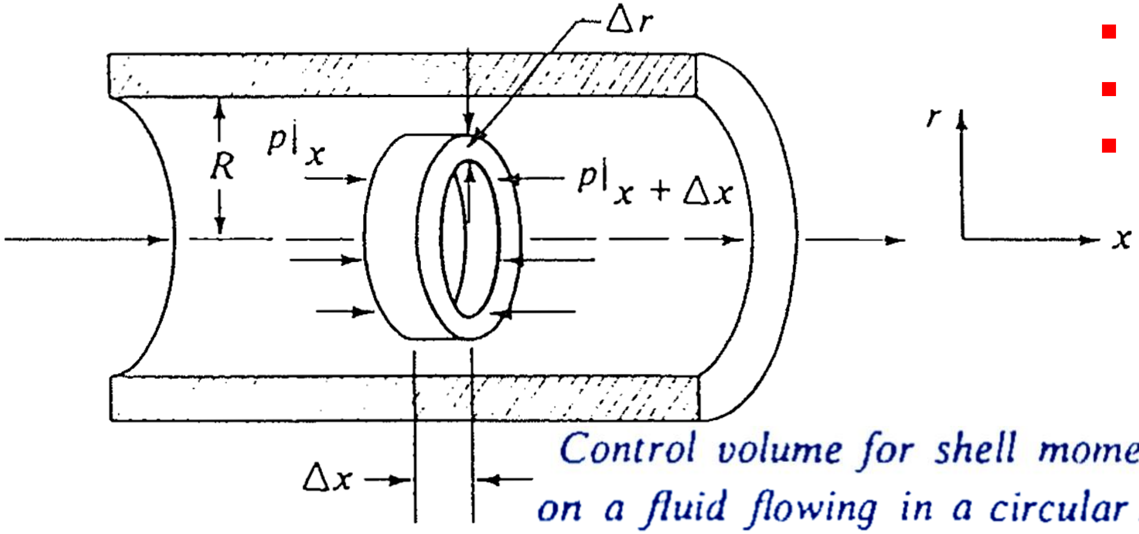
Shell Momentum Balance

INSIDE PIPES & FALLING FILM

SHELL MOMENTUM BALANCE AND VELOCITY PROFILE IN LAMINAR FLOW

- **Shell Momentum Balance Inside a Pipe**
- Consider a flow of fluid inside a circular conduit or pipe.
- Assume a control volume as shown in the figure.
- Assume fully developed flow i.e velocity, $\upsilon = f(r) \neq f(x)$. No net momentum flux across the annular volume at x and x+ Δx since $\upsilon_x = \upsilon_{x+\Delta x}$.



Control volume for shell momentum balance on a fluid flowing in a circular tube.

Assumptions:

- incompressible Newtonian fluid.
- one-dimensional flow.
- steady-state, laminar flow.
- the flow is fully developed.

At steady state the conservation of momentum, Eq. (3) previous slides.
becomes as follows:

sum of forces acting on control volume =
= rate of momentum out − ratmomentum into volume.
.... (1)

The pressure forces become

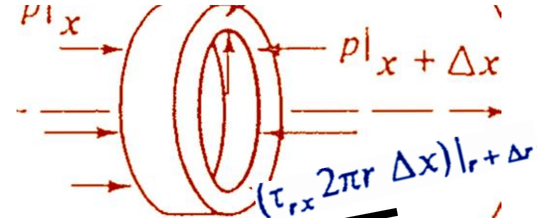
$$\text{pressure forces} = pA|_x - pA|_{x+\Delta x} = p(2\pi r\ \Delta r)|_x - p(2\pi r\ \Delta r)|_{x+\Delta x}$$
.... (2)

The shear force or drag force

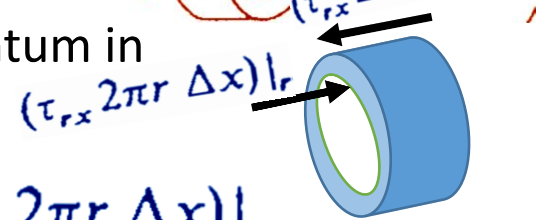
The shear force or drag force acting on the cylindrical surface at the radius r is the shear stress τ_{rx} times the area $2\pi r\ \Delta x$. This can also be considered as the rate of momentum flow into the cylindrical surface of the shell as described by

$$\tau_{yz} = \frac{\text{kg}\cdot\text{m/s}}{\text{m}^2\cdot\text{s}} = \frac{\text{momentum}}{\text{m}^2\cdot\text{s}}$$

∴ **The** net rate of momentum efflux is



the rate of momentum out − rate of momentum in



Or

$$\text{net efflux} = (\tau_{rx} 2\pi r\, \Delta x)|_{r+\Delta r} - (\tau_{rx} 2\pi r\, \Delta x)|_{r} \qquad \text{.................................. (3)}$$

Note:

The net momentum flux across the annular volume at x and x+ Δx = 0.

Now, substituting eqs. (2) and (3) into eq. (1) and making rearrangement

Hence,

$$\frac{(r\tau_{rx})|_{r+\Delta r} - (r\tau_{rx})|_{r}}{\Delta r} = \frac{r(p|_{x} - p|_{x+\Delta x})}{\Delta x} \qquad \text{.................................. (4)}$$

In fully developed flow, the pressure gradient $(\Delta p/\Delta x)$ is constant and becomes $(\Delta p/L)$, where Δp is the pressure drop for a pipe of length L. Letting Δr approach zero, we obtain

$$\frac{d(r\tau_{rx})}{dr} = \left(\frac{\Delta p}{L}\right) r \qquad \text{.................................. (5)}$$

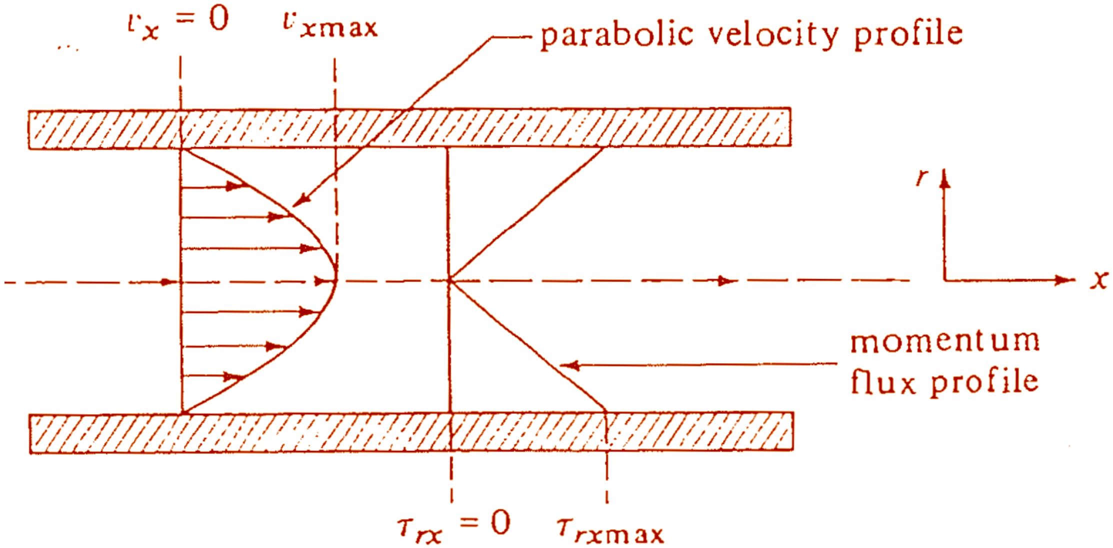
Separating variables and integrating,

$$\tau_{rx} = \left(\frac{\Delta p}{L}\right)\frac{r}{2} + \frac{C_1}{r} \qquad \text{.................................... (6)}$$

The constant of integration C_1 must be zero if the momentum flux is not infinite at $r = 0$.

$$\therefore \qquad \tau_{rx} = \left(\frac{\Delta p}{2L}\right)r = \frac{p_0 - p_L}{2L}\, r \qquad \text{.................... (7)}$$

This means that the momentum flux varies linearly with the radius as shown in the following figure.



Substituting Newton's law of viscosity **in the previous equation,**

$$\tau_{rx} = -\mu \frac{dv_x}{dr} \qquad \text{.................................. (8)}$$

We obtain

$$\frac{dv_x}{dr} = -\frac{p_0 - p_L}{2\mu L} r \qquad \text{.................................. (9)}$$

Integrating using the boundary condition that at the wall,

$v_x = 0$ at $r = R$, we obtain the equation for the velocity distribution.

$$v_x = \frac{p_0 - p_L}{4\mu L} R^2 \left[1 - \left(\frac{r}{R}\right)^2\right] \qquad \text{...................... (10)}$$

This result shows that the velocity distribution is parabolic as indicated in the previous figure.

To obtain avg velocity

- Use the average concept equation as given earlier.

$$v_{x\,av} = \frac{1}{A}\iint_A v_x\, dA = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R v_x\, r\, dr\, d\theta = \frac{1}{\pi R^2}\int_0^R v_x\, 2\pi r\, dr$$

.... (11)

- Note: $dA = r\ dr\ d\theta$ and $A = \pi R^2$
- Combining eqs. (10) and (11) and integrating

$$v_{x\,av} = \frac{(p_0 - p_L)R^2}{8\mu L} = \frac{(p_0 - p_L)D^2}{32\mu L}$$

.... (12)

- This eq. is called *Hagen–Poiseuille equation* which relates the pressure drop and the average velocity for laminar flow in a horizontal pipe.

Max velocity can be found from eq. 10 at r = 0

$$v_{x\,max} = \frac{p_0 - p_L}{4\mu L} R^2 \quad \text{.........................} (13)$$

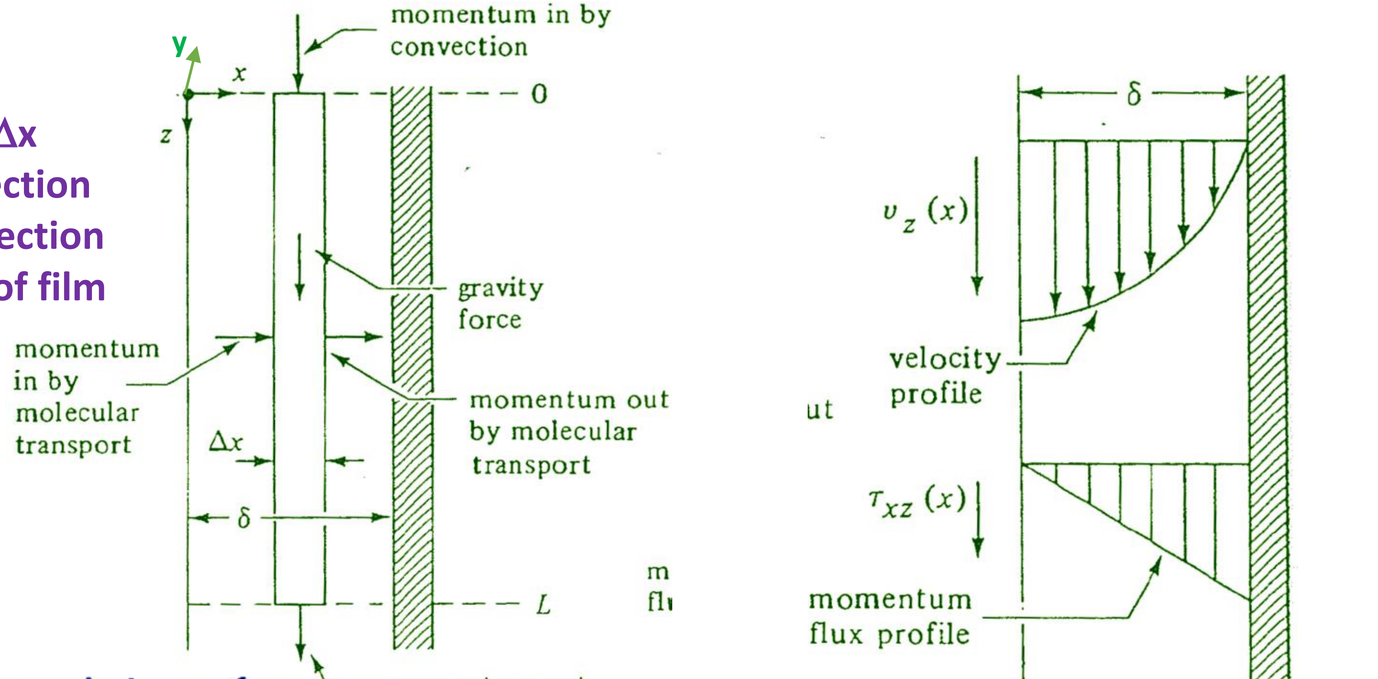
Combining eqs. (13) and (12), we obtain

$$v_{x\,av} = \frac{v_{x\,max}}{2} \quad \text{.........................} (13)$$

Shell Momentum Balance for Falling Film

- Meaning: a flow of a fluid as a film in laminar flow down a vertical surface.
- Applications: various phenomena in mass transfer; coatings on surfaces; condensation as filmwise condensation .. etc.
- Control volume: shown in figure (1).

Shell thickness = Δx
Length: L in z-direction
Width: W in y-direction
δ Total thickness of film



shell momentum balance for a control volume Δ x thick

velocity and momentum flux profiles

figure (1)

- momentum flux due to molecular transport

$$\text{net efflux} = LW(\tau_{xz})|_{x+\Delta x} - LW(\tau_{xz})|_{x}$$

.... (14)

- convective momentum flux

Assume Control volume is far from entrance and exit (no entrance and exit effects on shell flow), hence $v_z(x)$ is independent of z-direction.

net efflux is equal to 0 since v_z at $z = 0$ is equal to v_z at $z = L$ for each value of x.

$$\text{net efflux} = \Delta x W v_z(\rho v_z)|_{z=L} - \Delta x W v_z(\rho v_z)|_{z=0} = 0$$

.... (15)

- **Gravity force acting on the fluid.**

$$\text{gravity force} = \Delta x W L(\rho g)$$

.... (16)

- Then substitute in momentum conservation eq. at steady state:

$$\Delta x W L(\rho g) = LW(\tau_{xz})|_{x+\Delta x} - LW(\tau_{xz})|_{x} + 0$$

.... (17)

- Rearranging and letting $\Delta x \to 0$

$$\frac{\tau_{xz}|_{x+\Delta x} - \tau_{xz}|_x}{\Delta x} = \rho g \qquad \text{.........................} \ (18)$$

$$\frac{d}{dx}\tau_{xz} = \rho g \qquad \text{.........................} \ (19)$$

Integrating using the boundary conditions at $x = 0, \tau_{xz} = 0$ at the free liquid surface and at $x = x, \tau_{xz} = \tau_{xz}$,

$$\tau_{xz} = \rho g x \qquad \text{.........................} \ (20)$$

This means the momentum-flux profile is linear as shown in Fig. (1) and the max value at the wall.

Using Newton's Law of viscosity:

$$\tau_{xz} = -\mu \frac{dv_z}{dx} \qquad \text{.........................} \ (21)$$

- Combining eqs. (20) and (21)

$$\frac{dv_z}{dx} = -\left(\frac{\rho g}{\mu}\right)x \qquad \text{.......................... (22)}$$

- Separating variables and integrating gives

$$v_z = -\left(\frac{\rho g}{2\mu}\right)x^2 + C_1 \qquad \text{.......................... (23)}$$

- Using BC. At wall, x = δ; υ_z = 0. hence, $C_1 = (\rho g/2\mu)\delta^2$
- Then

$$v_z = \frac{\rho g \delta^2}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right] \qquad \text{.......................... (24)}$$

This means the velocity profile is parabolic as shown in Fig.(1)

- The max velocity occurs at x = 0; hence,

$$v_{z\,\max} = \frac{\rho g \delta^2}{2\mu} \qquad \text{.......................... (25)}$$

- Avg velocity as before

$$v_{z\,av} = \frac{1}{A}\iint_A v_z\,dA = \frac{1}{W\delta}\int_0^W\int_0^\delta v_z\,dx\,dy = \frac{W}{W\delta}\int_0^\delta v_z\,dx$$

.... (26)

- Substituting e. (24) into (26) and integrating

$$v_{z\,av} = \frac{\rho g \delta^2}{3\mu}$$

.... (27)

- Combing eqs. (25) and (27) we obtain,

$$v_{z\,av} = (2/3)v_{z\,max}$$

.... (28)

- The volumetric flow rate, q, can be obtained by multiplying the avg velocity times the cross-sectional area

$A_c = \delta W$

$$q = \frac{\rho g \delta^3 W}{3\mu}\ \mathrm{m^3/s}$$

.... (29)

Note

Often in falling films, the mass rate of flow per unit width of wall Γ in kg/s · m is defined

as $\Gamma = \rho \delta v_{z\,av}$ and a Reynolds number is defined as

$$N_{Re} = \frac{4\Gamma}{\mu} = \frac{4\rho\delta v_{z\,av}}{\mu} \qquad \text{.............................. (30)}$$

Laminar flow occurs for $N_{Re} < 1200$.

Laminar flow with rippling present occurs above a N_{Re} of 25.

Example

An oil is flowing down a vertical wall as a film 1.7 mm thick. The oil density is 820 kg/m^3 and the viscosity is 0.20 $Pa \cdot s$. Calculate the mass flow rate per unit width of wall, Γ, needed and the Reynolds number. Also calculate the average velocity.

Solution

- Substituting into the definition of Γ

$$\Gamma = \rho \delta v_{z\,av} = \frac{(\rho\delta)\rho g \delta^2}{3\mu} = \frac{\rho^2 \delta^3 g}{3\mu}$$

$$= \frac{(820)^2 (1.7 \times 10^{-3})^3 (9.806)}{3 \times 0.20} = 0.05399 \text{ kg/s} \cdot \text{m}$$

- Check Re

$$N_{Re} = \frac{4\Gamma}{\mu} = \frac{4(0.05399)}{0.20} = 1.080$$

Hence, the film is in laminar flow.

- Avg velocity

$$v_{z\,av} = \frac{\rho g \delta^2}{3\mu} = \frac{820(9.806)(1.7 \times 10^{-3})^2}{3(0.20)} = 0.03873 \text{ m/s}$$