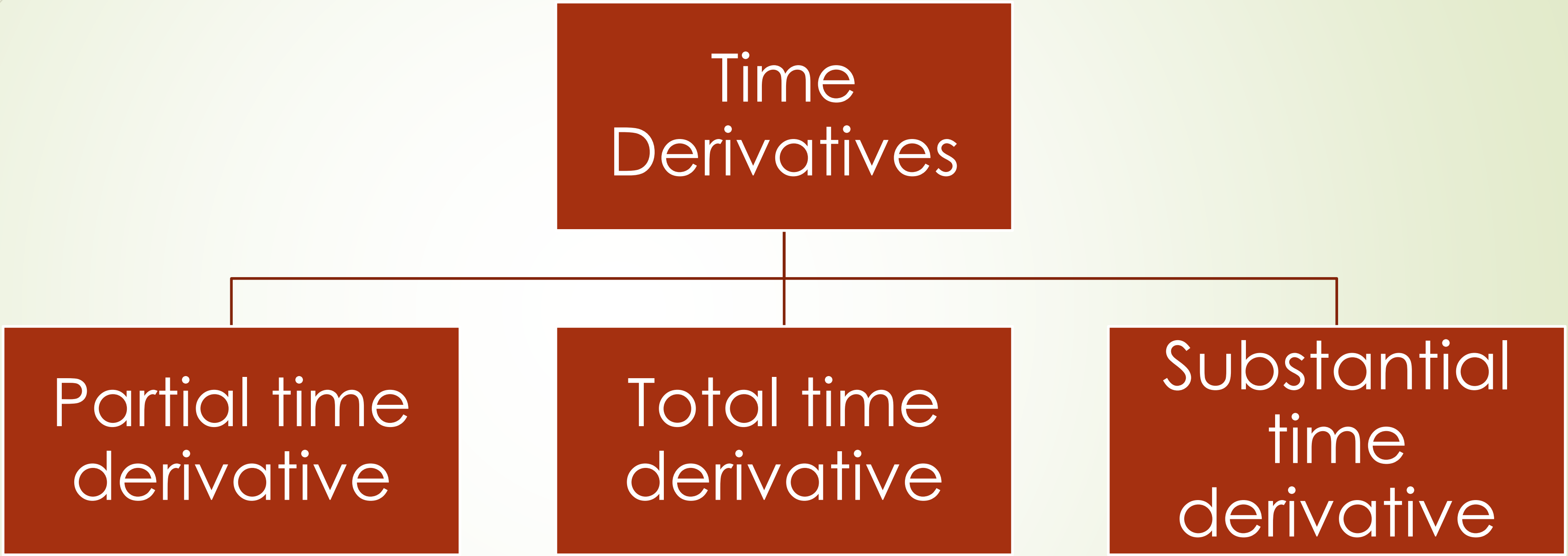
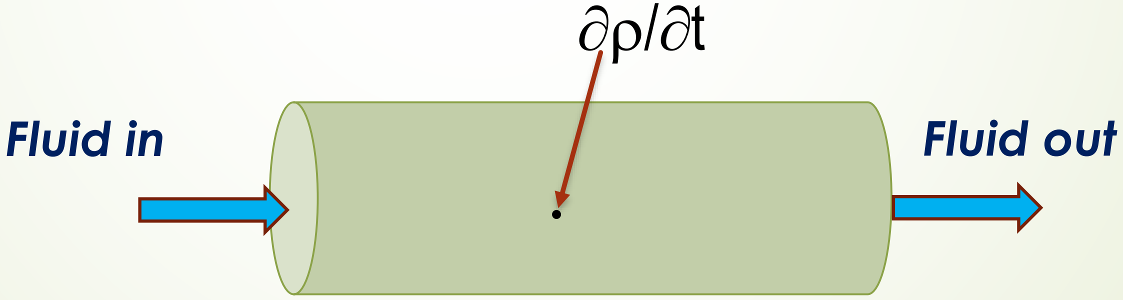
General Notes

Types of time derivatives & vectors and tensors


Time Derivatives
Partial time derivative
Total time derivative
Substantial time derivative

Partial time derivative

The most common type of derivative is the partial time derivative. For example, suppose that we are interested in the mass concentration or density ρ in kg/m^3 in a flowing stream as a function of position x, y, z and time t. The partial time derivative of ρ is $\partial \rho / \partial t$. This is the local change of density with time at a fixed point x, y, and z.



Total time derivative

Suppose that we want to measure the density in the stream while we are moving about in the stream with velocities in the x, y, and z directions of dx/dt, dy/dt and dz/dt, respectively. The total derivative $d\rho/dt$ is

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt}$$

This means that the density is a function of t and of the velocity components dx/dt, dy/dt, and dz/dt at which the observer is moving.

Substantial time derivative

Another useful type of time derivative is obtained if the observer floats along with the velocity **v** of the flowing stream and notes the change in density with respect to time. This is called the derivative that follows the motion, or the *substantial time derivative*, $D\rho/Dt$.

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla \rho)$$

where v_x, v_y, and v_z are the velocity components of the stream velocity **v**, which is a vector. This substantial derivative is applied to both scalar and vector variables.

Coordinate systems

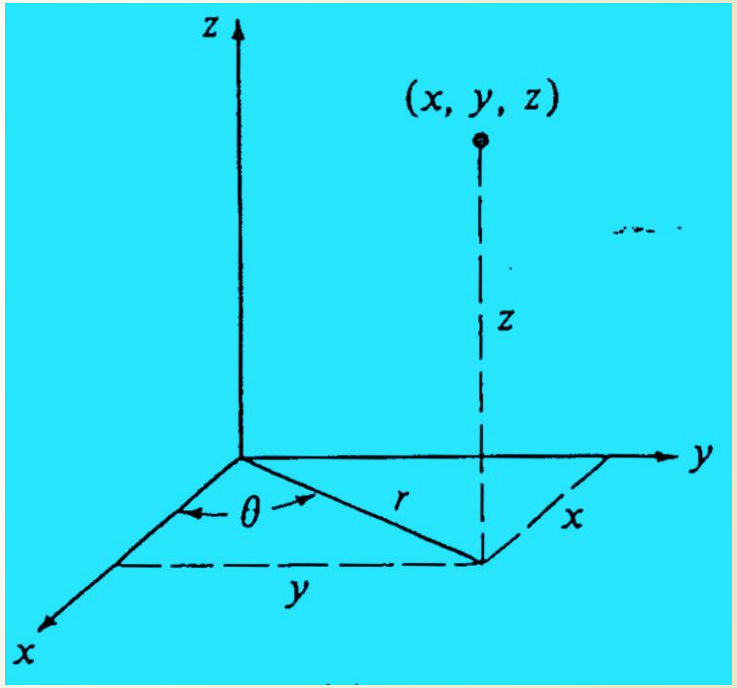
A. Cartesian coordinate system
B. Cylindrical coordinate system
C. Spherical coordinate system

Relation between Cylindrical system and Cartesian system

The relations between rectangular x, y, z and cylindrical r, θ, z coordinates are

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r = +\sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$



General notes on tensors

Temperature and mass are scalar quantities. The gradients of these (∇T or ∇C_A) and the flux terms (q/A or $\mathbf{J}_A/A$ or $\mathbf{N}_A/A$) are vectors. In marked contrast, the velocity itself is a vector, and the gradient of this ($\nabla \mathbf{U}$) is a second-order **tensor**

Correspondingly, the momentum flux or shear stress is also a second order tensor. Instead of a simple vector equation as given before, the momentum equation in three dimensions is a tensor relation, which for an incompressible fluid is

$$\boldsymbol{\tau} = -\mu[\nabla \mathbf{U} + (\nabla \mathbf{U})^{T}] \qquad \text{..................(i)}$$

- The previous equation shows that the stress tensor τ is a function of the shear rate tensor ∇U and its transpose $(\nabla U)^T$.
- Velocity, which is a vector quantity, has three components. Any one of these components can vary in three directions. Consequently, there are three components taken three ways, or nine possible terms. In the form of an array, these terms are

$$\nabla U = \begin{pmatrix} \partial U_x/\partial x & \partial U_y/\partial x & \partial U_z/\partial x \\ \partial U_x/\partial y & \partial U_y/\partial y & \partial U_z/\partial y \\ \partial U_x/\partial z & \partial U_y/\partial z & \partial U_z/\partial z \end{pmatrix}$$

- The transpose tensor $(\nabla U)^T$ is just the previous equation with the rows and columns exchanged;

$$(\nabla U)^T = \begin{pmatrix} \partial U_x/\partial x & \partial U_x/\partial y & \partial U_x/\partial z \\ \partial U_y/\partial x & \partial U_y/\partial y & \partial U_y/\partial z \\ \partial U_z/\partial x & \partial U_z/\partial y & \partial U_z/\partial z \end{pmatrix}$$

- Since Eq. (i) must be homogeneous, the left hand side must also be a second-order tensor, i.e.,

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \qquad \ldots\ldots\ldots\ldots\text{(ii)}$$

Look !!!!

- Each row of the tensor has three terms. In the first row of Eq. (ii) there is one normal stress τ_{xx} , and two tangential stresses, τ_{xy} and τ_{xz} .

- The three normal stresses in Eq. (ii) (the diagonal elements) act in the x, y, and z directions, and each is the force per unit area on a plane perpendicular to the direction in which it acts.

Summary: As already indicated, Eq. (i) is a shorthand representation for nine equations. Several of these are'

$$\boldsymbol{\tau} = -\mu[\nabla U + (\nabla U)^{T}]$$

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = -\mu \left[\begin{pmatrix} \partial U_x/\partial x & \partial U_y/\partial x & \partial U_z/\partial x \\ \partial U_x/\partial y & \partial U_y/\partial y & \partial U_z/\partial y \\ \partial U_x/\partial z & \partial U_y/\partial z & \partial U_z/\partial z \end{pmatrix} + \begin{pmatrix} \partial U_x/\partial x & \partial U_x/\partial y & \partial U_x/\partial z \\ \partial U_y/\partial x & \partial U_y/\partial y & \partial U_y/\partial z \\ \partial U_z/\partial x & \partial U_z/\partial y & \partial U_z/\partial z \end{pmatrix} \right]$$

$$\tau_{xx} = -\mu(\partial U_x/\partial x + \partial U_x/\partial x) = -2\mu(\partial U_x/\partial x) \quad \text{...........(iii)}$$

$$\tau_{yx} = -\mu[(\partial U_x/\partial y) + (\partial U_y/\partial x)] \quad \text{............(iv)}$$

$$\tau_{xy} = -\mu[(\partial U_y/\partial x) + (\partial U_x/\partial y)] \quad \text{.............(v)}$$

- For the one-dimensional problem of Eq. given in the beginning of the semester, **U_x, varies in the y direction only**, and both U_y and U_z are zero. Thus, most derivatives in ∇U are zero:

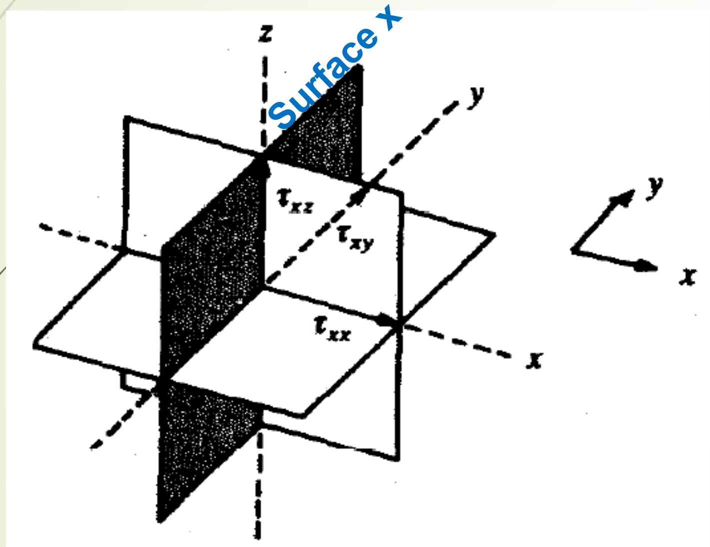
$$\partial U_x/\partial x = \partial U_x/\partial z = 0$$

$$\partial U_y/\partial x = \partial U_y/\partial y = \partial U_z/\partial z = 0$$

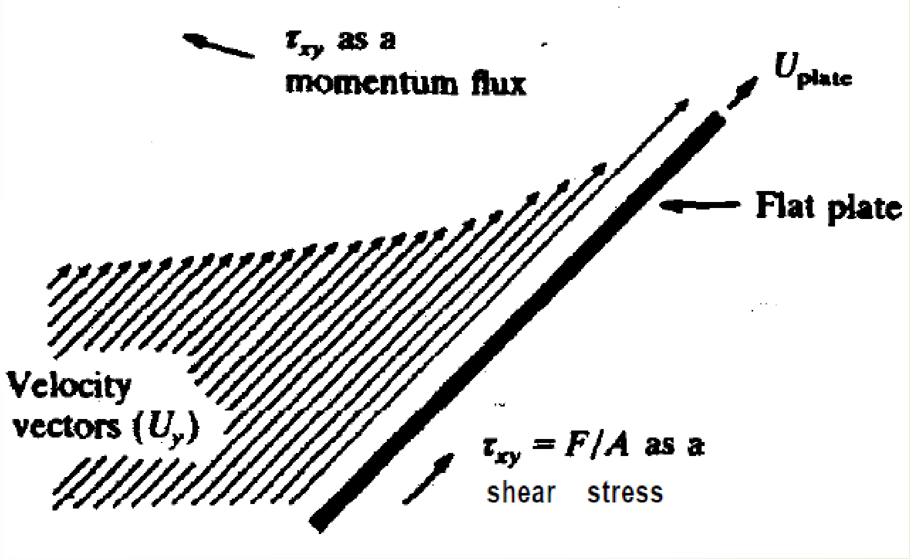
$$\partial U_z/\partial z = \partial U_z/\partial y = \partial U_z/\partial z = 0$$

- From the nine equations represented in shorthand by Eq. (i) only two equations remain, Eqs. (iv) and (v), both of which are identical to Eq. given for viscosity law since $\partial U_y/\partial x$ is zero and τ_{xy} equals τ_{yx}. It therefore follows that for the one-dimensional problem where given Eq. before is valid, there are only two non-zero shear stress terms, which are τ_{yx} and τ_{xy}.

The momentum flux τ_{xy}.

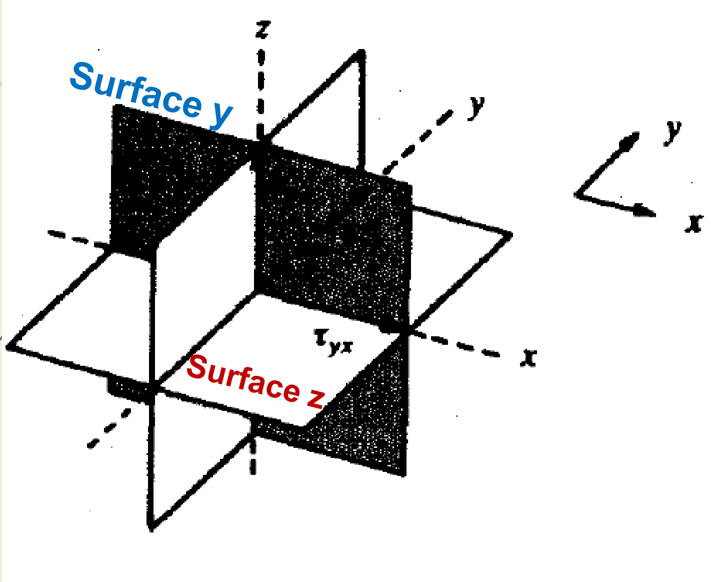


Stress tensor on the *yz* plane

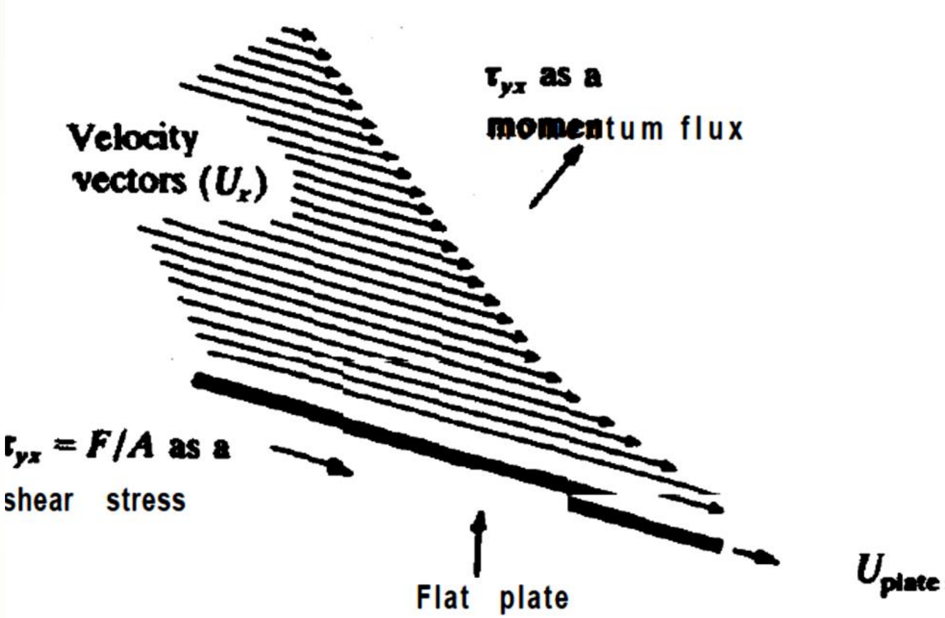


The *xy* component as a momentum flux

The momentum flux τ_{yx}



Stress tensor on the *xz* plane



The yx component as a momentum flux

Forces acting on a differential control volume

Double subscript is used to specify the stress components. The 1st indicates the surface orientation by providing the direction of its outward normal, and the 2nd indicates the direction of the force component.

In short $\sigma_{\text{surface, direction}}$