General Heat diffusion Equations

Temp. Distribution & Heat flux

Shapes: Cube, Cylinder, Sphere

Heat diffusion Equations methodology

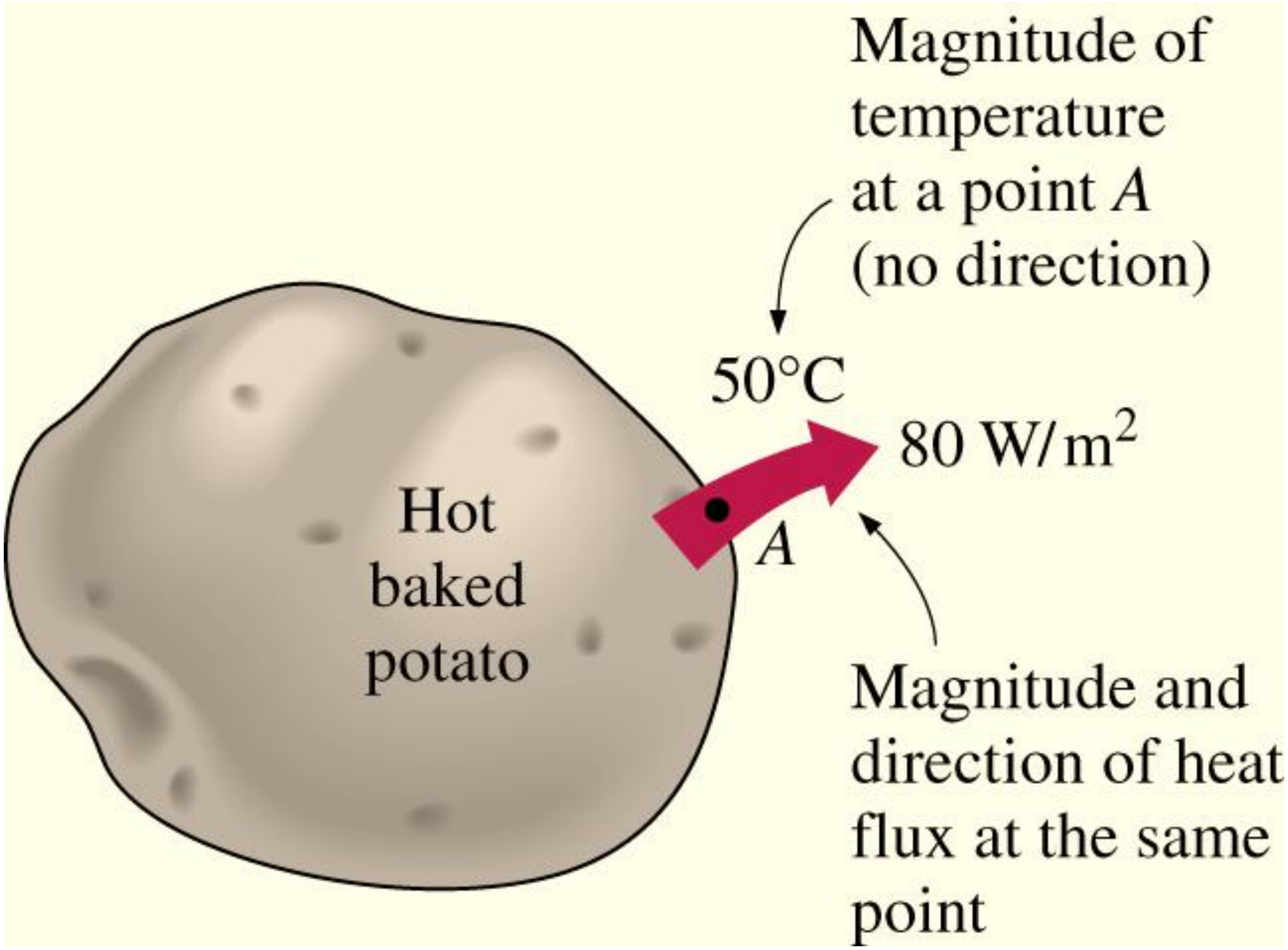
1. Equations of heat diffusion can be obtained by applying the energy balance equation over a differential element or a differential control volume.
2. Solving these differential equations subjected to certain boundary conditions will lead to the temperature distribution in a medium.
3. Once this distribution is known, the conduction heat flux at any point in the medium or on its surface could be computed from Fourier’s law.

Heat Transfer_ Basic information

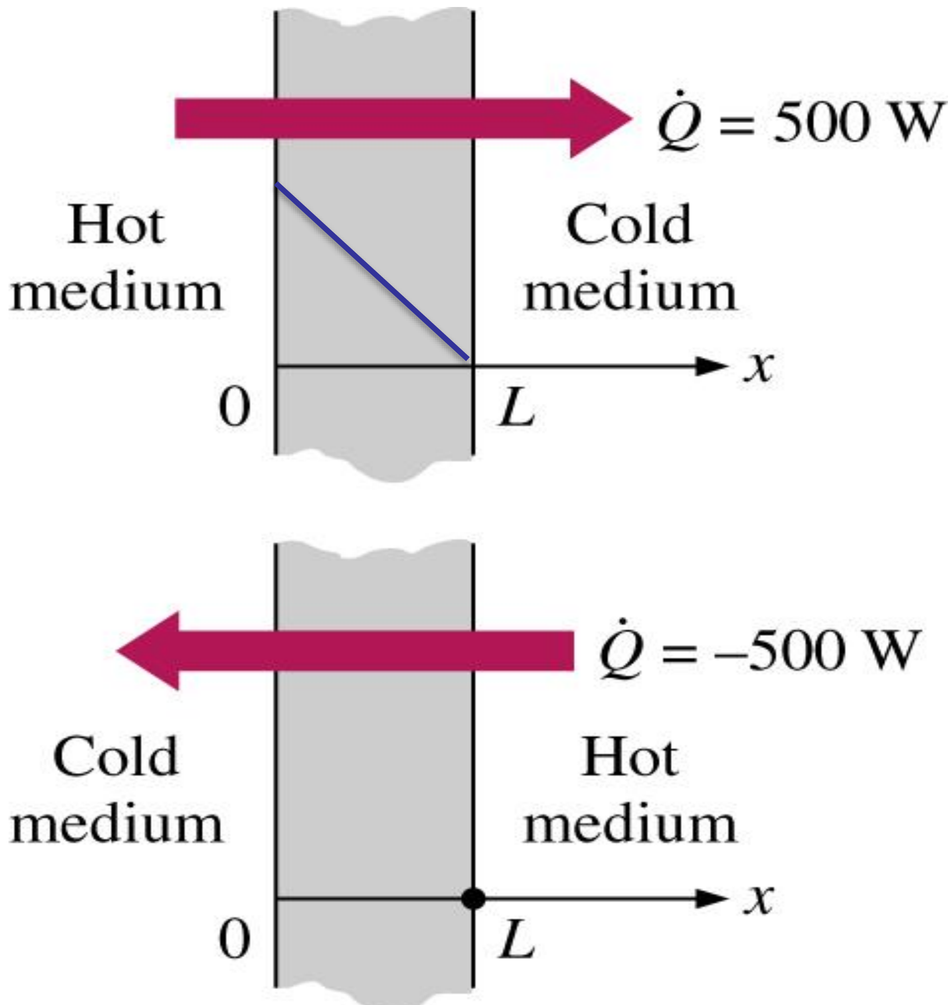
- Heat transfer is A vector

Magnitude
+
Direction
{coordinate system}

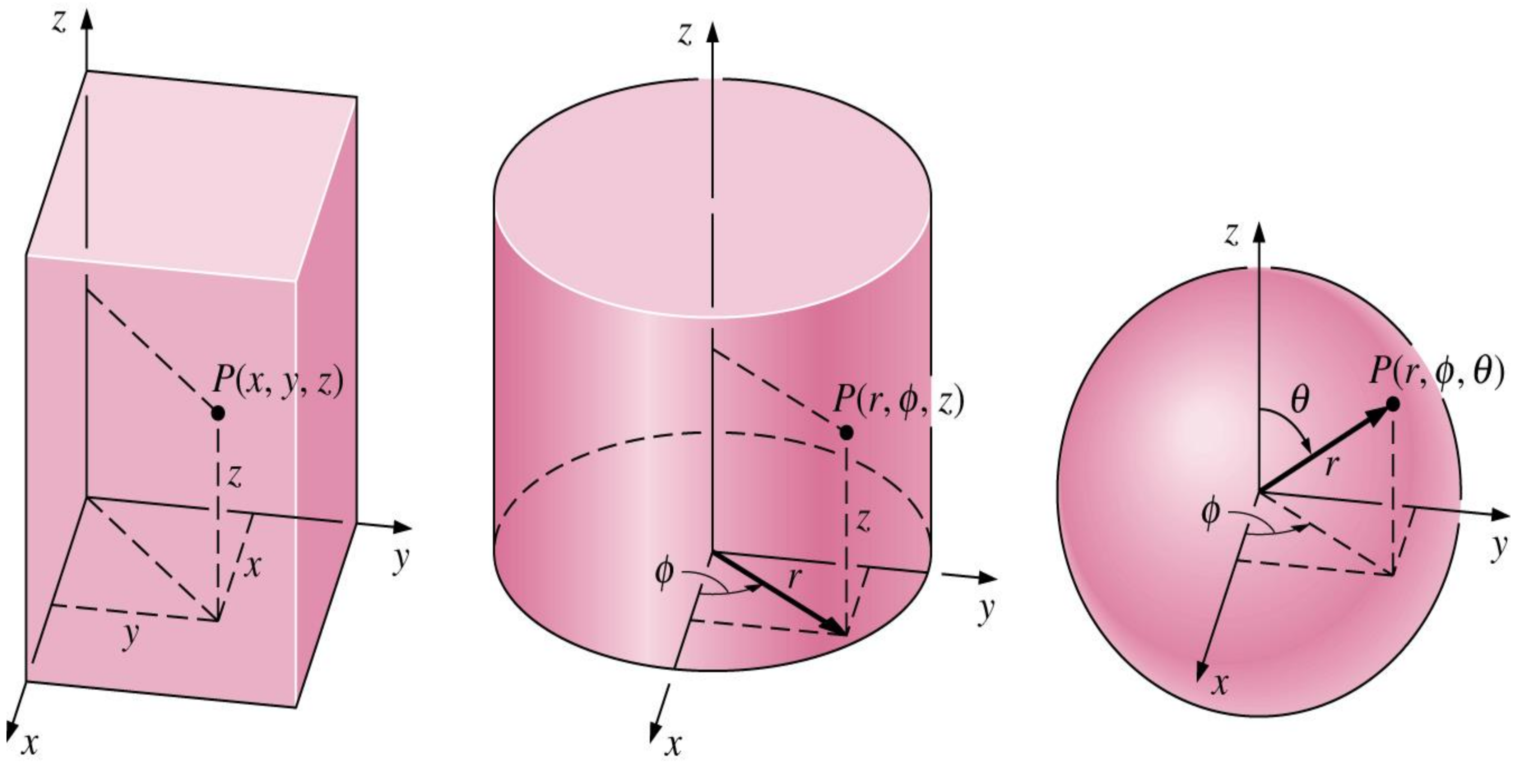
Heat transfer has direction plus magnitude and therefore it is a vector quantity



Indicating direction for heat transfer (+ve in the +ve direction and –ve in the –ve direction)

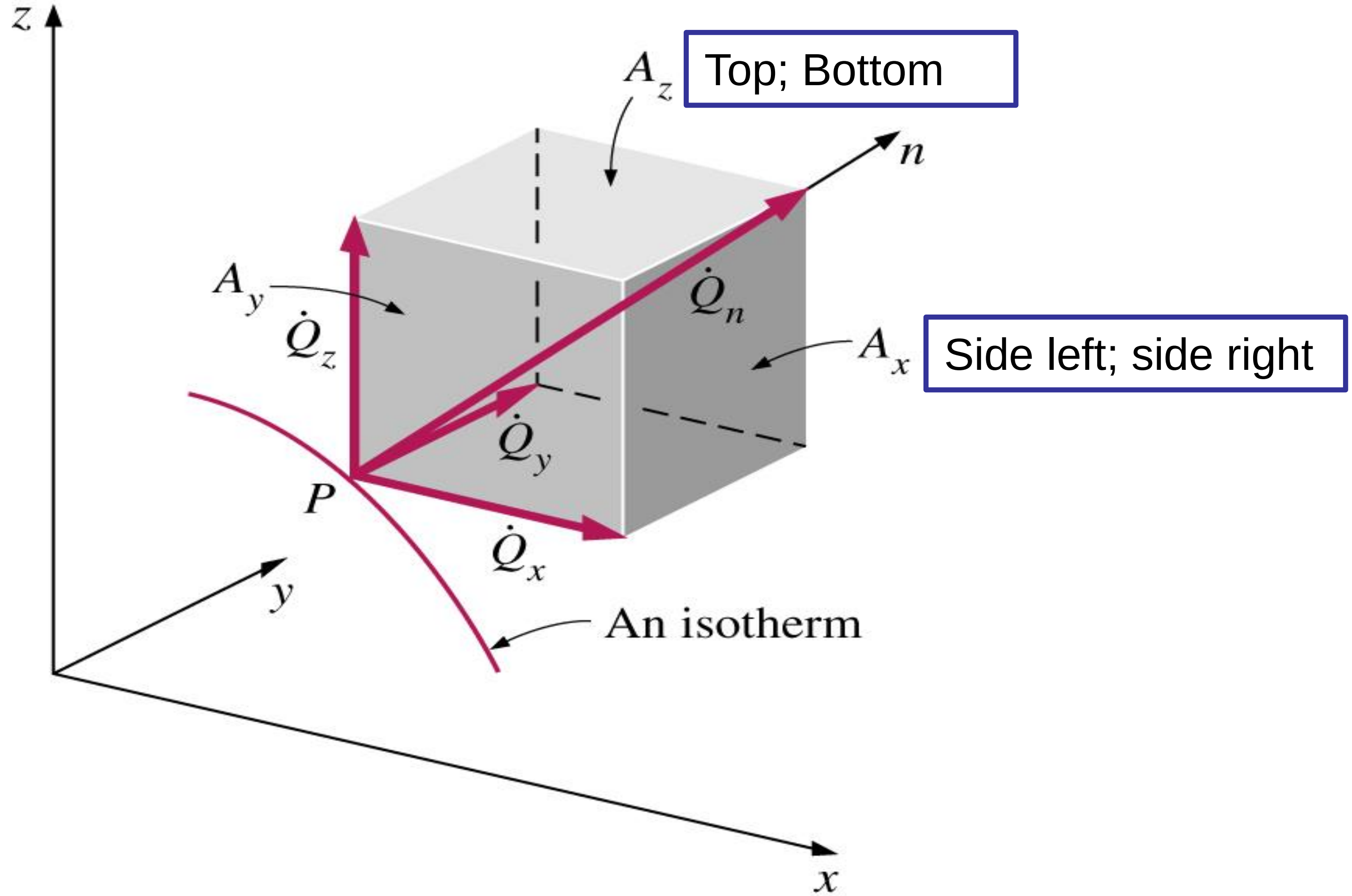


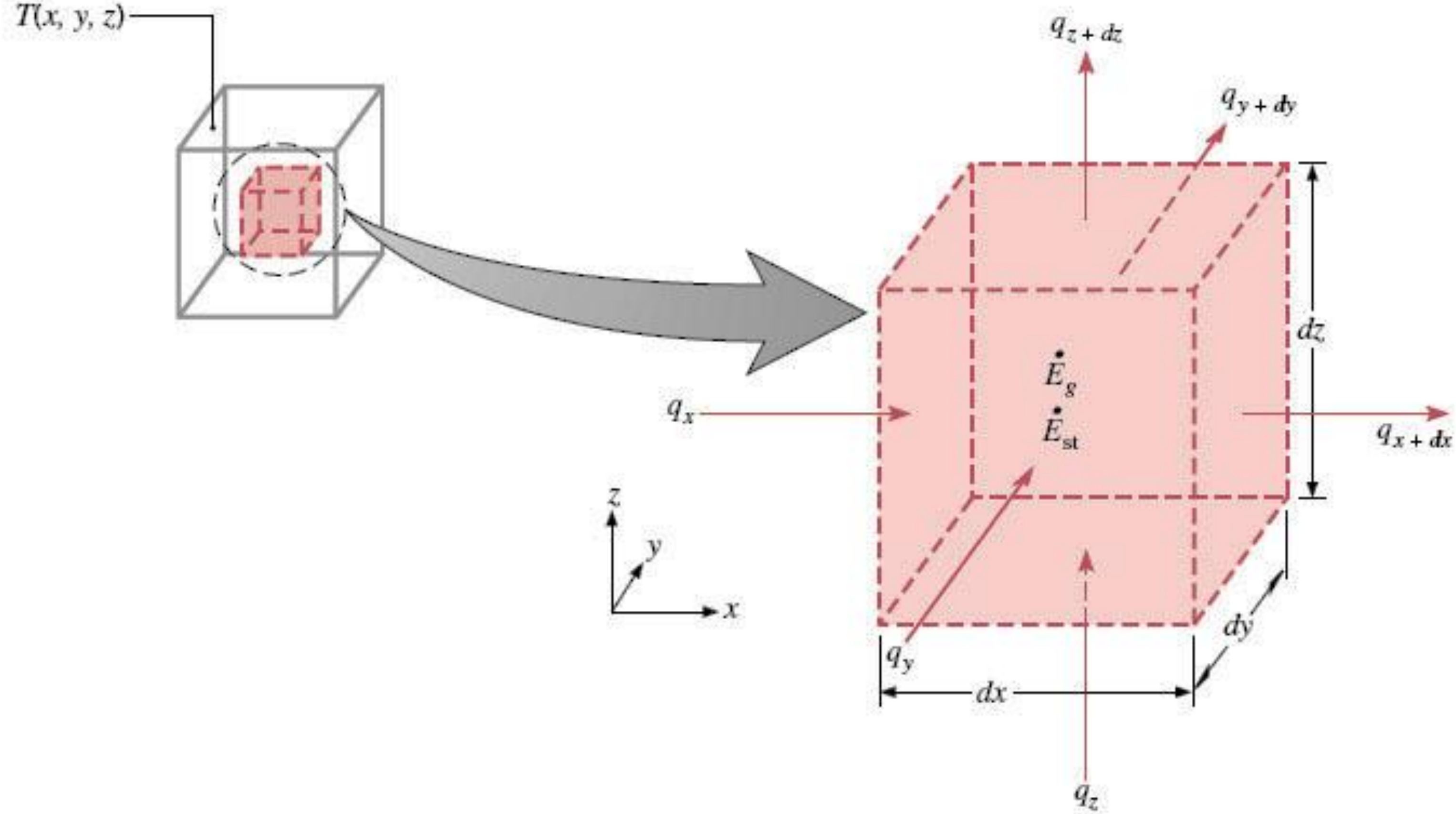
The various distance and angles involved when describing the location of a point in different coordinate system



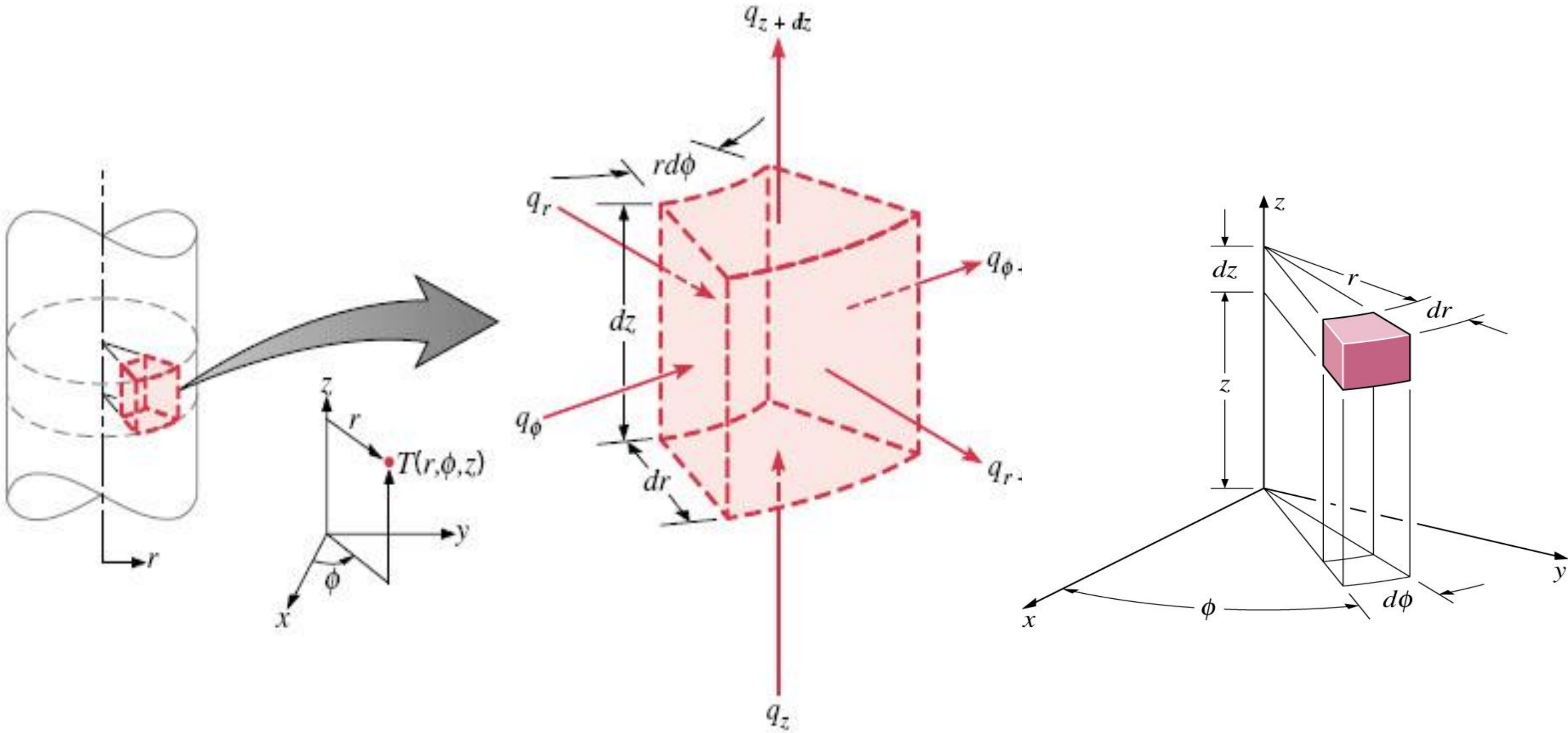
(*a*) Rectangular coordinates (*b*) Cylindrical coordinates (*c*) Spherical coordinates

The heat transfer vector is always normal to an isothermal surface and can be resolved into its components like any other vector

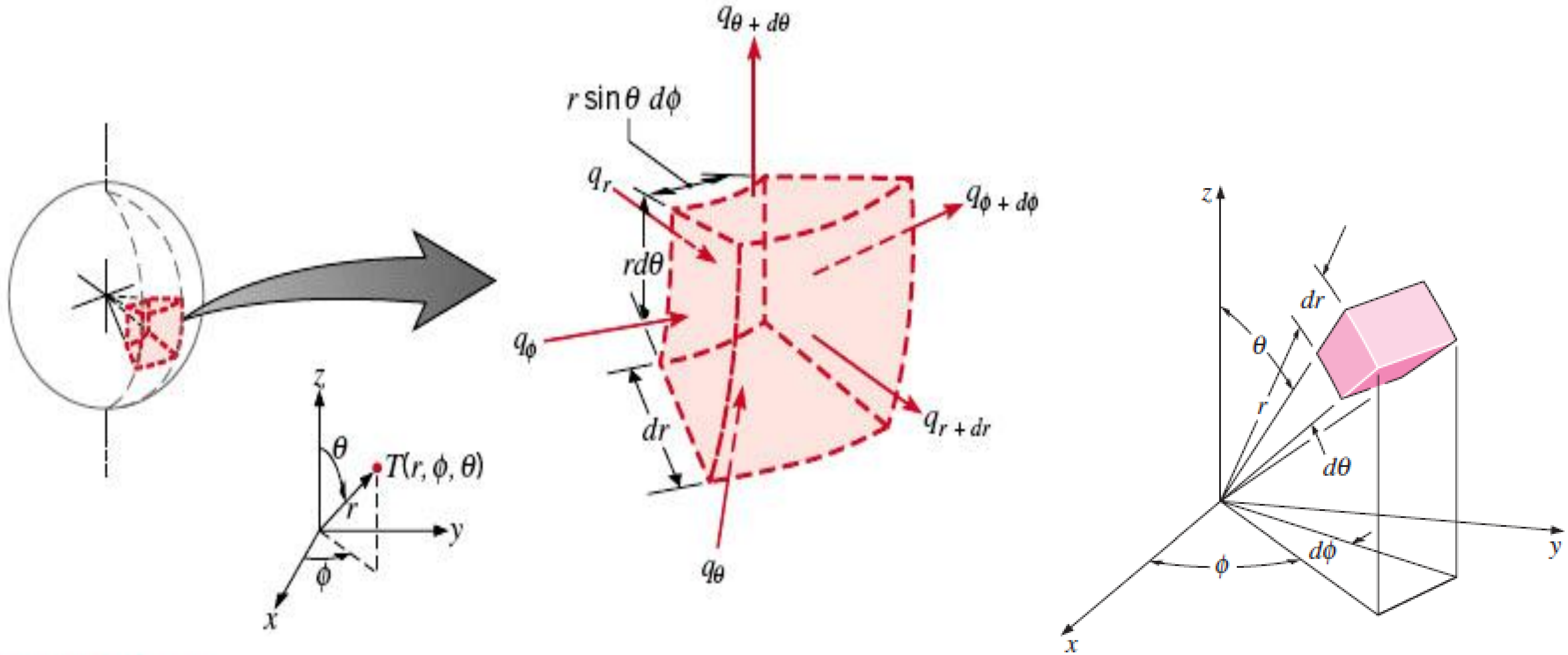




Differential control volume, $dx\ dy\ dz$, for conduction analysis in Cartesian coordinates.



Differential control volume, $dr \cdot r\, d\phi \cdot dz$, for conduction analysis in cylindrical coordinates (r, ϕ, z).



Differential control volume, $dr \cdot r \sin\theta \, d\phi \cdot r \, d\theta$, for conduction analysis in spherical coordinates (r, ϕ, θ).

Fourier's Law

- A rate equation that allows determination of the conduction heat flux from knowledge of the temperature distribution in a medium

- Its most general (vector) form for multidimensional conduction is:

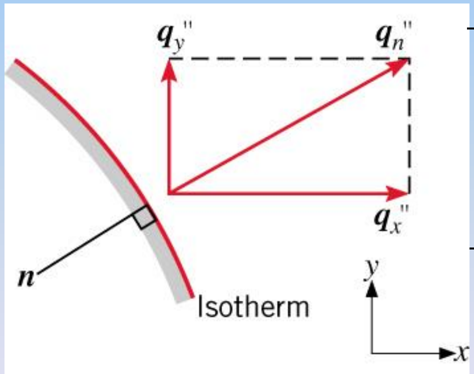
$$\vec{q}'' = -k \vec{\nabla} T$$

Implications:

– Heat transfer is in the direction of decreasing temperature (basis for minus sign).

– Fourier's Law serves to define the thermal conductivity of the medium

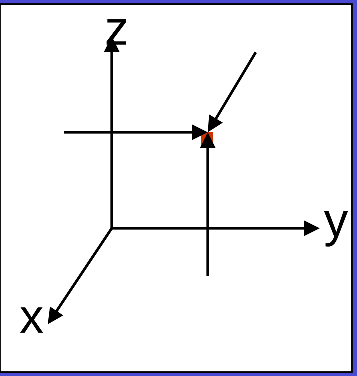
$$\left(k \equiv -\vec{q}'' / \vec{\nabla} T \right)$$



– Direction of heat transfer is perpendicular to lines of constant temperature (isotherms).

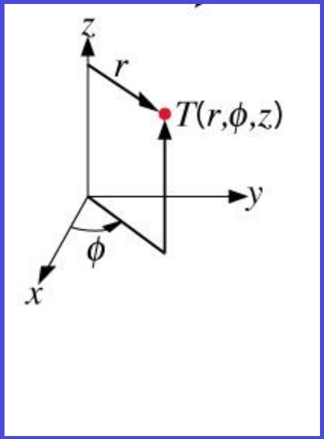
– Heat flux vector may be resolved into orthogonal components.

Heat Flux Components



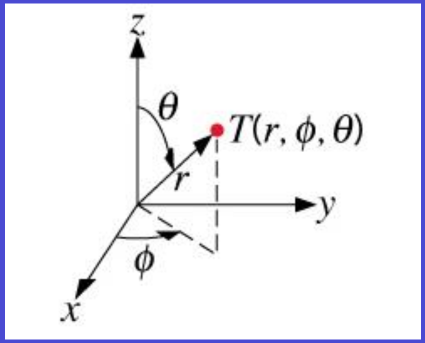
- Cartesian Coordinates: $T(x, y, z)$

$$\vec{q}'' = \underbrace{-k\frac{\partial T}{\partial x}}_{q''_x}\vec{i} \underbrace{-k\frac{\partial T}{\partial y}}_{q''_y}\vec{j} \underbrace{-k\frac{\partial T}{\partial z}}_{q''_z}\vec{k} \qquad (2.3)$$



- Cylindrical Coordinates: $T(r, \phi, z)$

$$\vec{q}'' = \underbrace{-k\frac{\partial T}{\partial r}}_{q''_r}\vec{i} \underbrace{-k\frac{\partial T}{r\partial \phi}}_{q''_\phi}\vec{j} \underbrace{-k\frac{\partial T}{\partial z}}_{q''_z}\vec{k} \qquad (2.22)$$

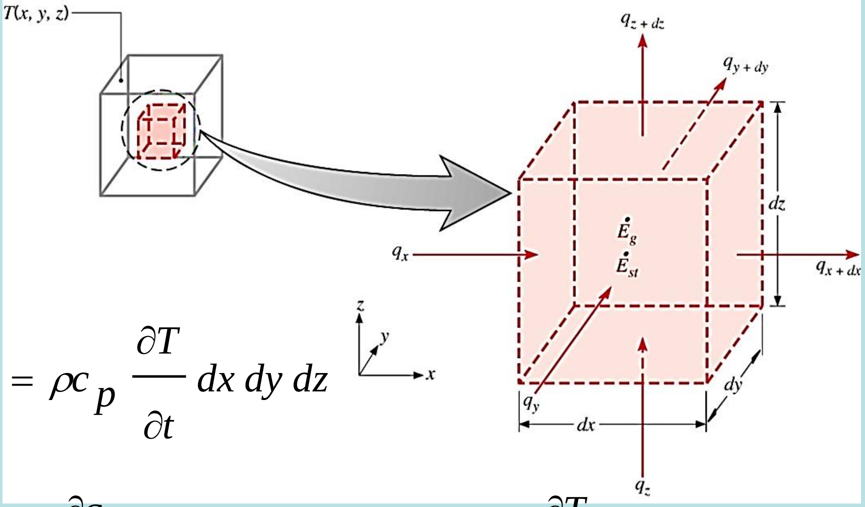


- Spherical Coordinates: $T(r, \phi, \theta)$

$$\vec{q}'' = \underbrace{-k\frac{\partial T}{\partial r}}_{q''_r}\vec{i} \underbrace{-k\frac{\partial T}{r\partial \theta}}_{q''_\theta}\vec{j} \underbrace{-k\frac{\partial T}{r\sin\theta\,\partial \phi}}_{q''_\phi}\vec{k} \qquad (2.25)$$

Energy Balance:

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = \dot{E}_{st}$$



$$q_x + q_y + q_z - q_{x+dx} - q_{y+dy} - q_{z+dz} + \dot{q}\, dx\, dy\, dz = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz$$

$$q_x + q_y + q_z - (q_x + \frac{\partial q_x}{\partial x} dx) - (q_y + \frac{\partial q_y}{\partial y} dy) - (q_z + \frac{\partial q_z}{\partial z} dz) + \dot{q}\, dx\, dy\, dz = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz$$

$$-\frac{\partial q_x}{\partial x} dx - \frac{\partial q_y}{\partial y} dy - \frac{\partial q_z}{\partial z} dz + \dot{q}\, dx\, dy\, dz = \rho c_p \frac{\partial t}{\partial t} dx\, dy\, dz$$

Use Fourier's Law

$$q_x = -k\, dy\, dz \frac{\partial T}{\partial x} \quad , \quad q_y = -k\, dx\, dz \frac{\partial T}{\partial y} \quad , \quad q_z = -k\, dy\, dx \frac{\partial T}{\partial z}$$

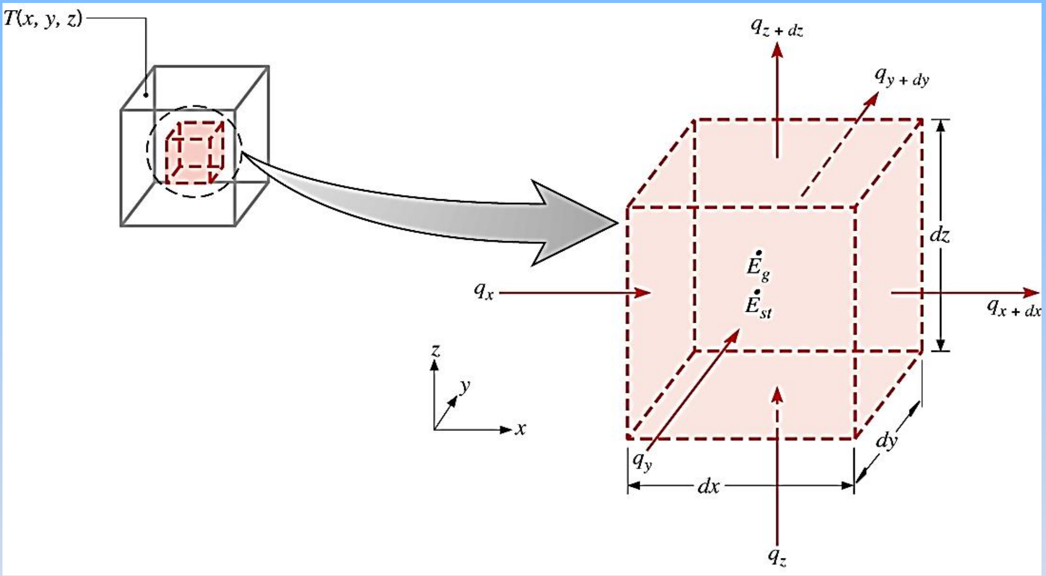
$\therefore$ *After substitution and dividing out the volume dx dy dz, we get*

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

This is the general thermal diffusion equation for Cartesian coordinate system.

Summary: The Heat Equation

- A differential equation whose solution provides the temperature distribution in a stationary medium.
- Based on applying conservation of energy to a differential control volume through which energy transfer is exclusively by conduction.
- Cartesian Coordinates:



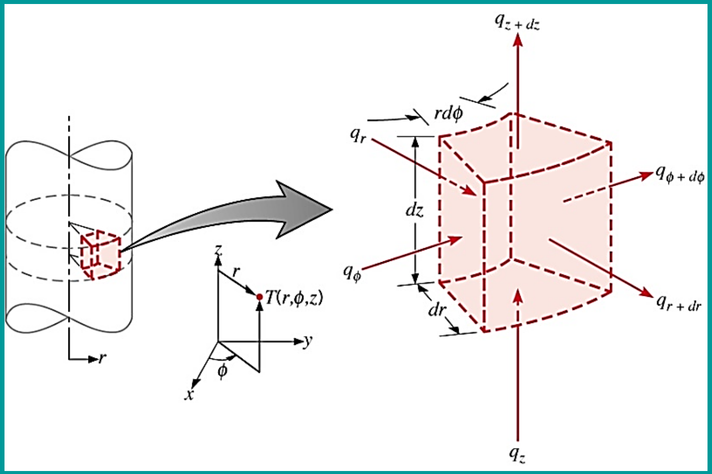
$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Net transfer of thermal energy into the control volume (inflow-outflow)

Thermal energy generation

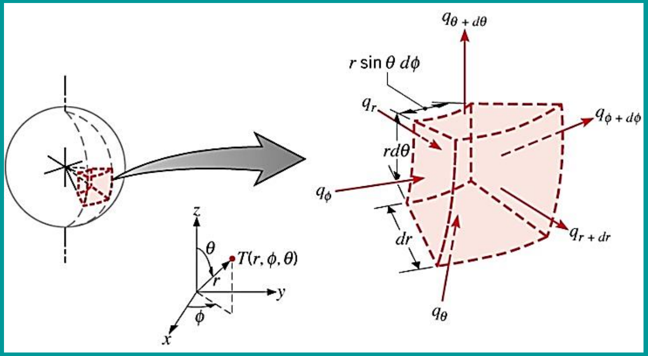
Change in thermal energy storage

- Cylindrical Coordinates:



$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

- Spherical Coordinates:



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Heat Equation (Special Case)

- One-Dimensional Conduction in a Planar Medium with Constant Properties and No Generation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

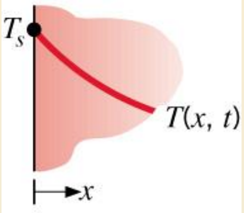
$\alpha \equiv \dfrac{k}{\rho c_p} \rightarrow$ thermal diffusivity of the medium

Boundary and Initial Conditions

- For transient conduction, heat equation is first order in time, requiring specification of an initial temperature distribution: $T(x,t)_{t=0} = T(x,0)$
- Since heat equation is second order in space, two boundary conditions must be specified. Some common cases:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

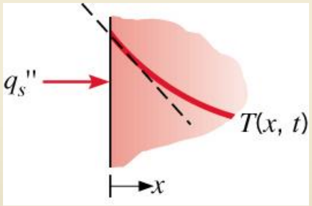
Constant Surface Temperature:



$$T(0,t) = T_s$$

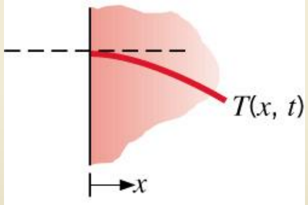
Constant Heat Flux:

Applied Flux



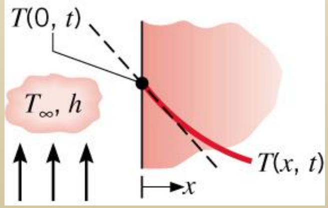
$$-k\frac{\partial T}{\partial x}\Big|_{x=0} = q_s''$$

Insulated Surface



$$\frac{\partial T}{\partial x}\Big|_{x=0} = 0$$

Convection



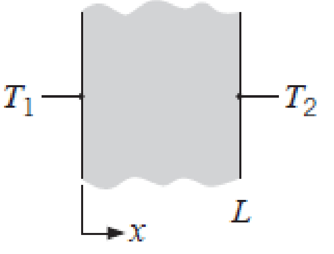
$$-k\frac{\partial T}{\partial x}\Big|_{x=0} = h\left[T_\infty - T(0,t)\right]$$

Problem 1

Consider steady-state conditions for one-dimensional conduction in a plane wall having a thermal conductivity, k = 50 W/m K and a thickness, L = 0.25 m, with no internal heat generation.

Determine the heat flux and the unknown quantity for each case and sketch the temperature distribution, indicating the direction of the heat flux

Case	T_1(°C)	T_2(°C)	dT/dx (K/m)
1	50	−20	
2	−30	−10	
3	70		160

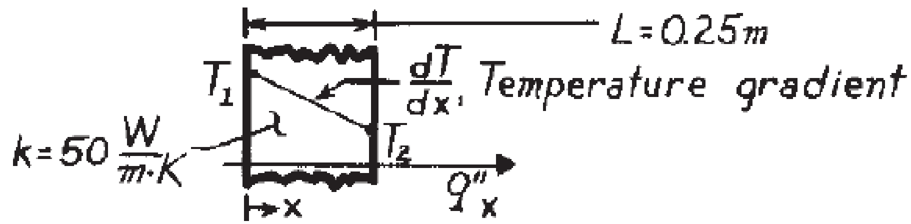


SOLUTION

KNOWN: One-dimensional system with prescribed thermal conductivity and thickness.

FIND: Unknowns for various temperature conditions and sketch distribution.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction, (3) No internal heat generation, (4) Constant properties.

ANALYSIS: The rate equation and temperature gradient for this system are

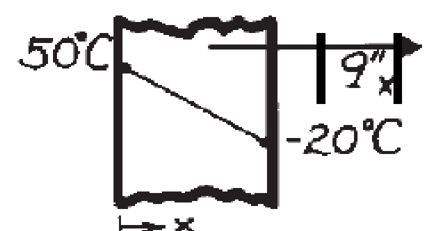
$$q''_x = -k\frac{dT}{dx} \qquad \text{and} \qquad \frac{dT}{dx} = \frac{T_2 - T_1}{L}.$$

Using Eqs. (1) and (2), the unknown quantities for each case can be determined.

(a)

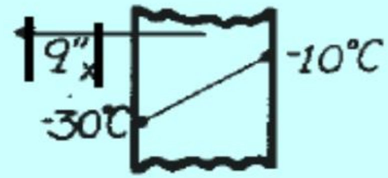
$$\frac{dT}{dx} = \frac{(-20-50)\,K}{0.25m} = -280\ K/m$$

$$q''_x = -50\frac{W}{m\cdot K} \times \left[-280\frac{K}{m}\right] = 14.0\ kW/m^2.$$



(b) $$\frac{dT}{dx} = \frac{(-10-(-30))\text{K}}{0.25\text{m}} = 80\ \text{K/m}$$

$$q''_x = -50\frac{\text{W}}{\text{m}\cdot\text{K}} \times \left[80\frac{\text{K}}{\text{m}}\right] = -4.0\ \text{kW/m}^2.$$



(c) $$q''_x = -50\frac{\text{W}}{\text{m}\cdot\text{K}} \times \left[160\frac{\text{K}}{\text{m}}\right] = -8.0\ \text{kW/m}^2$$

$$T_2 = L\cdot\frac{dT}{dx} + T_1 = 0.25\text{m} \times \left[160\frac{\text{K}}{\text{m}}\right] + 70^\circ\text{C}.$$

$$T_2 = 110^\circ\text{C}.$$