Discussion _ Heat

Applications on Heat Diffusion Equations

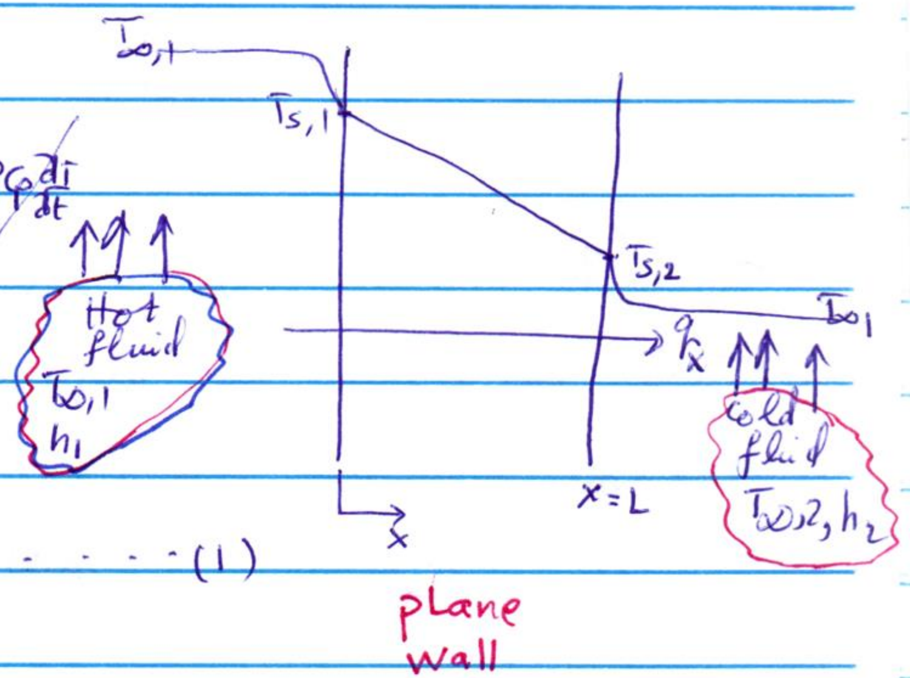
1-D, S.S conduction, no internal generation

- Temp distribution + heat transfer rate (planar, cylindrical, spherical)
- thermal resistance

Temp. distribution

The plan wall ✓

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{dT}{dt}$$



Assume k = cons.

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \quad \cdots\cdots (1)$$

Integrating twice

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \quad \cdots\cdots (1)$$

plane wall

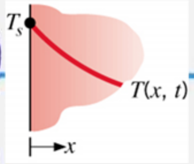
Integrating twice

$$\frac{dT}{dx} = C_1 \quad \cdots\cdots (2)$$

$$T(x) = C_1 x + C_2 \quad \cdots\cdots (3)$$

To obtain C_1, C_2 we need to apply B.C's

Constant Surface Temperature:



$$T(0,t) = T_s$$

at $x = 0$ $\quad T = T_{s,1}$ $\quad$ ✱ $\quad$ B.C(1)

at $x = L$ $\quad T(L) = T_{s,2}$ $\quad$ ✱✱ $\quad$ B.C(2) apply BC (1)

$T_{s,1} = C_2 \quad \therefore \quad C_2 = T_{s,1} \quad \cdots (4)$

apply BC (2)

$T_{s,2} = C_1 L + T_{s,1} \quad \therefore \quad C_1 = (T_{s,2} - T_{s,1})/L \quad \cdots (5)$

$\therefore$ Temp. distribution

$$T(x) = \frac{(T_{s,2} - T_{s,1})}{L} x + T_{s,1} \quad (6)$$

$\therefore$ linear distribution "at the previous conditions".

Note Rearrangement of Eq.(6)

$$\frac{T(x) - T_{s,1}}{T_{s,2} - T_{s,1}} = \left(\frac{x}{L}\right)$$

To obtain H.T.R

apply Fourier's Law

$$q_x = -kA\frac{dT}{dx}$$ ← from eq.(2) $\frac{dT}{dx} = C_1 = (T_{s,2} - T_{s1})/L$

$$= -kA\frac{T_{s,2} - T_{s,1}}{L}$$

$$q''_x = \frac{q_x}{A} = -\frac{k}{L}(T_{s,2} - T_{s,1}) = \frac{k}{L}(T_{s,1} - T_{s2})$$

In case convection B. C.^{s}

Follow the same procedure

Thermal resistance

Thermal Resistance

$$\therefore \quad q_x = kA\frac{(T_{s,1} - T_{s,2})}{L} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,1} - T_{s,2}}{R_{t,cond.}} = \frac{\text{driving Potential)}}{\text{Resistances}}$$

$$\therefore \quad R_{t,cond} = L/kA \quad \#$$

Note: Similarly for electrical conduction in the same system.

Ohm's Law provides an electrical resis.

$$R_e = \frac{E_{s,1} - E_{s,2}}{I} = \frac{L}{\sigma A} \qquad \text{OR} \quad \sigma = 1/\rho$$

($E_{s,1} - E_{s,2}$: electrical potential)

σ: electrical conductivity $1/\Omega\cdot cm$

ρ: resistivity $\Omega\cdot cm$

$$R_e = \frac{\rho L}{A}$$

A: wire cross section $\cdot cm^2$

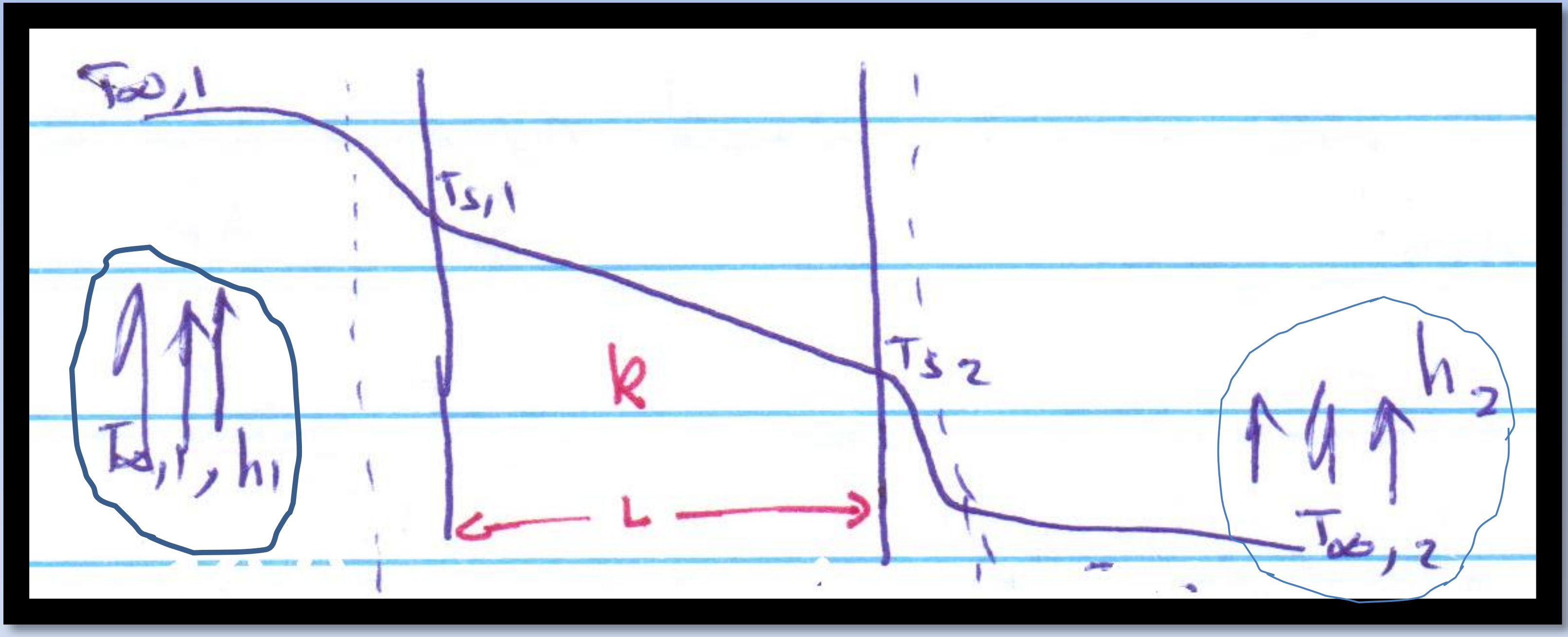
$$\text{Current } I = \frac{E_{s,1} - E_{s,2}}{R_e}$$

For convection Heat Transfer ; Newton's Law

$$q = h\,A\,(T_s - T_\infty) = \frac{(T_s - T_\infty)}{1/hA} = \frac{(T_s - T_\infty)}{R_{t,conv}}$$

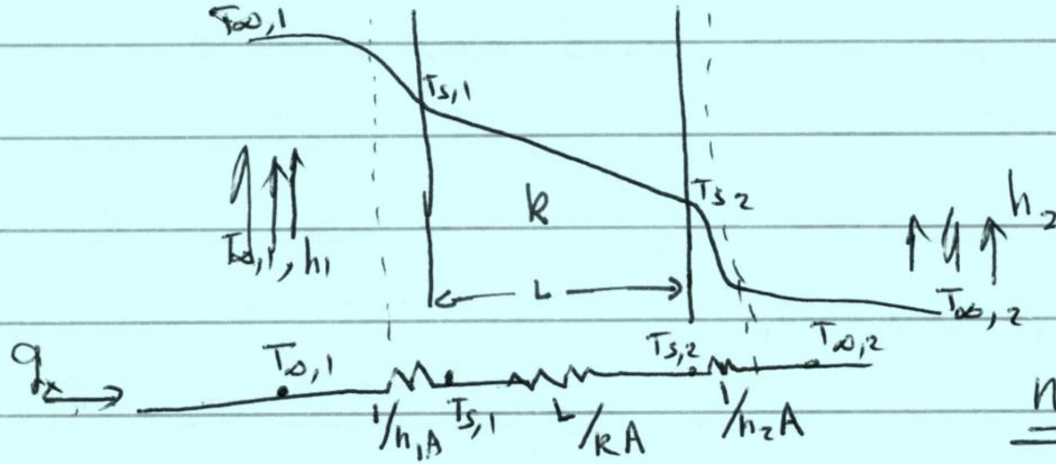
$$\therefore \quad R_{t,conv} = 1/hA$$

Find the Equivalent thermal circuit 'Network'



Equivalent thermal circuit

Now, Circuit representation or equivalent thermal circuit for the plane wall with convection surface conditions as shown below



cross sec. area = A

q_x cons. through the network

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

Overall temperature difference

In terms the overall temp diff.

$$q_x = \frac{T_{\infty,1} - T_{\infty 2}}{R_{total}}$$

because the conduction and convection resistances are in series

$$R_{total} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

Note

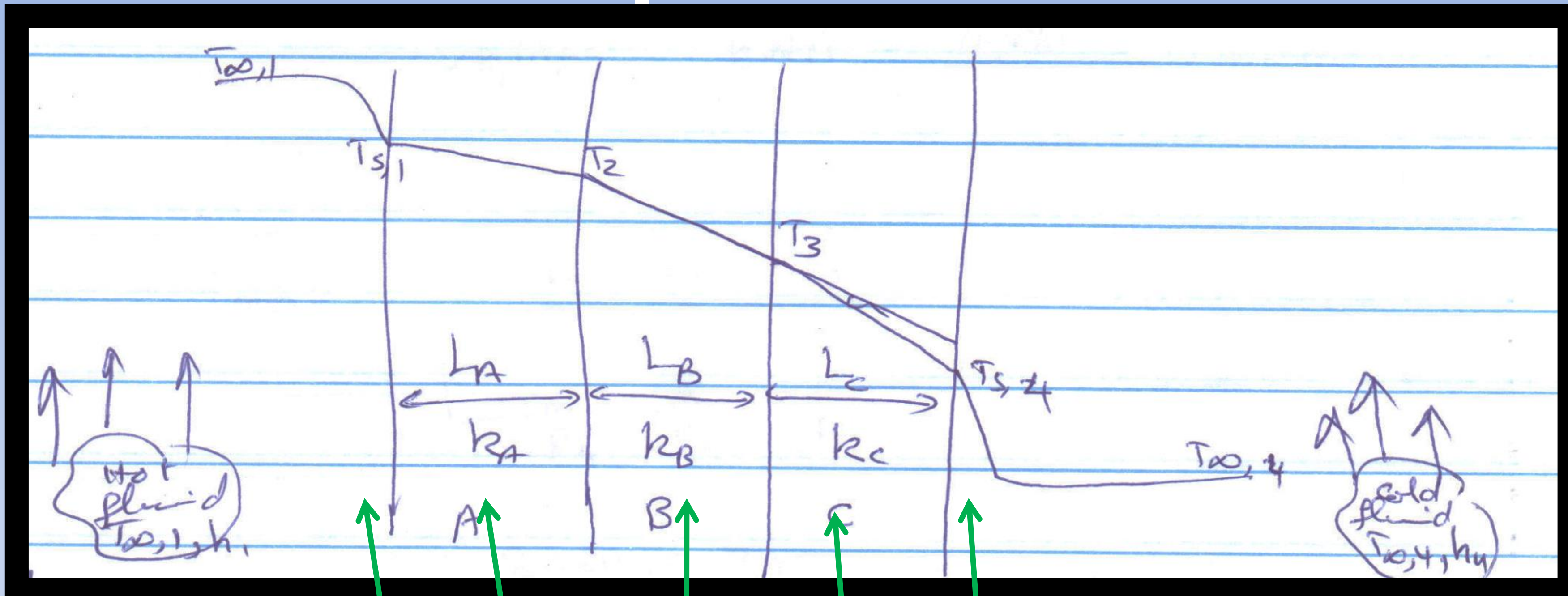
Thermal resis. for rad. could be defined via

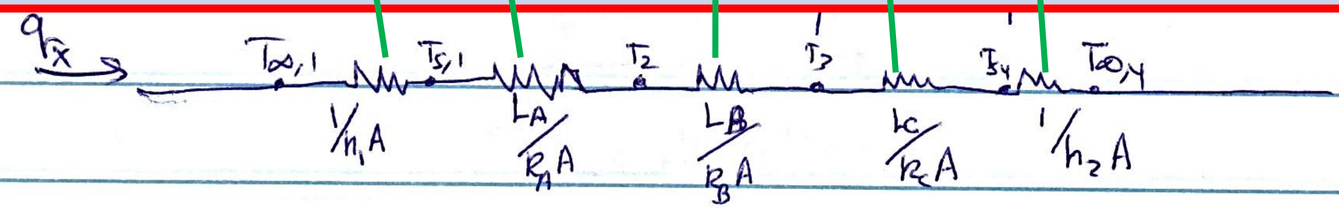
$$q_{rad} = h_r A (T_s - T_{sur}) = \frac{(T_s - T_{sur})}{1/h_r A} = \frac{T_s - T_{sur}}{R_{t,rad}}$$

where $R_{t,rad} = 1/h_r A$

Bearing in mind that $R_{t,rad}$ and $R_{t,conv}$ act in parallel.

Composite Wall





$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R} = \frac{T_{\infty,1} - T_{\infty,4}}{\left[(1/h_1A) + (L_A/k_AA) + (L_B/k_BA) + (L_C/k_CA) + (1/h_2A)\right]}$$

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{(1/h_1A)} = \frac{T_{s,1} - T_2}{(L_A/k_AA)} = \ldots\ldots$$

Overall Heat transfer Coefficient 'U'

Overall Heat Transfer Coeff. U

* U can be defined by an expression analogous to Newton's law of Cooling

$$q_x = U A \Delta T \quad \text{—— II}$$

where

ΔT : overall temp diff.

A : surface area

U overall is related to the total ~~Res~~ thermal resistance, R_t which given in the previous eqs i.e

$$q_x = \frac{\Delta T}{R_{total}} = \frac{T_{\infty,1} - T_{\infty 2}}{\sum R_t} \quad \text{—— III}$$

Compare the previous eqs:

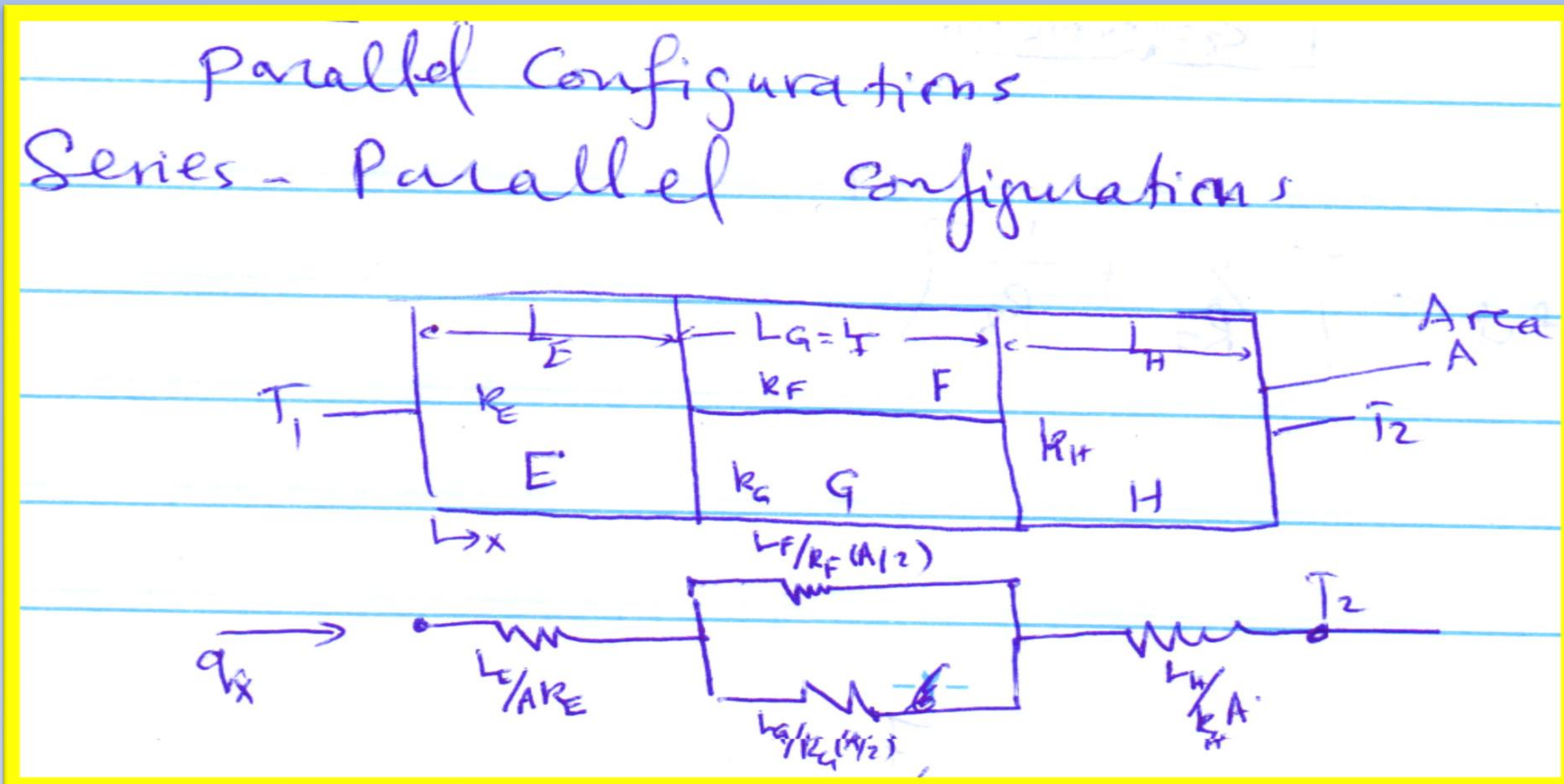
$$\therefore \quad UA = \frac{1}{R_{total}}$$

$$\therefore \quad U = \frac{1}{R_{total} A} = \frac{1}{\left[1/h_1 + L_A/k_A + L_B/k_B + L_C/k_C + 1/h_2 \right]} \quad \#$$

In general

$$\boxed{R_{tot} = \sum R_t = \frac{\Delta T}{q} = \frac{1}{UA}} \quad \#$$

Note


Parallel Configurations
Series - Parallel configurations
L_E
L_G = L_F
L_H
Area
A
T_1
T_2
k_E
E
k_F
F
k_G
G
k_H
H
x
L_F/k_F (A/2)
T_2
q_x
L_E/Ak_E
L_G/k_G (A/2)
L_H/k_H A

The Plane Wall

Consider one-dimensional, steady-state conduction in a plane wall of constant k, with uniform generation, and asymmetric surface conditions:

- Heat diffusion equation (eq. 2.3) :

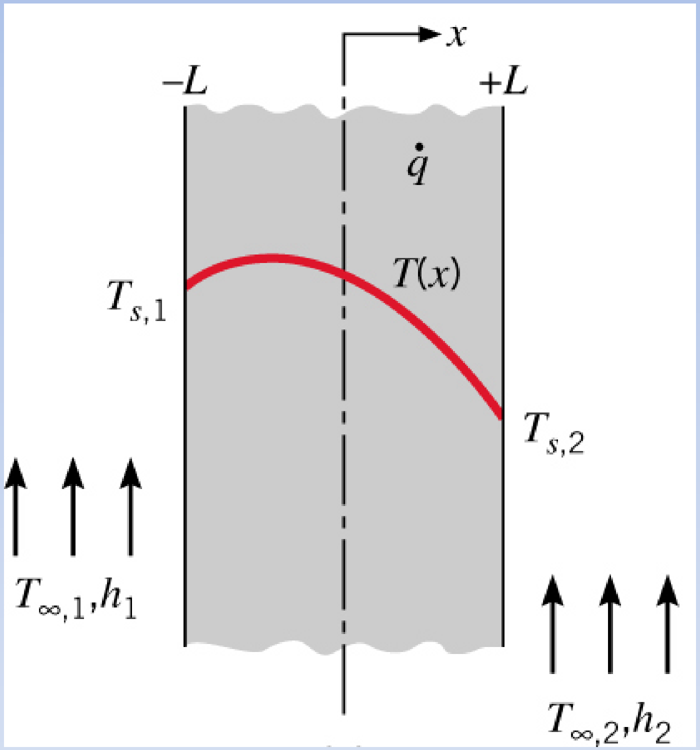
$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

- General Solution after taking double integration:

$$T = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

- Boundary Conditions:

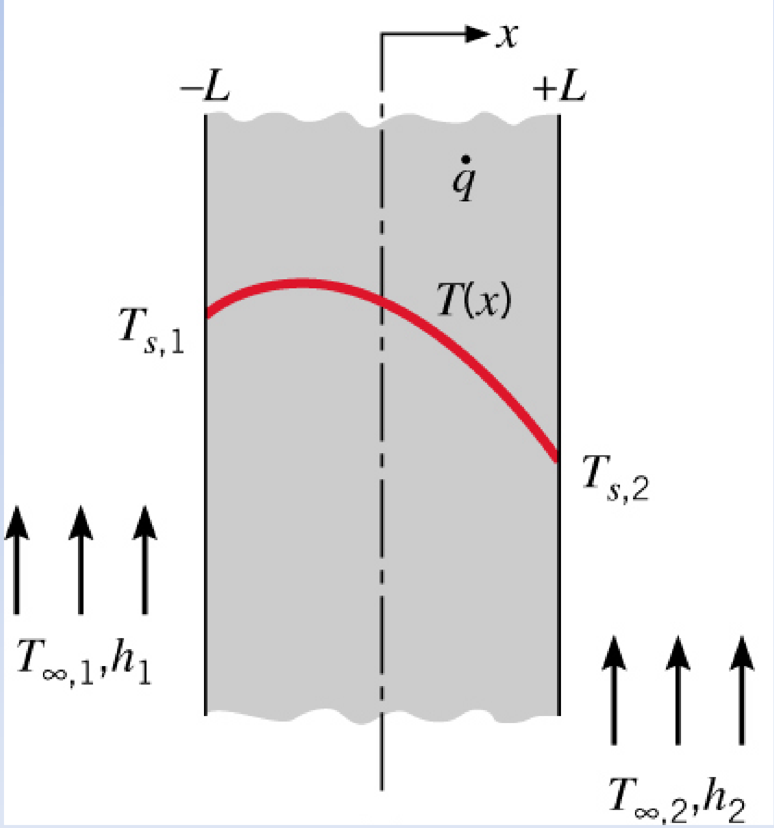
$$T(-L) = T_{s,1}, T(L) = T_{s,2}$$



Temperature Profile

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{(T_{s,2} - T_{s,1})}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$$

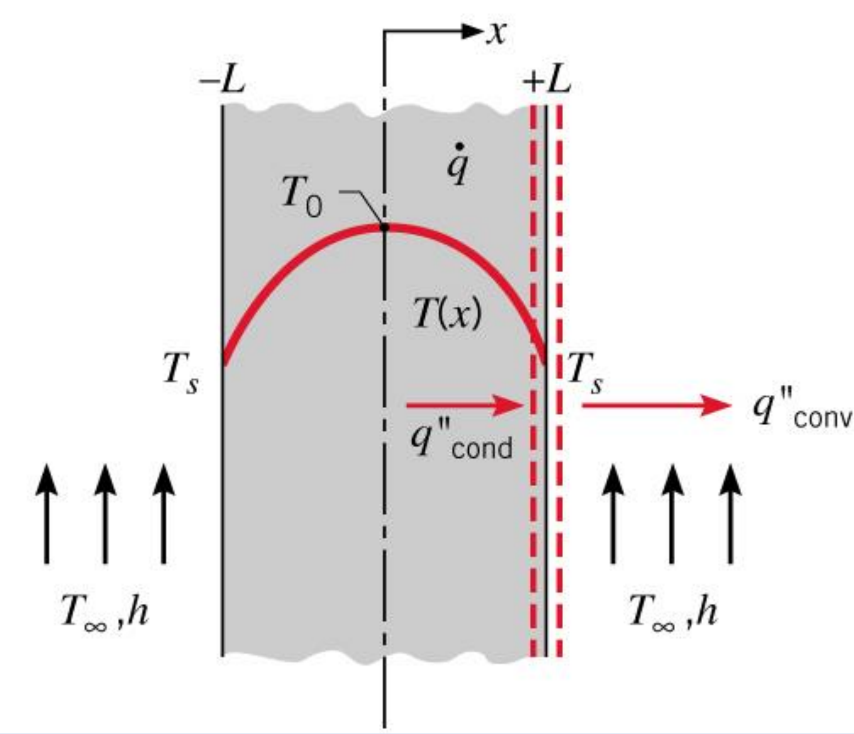
- Profile is parabolic.
- Heat flux is not independent of x



Symmetrical Distribution

- When both surfaces are maintained at a common temperature, $T_{s,1} = T_{s,2} = T_s$
- Previous equation becomes

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$



? What is the location of the maximum temperature?

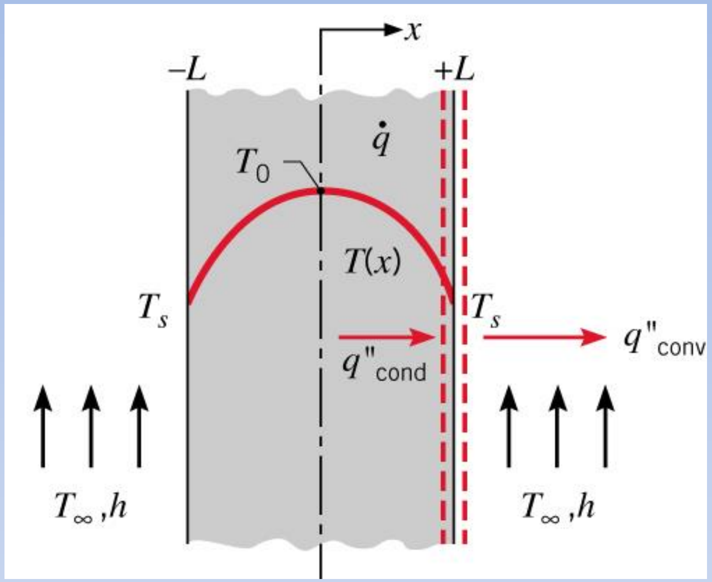
$$\therefore \frac{T(x) - T_{max}}{T_s - T_{max}} = \left(\frac{x}{L}\right)^2$$

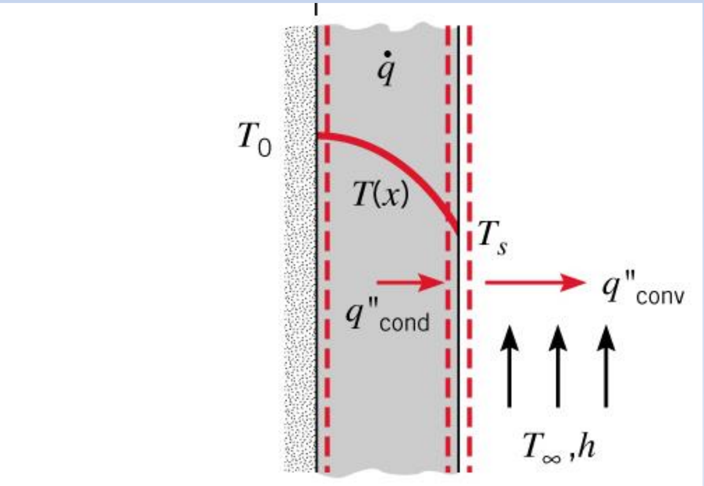
Symmetrical Distribution

- Note that at the plane of symmetry:

$$\left(\frac{dT}{dx}\right)_{x=0} = 0 \quad \Rightarrow q''\big|_{x=0} = 0$$

- Equivalent to adiabatic surface





Calculation of surface temperature T_s

In previous two equations the surface temperature, T_s is needed.

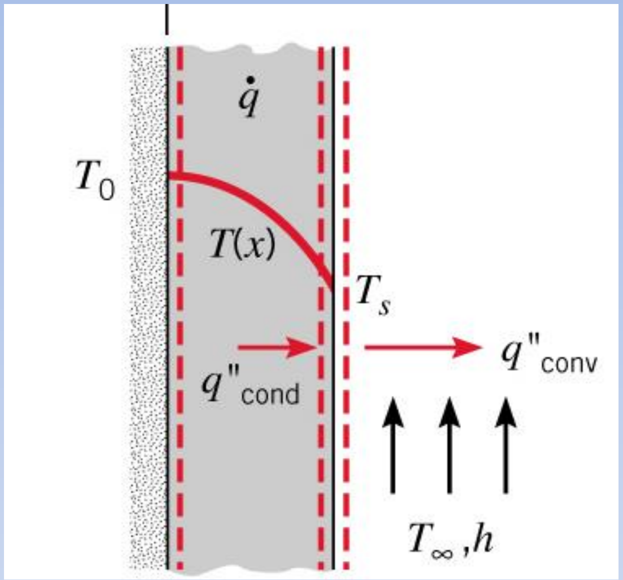
- Boundary condition at the wall:

$$-k\left.\frac{dT}{dx}\right|_{x=L} = h(T_s - T_\infty)$$

Substituting $(dT/dx)_{x=L}$ from equation of temp. distribution i.e

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

$$\therefore \quad T_s = T_\infty + \frac{\dot{q}L}{h}$$



Note: You can obtain the same results by applying energy balance;

$$\dot{q}\,AL = hA(T_s - T_\infty)$$

$$T_s = \frac{\dot{q}\,L}{h} + T_\infty$$

Example 1

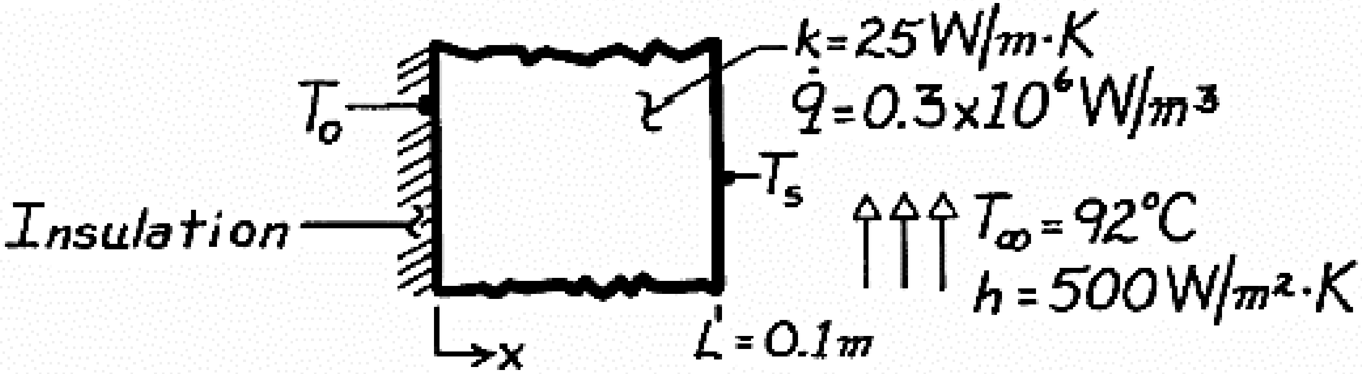
A plane wall of thickness 0.1 m and thermal conductivity 25 W/m·K having uniform volumetric heat generation of 0.3 MW/m^3 is insulated on one side, while the other side is exposed to a fluid at 92°C. The convection heat transfer coefficient between the wall and the fluid is 500 $W/m^2.K$. Determine the maximum temperature in the wall.

Solution

KNOWN: Plane wall with internal heat generation which is insulated at the inner surface and subjected to a convection process at the outer surface.

FIND: Maximum temperature in the wall.

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction with uniform volumetric heat generation, (3) Inner surface is adiabatic.

ANALYSIS: From previous equations for symmetrical wall, the max temp. is:

$T_o = \dot{q}L^2/2k + T_s.$ But T_s not given

Solution Continue

The outer surface temperature follows from

Previous equation of symmetrical

Apply Overall heat balance

$E_{in} - E_{out} + E_{gen} = E_{st}$ $\therefore E_{gen} = E_{out} = q_{con}$

$$T_s = T_\infty + \dot{q}L/h$$

$$T_s = 92^\circ C + 0.3\times10^6 \frac{W}{m^3} \times 0.1m/500W/m^2\cdot K = 92^\circ C + 60^\circ C = 152^\circ C.$$

It follows that

$$T_o = 0.3\times10^6\, W/m^3 \times (0.1m)^2 / 2\times 25 W/m\cdot K + 152^\circ C$$

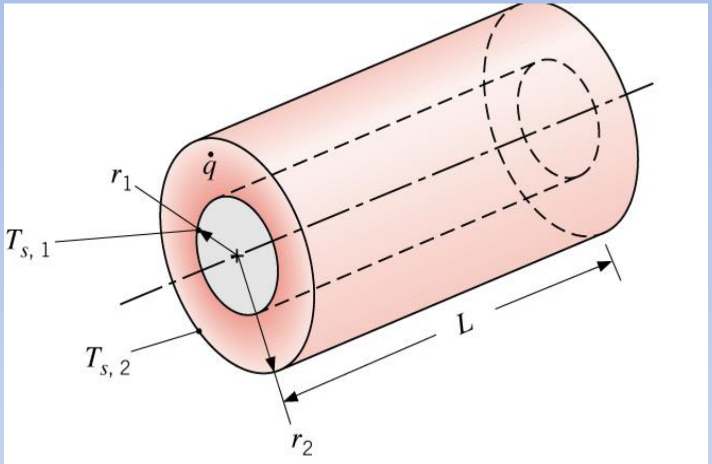
$$T_o = 60^\circ C + 152^\circ C = 212^\circ C.$$

Exercise
How can you find the heat flux leaving the wall?

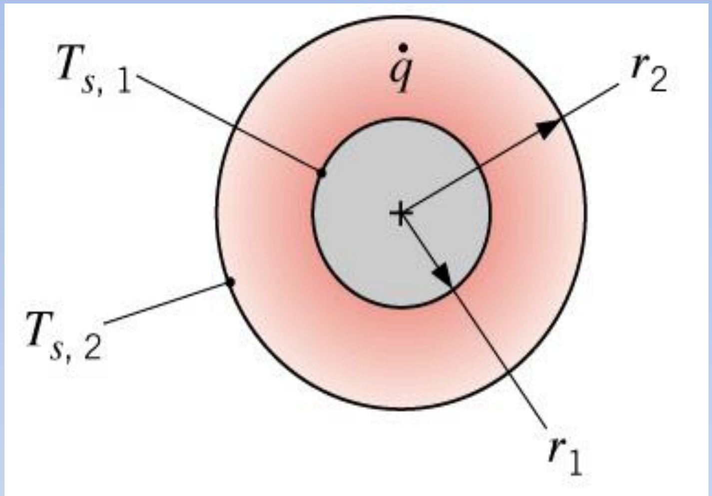
Method 1
Method 2

Radial Systems

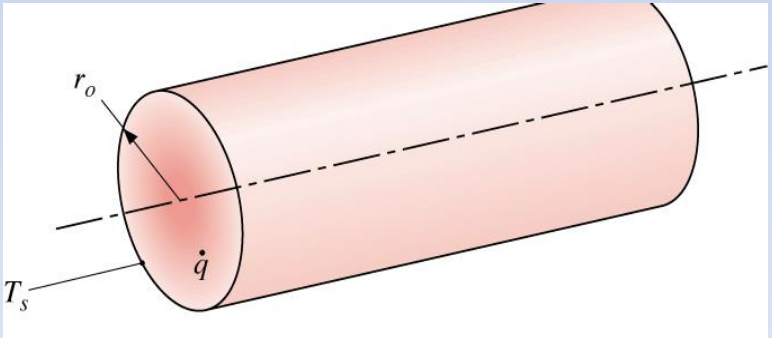
Cylindrical (Tube) Wall



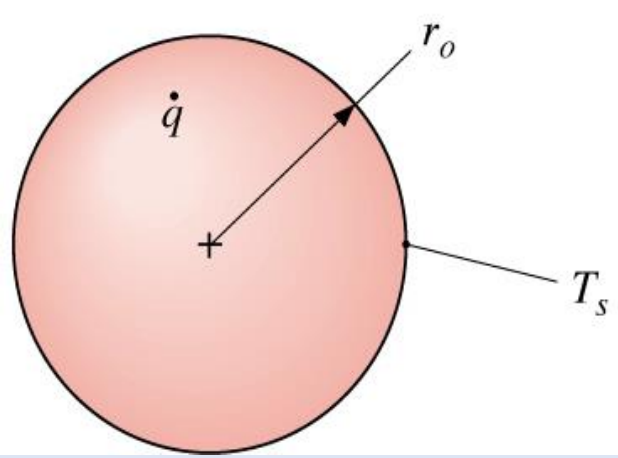
Spherical Wall (Shell)



Solid Cylinder (Circular Rod)

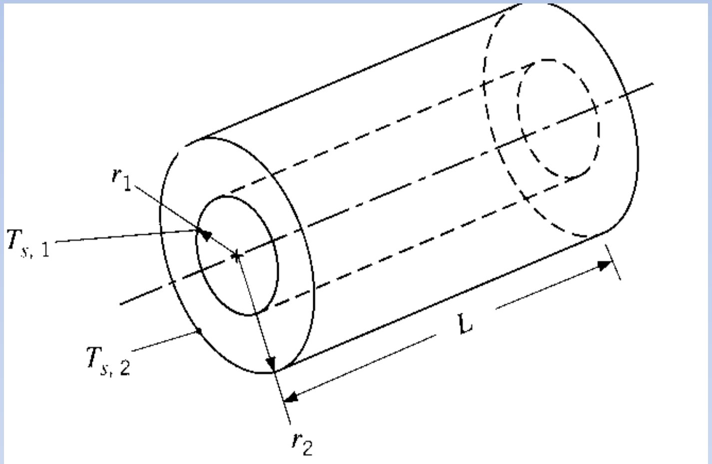


Solid Sphere



Hollow cylinder no generation only r-direction

Cylindrical (Tube) Wall



Find the temp. distribution and the heat rate.

Follow the same procedure.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0$$

To obtain temp. distribution

- Solve the previous eq. and applying suitable B.Cs.
- Assuming the value of k to be constant. Double integration gives the general solution:

$$T(r) = C_1 \ln r + C_2$$

$$T(r_1) = T_{s,1} \quad \text{and} \quad T(r_2) = T_{s,2}$$

Applying these conditions to the general solution, we then obtain

$$T_{s,1} = C_1 \ln r_1 + C_2 \quad \text{and} \quad T_{s,2} = C_1 \ln r_2 + C_2$$

Solving for C_1 and C_2 and substituting into the general solution, we then obtain

$$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}$$

Apply Fourier's Law

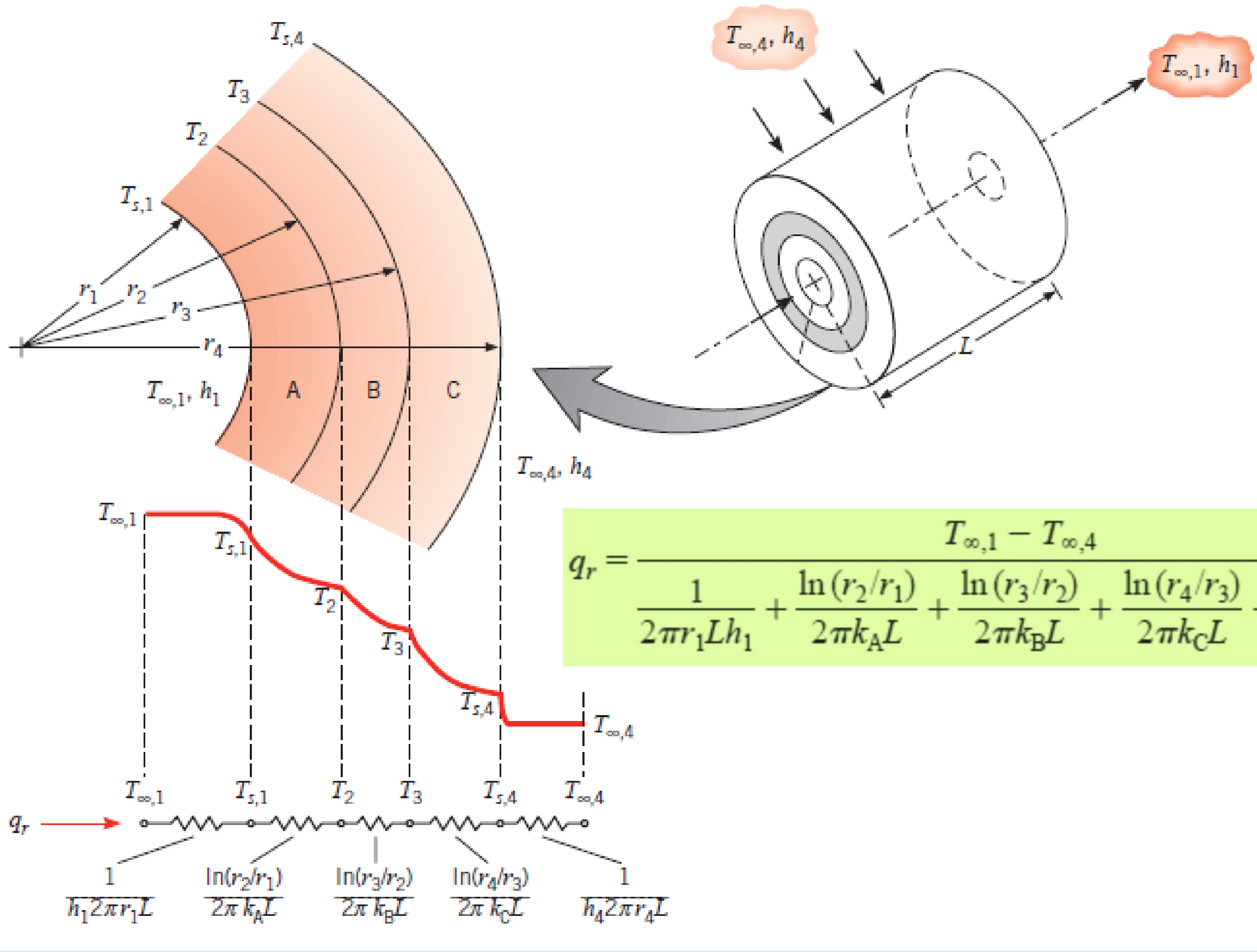
$$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr}$$

$$q_r = \frac{2\pi Lk(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)}$$

The thermal resistance is

$$R_{t,\mathrm{cond}} = \frac{\ln(r_2/r_1)}{2\pi Lk}$$

Composite cylindrical wall



$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\dfrac{1}{2\pi r_1 L h_1} + \dfrac{\ln(r_2/r_1)}{2\pi k_A L} + \dfrac{\ln(r_3/r_2)}{2\pi k_B L} + \dfrac{\ln(r_4/r_3)}{2\pi k_C L} + \dfrac{1}{2\pi r_4 L h_4}}$$

Note

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{2\pi r_1 L h_1} + \frac{\ln(r_2/r_1)}{2\pi k_A L} + \frac{\ln(r_3/r_2)}{2\pi k_B L} + \frac{\ln(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}} \quad \text{(a)}$$

Overall heat transfer coefficient, U, is obtained as follows

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{tot}} = UA(T_{\infty,1} - T_{\infty,4}) \quad \text{(b)}$$

If U is defined in terms of the inside area, $A_1 = 2\pi r_1 L$, the above eqs. (a) and (b) yield

$$U_1 = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A}\ln\frac{r_2}{r_1} + \frac{r_1}{k_B}\ln\frac{r_3}{r_2} + \frac{r_1}{k_C}\ln\frac{r_4}{r_3} + \frac{r_1}{r_4}\frac{1}{h_4}}$$

Look!!

This definition is *arbitrary,* and the overall coefficient may also be defined in terms of the outside surface area, A_4 or any of the intermediate areas. Note that

$$U_1A_1 = U_2A_2 = U_3A_3 = U_4A_4 = (\Sigma R_t)^{-1}$$

and the specific forms of U_2, U_3, and U_4 may be inferred from Equations (a) and (b).

Radial Systems

- Heat diffusion equation in the r-direction for steady-state conditions:

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)+\dot{q}=0$$

- Assume k = constant
- General Solution:

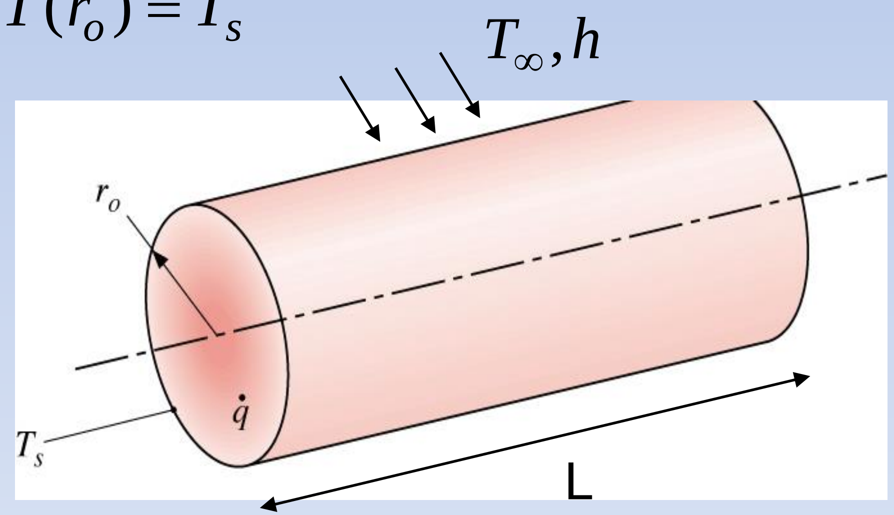
$$T=-\frac{\dot{q}}{4k}r^2+C_1\ln r+C_2$$

- Boundary Conditions:

$$\left.\frac{dT}{dr}\right|_{r=0}=0,\quad T(r_o)=T_s$$

- Temperature profile:

$$T(r)=\frac{\dot{q}r_o^2}{4k}\left(1-\frac{r^2}{r_o^2}\right)+T_s \qquad (3.6)$$



- Calculation of surface temperature:

$$\dot{q}(\pi r_o^2 L)=h(2\pi r_o L)(T_s-T_\infty)$$ and $$T_s=T_\infty+\frac{\dot{q}r_o}{2h} \qquad (3.7)$$

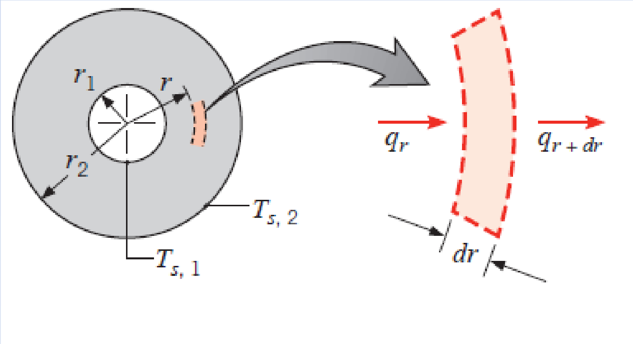
For Spherical shape

- Fourier's law, for steady-state, one-dimensional conditions with no heat generation

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$

- where $A = 4\pi r^2$ is the area normal to the direction of heat transfer

$$\frac{q_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{T_{s,1}}^{T_{s,2}} k(T)\,dT$$



- Assuming k = cons.

$$q_r = \frac{4\pi k(T_{s,1} - T_{s,2})}{(1/r_1) - (1/r_2)}$$

- Thermal resistance

$$R_{t,\text{cond}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

- For temp. distribution follow the common procedure as explained before.

Summary

One-dimensional, steady-state solutions to the heat equation with no generation

	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\mathrm{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$