TRANSIENT CONDUCTION

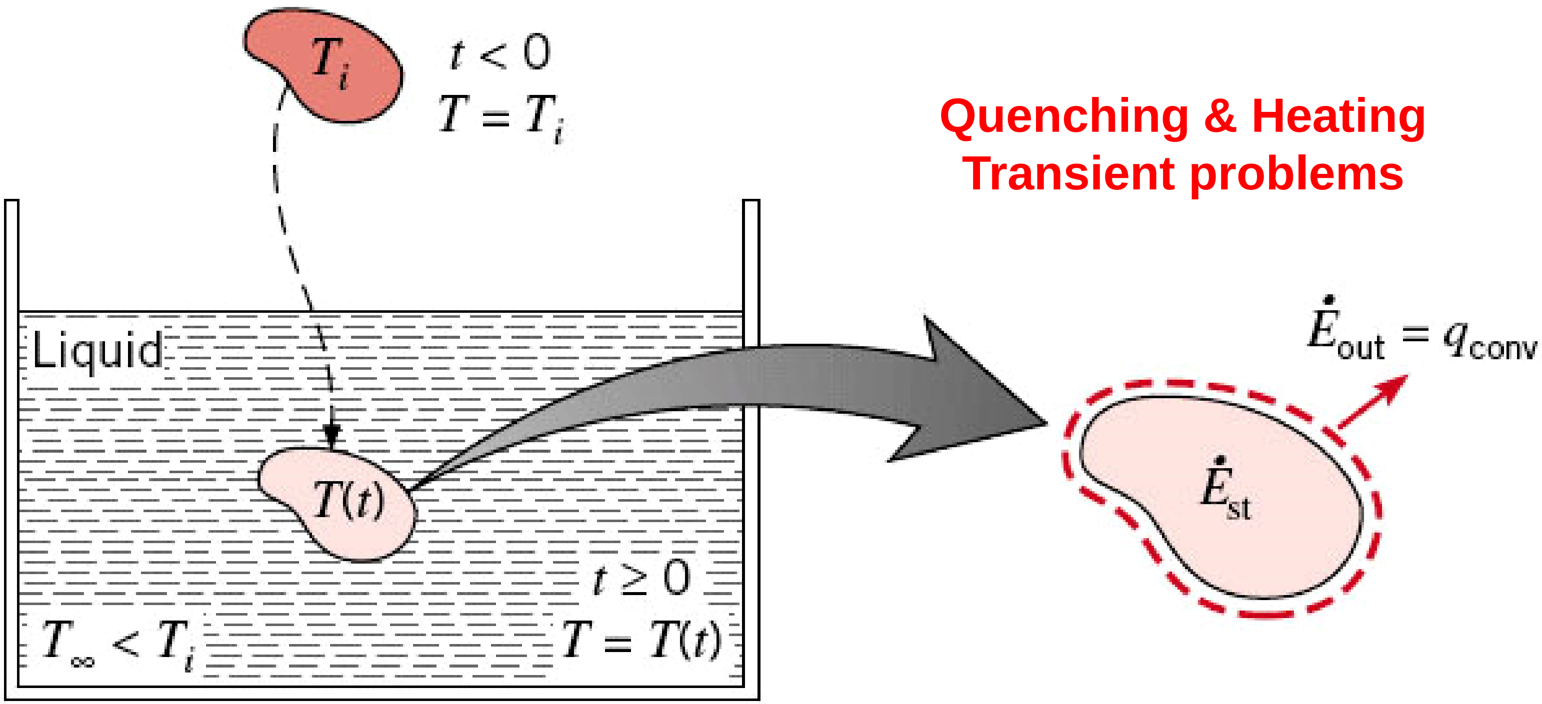
The lumped capacitance Method

INTRODUCTION

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

This term accounts the variation of temperature with time for unsteady state problems

Examples

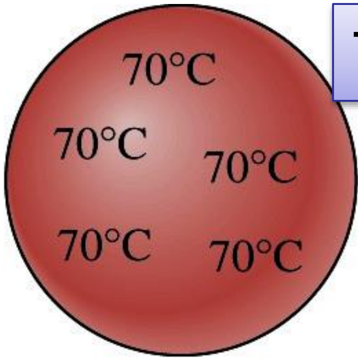


Quenching & Heating Transient problems

Cooling of a hot metal forging.

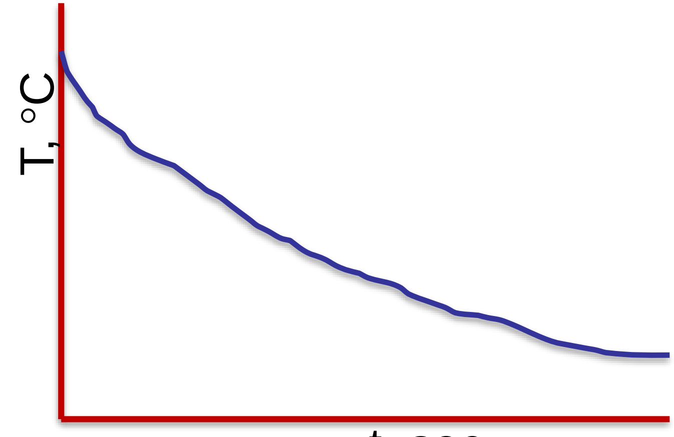
Could be a small copper ball

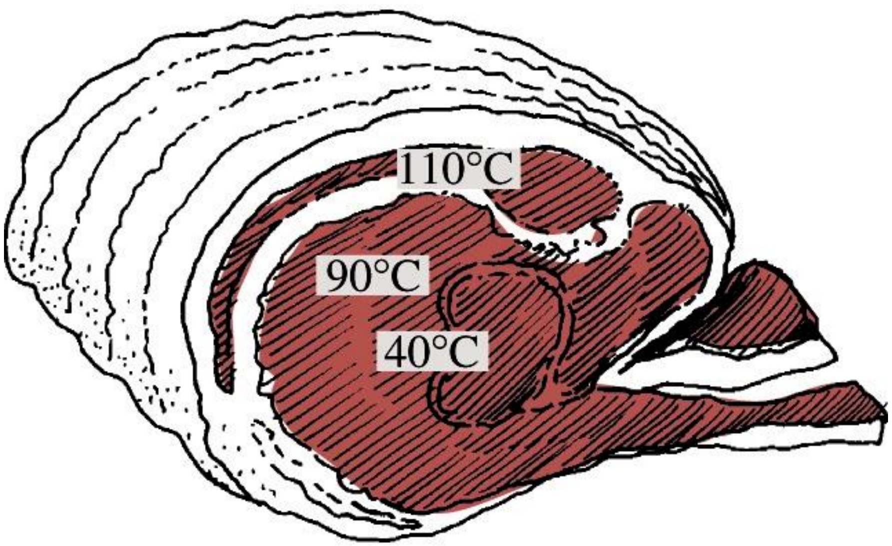
Examples and comparison



(a) Copper ball

$T=f(t), T \neq f(dis)$





(b) Roast beef

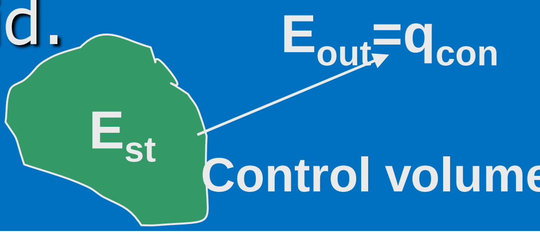
$T=f(t, dis)$

Analysis and general formulation

- Basic assumption:

The lumped capacitance method assumes that the temperature of solid is spatially uniform at any instant of time. This means negligible temp gradients within the solid.

Energy balance: $E_{in} - E_{out} = E_{st}$ (E_{in} = 0)



$$-\dot{E}_{out} = \dot{E}_{st}$$

or

$$-hA_s(T - T_\infty) = \rho V c \frac{dT}{dt} \quad \text{.................(1)}$$

Introducing the temperature difference

$$\theta = T - T_\infty \quad \text{.................(2)}$$

and recognizing that $(d\theta/dt) = (dT/dt)$ if T_∞ is constant, it follows that

$$\frac{\rho V c}{h A_s}\frac{d\theta}{dt} = -\theta \qquad \text{……………………..(3)}$$

Separating variables and integrating from the initial condition, for which $t = 0$ and $T(0) = T_i$, we then obtain

$$\frac{\rho V c}{h A_s}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = -\int_0^t dt \qquad \text{……………………..(4)}$$

where

$$\theta_i \equiv T_i - T_\infty \qquad \text{……………………..(5)}$$

Evaluating the integrals, it follows that

$$\frac{\rho Vc}{hA_s}\ln\frac{\theta_i}{\theta} = t \quad \text{.......................(6)}$$

Or

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho Vc}\right)t\right] \quad \text{.....................(7)}$$

Note

The temp. of a lumped system approaches the environment temp. as time gets larger. This is can be checked by the previous exponential equation (eq.7).

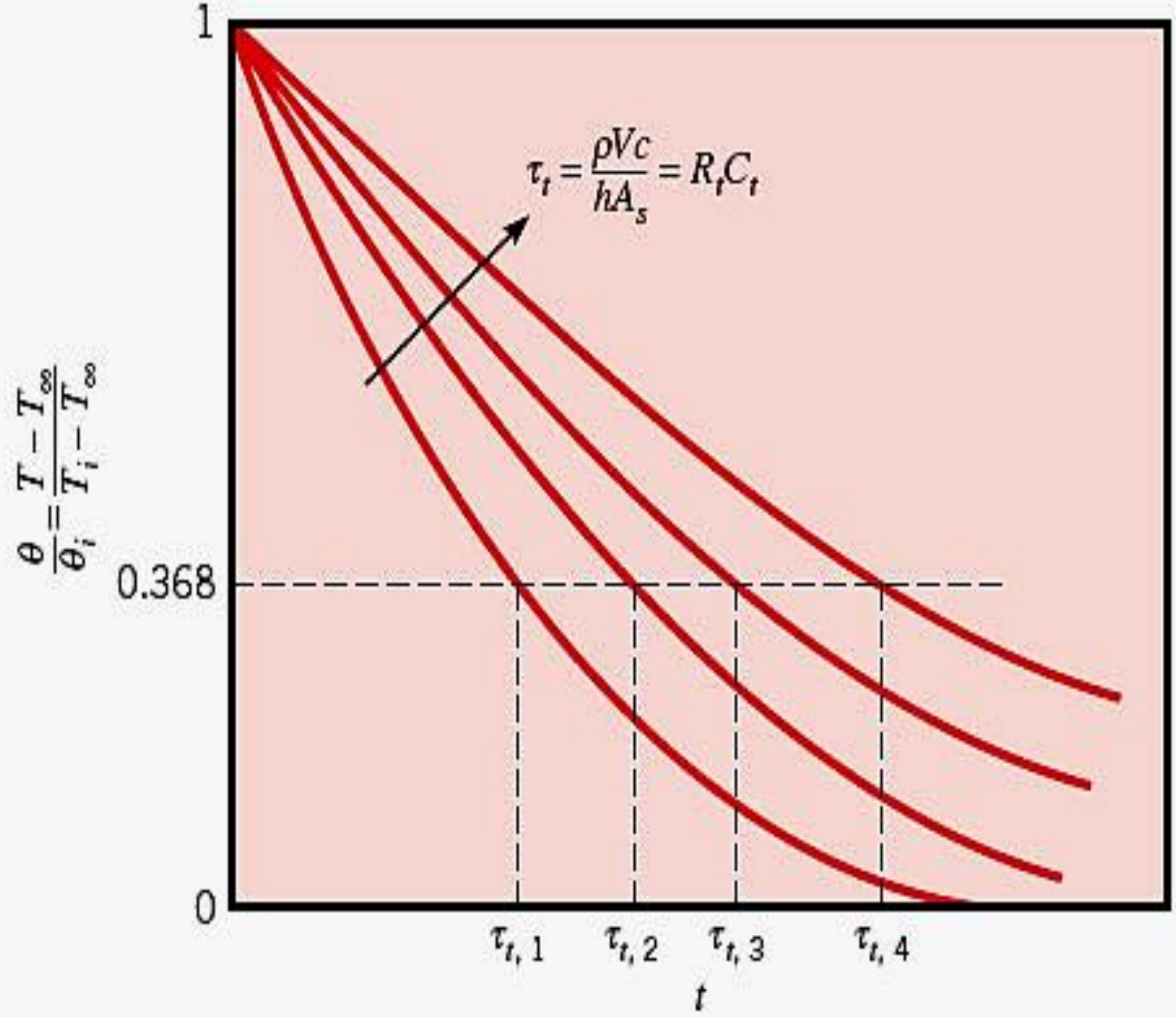
Thermal time constant,

$$\tau_t = \rho V c / h A_s \qquad \text{……………………..(8)}$$

Or

$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho V c) = R_t C_t \qquad \text{……………………..(9)}$$

Where R_t is the resistance to convection heat transfer and C_t is the lumped thermal capacitance of the solid.



Transient temperature response of lumped capacitance solids for different thermal time constants τ_t.

Total energy transfer Q

- The Total energy transfer Q occurring up to some time t can be obtained from:

$$Q = \int_0^t q\, dt = hA_s \int_0^t \theta\, dt \qquad \text{……………………..(10)}$$

Use the eq. (7) of θ

$$\frac{\theta}{\theta_i} = \exp\left[-\left(\frac{hA_s}{\rho V c} \right) t \right] \qquad \text{……………………..(7)}$$

Integrating the previous equation yields

$$Q = (\rho V c)\theta_i \left[1 - \exp\left(-\frac{t}{\tau_t}\right)\right] \quad \text{.....................(11)}$$

The quantity Q is, of course, related to the change in the internal energy of the solid,

Energy balance

$E_{in}^{\,0} - E_{out} = \Delta E_{st}$

Q +ve

Hot body

$$-Q = \Delta E_{st} \quad \text{..................(12)}$$

For quenching Q is positive and the solid experiences a decrease in energy. Equa-

For heating Q is negative and the solid experiences an increase in the internal energy of the solid. $(\theta < 0)$,

Heating

$T_\infty > T$

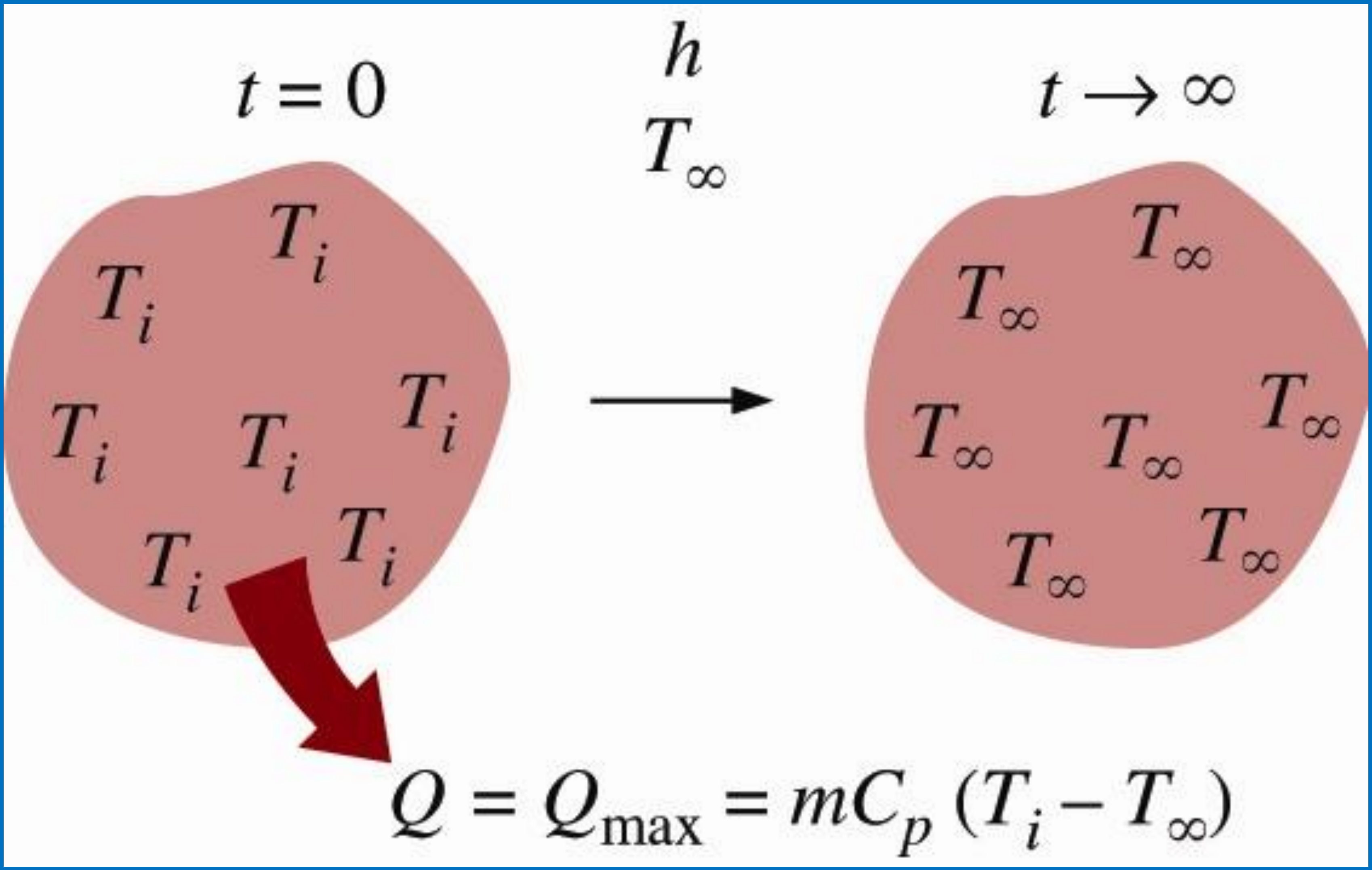
$\theta = T - T_\infty$

$\theta = -ve$

cold body

Q -ve

Maximum heat quantity



Validity of the Lumped Capacitance Method

To develop a suitable criterion consider steady-state conduction through the plane wall of area A (Fig. A). Although we are assuming steady-state conditions, this criterion is readily extended to transient processes. One surface is maintained at a temperature $T_{s,1}$ and the other surface is exposed to a fluid of temperature $T_\infty < T_{s,1}$. The temperature of this surface will be some intermediate value, $T_{s,2}$, for which $T_\infty < T_{s,2} < T_{s,1}$. Hence under steady-state conditions the surface energy balance, is

$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)$$

Or

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{cond}}{R_{conv}} = \frac{hL}{k} \equiv Bi$$

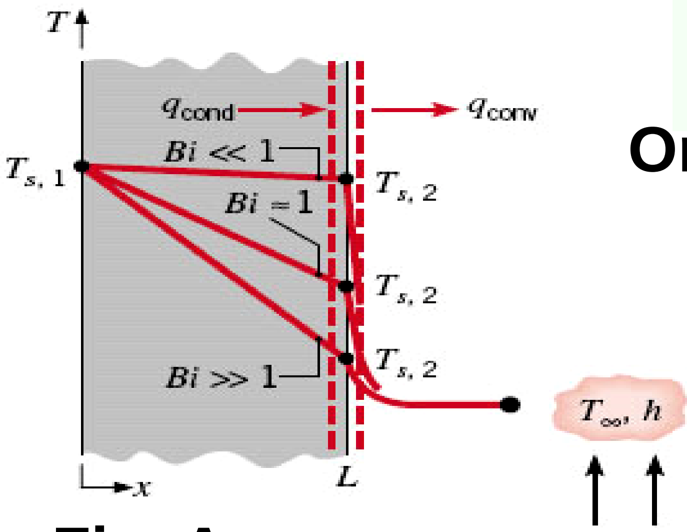


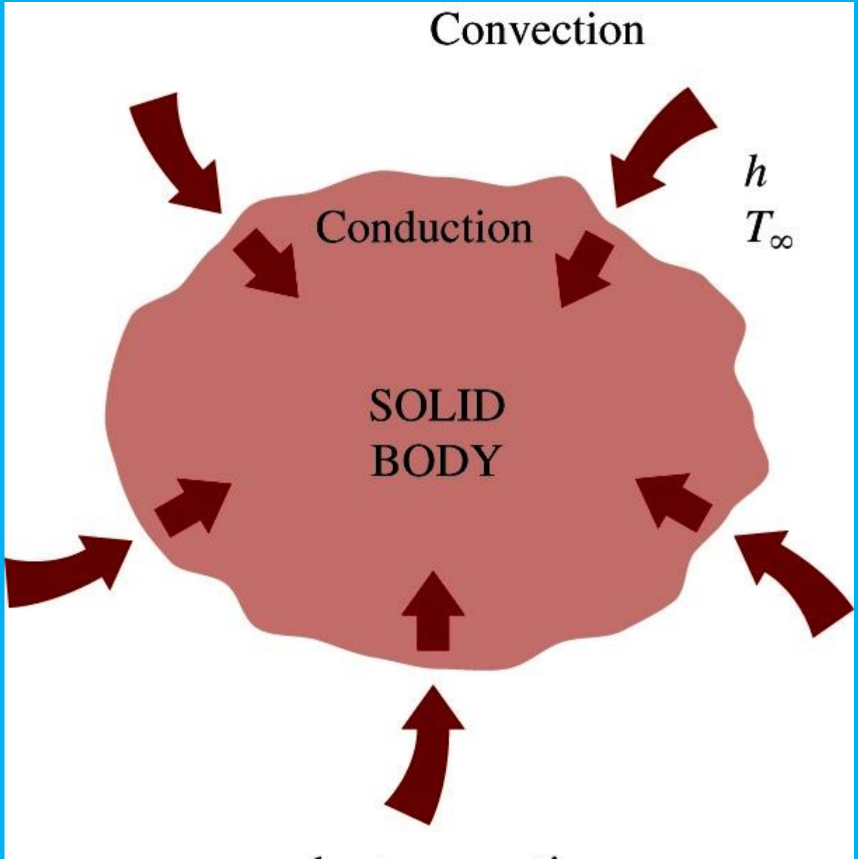
Fig. A

Effect of Biot number on steady-state temperature distribution in a plane wall with surface convection.

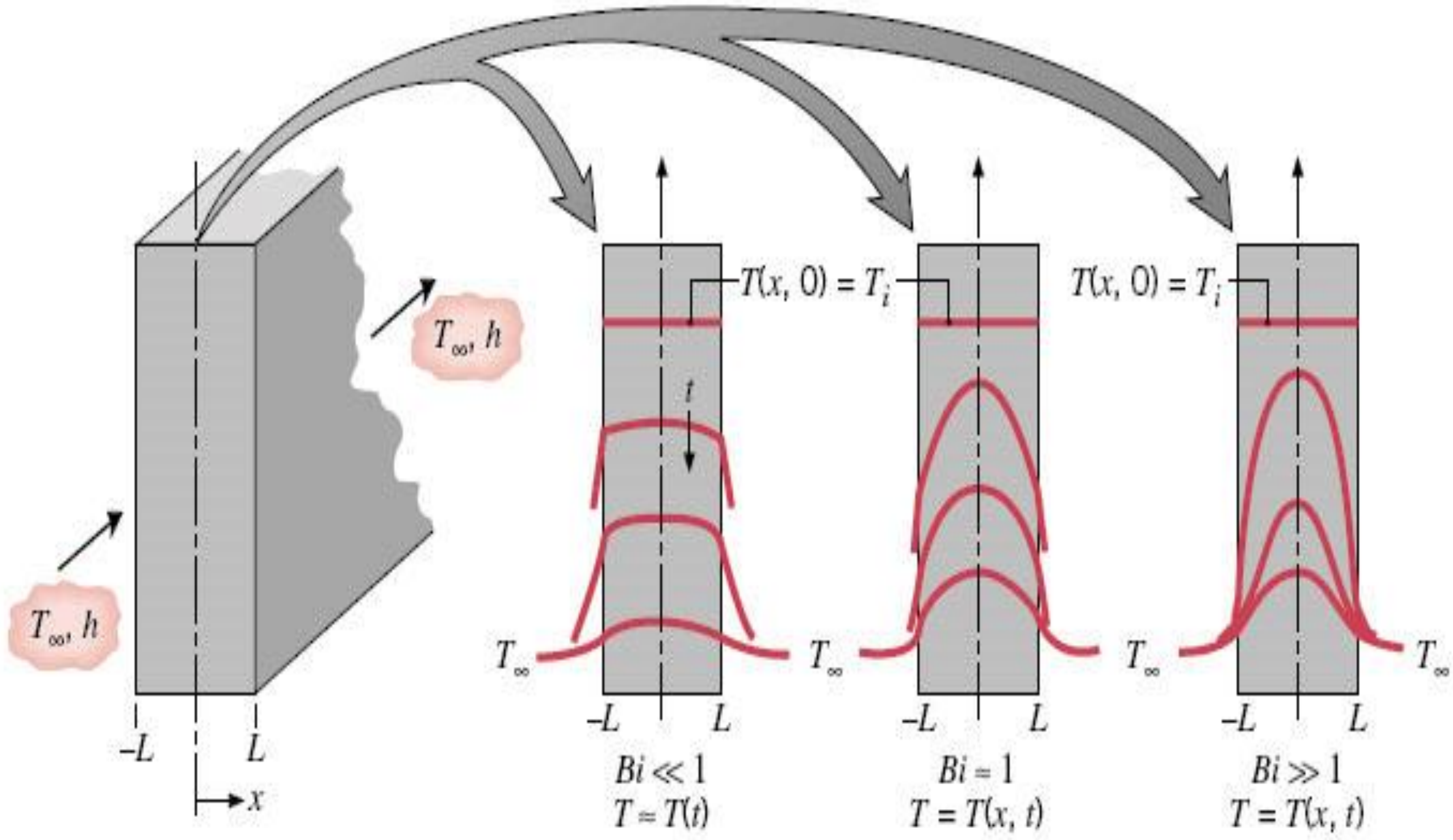
Notes

- Biot number plays a fundamental role in conduction problems that involve surface convection effect.
- Biot number provides a measure of the temperature drop in the solid relative to the temperature difference between the solid's surface and the fluid.
- Biot number may be interpreted as a ratio of thermal resistances.
- *if* $Bi \ll 1$, *the resistance to conduction within the solid is much less than the resistance to convection across the fluid boundary layer. Hence, the assumption of a uniform temperature distribution within the solid is reasonable if the Biot number is small.*

The Biot number can be viewed as the ratio of the convection at the surface to conduction within the body



$$\mathrm{Bi} = \frac{\text{heat convection}}{\text{heat conduction}}$$



Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection.

Conclusion

- If the following condition is satisfied

$$Bi = \frac{hL_c}{k} < 0.1 \quad \text{.......................(13)}$$

the error associated with using the lumped capacitance method is small.

Where L_c is the characteristic length. It is defined as the ratio of the solid's volume to surface area,

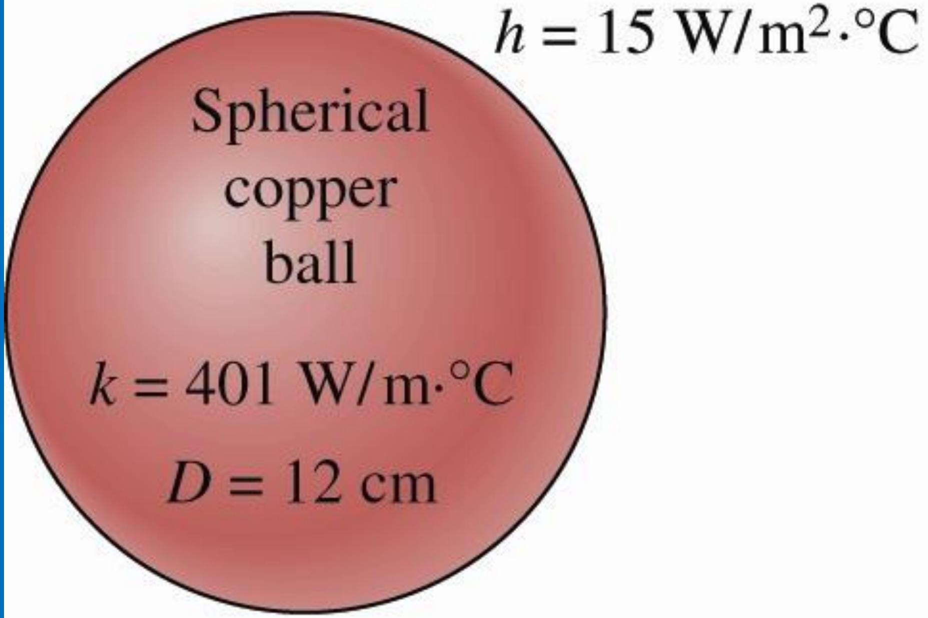
$$L_c = V/A_s$$

<u>Note</u>

For symmetrical heated or cooled plane wall of thickness 2L, $L_c = L$.

For a long cylinder $L_c = r_o/2$ and for a sphere, $L_c = r_o/3$

Example: small bodies with high thermal conductivities and low convection coefficients are most likely to satisfy the criterion for lumped system analysis



$$L_c = \frac{V}{A_s} = \frac{\frac{1}{6}\pi D^3}{\pi D^2} = \frac{1}{6}D = 0.02\ \text{m}$$

$$\text{Bi} = \frac{hL_c}{k} = \frac{15 \times 0.02}{401} = 0.00075 < 0.1$$

- Using $L_c=V/A_s$, the exponent of the eq. 7 may be written as

$$\frac{\theta}{\theta_i}=\frac{T-T_\infty}{T_i-T_\infty}=\exp\left[-\left(\frac{hA_s}{\rho Vc}\right)t\right]$$

$$\frac{hA_st}{\rho Vc}=\frac{ht}{\rho cL_c}=\frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2}=\frac{hL_c}{k}\frac{\alpha t}{L_c^2} \qquad \text{..................(14)}$$

or

$$\frac{hA_st}{\rho Vc}=Bi\cdot Fo$$

where

$$Fo\equiv\frac{\alpha t}{L_c^2}$$

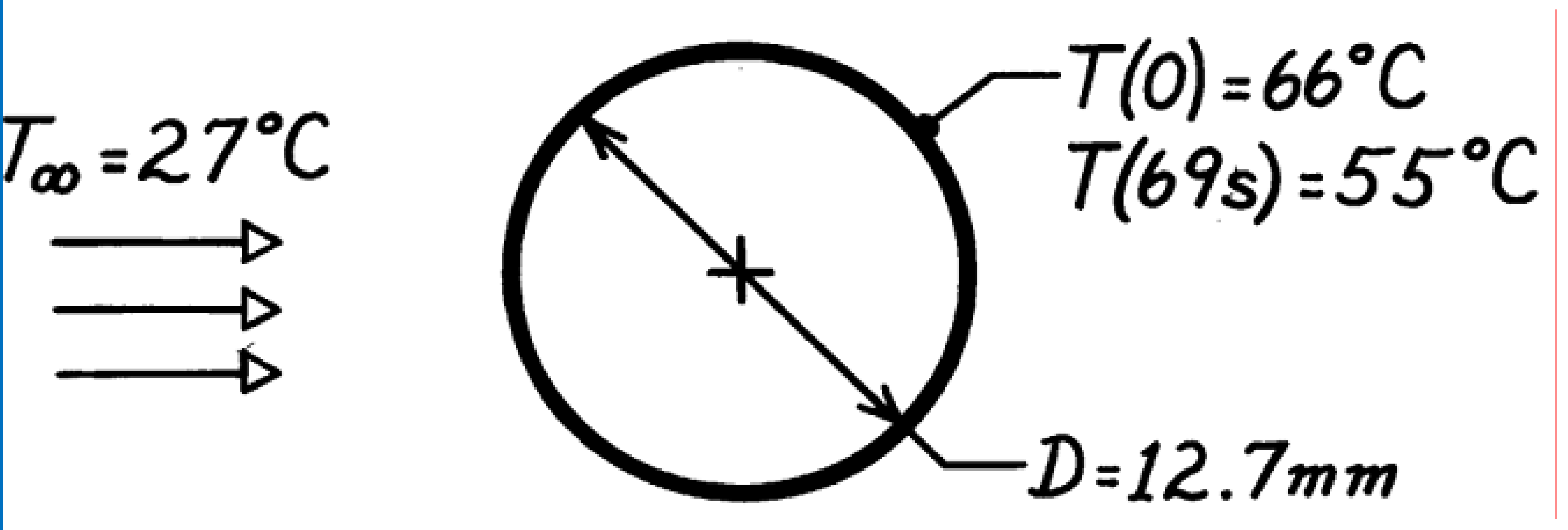
is termed the Fourier number. It is a *dimensionless time*, which, with the Biot number, characterizes transient conduction problems. Substituting the previous eq. in the original eq. (7), we obtain

$$\frac{\theta}{\theta_i}=\frac{T-T_\infty}{T_i-T_\infty}=\exp(-Bi\cdot Fo) \qquad \text{......................(15)}$$

Example

The heat transfer coefficient for air flowing over a sphere is to be determined by observing the temperature–time history of a sphere fabricated from pure copper. The sphere, which is 12.7 mm in diameter, is at 66°C before it is inserted into an airstream having a temperature of 27°C. A thermocouple on the outer surface of the sphere indicates 55°C 69 s after the sphere is inserted in the airstream. Assume, and then justify, that the sphere behaves as a spacewise isothermal object and calculate the heat transfer coefficient.

Schematic



Steps of calculation

1. Find from appendices the properties of copper (density ρ, heat capacity C and thermal conductivity k) at average temperature 333.5 K.
2. Assume Lumped Capacitance
3. Apply the Eq. 7
4. Find h
5. Check Bi number