Transient Conduction: Spatial Effects and the Role of Analytical Solutions

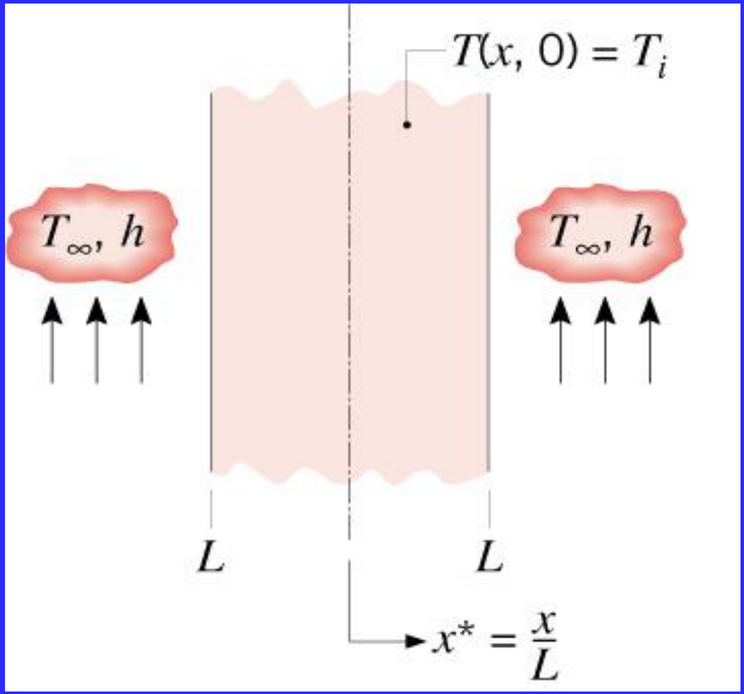
Solution to the Heat Equation for a Plane Wall with Symmetrical Convection Conditions

- If the lumped capacitance approximation can not be made, consideration must be given to spatial, as well as temporal, variations in temperature during the transient process.
  - For a plane wall with symmetrical convection conditions and constant properties, the **heat equation** and **initial/boundary** conditions are:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (5.26)$$

$$T(x,0) = T_i \qquad (5.27)$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \qquad (5.28)$$



$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = h\left[T(L,t) - T_{\infty}\right] \qquad (5.29)$$

Existence of seven independent variables:

$$T = T\left(x, t, T_i, T_{\infty}, k, \alpha, h\right) \qquad (5.30)$$

How may the functional dependence be simplified?

Non-dimensionalization of Heat Equation and Initial/Boundary Conditions:

Dimensionless temperature difference: $\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_{\infty}}{T_i - T_{\infty}}$

Dimensionless coordinate:

$$x^* \equiv \frac{x}{L}$$

Dimensionless time:

$$t^* \equiv \frac{\alpha t}{L^2} \equiv Fo$$

$Fo \rightarrow$ the Fourier Number

The Biot Number:

$$Bi \equiv \frac{hL}{k_{solid}}$$

$$\theta^* = f\left(x^*, Fo, Bi\right)$$

Substituting the definition of Eq.s 5.31 through 5.33 into Eq.s 5.26 through 5.29 the heat equation becomes

$$\frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial Fo} \tag{5.34}$$

and the initial and boundary conditions become

$$\theta^*(x^*, 0) = 1 \tag{5.35}$$

$$\left. \frac{\partial \theta^*}{\partial x^*} \right|_{x^*=0} = 0 \tag{5.36}$$

and

$$\left. \frac{\partial \theta^*}{\partial x^*} \right|_{x^*=1} = -Bi\,\theta^*(1, t^*) \tag{5.37}$$

In dimensionless form the functional dependence may now be expressed as

$$\theta^* = f\left(x^*, Fo, Bi\right)$$

This equation implies that *for a prescribed geometry, the transient temperature distribution is a universal function of x*, Fo, and Bi. That is, the dimensionless solution has a prescribed form that does not depend* on the particular value of *T_i, T_∞, L, k, , or h.*

Exact Solution:

Consider the *plane wall of thickness 2L (Figure 5.6a). If the thickness is small relative to the* width and height of the wall, it is reasonable to assume that conduction occurs exclusively in the *x-direction.*

If the wall is initially at a uniform temperature, *T(x, 0) = Ti, and is suddenly* immersed in a fluid of $T_\infty \neq T_i$*, the resulting temperatures may be obtained by solving Equation* 5.34 subject to the conditions of Equations 5.35 through 5.37. Since the convection conditions for the surfaces at *x*=±1 are the same, the temperature distribution at any instant must be* symmetrical about the midplane (*x* = 0). An exact solution to this problem is of the form [4]*

The solution of Eq. (5.34) is an infinite Fourier series

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right)\cos\left(\zeta_n x^*\right) \qquad (5.39a)$$

$$C_n = \frac{4\sin\zeta_n}{2\zeta_n + \sin\left(2\zeta_n\right)} \qquad \zeta_n \tan\zeta_n = Bi \qquad (5.39b,c)$$

See Appendix B.3 for first four roots (eigenvalues $\zeta_1,\ldots,\zeta_4$) of Eq. (5.39c)

Appendix B.3 for first four roots (eigenvalues $\zeta_1, \ldots\ldots, \zeta_4$) of Eq. (5.39c)

$Bi = \frac{hL}{k}$	ξ_1	ξ_2	ξ_3	ξ_4
0	0	3.1416	6.2832	9.4248
0.001	0.0316	3.1419	6.2833	9.4249
0.002	0.0447	3.1422	6.2835	9.4250
0.004	0.0632	3.1429	6.2838	9.4252
0.006	0.0774	3.1435	6.2841	9.4254
0.008	0.0893	3.1441	6.2845	9.4256
0.01	0.0998	3.1448	6.2848	9.4258

$\xi_n \tan \xi_n = Bi$, *for Transient Conduction in a Plane Wall*

Approximate Solution

It was shown that for Fo > 0.2 the infinite series solution eq. 5.39a can be approximated by the 1st term of the series

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x^*) \qquad (5.40a)$$

or

$$\theta^* = \theta_o^* \cos(\zeta_1 x^*) \qquad (5.40b)$$

where $\theta_o^* \equiv (T_o - T_\infty)/(T_i - T_\infty)$ represents the midplane ($x^* = 0$) temperature

$$\theta_o^* = C_1 \exp(-\zeta_1^2 Fo) \qquad (5.41)$$

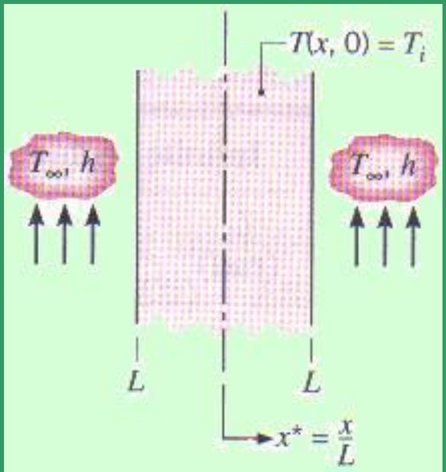
C_1 and ξ_1 are given in Table 5.1 for a range of Biot numbers.

Bi=hL/k for Plane wall and hr_o/k for the infinite cylinder and sphere

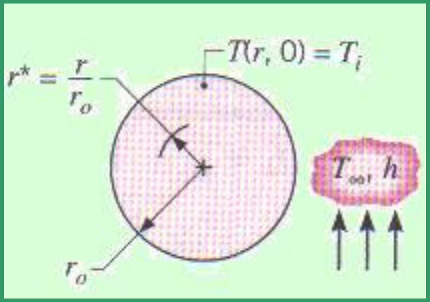
TABLE 5.1 Coefficients used in the one-term approximation to the series solutions for transient one-dimensional conduction

	Plane Wall		Infinite Cylinder		Sphere	
Bi^a	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.03	0.1723	1.0049	0.2440	1.0075	0.2991	1.0090
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.05	0.2218	1.0082	0.3143	1.0124	0.3854	1.0149
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.07	0.2615	1.0114	0.3709	1.0173	0.4551	1.0209
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.09	0.2956	1.0145	0.4195	1.0222	0.5150	1.0268
0.10	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.15	0.3779	1.0237	0.5376	1.0365	0.6609	1.0445
0.20	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.25	0.4801	1.0382	0.6856	1.0598	0.8447	1.0737
0.30	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0932	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0919	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5994	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2402	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	1.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2881	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5708	1.2733	2.4050	1.6018	3.1415	2.0000

[a] $Bi = hL/k$ for the plane wall and hr_o/k for the infinite cylinder and sphere. See Figure 5.6.



Plane wall



Infinite Cylinder Or sphere

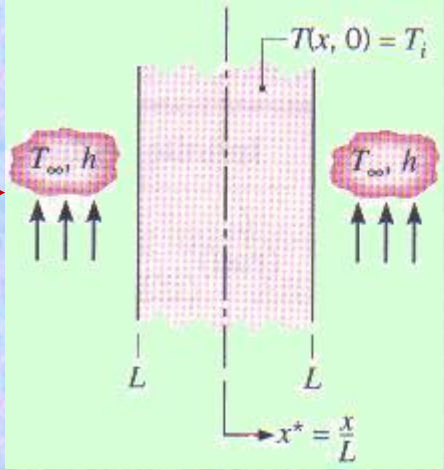
1-D system with an initial uniform temp subjected to sudden convection condition

Total Energy transfer Q "left" or "entered" the wall up to any time t in transient process

- Energy equation cab be applied over a time interval t=0 to any time t>0

$$E_{in} - E_{out} = \Delta E_{st} \qquad (5.42)$$

Zero =Q Look!



$\Delta E_{st} = E(t) - E(0)$

Eq. 5.42 becomes

$Q = -[E(t) - E(0)]$

Or

$$Q = -\int \rho C \, [T(x,t) - T_i] \, dV \qquad (5.43)$$

It is convenient to nondimensionalize the result of integration by adopting this quantity

$$Q_0 = \rho C V (T_i - T_\infty) \qquad (5.44)$$

which may be interpreted as the initial internal energy of the wall relative to the fluid temperature. It is also the *maximum* amount of energy transfer that could occur if the process were continued to time $t = \infty$. Hence, assuming constant properties, the ratio of the total energy transferred from the wall over the time interval t to the maximum possible transfer is

$-[(T - T_\infty) - (T_i - T_\infty)]/\theta_i$

$-[\theta - \theta_i]/\theta_i = -(\theta^* - 1)$

$$\frac{Q}{Q_o} = \int \frac{-[T(x,t) - T_i]}{T_i - T_\infty} \frac{dV}{V} = \frac{1}{V} \int (1 - \theta^*) dV \qquad (5.45)$$

$\theta^* = \theta_o^* \cos(\zeta_1 x^*)$

Employing the approximate form of the temperature distribution for the plane wall, Equation 5.40b, the integration prescribed by Equation 5.45 can be performed to obtain

$$\frac{Q}{Q_o} = 1 - \frac{\sin \zeta_1}{\zeta_1} \theta_o^* \qquad (5.46)$$

where θ_o^* can be determined from Equation 5.41, using Table 5.1 for values of the coefficients C_1 and ζ_1.

Approximate Solution infinite cylinder

Infinite Cylinder The one-term approximation to Equation 5.47a is

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) J_0(\zeta_1 r^*) \tag{5.49a}$$

or

$$\theta^* = \theta_o^* J_0(\zeta_1 r^*) \tag{5.49b}$$

where θ_o^* represents the centerline temperature and is of the form

$$\theta_o^* = C_1 \exp(-\zeta_1^2 Fo) \tag{5.49c}$$

Values of the coefficients C_1 and ζ_1 have been determined and are listed in Table 5.1 for a range of Biot numbers.

Bessel Functions of the First Kind

x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960

Approximate Solution
Sphere

Sphere From Equation 5.48a, the one-term approximation is

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*) \qquad (5.50a)$$

or

$$\theta^* = \theta_o^* \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*) \qquad (5.50b)$$

where θ_o^* represents the center temperature and is of the form

$$\theta_o^* = C_1 \exp(-\zeta_1^2 Fo) \qquad (5.50c)$$

Values of the coefficients C_1 and ζ_1 have been determined and are listed in Table 5.1 for a range of Biot numbers.

Total Heat transfer

As in Section 5.5.3, an energy balance may be performed to determine the total energy transfer from the infinite cylinder or sphere over the time interval $\Delta t = t$. Substituting from the approximate solutions, Equations 5.49b and 5.50b, and introducing Q_o from Equation 5.44, the results are as follows.

Infinite Cylinder

$$\frac{Q}{Q_o} = 1 - \frac{2\theta_o^*}{\zeta_1} J_1(\zeta_1) \tag{5.51}$$

Sphere

$$\frac{Q}{Q_o} = 1 - \frac{3\theta_o^*}{\zeta_1^3} [\sin(\zeta_1) - \zeta_1 \cos(\zeta_1)] \tag{5.52}$$