Boundary Layer Similarity

The normalized Boundary Layer Equations & The Convective Heat Transfer Coefficients

With the foregoing simplification and approximations, the overall continuity equation and the x-momentum equation reduce to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Net rate of momentum from Control volume

Net viscous + pressure forces + body force

≈ 0.0

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

Also, the energy equation reduces to

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

And the species continuity equation becomes

$$u\frac{\partial C_A}{\partial x} + v\frac{\partial C_A}{\partial y} = D_{AB}\frac{\partial^2 C_A}{\partial y^2}$$

Note 1

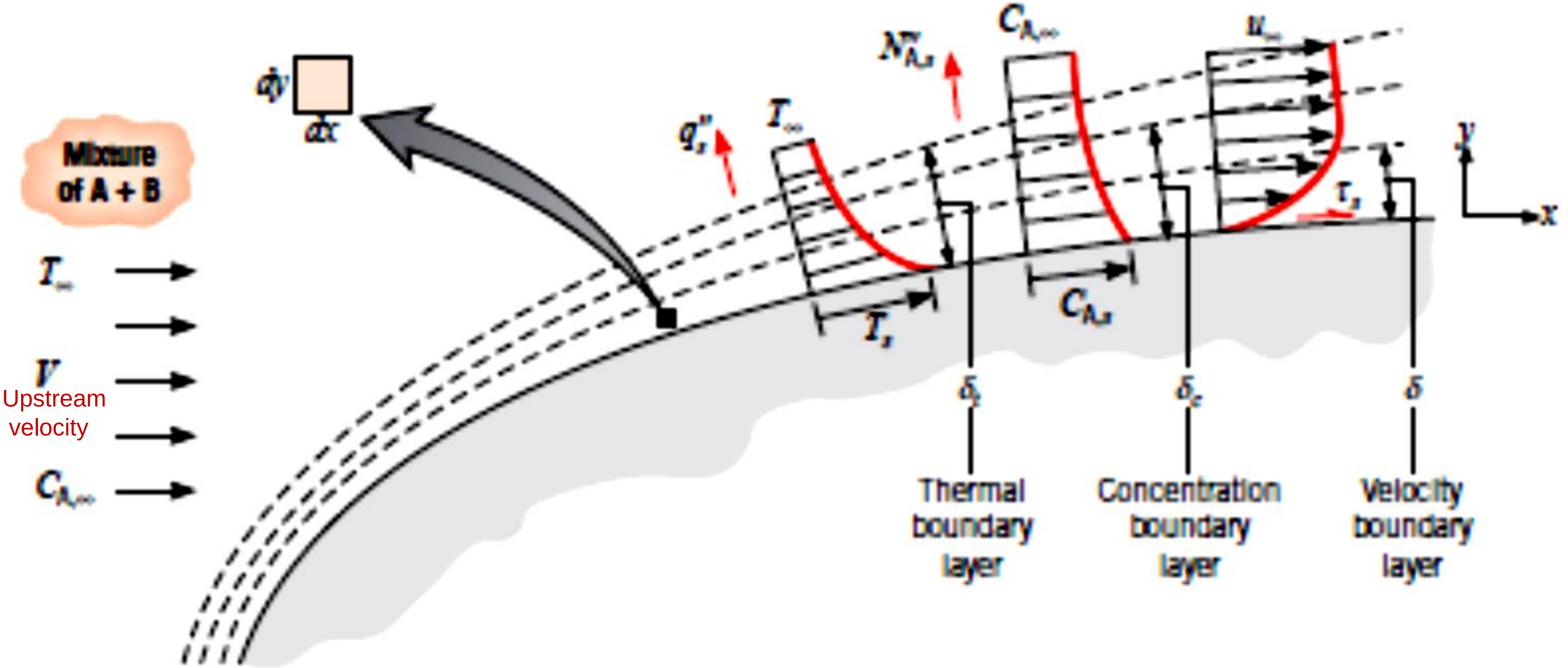
$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

the terms on the left-hand side account for the net rate at which thermal energy leaves the control volume due to bulk fluid motion (advection), while the terms on the right-hand side account for net inflow of energy due to conduction, viscous dissipation.

Note 2

$$u\frac{\partial C_A}{\partial x} + v\frac{\partial C_A}{\partial y} = D_{AB}\frac{\partial^2 C_A}{\partial y^2}$$

where C_A is the molar concentration of species A, D_{AB} is the binary diffusion coefficient. Again, this equation has been derived assuming steady, two-dimensional flow of an incompressible fluid with constant properties. Terms on the left-hand side account for net transport of species A due to bulk fluid motion (advection), while terms on the right-hand side account for net inflow due to diffusion.



Development of the velocity, thermal, and concentration boundary layers for an arbitrary surface.

Boundary Layers Similarity Parameters

- Define the following dimensionless variables:

$$x^* = \frac{x}{L} \quad , \quad y^* = \frac{y}{L} \; , \; P^* = \frac{P_\infty}{\rho V^2}$$

$$u^* = \frac{u}{V} \quad , \quad v^* = \frac{v}{V}$$

$$T^* = \frac{T - T_s}{T_\infty - T_s} \quad , \quad C_A^* = \frac{C_{\mathrm{A}} - C_{A,s}}{C_{\mathrm{A},\infty} - C_{A,s}}$$

Where L is the characteristic length of the surface, and V is the velocity upstream of the surface.

- Using the above definitions, the velocity and temperature equations become as shown in the next table. Neglect viscous dissipation term.

Similarity Parameters and the dimensionless form of the B.L. Equations

The boundary layer equations and their y-direction boundary conditions in nondimensional form

Boundary Layer	Conservation Equation	Boundary Conditions: Wall	Boundary Conditions: Free Stream	Similarity Parameter(s)
Velocity	$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{dp^*}{dx^*}+\frac{1}{Re_L}\frac{\partial^2 u^*}{\partial y^{*2}}$	$u^*(x^*,0)=0$ $v^*(x^*,0)=0$	$u^*(x^*,\infty)=\frac{u_\infty(x^*)}{V}$	$Re_L=\frac{VL}{\nu}$
Thermal	$u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}=\frac{1}{Re_L Pr}\frac{\partial^2 T^*}{\partial y^{*2}}$	$T^*(x^*,0)=0$	$T^*(x^*,\infty)=1$	$Re_L, Pr=\frac{\nu}{\alpha}$
Concentration	$u^*\frac{\partial C_A^*}{\partial x^*}+v^*\frac{\partial C_A^*}{\partial y^*}=\frac{1}{Re_L Sc}\frac{\partial^2 C_A^*}{\partial y^{*2}}$	$C_A^*(x^*,0)=0$	$C_A^*(x^*,\infty)=1$	$Re_L, Sc=\frac{\nu}{D_{AB}}$

See next slide

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp_\infty}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

Note !!!

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$\therefore$ u* = u / V or u = u* V ; x = x* L

V^2 u* du*/ Ldx*

And do the same transformation for the other terms. The final results should lead to the equations as given in the table.

- The following dimensionless parameters are given in the previous table

Re_L	Reynolds No.
Pr	Prandtl No.
Sc	Schmidt No.

- These parameters allow us to apply results obtained for a surface experiencing one set of convective conditions to geometrically similar surfaces experiencing entirely different conditions.

Functional form of the Solution

- The velocity eq. suggest the following functional forms of solution

Depends on geometry

$$u^* = f\left(x^*, y^*, \mathrm{Re}_L, \frac{dp^*}{dx^*}\right) \qquad (i)$$

$$\therefore \quad \tau_s = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \left(\frac{\mu V}{L}\right)\left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0}$$

$$\text{and} \qquad C_f = \frac{\tau_s}{\rho V^2/2} = \frac{2}{\mathrm{Re}_L}\left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} \qquad (ii)$$

From eq. (i)

$$\left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} = f\left(x^*, \mathrm{Re}_L, \frac{dp^*}{dx^*}\right)$$

Assume prescribed geometry, eq. (ii) becomes

$$C_f = \frac{2}{\mathrm{Re}_L} f\left(x^*, \mathrm{Re}_L\right) \quad \Rightarrow \text{Average value, } \overline{C}_f = \frac{2}{\mathrm{Re}_L} f\left(\mathrm{Re}_L\right)$$

- The thermal eq. suggests the following functional forms of solution

$$T^* = f\left(x^*, y^*, \mathrm{Re}_L, \mathrm{Pr}, \frac{dp^*}{dx^*}\right) \qquad \text{(i)}$$

- Where the dependence on *dp*/dx** originates from the effect of the geometry on the fluid motion (u* and *v**),which, hence, affects the thermal conditions.

$$\therefore \quad h = \frac{-k_f \, \partial T / \partial y\big|_{y=0}}{T_s - T_\infty}$$

$$= -\frac{k_f}{L}\frac{T_\infty - T_s}{T_s - T_\infty}\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0} = \frac{k_f}{L}\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0} \qquad \text{(ii)}$$

$\therefore$ equation (ii) can be rearranged as

$$\frac{hL}{k_f} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} \quad \text{this called Nusselt number}$$

$$\therefore \quad Nu = \frac{hL}{k_f} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} \quad \text{dimensionl ess temp gradient at surface}$$

For a prescribed geometry equqtion (i) becomes

$$Nu = f\,(x^*, \mathrm{Re}_L, \mathrm{Pr}) \qquad \text{(iii)}$$

Equation (iii) shows that Nu is a function of x*, Re_L, and Pr. If this function is known, hence Nu can be computed for various fluids and for various values of V and L. Consequently, the coefficient h can be found from the computed value of Nu.

Average Nusselt number

- As given before the average value for heat transfer coefficient h is evaluated by integrating over the entire surface.
- Therefore, the average coefficient is independent of the spatial variable x*.
- The functional dependence of the average Nusselt number is

$$\overline{Nu} = \frac{\overline{h}L}{k_f} = f(\mathrm{Re}_L, \mathrm{Pr})$$

Note: Physical interpretation of Prandtal number "Pr"

Since $Pr = c_p\mu/k = \nu/\alpha$

=momentum /thermal diffusivity

This number gives a measure of the relative effectiveness of momentum diffusion in the velocity B.L. and energy transport by diffusion in the thermal B.L.

For gases $Pr \approx 1.0$, this means momentum transfer=energy transfer

- For liquid metal Pr <<1 ; this means energy diffusion rate exceeds the momentum diffusion rate
- For oil Pr >>1; this means momentum diffusion rate exceeds the energy diffusion rate.
- In sum, the value of Pr number influences the relative growth rate of the velocity and thermal boundary layers.

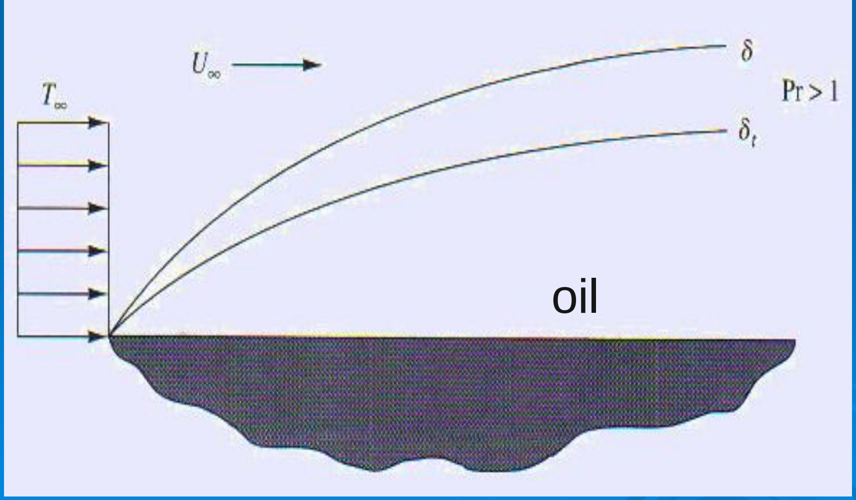
Conclusion: “In General”

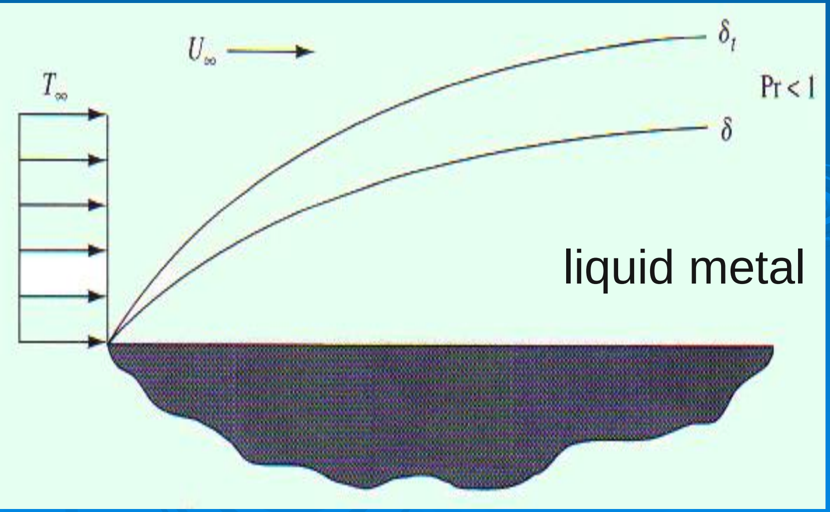
- For Laminar flow (transport by diffusion) it is reasonable to assume that

$$\frac{\delta}{\delta_t} \approx \mathrm{Pr}^n$$

where n is a positive exponent. for a gas $\delta_t \approx \delta$

for a liquid metal $\delta_t >> \delta$, for an oil $\delta_t << \delta$





Boundary Layer analogies
Heat and Mass transfer Analogy

- Definition:

  'If two or more processes are governed by dimensionless equations of the same form, the processes are said to be analogous'.

- The next table shows the analogies between Heat and Mass transfer via eq.^{s} (4 & 8), (5 & 9), (6 & 10) and (7 & 11)

Summary of the functional relations and B.L. analogies

Functional relations pertinent to the boundary layer analogies

Fluid Flow		Heat Transfer		Mass Transfer	
$u^* = f\left(x^*, y^*, Re_L, \frac{dp^*}{dx^*}\right)$	(1)	$T^* = f\left(x^*, y^*, Re_L, Pr, \frac{dp^*}{dx^*}\right)$	(4)	$C_A^* = f\left(x^*, y^*, Re_L, Sc, \frac{dp^*}{dx^*}\right)$	(8)
$C_f = \frac{2}{Re_L}\frac{\partial u^*}{\partial y^*}\Big\vert_{y^*=0}$	(2)	$Nu = \frac{hL}{k} = +\frac{\partial T^*}{\partial y^*}\Big\vert_{y^*=0}$	(5)	$Sh = \frac{h_m L}{D_{AB}} = +\frac{\partial C_A^*}{\partial y^*}\Big\vert_{y^*=0}$	(9)
$C_f = \frac{2}{Re_L} f(x^*, Re_L)$	(3)	$Nu = f(x^*, Re_L, Pr)$	(6)	$Sh = f(x^*, Re_L, Sc)$	(10)
		$\overline{Nu} = f(Re_L, Pr)$	(7)	$\overline{Sh} = f(Re_L, Sc)$	(11)

Conclusion

- If one has performed a set of heat experiments to find the functional form of equation (7), for example, the results may be used for the convective mass transfer involving the same geometry. This could be obtained by replacing Nu with Sh and Pr with Sc.
- In general, Nu and Sh are proportional to Pr^n and Sc^n , respectively.

- Use the following analogy equations:

  Nu= $f(x^*, Re_L)$ Pr^n, Sh= $f(x^*, Re_L)$ Sc^n

  in which case, with equivalent functions, $f(x^*, Re_L)$,

$$\frac{Nu}{\Pr^n} = \frac{Sh}{Sc^n}$$

$$\frac{hL/k}{\Pr^n} = \frac{h_m L / D_{AB}}{Sc^n}$$

Or

$$\frac{h}{h_m} = \frac{k}{D_{AB} Le^n} = \rho c_p Le^{1-n}$$

Note

For most engineering applications, assume a value of n=1/3

$$Le = \frac{\alpha}{D_{AB}} = \frac{\rho C_p}{k D_{AB}}$$

$$Sc = \frac{\nu}{D_{AB}}$$

Reynolds Analogy

- This analogy assumes the following:
  $dp^*/dx^*=0$ and $Pr = Sc = 1$.
  and for a flat surface $u_\infty = V$
- Hence, the velocity, the thermal and the concentration Equations and boundary conditions become analogous and the functional form of the solutions for u^*, T^*, and C^*, eqs. 1, 4, and 8 are equivalent.

- From eqs. 3, 6 and 10 it follows that (see the previous table)

$$C_f \frac{\mathrm{Re}_L}{2} = Nu = Sh \qquad \text{............(12)}$$

- Replacing *Nu* and *Sh* by the Stanton number, St, and the mass transfer Stanton number, St_m, respectively,

$$St = \frac{h}{\rho V c_p} = \frac{Nu}{\mathrm{Re}\,\mathrm{Pr}}$$

$$St_m = \frac{h_m}{V} = \frac{Sh}{\mathrm{Re}\,Sc}$$

$$C_f \frac{\mathrm{Re}_L}{2} = St\,\mathrm{Re} = St_m\,\mathrm{Re} \quad \{Note: \mathrm{Pr} = Sc = 1\}$$

$$\therefore \quad \frac{C_f}{2} = St = St_m$$

- Eq. 12 may be expressed as

$$\frac{C_f}{2} = St = St_m \qquad \Pr = Sc = 1$$

$$\text{and} \quad dp^*/dx^* = 0$$

- The modified Reynolds, or Chilton-Colburn, analogies

$$\frac{C_f}{2} = St\,Pr^{2/3} = j_H \qquad 0.6 < Pr < 60$$

$$\frac{C_f}{2} = St_m\,Sc^{2/3} = j_m \qquad 0.6 < Sc < 3000$$

Selected dimensionless groups of heat and mass transfer

Group	Definition	Interpretation
Biot number (Bi)	$\frac{hL}{k_s}$	Ratio of the internal thermal resistance of a solid to the boundary layer thermal resistance.
Mass transfer Biot number (Bi_m)	$\frac{h_m L}{D_{AB}}$	Ratio of the internal species transfer resistance to the boundary layer species transfer resistance.
Bond number (Bo)	$\frac{g(\rho_l - \rho_v)L^2}{\sigma}$	Ratio of gravitational and surface tension forces.
Coefficient of friction (C_f)	$\frac{\tau_s}{\rho V^2/2}$	Dimensionless surface shear stress.
Eckert number (Ec)	$\frac{V^2}{c_p(T_s - T_\infty)}$	Kinetic energy of the flow relative to the boundary layer enthalpy difference.
Fourier number (Fo)	$\frac{\alpha t}{L^2}$	Ratio of the heat conduction rate to the rate of thermal energy storage in a solid. Dimensionless time.
Mass transfer Fourier number (Fo_m)	$\frac{D_{AB} t}{L^2}$	Ratio of the species diffusion rate to the rate of species storage. Dimensionless time.
Friction factor (f)	$\frac{\Delta p}{(L/D)(\rho u_m^2/2)}$	Dimensionless pressure drop for internal flow.
Grashof number (Gr_L)	$\frac{g\beta(T_s - T_\infty)L^3}{\nu^2}$	Measure of the ratio of buoyancy forces to viscous forces.
Colburn j factor (j_H)	$St\, Pr^{2/3}$	Dimensionless heat transfer coefficient.
Colburn j factor (j_m)	$St_m\, Sc^{2/3}$	Dimensionless mass transfer coefficient.
Jakob number (Ja)	$\frac{c_p(T_s - T_{sat})}{h_{fg}}$	Ratio of sensible to latent energy absorbed during liquid–vapor phase change.
Lewis number (Le)	$\frac{\alpha}{D_{AB}}$	Ratio of the thermal and mass diffusivities.
Nusselt number (Nu_L)	$\frac{hL}{k_f}$	Ratio of convection to pure conduction heat transfer.
Peclet number (Pe_L)	$\frac{VL}{\alpha} = Re_L Pr$	Ratio of advection to conduction heat transfer rates.
Prandtl number (Pr)	$\frac{c_p \mu}{k} = \frac{\nu}{\alpha}$	Ratio of the momentum and thermal diffusivities.

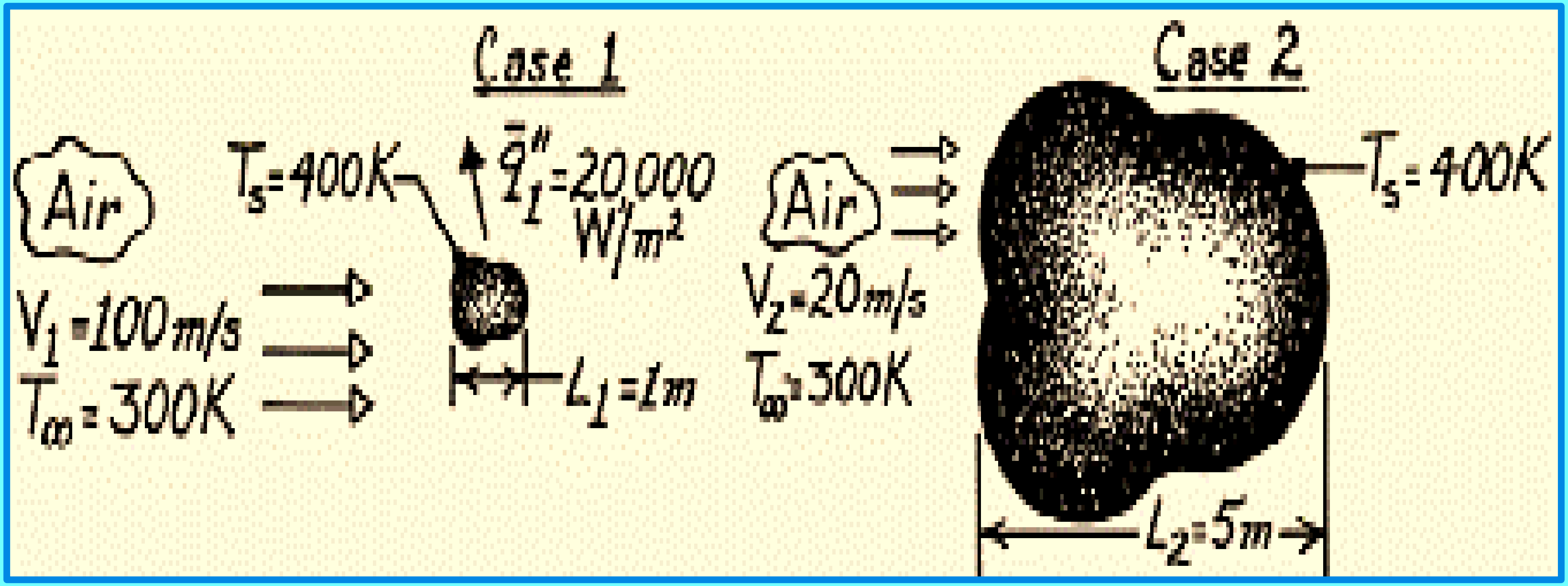
Continued

Group	Definition	Interpretation
Reynolds number (Re_L)	$\frac{VL}{\nu}$	Ratio of the inertia and viscous forces.
Schmidt number (Sc)	$\frac{\nu}{D_{AB}}$	Ratio of the momentum and mass diffusivities.
Sherwood number (Sh_L)	$\frac{h_m L}{D_{AB}}$	Dimensionless concentration gradient at the surface.
Stanton number (St)	$\frac{h}{\rho V c_p} = \frac{Nu_L}{Re_L\, Pr}$	Modified Nusselt number.
Mass transfer Stanton number (St_m)	$\frac{h_m}{V} = \frac{Sh_L}{Re_L Sc}$	Modified Sherwood number.
Weber number (We)	$\frac{\rho V^2 L}{\sigma}$	Ratio of inertia to surface tension forces.

Example

An object of irregular shape has a characteristic length of $L = 1$ m and is maintained at a uniform surface temperature of $T_s = 400$ K. When placed in atmospheric air at a temperature of $T_\infty = 300$ K and moving with a velocity of $V = 100$ m/s, the average heat flux from the surface to the air is 20,000 W/m^2. If a second object of the same shape, but with a characteristic length of $L = 5$ m, is maintained at a surface temperature of $T_s = 400$ K and is placed in atmospheric air at $T_\infty = 300$ K, what will the value of the average convection coefficient be if the air velocity is $V = 20$ m/s?

Schematic



Case 1
Case 2
Air
$T_s = 400K$
$\bar{q}''_1 = 20,000$ W/m²
$V_1 = 100$ m/s
$T_\infty = 300K$
$L_1 = 1m$
Air
$V_2 = 20$ m/s
$T_\infty = 300K$
$T_s = 400K$
$L_2 = 5m$

Solution

ASSUMPTIONS: (1) Steady-state conditions, (2) Constant properties.

ANALYSIS: For a particular geometry,

$$\overline{Nu}_L = f\left(Re_L, Pr\right).$$

The Reynolds numbers for each case are

Case 1: $$Re_{L,1} = \frac{V_1 L_1}{\nu_1} = \frac{(100\,m/s)\,1m}{\nu_1} = \frac{100\ m^2/s}{\nu_1}$$

Case 2: $$Re_{L,2} = \frac{V_2 L_2}{\nu_2} = \frac{(20\,m/s)\,5m}{\nu_2} = \frac{100\ m^2/s}{\nu_2}.$$

Hence, with $v_1 = v_2$, $Re_{L,1} = Re_{L,2}$. Since $Pr_1 = Pr_2$, it follows that

$$\overline{Nu}_{L,2} = \overline{Nu}_{L,1}.$$

Hence,

$$\bar{h}_2 L_2 / k_2 = \bar{h}_1 L_1 / k_1$$

$$\bar{h}_2 = \bar{h}_1 \frac{L_1}{L_2} = 0.2\, \bar{h}_1.$$

For *Case 1*, using the rate equation, the convection coefficient is

$$q_1 = \bar{h}_1 A_1 \left(T_s - T_\infty\right)_1$$

$$\bar{h}_1 = \frac{\left(q_1 / A_1\right)}{\left(T_s - T_\infty\right)_1} = \frac{q_1''}{\left(T_s - T_\infty\right)_1} = \frac{20{,}000\ \text{W/m}^2}{(400 - 300)\,\text{K}} = 200\ \text{W/m}^2 \cdot \text{K}.$$

Hence, it follows that for *Case 2*

$$\bar{h}_2 = 0.2 \times 200\ \text{W/m}^2 \cdot \text{K} = 40\ \text{W/m}^2 \cdot \text{K}.$$

Exercise

A thin, flat plate that is 0.2 m by 0.2 m on a side is oriented parallel to an atmospheric airstream having a velocity of 40 m/s. The air is at a temperature of $T_\infty = 20°C$, while the plate is maintained at $T_s = 120°C$. The air flows over the top and bottom surfaces of the plate, and measurement of the drag force reveals a value of 0.075 N. What is the rate of heat transfer from both sides of the plate to the air?

Hints

- Use Chilton-Colburn Analogy factor to obtain the convection heat transfer coefficient, *h*.

$$\frac{C_f}{2} = St\,Pr^{2/3} \equiv j_H \qquad 0.6 < Pr < 60$$

$$C_f = \frac{\tau_s}{\rho V^2/2}$$

$$St = \frac{h}{\rho V c_p}$$