Convection Mass Transfer

Exact Solution: Blausius or Similarity solution

In Brief: Exact solution(Blausius or Similarity solution)

The major convection parameters may be obtained by solving the appropriate form of the boundary layer equations. Assuming *steady, incompressible, laminar* flow with *constant fluid properties* and *negligible viscous dissipation* and recognizing that *dp/dx* = 0, the boundary layer equations reduce to

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (7.4)$$

Momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (7.5)$$

Energy:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (7.6)$$

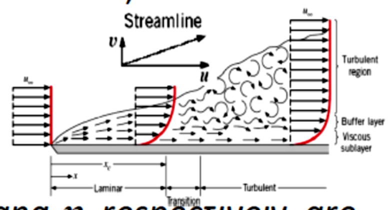
Species:

$$u\frac{\partial \overset{C_A}{\rho_A}}{\partial x} + v\frac{\partial \overset{C_A}{\rho_A}}{\partial y} = D_{AB}\frac{\partial^2 \overset{C_A}{\rho_A}}{\partial y^2} \qquad (7.7)$$

Hydrodynamic Solution _ similarity solution

The hydrodynamic solution follows the method of Blasius;

- Define a stream function ψ *(x, y), such that*



$$u \equiv \frac{\partial \psi}{\partial y} \quad \text{and} \quad v \equiv -\frac{\partial \psi}{\partial x}$$

- *Hence continuity eq. will be satisfied.*
- New dependent and independent variables, *f and* η*, respectively, are then defined such that*

$$f(\eta) \equiv \frac{\psi}{u_\infty \sqrt{\nu x / u_\infty}} \qquad \eta \equiv y\sqrt{u_\infty / \nu x}$$

- Use of these variables, reducing the partial differential equation, momentum eq., to an ordinary differential equation.
- The Blasius solution is termed a *similarity solution, and* η *is a similarity variable. This* terminology is used because, despite growth of the boundary layer with distance *x from the* leading edge, the velocity profile u/u_∞ *remains geometrically similar. This similarity is of* the functional form

$$\frac{u}{u_\infty} = \phi\left(\frac{y}{\delta}\right)$$

where δ is the boundary layer thickness. We will find from the Blasius solution that δ varies as $(\nu x/u_\infty)^{1/2}$; thus, it follows that

$$\frac{u}{u_\infty} = \phi(\eta)$$

Hence the velocity profile is uniquely determined by the similarity variable η, which depends on both *x and y.*

- Conversion of momentum eq. into ordinary differential eq.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\infty \sqrt{\frac{\nu x}{u_\infty}}\frac{df}{d\eta}\sqrt{\frac{u_\infty}{\nu x}} = u_\infty \frac{df}{d\eta}$$

Or $$\frac{u}{u_\infty} = \frac{df}{d\eta}$$

- For details, See the text Incropera.

Definition& transformation

- Similarity variable, η, is defined as

$$\eta = y\sqrt{\frac{u_\infty}{\nu x}}$$

- Using this variable, transform the momentum equation into the following nonlinear, 3^{rd} order ordinary differential equation

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0$$

- With appropriate B.Cs, the solution may be obtained by a series expansion or numerical integration. Selected results are given in the following table or graph.
- $B.C^s$ $\quad u(x,0) = v(x,0) = 0 \quad$ and $\quad u(x,\infty) = u_\infty$

in terms of the similarity variables

$$\left.\frac{df}{d\eta}\right|_{\eta=0} = f(0) = 0 \quad \text{and} \quad \left.\frac{df}{d\eta}\right|_{\eta\to\infty} = 1$$

TABLE 7.1 Flat plate laminar boundary layer functions [3]

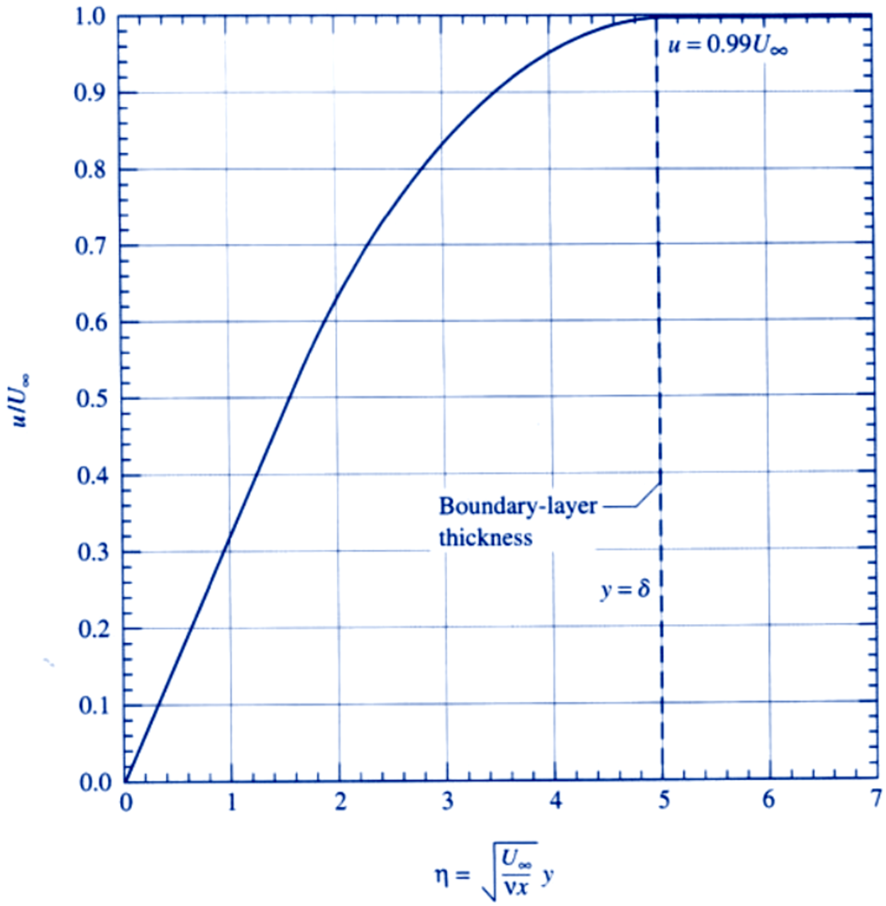
$\eta = y\sqrt{\frac{u_\infty}{\nu x}}$	f	$\frac{df}{d\eta} = \frac{u}{u_\infty}$	$\frac{d^2f}{d\eta^2}$
0	0	0	0.332
0.4	0.027	0.133	0.331
0.8	0.106	0.265	0.327
1.2	0.238	0.394	0.317
1.6	0.420	0.517	0.297
2.0	0.650	0.630	0.267
2.4	0.922	0.729	0.228
2.8	1.231	0.812	0.184
3.2	1.569	0.876	0.139
3.6	1.930	0.923	0.098
4.0	2.306	0.956	0.064
4.4	2.692	0.976	0.039
4.8	3.085	0.988	0.022
5.2	3.482	0.994	0.011
5.6	3.880	0.997	0.005
6.0	4.280	0.999	0.002
6.4	4.679	1.000	0.001
6.8	5.079	1.000	0.000

The laminar boundary layer thickness is defined by velocity that is equal to 99% of the free-stream velocity, U∞

$\eta \equiv y\sqrt{u_\infty / \nu x} = 5.0$

$\delta \sqrt{u_\infty / \nu x} = 5.0$

$\delta = \frac{5.0}{\sqrt{u_\infty / \nu x}} = \frac{5x}{\sqrt{Re_x}}$



Exact laminar boundary-layer analysis

- The similarity in the three differential equations, (6-35), (6-36), and (6-37), and the boundary conditions suggests that similar solutions should be obtained for the three transfer phenomena.
- In Chapter 7, the Blasius solution for equation (7-17) was modified and successfully applied to explain convective heat transfer when the ratio of the momentum to thermal diffusivity $\nu/\alpha = Pr = 1$.
- The same type of solution should also describe convective mass transfer when the ratio of the momentum to mass diffusivity $\nu / D_{AB} = Sc = 1$.

- The Blasius solution to the momentum boundary layer

$$\tau_s = \mu \frac{\partial u}{\partial y}\bigg|_{y=0}$$

$$= \mu u_\infty \sqrt{u_\infty / \nu x} \frac{d^2 f}{d\eta^2}\bigg|_{\eta=0}$$

NOTE

$= 0.332$ (table)

$$\therefore \quad \tau_s = 0.332 u_\infty \sqrt{\rho \mu u_\infty / x}$$

$$C_{f,x} \equiv \frac{\tau_{s,x}}{\rho u_\infty^2/2} = 0.664 \, Re_x^{-1/2}$$

$$\frac{\partial u}{\partial x} = -\frac{u_\infty}{2x} \eta \frac{d^2 f}{d\eta^2}$$

$$\frac{\partial u}{\partial y} = u_\infty \sqrt{\frac{u_\infty}{\nu x}} \frac{d^2 f}{d\eta^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_\infty^2}{\nu x} \frac{d^3 f}{d\eta^3}$$

- Mass Transfer Solution

$$\frac{d^2 C_A^*}{d\eta^2} + \frac{Sc}{2} f \frac{dC_A^*}{d\eta} = 0 \quad \text{B.C}^s\text{:} \quad \eta = 0 : \; C_A^* = 0 \qquad \eta = \infty : \; C_A^* = 1$$

- The solution is

$$\frac{dC_A^*}{d\eta}\bigg|_{\eta=0} = 0.332 \, Sc^{1/3} \qquad \text{(i)}$$

$$\therefore \quad k_c = \frac{-D_{AB} \, \partial C_A / \partial y\big|_{y=0}}{C_{As} - C_{A\infty}}$$

$$C_A^* = \frac{C_A - C_{A,s}}{C_{A,\infty} - C_{A,s}}$$

$$k_c = D_{AB} \frac{\partial C_A^*}{\partial y}\bigg|_{y=0}$$

$$k_c = D_{AB} \left(\frac{u_\infty}{\nu x}\right)^{1/2} \frac{\partial C_A^*}{\partial \eta}\bigg|_{\eta=0} \qquad (ii)$$

Combining (i) and (ii) gives

$$k_c = D_{AB} \left(\frac{u_\infty}{\nu x}\right)^{1/2} 0.332 \, Sc^{1/3} \qquad \text{"local coefficient"}$$

$$k_c = D_{AB}\frac{x}{x}\left(\frac{u_\infty}{\nu x}\right)^{1/2} 0.332\, Sc^{1/3}$$

$$k_c = D_{AB}\frac{1}{x}\left(\frac{u_\infty x}{\nu}\right)^{1/2} 0.332\, Sc^{1/3}$$

$$\frac{k_c x}{D_{AB}} = 0.332\,\mathrm{Re}_x^{1/2}\; Sc^{1/3}$$

$$Sh_x = 0.332\,\mathrm{Re}_x^{1/2}\; Sc^{1/3} \qquad Sc \geq 0.6 \qquad "Sh_{local}"$$

$$\overline{Sh} = 0.664\,\mathrm{Re}^{1/2}\; Sc^{1/3} \qquad Sc \geq 0.6 \qquad \text{(iii)}$$

$$Note \quad \bar{k}_c = 2k_c$$

Also, similar to heat transfer, the ratio of boundary layer thicknesses is

$$\frac{\delta}{\delta_c} = Sc^{1/3}$$

Equation (iii) has been experimentally verified and shown to give accurate results for values of Re, $< 3 \times 10^5$.

Sherwood number relations in mass convection for specified concentration at the surface corresponding to the Nusselt number relations in heat convection for specified surface temperature

Convective Heat Transfer	Convective Mass Transfer "Average"
1. *Forced Convection over a Flat Plate*	
(*a*) Laminar flow ($\mathrm{Re} < 5 \times 10^5$)	
$\mathrm{Nu} = 0.664\,\mathrm{Re}_L^{0.5}\,\mathrm{Pr}^{1/3}$, $\quad \mathrm{Pr} > 0.6$	$\mathrm{Sh} = 0.664\,\mathrm{Re}_L^{0.5}\,\mathrm{Sc}^{1/3}$, $\quad \mathrm{Sc} > 0.5$
(*b*) Turbulent flow ($5 \times 10^5 < \mathrm{Re} < 10^7$)	
$\mathrm{Nu} = 0.037\,\mathrm{Re}_L^{0.8}\,\mathrm{Pr}^{1/3}$, $\quad \mathrm{Pr} > 0.6$	$\mathrm{Sh} = 0.037\,\mathrm{Re}_L^{0.8}\,\mathrm{Sc}^{1/3}$, $\quad \mathrm{Sc} > 0.5$
2. *Fully Developed Flow in Smooth Circular Pipes*	
(*a*) Laminar flow ($\mathrm{Re} < 2300$)	
$\mathrm{Nu} = 3.66$	$\mathrm{Sh} = 3.66$
(*b*) Turbulent flow ($\mathrm{Re} > 10{,}000$)	
$\mathrm{Nu} = 0.023\,\mathrm{Re}^{0.8}\,\mathrm{Pr}^{0.4}$, $\quad 0.7 < \mathrm{Pr} < 160$	$\mathrm{Sh} = 0.023\,\mathrm{Re}^{0.8}\,\mathrm{Sc}^{0.4}$, $\quad 0.7 < \mathrm{Sc}\ 160$
3. *Natural Convection over Surfaces*	
(*a*) Vertical plate	
$\mathrm{Nu} = 0.59(\mathrm{Gr\,Pr})^{1/4}$, $\quad 10^5 < \mathrm{Gr\,Pr} < 10^9$	$\mathrm{Sh} = 0.59(\mathrm{Gr\,Sc})^{1/4}$, $\quad 10^5 < \mathrm{Gr\,Sc} < 10^9$
$\mathrm{Nu} = 0.1(\mathrm{Gr\,Pr})^{1/3}$, $\quad 10^9 < \mathrm{Gr\,Pr} < 10^{13}$	$\mathrm{Sh} = 0.1(\mathrm{Gr\,Sc})^{1/3}$, $\quad 10^9 < \mathrm{Gr\,Sc} < 10^{13}$
(*b*) Upper surface of a horizontal plate	
Surface is hot ($T_s > T_\infty$)	Fluid near the surface is light ($\rho_s < \rho_\infty$)
$\mathrm{Nu} = 0.54(\mathrm{Gr\,Pr})^{1/4}$, $\quad 10^4 < \mathrm{Gr\,Pr} < 10^7$	$\mathrm{Sh} = 0.54(\mathrm{Gr\,Sc})^{1/4}$, $\quad 10^4 < \mathrm{Gr\,Sc} < 10^7$
$\mathrm{Nu} = 0.15(\mathrm{Gr\,Pr})^{1/3}$, $\quad 10^7 < \mathrm{Gr\,Pr} < 10^{11}$	$\mathrm{Sh} = 0.15(\mathrm{Gr\,Sc})^{1/3}$, $\quad 10^7 < \mathrm{Gr\,Sc} < 10^{11}$
(*c*) Lower surface of a horizontal plate	
Surface is hot ($T_s > T_\infty$)	Fluid near the surface is light ($\rho_s < \rho_\infty$)
$\mathrm{Nu} = 0.27(\mathrm{Gr\,Pr})^{1/4}$, $\quad 10^5 < \mathrm{Gr\,Pr} < 10^{11}$	$\mathrm{Sh} = 0.27(\mathrm{Gr\,Sc})^{1/4}$, $\quad 10^5 < \mathrm{Gr\,Sc} < 10^{11}$

Example

A large volume of pure water at 26.1°C is flowing parallel to a flat plate of solid benzoic acid, where $L = 0.244$ m in the direction of flow. The water velocity is 0.061 m/s. The solubility of benzoic acid in water is 0.02948 kg mol/m^3. The diffusivity of benzoic acid is 1.245×10^{-9} m^2/s. Calculate the mass-transfer coefficient k_L and the flux N_A.

Solution

- Since the solution is quite dilute, the physical properties of water at 26.1 °C from Appendix A.2 can be used.

$$\mu = 8.71 \times 10^{-4} \text{ Pa} \cdot \text{s}$$

$$\rho = 996 \text{ kg/m}^3$$

$$D_{AB} = 1.245 \times 10^{-9} \text{ m}^2/\text{s}$$

$$N_{Sc} = \frac{8.71 \times 10^{-4}}{996(1.245 \times 10^{-9})} = 702$$

$$N_{Re,L} = \frac{Lv\rho}{\mu} = \frac{0.244(0.0610)(996)}{8.71 \times 10^{-4}} = 1.700 \times 10^4$$

Laminar $< 3\text{x}10^5$

- Find the kc' from a suitable formula, for example , for liquid it is recommended to use

$$\frac{k_c'}{v}(N_{Sc})^{2/3} = J_D = 0.99 N_{Re,L}^{-0.5} = 0.99(1.700 \times 10^4)^{-0.5} = 0.00758$$

$$k'_c = 0.00758(0.0610)(702)^{-2/3} = 5.85 \times 10^{-6} \text{ m/s}$$

In this case, diffusion is for A through nondiffusing B, so k_c in Eq. (7.2-10) should be used.

$$N_A = \frac{k'_c}{x_{BM}}(c_{A1} - c_{A2}) = k_c(c_{A1} - c_{A2})$$

Since the solution is very dilute, $x_{BM} \cong 1.0$ and $k'_c \cong k_c$. Also, $c_{A1} = 2.948 \times 10^{-2}$ kg mol/m^3 (solubility) and $c_{A2} = 0$ (large volume of fresh water). Substituting into Eq. (7.2-10),

$$N_A = (5.85 \times 10^{-6})(0.02948 - 0) = 1.726 \times 10^{-7} \text{ kg mol/s} \cdot \text{m}^2$$