Laplace Transforms

➢Definition of Laplace transform:

$$F(s) = \mathbf{L}(f(t)) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$$

- *F(s)* is called ***Laplace transform*** of *f(t)*.
- *f(t)* must be piecewise continuous.
- *F(s)* contains no information on *f(t)* for *t* < 0.
- The past information on *f(t)* (for *t* < 0) is irrelevant.
- The ***s*** is a complex variable called "***Laplace transform variable***"

➢Inverse of Laplace transform: $f(t) = \mathbf{L}^{-1}(F(s))$

- $\mathbf{L}$ and $\mathbf{L}^{-1}$ are linear:

$$\mathbf{L}\left[af_1(t) + bf_2(t)\right] = a\mathbf{L}\left[f_1(t)\right] + b\mathbf{L}\left[f_2(t)\right]$$
$$= aF_1(s) + bF_2(s)$$

Laplace Transforms

➢Laplace transforms of some functions:

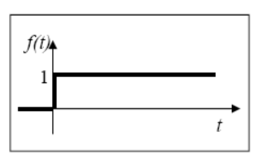
- **Constant function,** a

$$\mathbf{L}\{a\} = \int_0^\infty ae^{-st}dt = -\frac{a}{s}e^{-st}\Big|_0^\infty = 0 - \left(-\frac{a}{s}\right) = \frac{a}{s}$$

- **Step function,** $\mathrm{S}(t)$

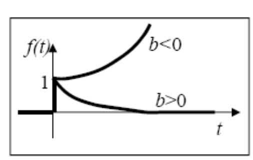
$$f(t) = \mathrm{S}(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\mathbf{L}\{\mathrm{S}(t)\} = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$



- **Exponential function,** e^{-bt}

$$\mathbf{L}\left\{e^{-bt}\right\} = \int_0^\infty e^{-bt}e^{-st}dt = \frac{-1}{s+b}e^{-(b+s)t}\Big|_0^\infty = \frac{1}{s+b}$$

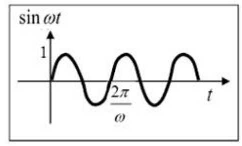


Laplace Transforms

➢Laplace transforms of some functions:

- **Trigonometric functions**
  - **Euler's Identity:** $e^{j\omega t} = \cos\omega t + j\sin\omega t$

$$\cos\omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \qquad \sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$



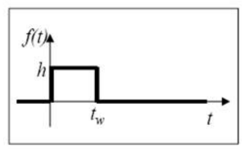
$$\mathbf{L}\{\sin\omega t\} = \mathbf{L}\left\{\frac{1}{2j}e^{j\omega t}\right\} - \mathbf{L}\left\{\frac{1}{2j}e^{-j\omega t}\right\} = \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}$$

$$\mathbf{L}\{\cos\omega t\} = \mathbf{L}\left\{\frac{1}{2}e^{j\omega t}\right\} + \mathbf{L}\left\{\frac{1}{2}e^{-j\omega t}\right\} = \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) = \frac{s}{s^2+\omega^2}$$

$j = \sqrt{-1}$

- **Rectangular pulse, $\mathrm{P}(t)$**

$$f(t) = \mathrm{P}(t) = \begin{cases} 0 & \text{for } t > t_w \\ h & \text{for } t_w \geq t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



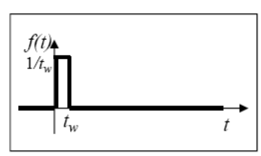
$$\mathbf{L}\{\mathrm{P}(t)\} = \int_0^{t_w} he^{-st}dt = -\frac{h}{s}e^{-st}\Big|_0^{t_w} = \frac{h}{s}\left(1-e^{-t_w s}\right)$$

Laplace Transforms

➢Laplace transforms of some functions:

- **Impulse function, $\delta(t)$**

$$f(t) = \delta(t) = \lim_{t_w\to 0}\begin{cases} 0 & \text{for } t > t_w \\ 1/t_w & \text{for } t_w \geq t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



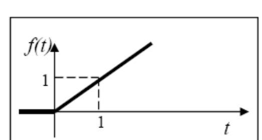
$$\mathbf{L}\{\delta(t)\} = \lim_{t_w\to 0}\int_0^{t_w}\frac{1}{t_w}e^{-st}dt = \lim_{t_w\to 0}\frac{1}{t_w s}\left(1-e^{-t_w s}\right) = 1$$

$$\left(\text{L'Hospital's rule: } \lim_{t\to 0}\frac{f(t)}{g(t)} = \lim_{t\to 0}\frac{f'(t)}{g'(t)}\right)$$

- **Ramp function, t**

$$\mathbf{L}\{t\} = \int_0^{\infty} te^{-st}dt$$

$$= \frac{t}{-s}e^{-st}\Big|_0^{\infty} - \int_0^{\infty}\frac{e^{-st}}{-s}dt = \frac{1}{s}\int_0^{\infty}e^{-st}dt = \frac{1}{s^2}$$



(Integration by part)

Laplace Transforms

➢**Refer to Table 3.1 (Seborg et al.) for Laplace transforms of other functions:**

f(*t*)	*F*(*s*)
1. $\delta(t)$ (unit impulse)	1
2. $S(t)$ (unit step)	$\frac{1}{s}$
3. t (ramp)	$\frac{1}{s^2}$
4. t^{n-1}	$\frac{(n-1)!}{s^n}$
5. e^{-bt}	$\frac{1}{s+b}$
6. $\frac{1}{\tau}e^{-t/\tau}$	$\frac{1}{\tau s+1}$
7. $\frac{t^{n-1}e^{-bt}}{(n-1)!}$ $(n>0)$	$\frac{1}{(s+b)^n}$
8. $\frac{1}{\tau^n(n-1)!}t^{n-1}e^{-t/\tau}$	$\frac{1}{(\tau s+1)^n}$
9. $\frac{1}{b_1-b_2}(e^{-b_2 t}-e^{-b_1 t})$	$\frac{1}{(s+b_1)(s+b_2)}$

Laplace Transforms

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a]

f(*t*)	*F*(*s*)
10. $\frac{1}{\tau_1-\tau_2}(e^{-t/\tau_1}-e^{-t/\tau_2})$	$\frac{1}{(\tau_1 s+1)(\tau_2 s+1)}$
11. $\frac{b_3-b_1}{b_2-b_1}e^{-b_1 t}+\frac{b_3-b_2}{b_1-b_2}e^{-b_2 t}$	$\frac{s+b_3}{(s+b_1)(s+b_2)}$
12. $\frac{1}{\tau_1}\frac{\tau_1-\tau_3}{\tau_1-\tau_2}e^{-t/\tau_1}+\frac{1}{\tau_2}\frac{\tau_2-\tau_3}{\tau_2-\tau_1}e^{-t/\tau_2}$	$\frac{\tau_3 s+1}{(\tau_1 s+1)(\tau_2 s+1)}$
13. $1-e^{-t/\tau}$	$\frac{1}{s(\tau s+1)}$
14. $\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
15. $\cos\omega t$	$\frac{s}{s^2+\omega^2}$
16. $\sin(\omega t+\phi)$	$\frac{\omega\cos\phi+s\sin\phi}{s^2+\omega^2}$
17. $e^{-bt}\sin\omega t$ (b, ω real)	$\frac{\omega}{(s+b)^2+\omega^2}$
18. $e^{-bt}\cos\omega t$ (b, ω real)	$\frac{s+b}{(s+b)^2+\omega^2}$
19. $\frac{1}{\tau\sqrt{1-\zeta^2}}e^{-\zeta t/\tau}\sin(\sqrt{1-\zeta^2}\,t/\tau)$ $(0\le\lvert\zeta\rvert<1)$	$\frac{1}{\tau^2 s^2+2\zeta\tau s+1}$

Laplace Transforms

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a]

$f(t)$	$F(s)$
20. $1+\frac{1}{\tau_2-\tau_1}\left(\tau_1 e^{-t/\tau_1}-\tau_2 e^{-t/\tau_2}\right)$ $(\tau_1 \neq \tau_2)$	$\frac{1}{s(\tau_1 s+1)(\tau_2 s+1)}$
21. $1-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin\left[\sqrt{1-\zeta^2}\, t/\tau+\psi\right]$ $\psi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}, \quad (0 \leq \lvert\zeta\rvert<1)$	$\frac{1}{s(\tau^2 s^2+2\zeta\tau s+1)}$
22. $1-e^{-\zeta t/\tau}\left[\cos\left(\sqrt{1-\zeta^2}\, t/\tau\right)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\, t/\tau\right)\right]$ $(0 \leq \lvert\zeta\rvert<1)$	$\frac{1}{s(\tau^2 s^2+2\zeta\tau s+1)}$
23. $1+\frac{\tau_3-\tau_1}{\tau_1-\tau_2} e^{-t/\tau_1}+\frac{\tau_3-\tau_2}{\tau_2-\tau_1} e^{-t/\tau_2}$ $(\tau_1 \neq \tau_2)$	$\frac{\tau_3 s+1}{s(\tau_1 s+1)(\tau_2 s+1)}$
24. $\frac{df}{dt}$	$sF(s)-f(0)$
25. $\frac{d^n f}{dt^n}$	$s^n F(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$
26. $f(t-t_0)S(t-t_0)$	$e^{-t_0 s}F(s)$

[a]Note that $f(t)$ and $F(s)$ are defined for $t \geq 0$ only.

Laplace Transforms

➢Properties of Laplace transform:

- **Differentiation**

$$\mathbf{L}\left\{\frac{df}{dt}\right\}=\int_0^\infty f'\cdot e^{-st}dt=f(t)e^{-st}\Big|_0^\infty-\int_0^\infty f\cdot(-s)e^{-st}dt \quad \left(\text{Integration by part}\right)$$

$$=s\int_0^\infty f\cdot e^{-st}dt-f(0)=sF(s)-f(0)$$

$$\mathbf{L}\left\{\frac{d^2 f}{dt^2}\right\}=\int_0^\infty f''\cdot e^{-st}dt=f(t)'e^{-st}\Big|_0^\infty-\int_0^\infty f'\cdot(-s)e^{-st}dt=s\int_0^\infty f'\cdot e^{-st}dt-f'(0)$$

$$=s\left(sF(s)-f(0)\right)-f'(0)=s^2F(s)-sf(0)-f'(0)$$

$$\vdots$$

$$\mathbf{L}\left\{\frac{d^n f}{dt^n}\right\}=\int_0^\infty f^{(n)}\cdot e^{-st}dt=f(t)^{(n-1)}e^{-st}\Big|_0^\infty-\int_0^\infty f^{(n-1)}\cdot(-s)e^{-st}dt$$

$$=s\int_0^\infty f^{(n-1)}\cdot e^{-st}dt-f^{(n-1)}(0)=s\left(\mathbf{L}\left\{\frac{d^{n-1}f}{dt^{n-1}}\right\}\right)-f^{(n-1)}(0)$$

$$=s^nF(s)-s^{n-1}f(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

Laplace Transforms

➢Properties of Laplace transform:

- **If** $f(0) = f'(0) = f''(0) = \ldots = f^{(n-1)}(0) = 0$,
  - **Initial condition effects are vanished.**
  - **It is very convenient to use deviation variables so that all the effects of initial condition vanish.**

$$\mathbf{L}\left\{\frac{df}{dt}\right\} = sF(s)$$

$$\mathbf{L}\left\{\frac{d^2 f}{dt^2}\right\} = s^2 F(s)$$

$$\vdots$$

$$\mathbf{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s)$$

- **Transforms of linear differential equations.**

$$y(t) \xrightarrow{\mathbf{L}} Y(s), \quad u(t) \xrightarrow{\mathbf{L}} U(s)$$

$$\frac{dy(t)}{dt} \xrightarrow{\mathbf{L}} sY(s) \quad (\text{if } y(0) = 0)$$

$$\tau\frac{dy(t)}{dt} = -y(t) + Ku(t) \quad (y(0) = 0) \xrightarrow{\mathbf{L}} (\tau s + 1)Y(s) = KU(s)$$

$$\frac{\partial T_L}{\partial t} = -v\frac{\partial T_L}{\partial z} + \frac{1}{\tau_{HL}}(T_w - T_L) \xrightarrow{\mathbf{L}} \tau_{HL} v \frac{\partial \tilde{T}_L(s)}{\partial z} + (\tau_{HL}s + 1)\tilde{T}_L(s) = \tilde{T}_w(s)$$

Laplace Transforms

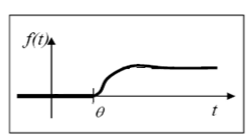
➢Properties of Laplace transform:

- **Integration**

$$\mathbf{L}\left\{\int_0^t f(\xi)d\xi\right\} = \int_0^\infty \left(\int_0^t f(\xi)d\xi\right)e^{-st}dt$$

$$= \cancelto{0}{\frac{e^{-st}}{-s}\int_0^t f(\xi)d\xi\Bigg|_0^\infty} + \frac{1}{s}\int_0^\infty f\cdot e^{-st}dt = \frac{F(s)}{s} \qquad (\text{by } i.b.p.)$$

- **Time delay (Translation in time)**



$$f(t) \xrightarrow{+\theta \text{ in } t} f(t-\theta)\mathrm{S}(t-\theta)$$

$$\mathbf{L}\{f(t-\theta)\mathrm{S}(t-\theta)\} = \int_\theta^\infty f(t-\theta)e^{-st}dt = \int_0^\infty f(\tau)e^{-s(\tau+\theta)}d\tau \quad (\text{let } \tau = t-\theta)$$

$$= e^{-\theta s}\int_0^\infty f(\tau)e^{-\tau s}d\tau = e^{-\theta s}F(s)$$

- **Derivative of Laplace transform**

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f\cdot e^{-st}dt = \int_0^\infty f\cdot\frac{d}{ds}e^{-st}dt = \int_0^\infty(-t\cdot f)e^{-st}dt = \mathbf{L}\left[-t\cdot f(t)\right]$$

Laplace Transforms

➢Properties of Laplace transform:

- **Final value theorem**

$$\lim_{s\to 0}\int_0^\infty \frac{df}{dt}\cdot e^{-st}dt = \lim_{s\to 0}\left[sF(s)-f(0)\right]$$

$$\int_0^\infty \frac{df}{dt}dt = f(\infty)-f(0) = \lim_{s\to 0} sF(s) - f(0) \Rightarrow \boxed{f(\infty) = \lim_{s\to 0} sF(s)}$$

  – **Limitation:** $f(\infty)$ **has to exist. If it diverges or oscillates, this theorem is not valid.**

- **Initial value theorem**

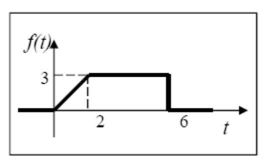
$$\lim_{s\to \infty}\int_0^\infty \frac{df}{dt}\cdot e^{-st}dt = \lim_{s\to \infty}\left[sF(s)-f(0)\right]$$

$$\lim_{s\to \infty}\int_0^\infty \frac{df}{dt}e^{-st}dt = 0 = \lim_{s\to \infty} sF(s) - f(0) \Rightarrow \boxed{f(0) = \lim_{s\to \infty} sF(s)}$$

Laplace Transforms

Example. **Find the Laplace transfer of:**

$$f(t) = \begin{cases} 1.5t & \text{for } 0 \le t < 2 \\ 3 & \text{for } 2 \le t < 6 \\ 0 & \text{for } 6 \le t \\ 0 & \text{for } t < 0 \end{cases}$$



$$f(t) = 1.5t\,\mathrm{S}(t) - 1.5(t-2)\mathrm{S}(t-2) - 3\mathrm{S}(t-6)$$

$$\therefore F(s) = \mathbf{L}\{f(t)\} = \frac{1.5}{s^2}(1-e^{-2s}) - \frac{3}{s}e^{-6s}$$

Example. For $F(s) = \dfrac{2}{s-5}$, find $f(0)$ and $f(\infty)$.

– **Using the initial and final value theorems**

$$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{2s}{s-5} = 2 \qquad f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0}\frac{2s}{s-5} = 0$$

– **But the final value theorem is not valid because**

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 2e^{5t} = \infty$$

Laplace Transforms

Example. **What is the final value of X given by the following system?** $x'' + x' + x = \sin t;\ x(0) = x'(0) = 0$

$$\Rightarrow\ s^2 X(s) + sX(s) + X = \frac{1}{s^2+1}\ \Rightarrow\ x(s) = \frac{1}{(s^2+1)(s^2+s+1)}$$

$$x(\infty) = \lim_{s\to 0} \frac{s}{(s^2+1)(s^2+s+1)} = 0$$

– **Actually, $x(\infty)$ cannot be defined due to sin t term.**

Example. **Find the Laplace transform for $(t \sin \omega t)$?**

$$\text{From } \frac{dF(s)}{ds} = \mathbf{L}\left[-t \cdot f(t)\right]$$

$$\mathbf{L}\left[t \cdot \sin \omega t\right] = -\frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{2\omega s}{(s^2+\omega^2)^2}$$

Laplace Transforms

➢ **Inverse Laplace Transform:**

$$f(t) = \mathbf{L}^{-1}(F(s))$$

- Used to recover the solution in time domain:
  - From the table
  - By partial fraction expansion

Laplace Transforms

➢ Partial fraction expansion

– After the partial fraction expansion, it requires to know some simple formula of inverse Laplace transform such as:

$$\frac{1}{(\tau s+1)},\ \frac{s}{(s+b)^2+\omega^2},\ \frac{(n-1)!}{s^n},\ \frac{e^{-\theta s}}{\tau^2 s^2+2\zeta\tau s+1},\ \text{etc.}$$

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+p_1)\cdots(s+p_n)}=\frac{\alpha_1}{(s+p_1)}+\cdots+\frac{\alpha_n}{(s+p_n)}$$

Laplace Transforms

- **Case I: All *pi's are distinct and real:***

– Find the coefficients for each fraction:
  - Comparison of the coefficients after multiplying the denominator
  - Replace some values for *s and solve linear algebraic equation*
  - **Use of Heaviside expansion**
    - ➢Multiply both side by a factor, ($s+p_i$)*, and replace s with* $-p_i$:

$$\alpha_i=(s+p_i)\frac{N(s)}{D(s)}\bigg|_{s=-p_i}$$

– Inverse LT: $f(t)=\alpha_1 e^{-p_1 t}+\alpha_2 e^{-p_2 t}+\cdots+\alpha_n e^{-p_n t}$

Laplace Transforms

- **Case II: Some roots are repeated**

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+p)^r}=\frac{b_{r-1}s^{r-1}+\cdots+b_0}{(s+p)^r}=\frac{\alpha_1}{(s+p)}+\cdots+\frac{\alpha_r}{(s+p)^r}$$

– Each repeated factor has to be separated first.
– Same methods as Case I can be applied.
– Heaviside expansion for repeated factors:

$$\alpha_{r-i}=\frac{1}{i!}\frac{d^{(i)}}{ds^{(i)}}\left(\frac{N(s)}{D(s)}(s+p)^r\right)\Bigg|_{s=-p}\qquad (i=0,\cdots,r-1)$$

– Inverse LT:

$$f(t)=\alpha_1 e^{-pt}+\alpha_2 te^{-pt}+\cdots+\frac{\alpha_r}{(r-1)!}t^{r-1}e^{-pt}$$

Laplace Transforms

- **Case III: Some roots are complex**

$$F(s)=\frac{N(s)}{D(s)}=\frac{c_1s+c_0}{s^2+d_1s+d_0}=\frac{\alpha_1(s+b)+\beta_1\omega}{(s+b)^2+\omega^2}$$

– Each repeated factor has to be separated first.
– Then,

$$\frac{\alpha_1(s+b)+\beta_1\omega}{(s+b)^2+\omega^2}=\alpha_1\frac{(s+b)}{(s+b)^2+\omega^2}+\beta_1\frac{\omega}{(s+b)^2+\omega^2}$$

$$\text{where}\quad b=d_1/2,\quad \omega=\sqrt{d_0-d_1^2/4}$$

$$\alpha_1=c_1,\quad \beta_1=\left(c_0-\alpha_1 b\right)/\omega$$

– **Inverse LT:** $f(t)=\alpha_1 e^{-bt}\cos\omega t+\beta_1 e^{-bt}\sin\omega t$

Laplace Transforms

Example. Perform partial fractions of:

$$F(s)=\frac{(s+5)}{s(s+1)(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}+\frac{D}{s+3}\quad \text{(distinct)}$$

– **Multiply each factor and insert the zero value**

$$\left.\frac{(s+5)}{(s+1)(s+2)(s+3)}\right|_{s=0}=\left.\left(A+s\frac{B}{s+1}+s\frac{C}{s+2}+s\frac{D}{s+3}\right)\right|_{s=0}\Rightarrow A=5/6$$

$$\left.\frac{(s+5)}{s(s+2)(s+3)}\right|_{s=-1}=\left.\left(\frac{A(s+1)}{s}+B+\frac{C(s+1)}{s+2}+\frac{D(s+1)}{s+3}\right)\right|_{s=-1}\Rightarrow B=-2$$

$$\left.\frac{(s+5)}{s(s+1)(s+3)}\right|_{s=-2}=\left.\left(\frac{A(s+2)}{s}+\frac{B(s+2)}{s+1}+C+\frac{D(s+2)}{s+3}\right)\right|_{s=-2}\Rightarrow C=3/2$$

$$\left.\frac{(s+5)}{s(s+1)(s+2)}\right|_{s=-3}=\left.\left(\frac{A(s+3)}{s}+\frac{B(s+3)}{s+1}+\frac{C(s+3)}{s+2}+D\right)\right|_{s=-3}\Rightarrow D=-1/3$$

$$\therefore f(t)=\mathbf{L}^{-1}\{F(s)\}=\frac{5}{6}-2e^{-t}+\frac{3}{2}e^{-2t}-\frac{1}{3}e^{-3t}$$

Laplace Transforms

Example. Perform partial fractions of:

$$F(s)=\frac{1}{(s+1)^3(s+2)}=\frac{As^2+Bs+C}{(s+1)^3}+\frac{D}{(s+2)}\quad \text{(repeated)}$$

$$\begin{aligned}1&=(As^2+Bs+C)(s+2)+D(s+1)^3\\&=(A+D)s^3+(2A+B+3D)s^2+(2B+C+3D)s+(2C+D)\end{aligned}$$

$$\therefore A=-D,\quad 2A+B+3D=0,\quad 2B+C+3D=0,\quad 2C+D=1$$

$$\Rightarrow A=1,\quad B=1,\quad C=1,\quad D=-1$$

– **Use of Heaviside expansion:**

$$\alpha_{r-i}=\left.\frac{1}{i!}\frac{d^{(i)}}{ds^{(i)}}\left(\frac{N(s)}{D(s)}(s+p)^r\right)\right|_{s=-p}\quad (i=0,\cdots,r-1)$$

In this example, r = 3 and p=1

Laplace Transforms

$$\frac{s^2+s+1}{(s+1)^3}=\frac{\alpha_1}{(s+1)}+\frac{\alpha_2}{(s+1)^2}+\frac{\alpha_3}{(s+1)^3}$$

$$(i=0):\ \alpha_3=\left(s^2+s+1\right)\Big|_{s=-1}=1$$

$$(i=1):\ \alpha_2=\frac{1}{1!}\frac{d}{ds}\left(s^2+s+1\right)\Big|_{s=-1}=-1$$

$$(i=2):\ \alpha_1=\frac{1}{2!}\frac{d^2}{ds^2}\left(s^2+s+1\right)\Big|_{s=-1}=1$$

$$\therefore f(t)=\{F(s)\}=\ \mathbf{L}^{-1}\left\{\frac{1}{(s+1)}+\frac{-1}{(s+1)^2}+\frac{1}{(s+1)^3}+\frac{-1}{(s+2)}\right\}$$

$$=e^{-t}-te^{-t}+\frac{1}{2}t^2e^{-t}-e^{-2t}$$

Laplace Transforms

Example. Perform partial fractions of:

$$F(s)=\frac{(s+1)}{s^2(s^2+4s+5)}=\frac{A(s+2)+B}{(s+2)^2+1}+\frac{Cs+D}{s^2}\qquad \text{(complex)}$$

$$s+1=A(s+2)s^2+Bs^2+(Cs+D)(s^2+4s+5)$$

$$=(A+C)s^3+(2A+B+4C+D)s^2+(5C+4D)s+5D$$

$$\therefore A=-C,\ \ 2A+B+4C+D=0,\ \ 5C+4D=1,\ \ 5D=1$$

$$\Rightarrow A=-1/25,\ \ B=-7/25,\ \ C=1/25,\ \ D=1/5$$

$$\frac{A(s+2)+B}{(s+2)^2+1}=-\frac{1}{25}\frac{(s+2)}{(s+2)^2+1}-\frac{7}{25}\frac{}{(s+2)^2+1}$$

$$\frac{Cs+D}{s^2}=\frac{1}{25}\frac{1}{s}+\frac{1}{5}\frac{1}{s^2}$$

$$\therefore f(t)=\mathbf{L}^{-1}\{F(s)\}=-\frac{1}{25}e^{-2t}\cos t-\frac{7}{25}e^{-2t}\sin t+\frac{1}{25}+\frac{1}{5}t$$

Laplace Transforms

Example. Perform partial fractions of:

$$F(s)=\frac{1+e^{-2s}}{(4s+1)(3s+1)}=\left(\frac{A}{4s+1}+\frac{B}{3s+1}\right)(1+e^{-2s})\quad \text{(Time delay)}$$

$$A=1/(3s+1)\big|_{s=-1/4}=4,\quad B=1/(4s+1)\big|_{s=-1/3}=-3$$

$$\therefore f(t)=\mathbf{L}^{-1}\{F(s)\}=\mathbf{L}^{-1}\left\{\frac{4}{4s+1}-\frac{3}{3s+1}\right\}+\mathbf{L}^{-1}\left\{\frac{4e^{-2s}}{4s+1}-\frac{3e^{-2s}}{3s+1}\right\}$$
$$=e^{-t/4}-e^{-t/3}+\left(e^{-(t-2)/4}-e^{-(t-2)/3}\right)S(t-2)$$

Laplace Transforms

- **Laplace transforms can be used in process control for:**

1. Solution of differential equations (linear)
2. Analysis of linear control systems (frequency response)
3. Prediction of transient response for different inputs

Laplace Transforms

- **Another useful feature of Laplace transform:**

  We can analyze the denominator of the transform to determine its dynamic behavior.

  - For example if: $Y(s)=\dfrac{1}{s^2+3s+2}$

  The denominator can be factored into (s+2)(s+1). Using the partial fraction technique:

$$Y(s)=\frac{\alpha_1}{s+2}+\frac{\alpha_2}{s+1}$$

  The process response will have exponential terms e^{-2t} and e^{-t}, which indicates y(t) approaches zero for large time (stable system).

Laplace Transforms

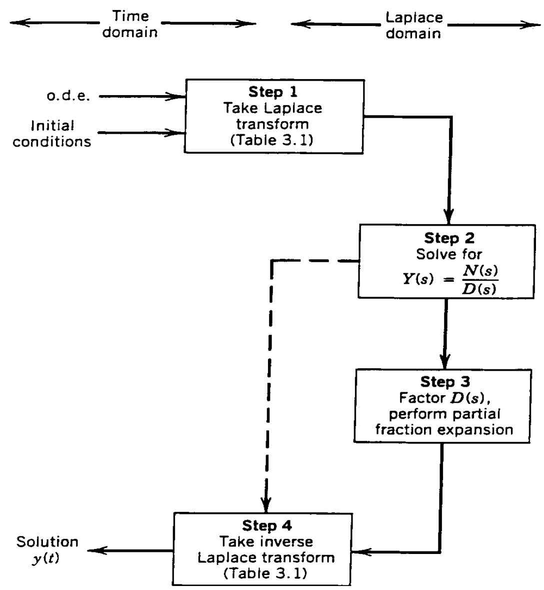
- **Another useful feature of Laplace transform:**

  - However, if for example: $Y(s)=\dfrac{1}{s^2-s-2}=\dfrac{1}{(s+1)(s-2)}$

  We know that the system is unstable and has a transient response involving e^{2t} and e^{-t}. e^{2t} is *unbounded* for large time. We shall use this concept later in the analysis of feedback system stability.

Laplace Transforms

➢ Solving ODE by Laplace Transform



Laplace Transforms

➢ Solving ODE by Laplace Transform

Example. **Solve the following ODE:** $5\frac{dy}{dt} + 4y = 2;\ y(0) = 1$

$$\mathbf{L}\left\{5\frac{dy}{dt}\right\} + \mathbf{L}\{4y\} = \mathbf{L}\{2\} \Rightarrow 5(sY(s) - y(0)) + 4Y(s) = \frac{2}{s}$$

$$(5s+4)Y(s) = \frac{2}{s} + 5 \Rightarrow Y(s) = \frac{5s+2}{s(5s+4)}$$

$$\therefore y(t) = \mathbf{L}^{-1}\{Y(s)\} = \mathbf{L}^{-1}\left\{\frac{0.5}{s} + \frac{2.5}{5s+4}\right\} = 0.5 + 0.5e^{-0.8t}$$

Laplace Transforms

➢ Solving ODE by Laplace Transform

Example. **Solve the following ODE:**

$$\frac{d^3y}{dt^3}+6\frac{d^2y}{dt^2}+11\frac{dy}{dt}+6y=4$$

$$y(0)=y'(0)=y''(0)=0$$

$$\mathbf{L}\left\{\frac{d^3y}{dt^3}+6\frac{d^2y}{dt^2}+11\frac{dy}{dt}+6y\right\}=\mathbf{L}\{4\}$$

$$s^3Y(s)+6s^2Y(s)+11sY(s)+6Y(s)=\frac{4}{s}$$

$$\therefore\ Y(s)=\frac{4}{(s^3+6s^2+11s+6)s}$$

Laplace Transforms

➢ Solving ODE by Laplace Transform

Using partial fraction decomposition:

$$\frac{4}{s(s+1)(s+2)(s+3)}=\frac{\alpha_1}{s}+\frac{\alpha_2}{s+1}+\frac{\alpha_3}{s+2}+\frac{\alpha_4}{s+3}$$

Multiply by s, set s = 0

$$\left.\frac{4}{(s+1)(s+2)(s+3)}\right|_{s=0}=\alpha_1+s\left[\frac{\alpha_2}{s+1}+\frac{\alpha_3}{s+2}+\frac{\alpha_4}{s+3}\right]\Bigg|_{s=0}$$

$$\frac{4}{1\cdot 2\cdot 3}=\alpha_1=\frac{2}{3}$$

For α_2, multiply by (s+1), set s=-1 (same procedure for α_3, α_4):

$$\alpha_2=-2,\ \alpha_3=2,\ \alpha_4=-\frac{2}{3}$$

Laplace Transforms

Solving ODE by Laplace Transform

Thus: $Y(s) = \frac{2}{3s} - \frac{2}{s+1} + \frac{2}{s+2} - \frac{2/3}{s+3}$

$$\therefore y(t) = \mathbf{L}^{-1}\{Y(s)\} = \mathbf{L}^{-1}\left\{ \frac{2}{3s} - \frac{2}{s+1} + \frac{2}{s+2} - \frac{2/3}{s+3} \right\}$$

$$= \frac{2}{3} - 2e^{-t} + 2e^{-2t} - \frac{2}{3}e^{-3t}$$

$$t \to \infty \quad y(t) \to \frac{2}{3} \qquad t = 0 \quad y(0) = 0.$$

Exercise. Resolve the dynamic system examples in topic II using Laplace Transform. Linearize the nonlinear ODE.