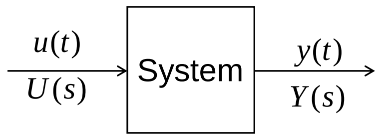
Transfer Function

➢ Definition of transfer function:

- It is an algebraic expression for the dynamic relation between the input and output of the process model



- Let *G(s)* denote the transfer function between an input, *u*, and an output, *y*. Then, by definition:

$$G(s) = \frac{Y(s)}{U(s)}$$

where:

$$Y(s) = \mathcal{L}\left[\, y(t) \,\right]$$
$$U(s) = \mathcal{L}\left[\, u(t) \,\right]$$

Transfer Function

➢ How to find transfer function; G(s):

1. If the dynamic model is nonlinear, linearize it around the desired steady-state point.
2. Write the steady-state Eq.
3. Define deviation variables and write the linearized dynamic model in terms of these deviation variables.
4. Take Laplace transform and rearrange to have the desired transfer function.

Transfer Function

Example. Find the transfer function that relates the output y with input u :

$$5\frac{dy}{dt}+4y=u; \boxed{y(0)=1}$$

Solution: $\bar{y}=y(0)=1$

- Steady-state Eq.: $0+4\bar{y}=\bar{u} \Rightarrow \bar{u}=4$
- Deviated variables: $\tilde{y}=y-\bar{y} \Rightarrow y=\tilde{y}+1$
  $\tilde{u}=u-\bar{u} \Rightarrow u=\tilde{u}+4$
- Dynamic model in terms of deviation variables:

$$5\frac{d\tilde{y}}{dt}+4(\tilde{y}+1)=\tilde{u}+4$$

$$\therefore 5\frac{d\tilde{y}}{dt}+4\tilde{y}=\tilde{u} \;;\; \tilde{y}(0)=\tilde{u}(0)=0$$

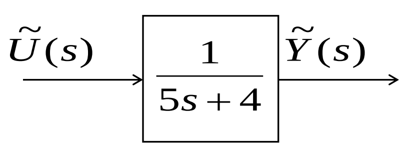
Transfer Function

- Take Laplace transform:

$$\mathbf{L}\left\{5\frac{d\tilde{y}}{dt}+4\tilde{y}\right\}=\mathbf{L}\{\tilde{u}\}$$

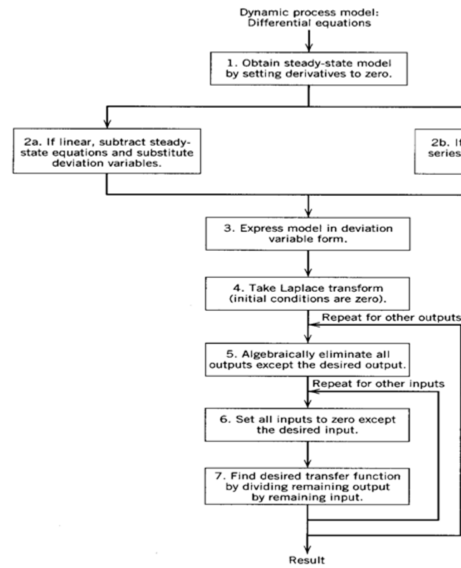
$$5s\tilde{Y}(s)+4\tilde{Y}(s)=\tilde{U}(s)$$

$$\Rightarrow G(s)=\frac{\tilde{Y}(s)}{\tilde{U}(s)}=\frac{1}{5s+4}$$



Transfer Function

General procedure for developing transfer function model:



Transfer Function

➢ **Properties of Transfer Function Models:**

- **The order of the transfer function(TF**) is defined to be the order of its denominator polynomial. Note that the **order of the TF is equal to the order of the ODE.**

  → **The TF of the previous example is 1st-order.**

- **Steady-state Gain (K**): the ratio between ultimate changes in output and input:

$$\text{Gain}=K=\frac{\Delta\text{ouput}}{\Delta\text{input}}=\frac{(y(\infty)-y(0))}{(u(\infty)-u(0))}$$

  - For a unit step change in input, the gain is the change in output
  - From the final value theorem, unit step change in input with zero initial condition gives:

$$K=\frac{y(\infty)}{1}=\lim_{s\to 0} sY(s)=\lim_{s\to 0} sG(s)\frac{1}{s}=\lim_{s\to 0} G(s)$$

Transfer Function

➢ Properties of Transfer Function Models:

- Some TF models do not have steady-state gain as for integrating processes and processes with sustaining oscillation in output.

Example. for the previous example find the static gain with unit step change in input. $\lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{1}{5s+4} = 0.25$

- **Physical realizability.** TF is physically unrealizable if the order of numerator (m) is greater than that of denominator(n). This means that the order of derivative for the input is higher than that of output.

Example: $a_0 y = b_1 \frac{du}{dt} + b_0 u$ and step change in u

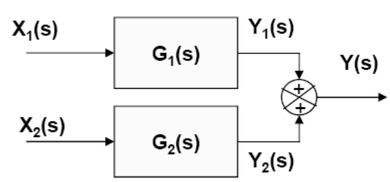
Physical realizability≡ requires future input values for current output!

Transfer Function

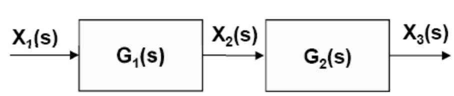
➢ Properties of Transfer Function Models:

- **Additive property:**

$$Y(s) = Y_1(s) + Y_2(s)$$
$$= G_1(s)X_1(s) + G_2(s)X_2(s)$$



- **Multiplicative property:**



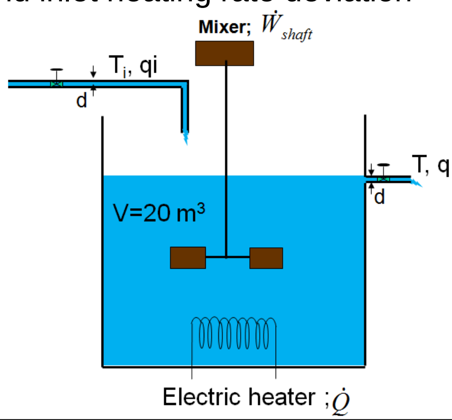
$$X_3(s) = G_2(s)X_2(s)$$
$$= G_2(s)\left[G_1(s)X_1(s)\right]$$
$$= G_2(s)G_1(s)X_1(s)$$

Transfer Function

Example: For the stirred-tank heating process with constant holdup, V, example in topic 2, find the transfer function between:

-*Outlet temperature deviation and inlet temperature deviation*.

- Outlet temperature deviation and inlet heating rate deviation



Transfer Function

See the dynamic model of this example in Topic II:

$$\Rightarrow \tau \frac{dy}{dt} = y_i - y + Ku \quad at\ t=0\ \text{(old desired st.st.)}:\ y=0$$

Where $K = \dfrac{1}{\rho q C}$; $\tau = \dfrac{V}{q}$

and "*deviation variables*" are:

$$y = T - \overline{T}$$
$$y_i = T_i - \overline{T_i}$$
$$u = \dot{Q} - \overline{\dot{Q}}$$

Take the Laplace transform of the dynamic model equation:

$$\mathbf{L}\left\{\tau \frac{dy}{dt}\right\} = \mathbf{L}\{y_i - y + Ku\}$$

$$\Rightarrow \tau s Y = Y_i - Y + KU$$

Transfer Function

Rearrange the Eq. as:

$$(\tau s+1)Y = Y_i + KU$$

$$Y = \frac{1}{(\tau s+1)}Y_i + \frac{K}{(\tau s+1)}U$$

- Transfer function between outlet temperature deviation and inlet temperature deviation is:

$$G_1(s) = \frac{Y(s)}{Y_i(s)} = \frac{1}{(\tau s+1)}$$

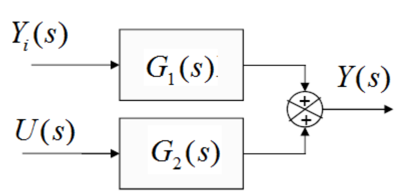
- Transfer function between outlet temperature deviation and inlet heating rate deviation is:

$$G_2(s) = \frac{Y(s)}{U(s)} = \frac{K}{(\tau s+1)}$$

$$Y = G_1(s)Y_i + G_2(s)U$$

Transfer Function

Using the addition property, this can be represented by the following block diagram:



$$G_1(s) = \frac{Y(s)}{Y_i(s)} = \frac{1}{(\tau s+1)}$$

$$G_2(s) = \frac{Y(s)}{U(s)} = \frac{K}{(\tau s+1)}$$

Transfer Function

Example. In the previous example if the liquid volume in the tank is 20 m^3 and the flow rate is 10 m^3/hr. , the liquid density is 1000 kg/m^3 and liquid heat capacity is 1 cal/g.^oC, Suppose that there is a step change in heating rate from 30 kcal/s to 99 kcal/s and there was no change in the inlet temperature. Determine the outlet deviated temperature as function of time

$$Y(s) = \frac{1}{(\tau s+1)} Y_i(s) + \frac{K}{(\tau s+1)} U(s)$$

No change in the inlet temperature: $Y_i(s) = 0$

$$Y(s) = \frac{K}{(\tau s+1)} U(s)$$

Step change in heating rate: $U(s) = \frac{99-30}{s} = \frac{69}{s}$

Transfer Function

$$K = \frac{1}{C\rho q} = \frac{1}{1000 \times 1000 \times 10} = 1 \times 10^{-7} \ {}^oC.hr/cal = 0.36^o C.s/kcal$$

$$Y(s) = \frac{0.36}{(2s+1)} \frac{69}{s} = \frac{24.8}{s(2s+1)} = 24.8\left[\frac{1}{s} - \frac{2}{2s+1}\right]$$

Take Laplace inverse transform

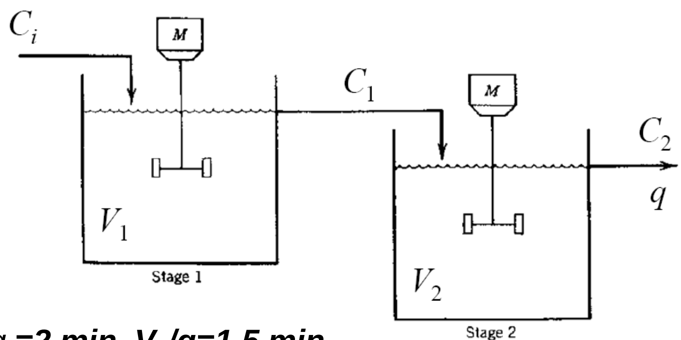
$$y(t) = 24.8\left(1 - e^{-t/2}\right)$$

If the initial steady state temperature in the tank is 50 ^oC, the new steady state values is 50+24.8=74.8 ^oC. And the steady state gain is

$$Gain = K = \frac{y(\infty) - y(0)}{u(\infty) - u(0)} = \frac{24.8 - 0}{(99-30) - 0} = 0.36^o C.s/kCal$$

Transfer Function

Example. Two stirred tanks in series for mixing of two liquid components (constant volumes, no reactions). Find the transfer function between component concentration out of 2nd tank and to the feed concentration to first tank.



Parameters: V_1/q **= 2 min,** V_2/q **= 1.5 min.**

Initial conditions: $c_1(0) = c_2(0) = 1 \text{ kg mol/m}^3$

Transfer Function

Dynamic model:

- **Apply component mole balance on tank 1:**

$$V_1 \frac{dc_1}{dt} + qc_1 = qc_i$$

- **Apply component mole balance on tank 2:**

$$V_2 \frac{dc_2}{dt} + qc_2 = qc_1$$

- **Use deviation variables, take the Laplace transforms, and rearrange to have the following transfer functions:**
  - TF between concentration out of the 1st tank to the feed concentration to the 1st tank:

$$\frac{\tilde{C}_1(s)}{\tilde{C}_i(s)} = \frac{1}{(V_1/q)s+1} = \frac{1}{(2s+1)} \quad \textbf{(1st-order TF)}$$

Transfer Function

- TF between concentration out of the 2nd tank and the feed concentration to the 2nd tank:

$$\frac{\tilde{C}_2(s)}{\tilde{C}_1(s)} = \frac{1}{(V_2/q)s+1} = \frac{1}{(1.5s+1)} \qquad \textbf{(1st-order TF)}$$

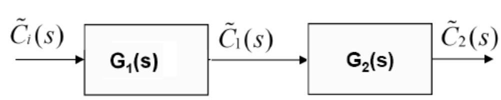
- TF between concentration out of the 2nd tank and the feed concentration to the 1st tank (**chain rule**):

$$\frac{\tilde{C}_2(s)}{\tilde{C}_i(s)} = \frac{\tilde{C}_2(s)}{\tilde{C}_1(s)}\frac{\tilde{C}_1(s)}{\tilde{C}_i(s)} = \frac{1}{\left((V_2/q)s+1\right)\left((V_1/q)s+1\right)}$$

$$= \frac{1}{(2s+1)(1.5s+1)} \qquad \textbf{(2nd -order TF)}$$

Transfer Function

Using the multiplication properly, this can be represented by the following block diagram:

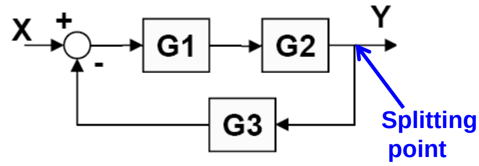


$$G_1(s) = \frac{1}{(2s+1)} \qquad G_2(s) = \frac{1}{(1.5s+1)}$$

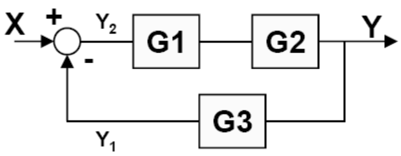
Transfer Function

Example. Use the TF block diagram shown below to find:

$Y(s)/X(s)$



Solution:



- **Y_1 and Y_2: intermediate variables**

$$Y = G_1G_2Y_2 = G_1G_2(X - Y_1)$$
$$= G_1G_2(X - G_3Y)$$
$$= G_1G_2X - G_1G_2G_3Y$$
$$\Rightarrow Y + G_1G_2G_3Y = G_1G_2X$$
$$Y(1 + G_1G_2G_3) = G_1G_2X$$
$$\therefore \frac{Y}{X} = \frac{G_1G_2}{1 + G_1G_2G_3}$$

Transfer Function

Exercise. **Determine the transfer functions between output and inputs for dynamic system examples in topic II. Draw the TF block diagram for each example.**