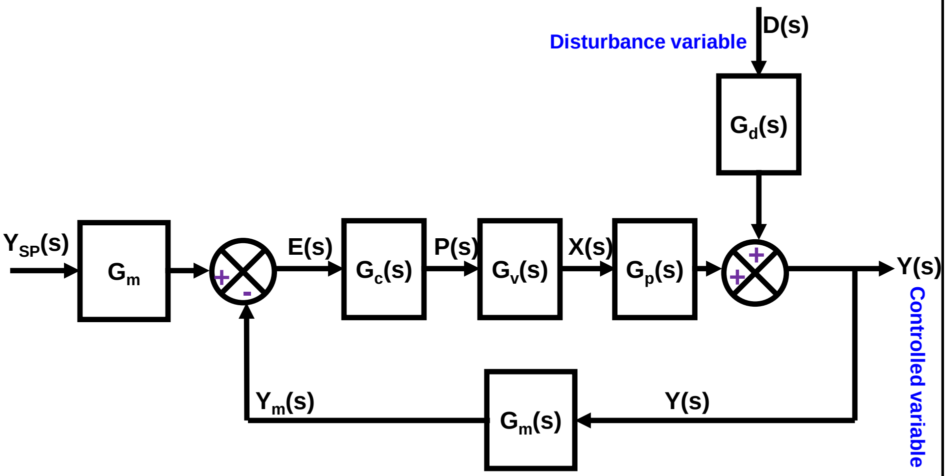
Dynamic Behavior and Stability of Closed-Loop Control System



“Standard block diagram of closed-loop feedback control system with one disturbance”

Dynamic Behavior and Stability of Closed-Loop Control System

- **Closed-Loop Transfer Functions:**
  - ➢ Using additive and multiplicative properties of transfer functions, previously explained:
    - Transfer function between controlled variable and its set point (**Servo problem**: change in set point; no changes disturbances):

$$\boxed{\frac{Y(s)}{Y_{SP}(s)}=\frac{G_mG_cG_vG_p}{1+G_mG_cG_vG_p}}$$

    - Transfer function between controlled variable and its disturbance/load (**Regulatory problem**: changes in set disturbance; no change in set point disturbances:

$$\boxed{\frac{Y(s)}{D(s)}=\frac{G_d}{1+G_cG_vG_pG_m}}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- The closed loop becomes open when the feedback path is broken. The open-loop transfer function is:

$$\boxed{G_{\mathrm{OL}} = G_m G_c G_v G_p}$$

$$\boxed{\frac{Y(s)}{Y_{\mathrm{SP}}(s)} = \frac{G_{OL}}{1+G_{OL}}} \qquad \boxed{\frac{Y(s)}{D(s)} = \frac{G_d}{1+G_{OL}}}$$

- For simultaneous changes in set point and disturbance:

$$\boxed{Y(s) = \frac{G_c G_v G_p G_m}{1+G_c G_v G_p G_m} Y_{SP}(s) + \frac{G_d}{1+G_c G_v G_p G_m} D(s)}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Mason's Rule:** for closed-loop control systems with negative feedback, the transfer function between Y and X is given by:

$$\frac{Y(s)}{X(s)} = \frac{\pi_f}{1+\pi_e}$$

π_f : Product of the transfer functions in the path from X to Y

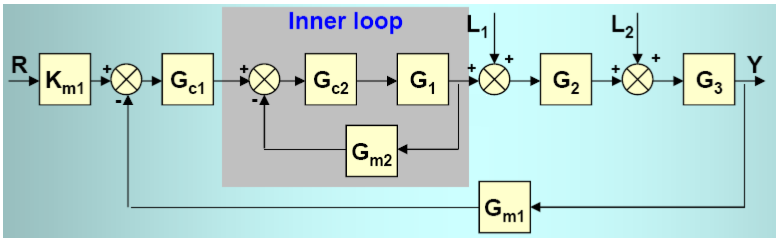
π_e : Product of all transfer functions in the entire feedback loop

→ If the control loop has **positive feedback**:

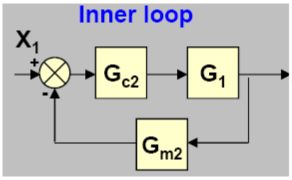
$$\frac{Y(s)}{X(s)} = \frac{\pi_f}{1-\pi_e}$$

Dynamic Behavior and Stability of Closed-Loop Control System

Example. For the control loop shown below, find the transfer functions Y/R, Y/L_1, and Y/L_2, :



Solution:



$$X_2 = \frac{G_1 G_{c2}}{1 + G_{m2} G_1 G_{c2}} X_1$$

Dynamic Behavior and Stability of Closed-Loop Control System

→Transfer function Y/R:

$$\pi_f = K_{m1} G_3 G_2 \frac{G_1 G_{c2}}{1 + G_{m2} G_1 G_{c2}} G_{c1} \qquad \pi_e = G_{m1} G_3 G_2 \frac{G_1 G_{c2}}{1 + G_{m2} G_1 G_{c2}} G_{c1}$$

$$\frac{Y}{R} = \frac{K_{m1} G_3 G_2 G_1 G_{c2} G_{c1}}{1 + G_{m2} G_1 G_{c2} + G_{m1} G_3 G_2 G_1 G_{c2} G_{c1}}$$

→Transfer function Y/L_1 :

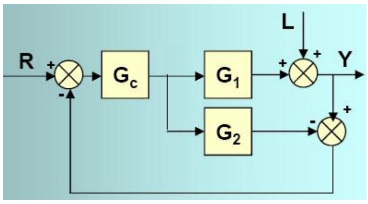
$$\frac{Y}{L_1} = \frac{G_3 G_2 (1 + G_{m2} G_1 G_{c2})}{1 + G_{m2} G_1 G_{c2} + G_{m1} G_3 G_2 G_1 G_{c2} G_{c1}}$$

→Transfer function Y/L_2:

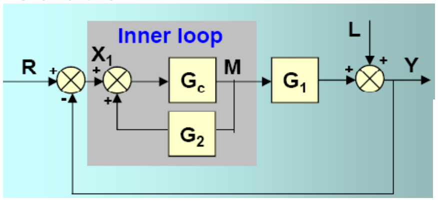
$$\frac{Y}{L_2} = \frac{G_3 (1 + G_{m2} G_1 G_{c2})}{1 + G_{m2} G_1 G_{c2} + G_{m1} G_3 G_2 G_1 G_{c2} G_{c1}}$$

Dynamic Behavior and Stability of Closed-Loop Control System

Example. For the control loop shown below, find the transfer functions Y/R and Y/L:



Solution:



$$M = \frac{G_c}{1-G_2G_c}X_1$$

$$\pi_f = \frac{G_c}{1-G_2G_c}G_1 \qquad \pi_e = \frac{G_c}{1-G_2G_c}G_1$$

$$\frac{Y}{R} = \frac{G_1G_c}{1-G_2G_c+G_1G_c} = \frac{G_1G_c}{1+(G_1-G_2)G_c}$$

$$\frac{Y}{L} = \frac{1}{1+\pi_f} = \frac{1-G_2G_c}{1+(G_1-G_2)G_c}$$

$$= \frac{1-G_2G_c}{1-G_2G_c+G_1G_c}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Stability of closed-loop control system:**

General stability criterion: A linear system is stable **if and only if** all roots (poles) of the denominator in the transfer function(TF) are negative or have negative real parts. Otherwise, the system is unstable.

→ To find the roots (poles) of the denominator in TF:

Denominator of TF=0 **"Characteristic Eq."**

- For standard closed-loop feedback control system, the characteristic Eq. is:

$$\boxed{1+G_mG_cG_vG_p = 0 \quad or \quad 1+G_{\mathrm{OL}} = 0}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Stability of closed-loop control system:**
  - The roots (poles) of the characteristic equation $(s - p_i)$ determine the type of response that occurs:

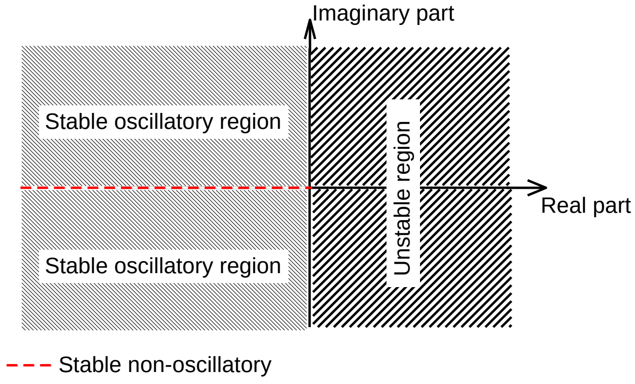
1. Real positive roots ⇒ Unstable response.
2. Real negative roots ⇒ Stable system without oscillation
3. Complex root with negative real part ⇒ Stable oscillatory response.
4. Complex roots with positive real parts ⇒ Unstable response.

Remark. Stability criterion help us to decide the action of controller whether **reverse or direct**.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Stability of closed-loop control system:**

Stability regions in the complex plane for the roots of characteristic Eq.:



→ If all roots in left half of complex plane → stable system

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example.** Standard closed-loop feedback control system has proportional controller, A/C control valve, and transmitter. The process has first-order transfer function with positive gain and space time of 9 min. Does the controller have reverse or direct action to achieve stable response?

Characteristic Eq.: $1+G_mG_cG_vG_p=0 \Rightarrow 1+K_mK_cK_v\dfrac{K_p}{\tau_p s+1}=0$

Multiply by $\tau_p s+1$:

$$(\tau_p s+1)+K_mK_cK_vK_p=0 \Rightarrow s=-\frac{1+K_mK_cK_vK_p}{\tau_p}<0$$

$$\Rightarrow (1+K_mK_cK_vK_p)>0 \Rightarrow K_mK_cK_vK_p>-1$$

Since : $K_v<0$ (A/C control valve); $K_m>0$; and $K_p>0$;
the controller gain must be negative($K_c<0$):
→ **Direct action**

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example.** Study the stability of standard closed-loop feedback control system with:

$$G_c=K_c;G_v=1/(2s+1);G_m=1;G_p=1/(5s+1)$$

Characteristic Eq.:

$$1+G_mG_cG_vG_p=0 \Rightarrow 1+(1)K_c\frac{1}{2s+1}\frac{1}{5s+1}=0$$

$$(2s+1)(5s+1)+K_c=0 \Rightarrow 10s^2+7s+K_c+1=0$$

$$s=\frac{-7\pm\sqrt{49-40(K_c+1)}}{20}<0 \Rightarrow \sqrt{49-40(K_c+1)}<7$$

$$49-40(K_c+1)<49 \Rightarrow -40(K_c+1)<0$$

$$K_c+1>0$$

$\therefore K_c>-1$ Forstability→**Reverse acting controller (K$_c$ > 0)satisfies this condition.**

Dynamic Behavior and Stability of Closed-Loop Control System

- **Stability of closed-loop control system:**
  - Sometimes it is difficult to determine the nature of the poles of characteristic equation. In such case, root-finding techniques can be used to estimate the roots.

- **Example.** Study the stability of standard closed-loop feedback control system with:

$$G_c = K_c ; G_v = 1/(2s+1); G_m = 1/(s+1); G_p = 1/(5s+1)$$

Characteristic Eq.:

$$1 + G_m G_c G_v G_p = 0 \Rightarrow 1 + \frac{1}{s+1} K_c \frac{1}{2s+1} \frac{1}{5s+1} = 0$$

$$(s+1)(2s+1)(5s+1) + K_c = 0 \Rightarrow 10s^3 + 17s^2 + 8s + K_c + 1 = 0$$

→ Difficult to determine values of K_c such that s<0.

→ Any alternative?! **Yes, there are other stability criteria.**

Dynamic Behavior and Stability of Closed-Loop Control System

- **Stability of closed-loop control system:**

A. Routh-Hurwitz stability criterion:

It is applicable for characteristic Eq. of the form:

$$a_n s^n + a_{n-1} s^{n-1} + ... + a_2 s^2 + a_1 s + a_0 = 0$$ **"Polynomial form"**

→ **Construct the Routh array:**

$$\begin{array}{c|cccc}
1 & a_n & a_{n-2} & a_{n-4} & \cdots \\
2 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
3 & b_1 & b_2 & b_3 & \cdots \\
4 & c_1 & c_2 & \cdots & \\
\vdots & \vdots & & & \\
n+1 & z_1 & & &
\end{array}$$

Coefficients determinations:

$$b_1 = (a_{n-1}a_{n-2} - a_n a_{n-3}) / a_{n-1}$$

$$b_2 = (a_{n-1}a_{n-4} - a_n a_{n-5}) / a_{n-1}$$

$$\vdots$$

$$c_1 = (b_1 a_{n-3} - a_{n-1} b_2) / b_1$$

$$c_2 = (b_1 a_{n-5} - a_{n-1} b_3) / b_1$$

$$\vdots$$

$$b_1 = -\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} / a_{n-1}$$

$$b_2 = -\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} / a_{n-1}$$

$$c_1 = -\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} / b_1$$

$$c_2 = -\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} / b_1$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Stability of closed-loop control system:**

A. Routh-Hurwitz stability criterion:

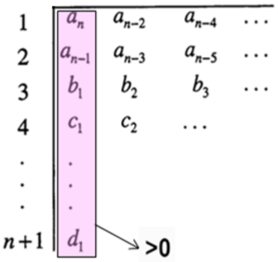
- **A necessary condition for stability**:

→ **all coefficients of characteristic Eq. (a_i's) *are positive:***

$$a_n s^n + a_{n-1} s^{n-1} + ... + a_2 s^2 + a_1 s + a_0 = 0 \ (a_i > 0 \ \ i = 0,...n)$$

- **A necessary and sufficient condition for stability:**

→ **All of the elements in the left column of the Routh array are positive."**



1	a_n	a_{n-2}	a_{n-4}	$\cdots$
2	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
3	b_1	b_2	b_3	$\cdots$
4	c_1	c_2	$\cdots$	
.	.			
.	.			
.	.			
$n+1$	d_1			

>0

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example.** Use Routh-Hurwitz stability criterion to study the stability of standard closed-loop feedback control system given in previous example:

Characteristic Eq.: $10s^3 + 17s^2 + 8s + K_c + 1 = 0$

– **Necessary condition:**

$$a_3 = 10 > 0$$
$$a_2 = 17 > 0$$
$$a_1 = 8 > 0$$
$$a_0 = K_c + 1 > 0 \Rightarrow K_c > -1$$

"For stability"

→ If any coefficient is not positive, stop and conclude the system is unstable.

Dynamic Behavior and Stability of Closed-Loop Control System

- Necessary and sufficient condition:

Routh array:

$$\begin{array}{c|cc} 1 & a_3 & a_1 \\ 2 & a_2 & a_0 \\ 3 & b_1 & b_2 \\ 4 & c_1 & \end{array} \rightarrow \begin{array}{c|cc} 1 & 10 & 8 \\ 2 & 17 & K_c+1 \\ 3 & 7.41-0.588K_c & 0 \\ 4 & 1+K_c & \end{array}$$

$$b_1 = \frac{17(8)-10(1+K_c)}{17} = 7.41-0.588K_c \qquad b_2 = \frac{17(0)-10(0)}{17} = 0$$

$$c_1 = \frac{b_1(1+K_c)-17(0)}{b_1} = 1+K_c$$

Stable region: $K_c + 1 > 0 \Rightarrow K_c > -1$

$$7.41 - 0.588K_c > 0 \Rightarrow K_c < 12.6$$

$\therefore -1 < K_c < 12.6$ **"For stability without oscillation"**

Dynamic Behavior and Stability of Closed-Loop Control System

- **Stability of closed-loop control system:**

B. Direct substitution stability criterion:

- This stability criterion is based on the fact that the imaginary axis is the dividing line between stable and unstable systems.
- **Procedure:**
  1. Substitute $s = j\omega$ into characteristic equation.
  2. Obtain two equations: one for real part and the another for imaginary part,
  3. Solve the two equations to obtain values of K_{cm} and ω. *Where* K_{cm} the maximum controller gain at which the roots of characteristic equation crosses the imaginary axis.
  4. Determine the stable region by trying test values of K_c in the characteristic Eq.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example.** Use direct substitution stability criterion to study the stability of standard closed-loop feedback control system given in previous example:

Characteristic Eq.: $10s^3+17s^2+8s+K_c+1=0$

→ **Set $s = j\omega$**

$$-10j\omega^3-17\omega^2+8j\omega+1+K_{cm}=(1+K_{cm}-17\omega^2)+j\omega(8-10\omega^2)=0$$

Real part Eq. : $(1+K_{cm}-17\omega^2)=0$

Imaginary part Eq. : $\omega(8-10\omega^2)=0$

Solve to obtain:

$\omega=0 \text{ or } \omega^2=0.8$

$\Rightarrow K_{cm}=-1 \text{ or}$

$K_{cm}=12.6$

→ **Try a test point such as: K$_c$=0**

$10s^3+17s^2+8s+1=0$ → Stable: All +ve coefficients:

→ **Thus, the stable /non-oscillation region is:** $-1<K_c<12.6$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example.** Use direct substitution stability criterion to study the stability of the system with the following characteristic Eq.:

$$1+5s+2K_c e^{-s}=0$$

set $s = j\omega$: $1+5j\omega+2K_{cm}\mathrm{e}^{-j\omega}=0$

But, $\mathrm{e}^{-j\omega}=\cos\omega-j\sin\omega$

$$\Rightarrow 1+5j\omega+2K_{cm}(\cos\omega-j\sin\omega)=0$$

Real part Eq.: $1+2K_{cm}\cos\omega=0\Rightarrow 2K_{cm}=-\dfrac{1}{\cos\omega}$

Imaginary part Eq.: $5\omega-2K_{cm}\sin\omega=0\Rightarrow 5\omega+\tan\omega=0$

Solve to obtain: $\omega=1.69$

$K_{cm}=4.25$

→ **Try a test point such as: K$_c$=0:** $1+5s=0$

→ **Stable: All +ve coefficients:** $K_c<4.25$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example.** Use Routh-Hurwitz stability criterion to study the stability of the system with the following characteristic Eq.:

$$1+5s+2K_c e^{-s}=0$$

This characteristic Eq. **does NOT have polynomial form** to use Routh-Hurwitz stability. It can be rewritten in a polynomial form using 1/1 Pade Approximation:

$$e^{-\theta s}\approx\frac{1-\frac{\theta}{2}s}{1+\frac{\theta}{2}s}$$

$$\Rightarrow 1+5s+2K_c\frac{1-0.5s}{1+0.5s}=0$$

$$\Rightarrow (1+0.5s)(1+5s)+2K_c(1-0.5s)=0$$

$$\Rightarrow 2.5s^2+(5.5-K_c)s+(1+2K_c)=0$$

Dynamic Behavior and Stability of Closed-Loop Control System

→ **Necessary condition:** $a_1=5.5-K_c>0\Rightarrow K_c<5.5$

$$a_0=1+2K_c>0\Rightarrow K_c>-0.5$$

$$\therefore -0.5<K_c<5.5$$ **"For stability without oscillation"**

→ **Necessary and sufficient condition:**

Routh array:

1	a_2	a_0
2	a_1	0
3	b_1	

→

1	2.5	$1+2K_c$
2	$5.5-K_c$	0
3	$1+2K_c$	0

$$5.5-K_c>0\Rightarrow K_c<5.5$$

$$\Rightarrow 1+2K_c>0\Rightarrow K_c>-0.5$$

$$\therefore -0.5<K_c<5.5$$

Remark. In this example, Routh array does not add additional information but it confirms the stable region.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Stability of closed-loop control system:**
  - Routh-Hurwitz stability criterion with 1/1 Pade approximation of the exponential term gives maximum controller gain of K_{cm}=5.5. The exact value resulted from direct substitution criterion is K_{cm}=4.25. The percent relative error is around 28%.
- **Exercise.** Resolve the previous example using 2/2 Pade approximation(more accurate than 1/1) given by:

$$e^{-\theta s} \approx \left[1-\frac{\theta}{2}s+\frac{\theta^2}{12}s^2\right] \Big/ \left[1+\frac{\theta}{2}s+\frac{\theta^2}{12}s^2\right]$$

  - Routh-Hurwitz stability criterion with 2/2 Pade approximation of the exponential term gives maximum controller gain of K_{cm}=4.29. The percent relative error is around 1%.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Stability of closed-loop control system:**

Root locus diagram: Complex plane diagram shows the location of closed-loop poles (roots of characteristic equation) depending on the parameter value such as controller gain K_c (single parametric study).

  - It can be built by finding the roots at a different values of the parameter under investigation such as K_c.

- **Example.** Consider a feedback control system with open-loop transfer function:

$$G_{OL}(s)=\frac{4K_C}{(s+1)(s+2)(s+3)}$$

Plot the root locus diagram for $0 \le K_c \le 20$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Stability of closed-loop control system:**

Root locus diagram

Characteristic Eq. :

$$1 + G_{OL}(s) = 0 \Rightarrow (s+1)(s+2)(s+3) + 4K_C = 0$$

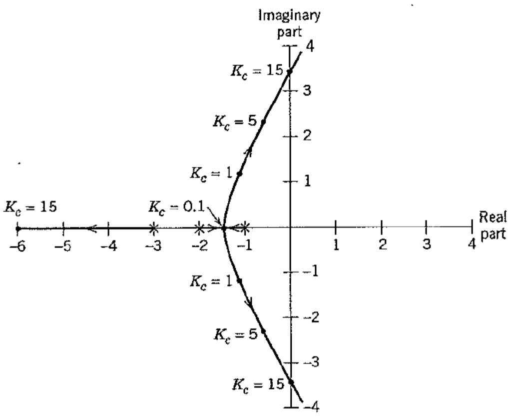
- At K_c = 0 (no controller; open loop): roots= -1,-2,-3
- At K_c =0.1: roots=……
- At K_c=1; roots=……
- At K_c=5; roots =…..
- At K_c=15, roots= -6,3.5j,-3.5j

- Localize these roots at each K_c on the complex plane to plot the root locus diagram.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Stability of closed-loop control system:**

Root locus diagram

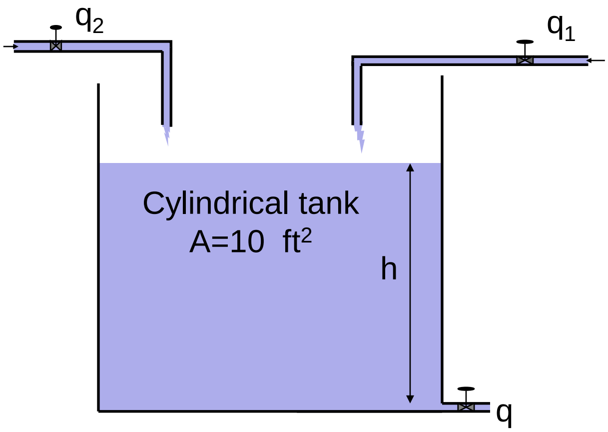


- **It is clear from root locus diagram that:**

1. The closed loop system is unstable for K_c >15.
2. The closed loop response will be stable for 0.1<K_c <15.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank** with two inlet streams and one outlet stream as shown below:



a) What is the dynamic model that describe the liquid height variation with time? Perform DOF analysis.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**
  - **DOF Analysis:**

→ **Parameters:** A, ρ, C_v , and C_c (**See topic II**).

→ $N_F = N_V - N_E$

N_V = 4 (q_1, q_2, q , h)
N_E = 2 (MB, MEB)

N_F = 4-2 = 2→ two input variables should be specified.

→ **The output variables**: q and h

→ **Input variables:** q_1 and q_2

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**
  - **Dynamic model:**
    - **Mass Balance (under constant density assumption):**

$$A\frac{dh}{dt} = q_1 + q_2 - q$$

    - **Mechanical Energy Balance(MEB); (see topic II):**

$$q = R\sqrt{h} \qquad \text{where } R = \frac{\pi d^2}{4}\sqrt{\frac{2g}{1 + C_c + C_v}}$$

$$\Rightarrow A\frac{dh}{dt} = q_1 + q_2 - R\sqrt{h}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

b) Write the model in deviated form:

Linearize the nonlinear terms in the dynamic equation around the desired steady state (denoted by overbar):
the only nonlinear term nonlinear term is $\sqrt{h}$:

$$f(h) = \sqrt{h} \rightarrow f(h) \approx f(\bar{h}) + \left.\frac{df}{dh}\right|_{h=\bar{h}}(h - \bar{h}) = \sqrt{\bar{h}} + \frac{1}{2\sqrt{\bar{h}}}(h - \bar{h})$$

$$\Rightarrow A\frac{dh}{dt} = q_1 + q_2 - R\sqrt{\bar{h}} - \frac{R}{2\sqrt{\bar{h}}}(h - \bar{h})....................(1)$$

Steady state Eq.: $0 = \bar{q}_1 + \bar{q}_2 - R\sqrt{\bar{h}} - 0..................(2)$

and $\bar{q} = R\sqrt{\bar{h}}$

Subtract Eq. 2 from Eq. 1 to have: $\Rightarrow A\dfrac{dH}{dt} = Q_1 + Q_2 - \dfrac{R}{2\sqrt{\bar{h}}}H$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

c) Write the transfer functions between the liquid height and flow rate of stream 1 and stream 2. Is this stable process?

Take Laplace transform of deviated dynamic model:

$$AsH(s)+\frac{R}{2\sqrt{\bar{h}}}H(s)=Q_1(s)+Q_2(s)$$

$$H(s)=\frac{1}{As+R/(2\sqrt{\bar{h}})}Q_1(s)+\frac{1}{As+R/(2\sqrt{\bar{h}})}Q_2(s)$$

$$\Rightarrow \frac{H(s)}{Q_1(s)}=\frac{H(s)}{Q_2(s)}=\frac{1/A}{s+\dfrac{R}{2\sqrt{\bar{h}}A}}=G(s)$$

"First-order TFs"

→It is stable process since the root of characteristic Eq. is always negative: $s=-R/(2\sqrt{\bar{h}}A)<0$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

d) If the outlet flow rate was 18 ft³/min at liquid height of 9 ft what is the value of the coefficient R?

→ **MEB:** $q=R\sqrt{h}\Rightarrow 18=R\sqrt{9}\Rightarrow R=6\ \text{ft}^{2.5}/\text{min}$

e) It is desired to operate the process steadily with liquid height of 4 ft. If the inlet flow rate of stream 2 is 5 ft³/min, find the steady-state flow rates of stream 1 and outlet stream.

→ **MEB:** $\bar{q}=6\sqrt{4}=12\,\text{ft}^3/\text{min}$

→ **Steady MB:**

$$0=\bar{q}_1+\bar{q}_2-\bar{q}$$

$$\bar{q}_2=5\,\text{ft}^3/\text{min}\ ;\ \bar{q}=12\,\text{ft}^3/\text{min}\Rightarrow \bar{q}_1=\bar{q}-\bar{q}_2=7\,\text{ft}^3/\text{min}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

f) Now a step change disturbance occurs suddenly in the flow rate of stream 2 to become q_2=8 ft^3/min and remains at this new value. While flow rate of stream 1 remains as before (q_1=7 ft^3/min). Find the liquid height response and the steady-state offset. Approximate the settling time.

$$H(s) = G(s)Q_1(s) + G(s)Q_2(s)$$

No change in flow rate of stream 1 $\rightarrow Q_1(s) = 0$

Step change in flow rate of stream 2$\rightarrow Q_2(s) \neq 0$

$$Q_2(t) = q_2(t) - \bar{q}_2 = 8 - 5 = 3\,\text{ft}^3/\text{min} \rightarrow Q_2(s) = \frac{3}{s}$$

$$G(s) = \frac{1/A}{s + R\big/(2\sqrt{\bar{h}}\,A)} = \frac{1/10}{s + 6\big/(2\times\sqrt{4}\times 10)} = \frac{0.1}{s + 0.15}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

$$\Rightarrow H(s) = G(s)Q_2(s) = \frac{0.1}{s+0.15}\frac{3}{s} = \frac{0.3}{s(s+0.15)}$$

$$= \frac{A}{s} + \frac{B}{(s+0.15)} = \frac{2}{s} - \frac{2}{s+0.15} \xrightarrow{L^{-1}} H(t) = 2 - 2e^{-0.15t}$$

$\rightarrow$ **Steady-state offset e(t→∞):**

$$e(t\rightarrow\infty) = \bar{H}_{sp} - H(t\rightarrow\infty) = 0 - 2\,\text{ft} \neq 0$$

$\Rightarrow$ When disturbance occurs, the liquid height will not return to the desired steady-state height. **This means that there is a need for controller.**

$\rightarrow$ The actual height, *h*, varies with time according:

$$H(t) = h(t) - \bar{h}$$

$$\Rightarrow h(t) = 6 - 2e^{-0.15t}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

→ **The new steady-state value(t→∞) of height is:**

$$h(t \to \infty) = 6\,\text{ft}$$

This is approximate value since it came from the linearized model. The exact ultimate steady-state value of the liquid height can be calculated from steady-state mass balance.

$$0 = \bar{q}_1 + \bar{q}_2 - R\sqrt{\bar{h}} \Rightarrow \bar{h}_{new} = \left(\frac{\bar{q}_1 + \bar{q}_2}{R}\right)^2 = 6.25\,\text{ft}$$

⇒ The percent error is 4 % (acceptable error, linearization is good approximation).

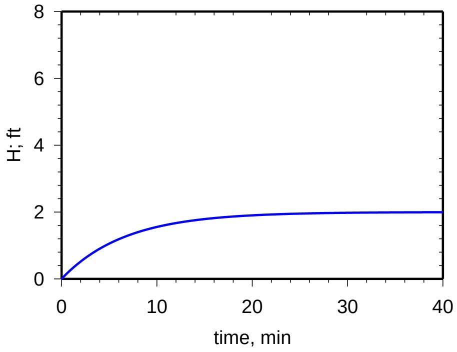
→ **Settling time:** to reach either 0.99 or 1.01 of the ultimate steady state value (choose a suitable value to avoid logarithm of negative value) : $0.99(6) \approx 6 - 2e^{-0.15t_s}$

$$\Rightarrow t_s \approx 23.4\,\text{min}$$

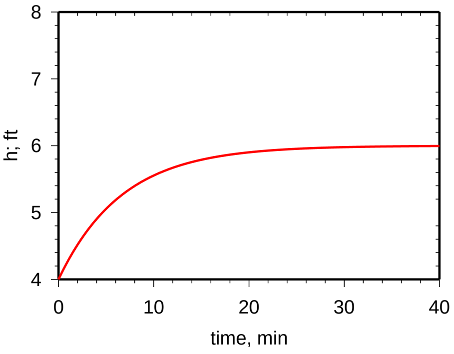
Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

- Response of deviated liquid height to step change:



- Response of liquid height to step change:



Offset = -2 ft ≠0 ⇒ A need for controller to keep height at 4 ft

Dynamic Behavior and Stability of Closed-Loop Control System

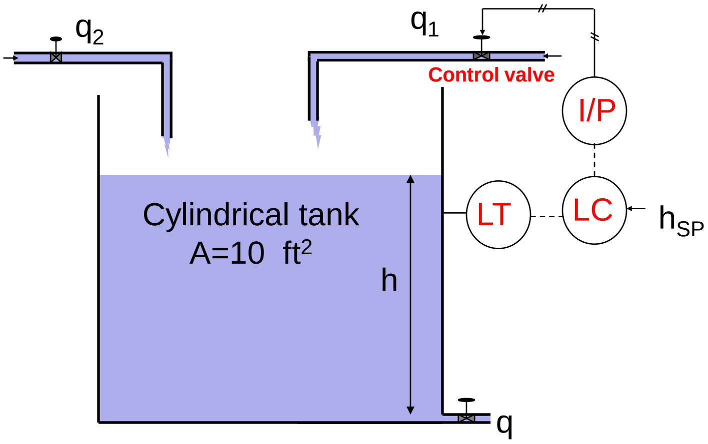
- **Example. Liquid storage tank**

g) **A feed-back closed-loop control system is built now to remove/reduce the offset in the liquid height. The flow rate of stream 1 is chosen as manipulated variable. The controller signal is electric. The control valve on stream 1 is A/O pneumatic one with a linear trim. When the control valve is fully opened, the flow rate is 24 ft^3/min. The level transmitter has a span of 8 ft.**

- ➢ Draw the liquid height closed-loop feedback control system.
- ➢ Draw the corresponding closed-loop transfer function block diagram.
- ➢ Write the closed-loop transfer functions between deviated liquid height, H and deviated manipulated variable, Q_1, and deviated disturbance Q_2.

Dynamic Behavior and Stability of Closed-Loop Control System

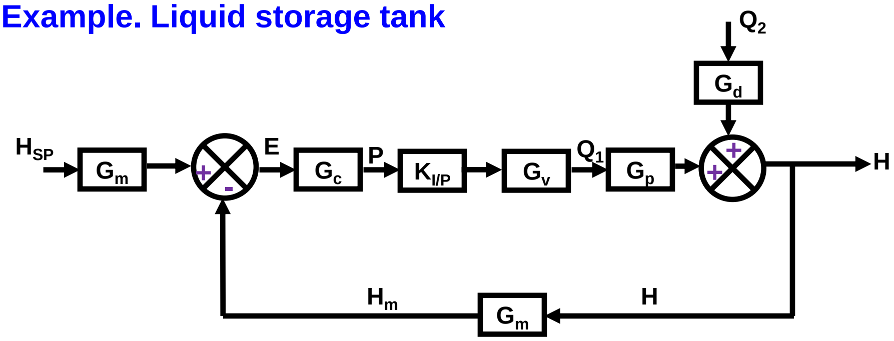
- **Example. Liquid storage tank**



“Liquid level closed-loop feedback control system”

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**



"Transfer function block diagram for liquid level closed-loop feedback control system"

$$H(s) = \frac{G_c G_v K_{IP} G_p G_m}{1 + G_c G_v K_{IP} G_p G_m} H_{SP}(s) + \frac{G_d}{1 + G_c G_v K_{IP} G_p G_m} Q_2(s)$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

→ **$K_{I/P}$**: Gain of current-to-pressure transducer

$$K_{IP} = \frac{15-3}{20-4} = 0.75\,\text{psig/mA}$$

→ **$G_m(s)$**: Assume zero –order level-to-current transmitter:

$$G_m(s) = K_m = \frac{20-4}{8} = 2\,\text{mA/ft}$$

→ **$G_v(s)$**: Assume zero–order control valve: $G_v(s) = K_v$

Valve with linear trim $\Rightarrow \dfrac{\Delta Q_1}{\Delta L} = q_{FO} = 24\,\text{ft}^3/\text{min}$

$$\therefore K_v = \frac{\Delta Q_1}{\Delta L}\frac{\Delta L}{\Delta P} = 24\frac{1-0}{15-3} = 2\,\frac{\text{ft}^3/\text{min}}{\text{psig}}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

→ **$G_p(s)$**: $G_p(s)=\dfrac{H(s)}{Q_1(s)}=\dfrac{0.1}{s+0.15}$

→ **$G_d(s)$**: $G_d(s)=\dfrac{H(s)}{Q_2(s)}=\dfrac{0.1}{s+0.15}$

$$H(s)=\frac{(0.75)(2)(2)G_cG_p}{1+(0.75)(2)(2)G_cG_p}H_{SP}(s)+\frac{G_d}{1+(0.75)(2)(2)G_cG_p}Q_2(s)$$

$$=\frac{3G_c\dfrac{0.1}{s+0.15}}{1+3G_c\dfrac{0.1}{s+0.15}}H_{SP}(s)+\frac{\dfrac{0.1}{s+0.15}}{1+3G_c\dfrac{0.1}{s+0.15}}Q_2(s)$$

$$=\frac{0.3G_c}{s+0.15+0.3G_c}H_{SP}(s)+\frac{0.1}{s+0.15+0.3G_c}Q_2(s)$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

h) Suppose that a proportional controller is used. Verify that the controller must work on the reverse mode.

→ **P controller: $G_c(s)=K_c$**

$$H(s)=\frac{0.3K_c}{s+0.15+0.3K_c}H_{SP}(s)+\frac{0.1}{s+0.15+0.3K_c}Q_2(s)$$

→ **Characteristic Eq. :** $s+0.15+0.3K_c=0$

→ Root of characteristic Eq. :

$$s=-(0.15+0.3K_c)<0\Rightarrow 0.3K_c>-0.15\Rightarrow K_c>-0.5$$

∴ To have regulatory or servo problem with stable non-oscillatory response, **K_c must be positive**. Thus, the controller must work on the **reverse mode**.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

i) If step change of 3 ft^3/min occurs in the disturbance and the liquid height set point remains the same, will the P controller remove the offset? Show controller gain effect on the offset.

→ **This is regulatory problem:** $H_{SP}(s) = 0$

→ Step change in q_2 of 3 → $Q_2(s) = 3/s$

$$H(s) = \frac{0.1}{s+0.15+0.3K_c} Q_2(s) = \frac{0.1}{s+0.15+0.3K_c}\frac{3}{s}$$

$$= \frac{0.3}{s(s+0.15+0.3K_c)} = \frac{A}{s} + \frac{B}{s+0.15+0.3K_c}$$

$$= \frac{0.3/(0.15+0.3K_c)}{s} - \frac{0.3/(0.15+0.3K_c)}{s+0.15+0.3K_c}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

$$H(t) = \mathbf{L}^{-1}[H(s)] = \mathbf{L}^{-1}\left[\frac{0.3/(0.15+0.3K_c)}{s}\right] - \mathbf{L}^{-1}\left[\frac{0.3/(0.15+0.3K_c)}{s+0.15+0.3K_c}\right]$$

$$= \frac{0.3}{0.15+0.3K_c}\left(1-e^{-(0.15+0.3K_c)t}\right)$$

→ For regulatory problem; steady-state offset e(t→∞) is given by:

$$e(t\to\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s(H_{sp}(s) - H(s)) = -\lim_{s\to 0} s(H(s))$$

$$= -H(t\to\infty) = -\frac{0.3}{0.15+0.3K_c}$$

⇒ For **regulatory problem**:

- as K_c increases the steady-state offset decreases.
- For regulatory problem with first-order open-loop TF: $K_c\to\infty$, offset = 0

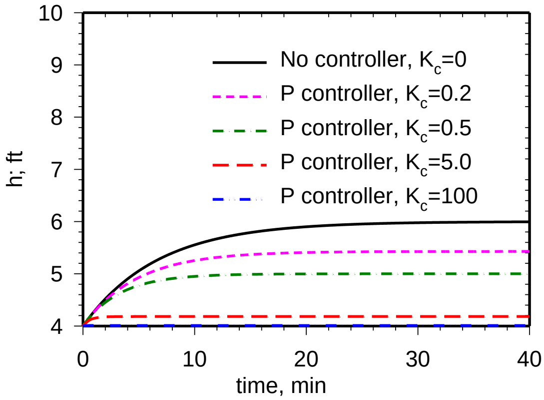
Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

→ **The actual height varies with time according**:

$$h(t) = 4 + \frac{0.3}{0.15+0.3K_c}\left(1-e^{-(0.15+0.3K_c)t}\right)$$

Kc	Offset; ft
0	-2.000
0.2	-1.429
0.5	-1.000
5	-0.182
10	-0.095
100	- 0.010



- It is clear that as K_c increases settling time and offset decreases.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

j) For step change in the set point from 4 to 9 ft and flow rate of stream 2 remains at 5 ft^3/min, Will the P controller remove the offset? Study the effect of controller gain on the offset.

→ **This is servo problem:** $Q_2(s) = 0$

→ **Step change in h of 3 :** $H_{SP}(s) = (9-4)/s = 5/s$

$$H(s) = \frac{0.3K_c}{s+0.15+0.3K_c}H_{SP}(s) = \frac{0.3K_c}{s+0.15+0.3K_c}\,\frac{5}{s}$$

$$= \frac{1.5K_c}{s(s+0.15+0.3K_c)} = \frac{1.5K_c/0.15+0.3K_c}{s} - \frac{1.5K_c/0.15+0.3K_c}{(s+0.15+0.3K_c)}$$

$$H(t) = \mathbf{L}^{-1}\left[H(s)\right] = \frac{1.5K_c}{0.15+0.3K_c}\left(1-e^{-(0.15+0.3K_c)t}\right)$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

→ For servo problem; steady-state offset e(t→∞) is given by:

$$e(t\to\infty)=\lim_{s\to 0} sE(s)=\lim_{s\to 0} s(\frac{5}{s}-H(s))=5-\lim_{s\to 0} s(H(s))$$

$$=5-H(t\to\infty)=5-\frac{1.5K_c}{0.15+0.3K_c}$$

→ Without controller: $K_c=0:e(t\to\infty)=5$

→ With controller: $K_c\to\infty:e(t\to\infty)=5-1.5/0.3=0$

⇒ For **servo problem**:

- As regulatory problem, as K_c increases the offset decreases. Offset becomes zero when K_c→∞.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

k) Suppose that the control valve transfer function is first-order with time constant of 0.1 min and there is step change in the set point of 5 ft. With P-controller only, minimize the offset such that the decay ratio does not exceed 0.25:

→ Now: $G_v(s)=K_v/(\tau_v s+1)=2/(0.1s+1)$

→ This is servo problem: $H_{SP}(s)=5/s \;\; ; \;\; Q_2(s)=0$

$$\frac{H(s)}{H_{SP}(s)}=\frac{G_cG_vK_{IP}G_pG_m}{1+G_cG_vK_{IP}G_pG_m}$$

$$=\frac{K_c\frac{2}{0.1s+1}(0.75)\frac{0.1}{s+0.15}(2)}{1+K_c\frac{2}{0.1s+1}(0.75)\frac{0.1}{s+0.15}(2)}=\frac{0.3K_c}{(0.1s+1)(s+0.15)+0.3K_c}$$

"2nd-order open-loop TF"

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

$$H(s)=\frac{0.3K_c}{(0.1s+1)(s+0.15)+0.3K_c}\frac{5}{s}$$

$$e(t\to\infty)=\lim_{s\to 0} sE(s)=\lim_{s\to 0} s(\frac{5}{s}-H(s))=5-\lim_{s\to 0} s(H(s))$$

$$\lim_{s\to 0} s(H(s))=\lim_{s\to 0}\left[s\frac{0.3K_c}{(0.1s+1)(s+0.15)+0.3K_c}\frac{5}{s}\right]=\frac{1.5K_c}{0.15+0.3K_c}$$

$$e(t\to\infty)=5-\frac{1.5K_c}{0.15+0.3K_c}$$

$$K_c\to\infty : offset=0$$

→ Again, as controller gain increases the offset decreases. However higher value of K_c will give oscillatory response.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

→ We have already studied the dynamic behavior of 2nd-order TF with step change of the general form:

$$H(s)=\frac{K}{s(\tau^2s^2+2\tau\xi s+1)}$$

→ Let us rewrite the response equation to have this form:

$$H(s)=\frac{0.3K_c}{(0.1s+1)(s+0.15)+0.3K_c}\frac{5}{s}$$

$$=\frac{1.5K_c/(0.15+0.3K_c)}{s(\frac{0.1}{0.15+0.3K_c}s^2+\frac{1.015}{0.15+0.3K_c}s+1)}\equiv\frac{K}{s(\tau^2s^2+2\tau\xi s+1)}$$

$$\Rightarrow K=\frac{1.5K_c}{0.15+0.3K_c};\tau=\sqrt{\frac{0.1}{(0.15+0.3K_c)}};\xi=\frac{1.6049}{\sqrt{(0.15+0.3K_c)}}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

→ **At critical damping coefficient; ζ=1:**

$$\xi = 1 = \frac{1.6049}{\sqrt{(0.15+0.3K_c)}} \Rightarrow K_c = 8.086$$

⇒ **The response is:**

→ **Critically damped (ζ=1 :K_c=8.086):** $H(t) = K\left[1-(1+t/\tau)e^{-t/\tau}\right]$

→ **Overdapmed (ζ>1 :0 ≤ K_c< 8.086):**

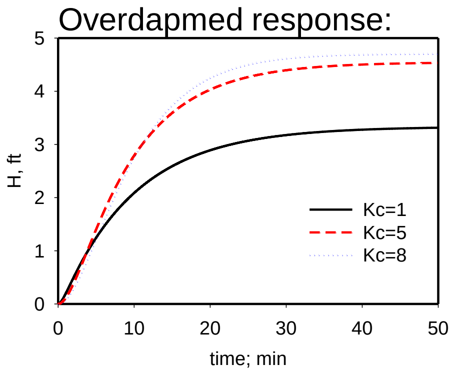
$$H(t) = K\left[1-\frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1-\tau_2}\right] \qquad ;\tau = \sqrt{\tau_1\tau_2};\zeta = \frac{\tau_1+\tau_2}{2\sqrt{\tau_1\tau_2}}$$

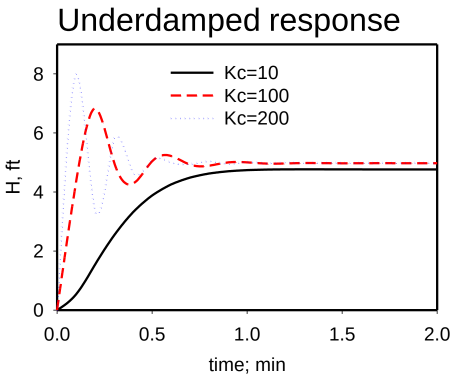
→ **Underdapmed (ζ<1: K_c> 8.086):**

$$H(t) = K\left[1-e^{\frac{-\zeta t}{\tau}}\left\{\cos(\alpha t)+\frac{\zeta\sin(\alpha t)}{\alpha\tau}\right\}\right] \; ; \; \alpha = \frac{\sqrt{1-\zeta^2}}{\tau}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**





→ For underdamped response: as K_c increases oscillations and overshooting increase.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

→ **At decay ratio, DR=0.25:**

$$\mathrm{DR} = 0.25 = e^{(-2\pi\xi/\sqrt{1-\xi^2})} \Rightarrow \xi = 0.215$$

$$\xi = 0.215 = \frac{1.6049}{\sqrt{(0.15+0.3K_c)}} \Rightarrow K_c = 185.2$$

→ Therefore, the controller gain must not exceed 185.2 to have decay ratio less than or equals to 0.25. At this gain value, the offset is:

$$e(t\to\infty) = 5 - \frac{1.5K_c}{0.15+0.3K_c} = 5 - \frac{1.5(185.2)}{0.15+(0.3)(185.2)} = 0.013\,\mathrm{ft}$$

$$\tau = \sqrt{\frac{0.1}{(0.15+0.3K_c)}} = 0.0424\,\mathrm{min}$$

→ **Settling time:** $t_s \approx -\tau \ln 0.05/\xi = 0.59\,\mathrm{min}$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

L) If PI controller is used instead of P controller. Do you think that this controller is able to eliminate the liquid height offset for step changes in either set point or flow rate of stream 2? Assume zero-order TF for control valve.

→**PI controller:** $G_c(s) = K_c(1+\frac{1}{\tau_I s}) = K_c\frac{\tau_I s+1}{\tau_I s}$

$$H(s) = \frac{0.3G_c}{s+0.15+0.3G_c}H_{SP}(s) + \frac{0.1}{s+0.15+0.3G_c}Q_2(s)$$

$$= \frac{0.3K_c\frac{\tau_I s+1}{\tau_I s}}{s+0.15+0.3K_c\frac{\tau_I s+1}{\tau_I s}}H_{SP}(s) + \frac{0.1}{s+0.15+0.3K_c\frac{\tau_I s+1}{\tau_I s}}Q_2(s)$$

$$= \frac{0.3K_c(\tau_I s+1)}{\tau_I s(s+0.15)+0.3K_c(\tau_I s+1)}H_{SP}(s) + \frac{0.1\tau_I s}{\tau_I s(s+0.15)+0.3K_c(\tau_I s+1)}Q_2(s)$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid Storage tank**

$$H(s)=\frac{0.3K_c\tau_I s+0.3K_c}{\tau_I s^2+(0.15+0.3K_c)\tau_I s+0.3K_c}H_{SP}(s)+\frac{0.1\tau_I s}{\tau_I s^2+(0.15+0.3K_c)\tau_I s+0.3K_c}Q_2(s)$$

→**Step change in set point:** $H_{SP}(s)=a/s; Q_2(s)=0$

→ **Using final value theorem:**

$$H(t\to\infty)=\lim_{s\to 0}sH(s)=\lim_{s\to 0}\left[\frac{a(0.3K_c\tau_I s+0.3K_c)}{\tau_I s^2+(0.15+0.3K_c)\tau_I s+0.3K_c}\right]=a$$

$$e(t\to\infty)=H_{sp}(t\to\infty)-H(t\to\infty)=a-a=0$$

⇒ **For servo problem**, PI controller eliminates the offset for any value of gain and integral time.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

$$H(s)=\frac{0.3K_c\tau_I s+0.3K_c}{\tau_I s^2+(0.15+0.3K_c)\tau_I s+0.3K_c}H_{SP}(s)+\frac{0.1\tau_I s}{\tau_I s^2+(0.15+0.3K_c)\tau_I s+0.3K_c}Q_2(s)$$

→**Step change in q_2:** $H_{SP}(s)=0; Q_2(s)=b/s$

→ **Using final value theorem:**

$$H(t\to\infty)=\lim_{s\to 0}sH(s)=\lim_{s\to 0}\left[\frac{b0.1\tau_I s}{\tau_I s^2+(0.15+0.3K_c)\tau_I s+0.3K_c}\right]=0$$

$$e(t\to\infty)=H_{sp}(t\to\infty)-H(t\to\infty)=0-0=0$$

⇒ **For regulatory problem**, PI controller eliminates also the offset for any value of gain and integral time.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

  m) For regulatory problem in the previous part, how the integral time affects the response for step change in q_2 of 3 ft^3/min:

  → **This is regulatory problem:** $H_{SP}(s) = 0$

  → **Step change in q_2 of 3 :** $Q_2(s) = 3/s$

$$H(s) = \frac{0.1 s \tau_I}{\tau_I s^2 + (0.15 + 0.3K_c)\tau_I s + 0.3K_c}\frac{3}{s}$$

$$= \frac{0.3}{s^2 + (0.15 + 0.3K_c)s + 0.3\frac{K_c}{\tau_I}}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

$$H(s) = \frac{0.1 s \tau_I}{\tau_I s^2 + (0.15 + 0.3K_c)\tau_I s + 0.3K_c}\frac{3}{s}$$

$$= \frac{0.3\tau_I}{\tau_I s^2 + (0.15 + 0.3K_c)\tau_I s + 0.3K_c}$$

$$= \frac{\tau_I / K_c}{\frac{\tau_I}{0.3K_c}s^2 + \frac{(0.15 + 0.3K_c)}{0.3K_c}\tau_I s + 1} \equiv \frac{K}{(\tau^2 s + 2\tau\xi s + 1)}$$

$$\tau = \sqrt{\frac{\tau_I}{0.3K_c}} \; ; \; \xi = \frac{(0.15 + 0.3K_c)}{0.6K_c\sqrt{\frac{\tau_I}{0.3K_c}}} ; K = \frac{\tau_I}{K_c}$$

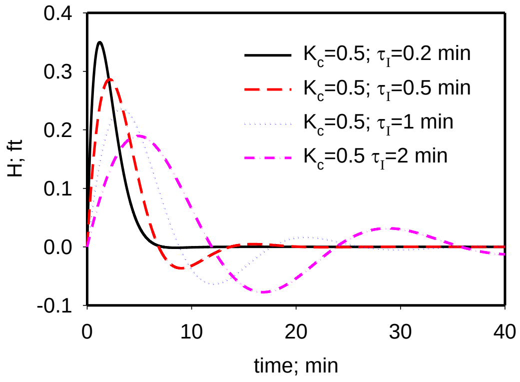
→ **From Laplace transform tables:**

$$H(t) = \frac{K}{\tau\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin(\sqrt{1-\zeta^2}\, t/\tau); \;\; 0 \le |\zeta| < 1$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

  - Response for regulatory problem with PI controller:

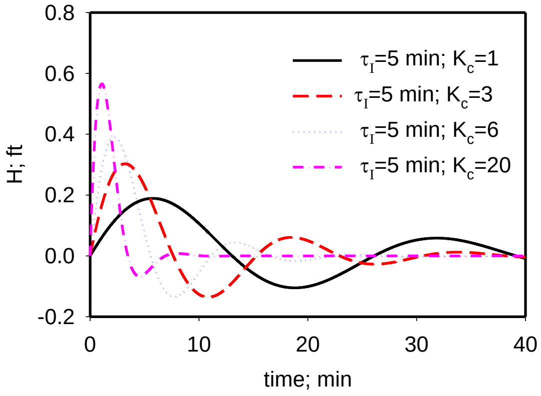


→ **With PI controller**, decreasing τ_I lead to more overshooting, less oscillation and less settling time(faster response).

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

  - Response for regulatory problem with PI controller:



→ **With PI controller**, increasing K_c lead to more overshooting, less oscillation and less settling time (faster response).

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

n) If PID controller is used. How the derivative time constant will affect the response of regulatory problem with step change disturbance. The control valve TF is zero order.

→PID controller: $G_c(s) = K_c + \frac{K_c}{\tau_I s} + K_c \tau_D s = K_c \frac{\tau_I \tau_D s^2 + \tau_I s + 1}{\tau_I s}$

$$H(s) = \frac{0.1}{s + 0.15 + 0.3G_c} Q_2(s)$$

$$= \frac{0.1}{s + 0.15 + 0.3K_c \frac{\tau_I \tau_D s^2 + \tau_I s + 1}{\tau_I s}} Q_2(s)$$

$$= \frac{0.1\tau_I s}{\tau_I s(s + 0.15) + 0.3K_c(\tau_I \tau_D s^2 + \tau_I s + 1)} Q_2(s)$$

$$= \frac{0.1\tau_I s}{(\tau_I + 0.3K_c \tau_I \tau_D)s^2 + (0.15\tau_I + 0.3K_c \tau_I)s + 0.3K_c)} Q_2(s)$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

→ This is regulatory problem: $H_{SP}(s) = 0$

→ Step change in q₂ of , for example, 3 : $Q_2(s) = 3/s$

$$H(s) = \frac{0.1s\tau_I}{(\tau_I + 0.3K_c \tau_I \tau_D)s^2 + (0.15\tau_I + 0.3K_c \tau_I)s + 0.3K_c} \frac{3}{s}$$

$$= \frac{0.3\tau_I / 0.3K_c}{\frac{\tau_I + 0.3K_c \tau_I \tau_D}{0.3K_c} s^2 + \frac{(0.15\tau_I + 0.3K_c \tau_I)}{0.3K_c} s + 1} \equiv \frac{K}{(\tau^2 s + 2\tau\xi s + 1)}$$

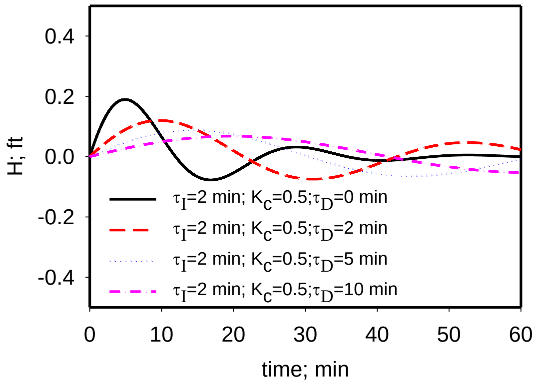
$$\tau = \sqrt{\frac{\tau_I + 0.3K_c \tau_I \tau_D}{0.3K_c}} \; ; \; \xi = \frac{(0.15 + 0.3K_c)}{0.6K_c \tau} ; K = \frac{\tau_I}{K_c}$$

→From Laplace transform tables:

$$H(t) = \frac{1}{\tau\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin(\sqrt{1-\zeta^2} t/\tau); \;\; 0 \le |\zeta| < 1$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**
  - Response for regulatory problem with PID controller:



→ **With PID controller**, increasing τ_D lead to less overshooting, less oscillation and large settling time (slower response).

Dynamic Behavior and Stability of Closed-Loop Control System

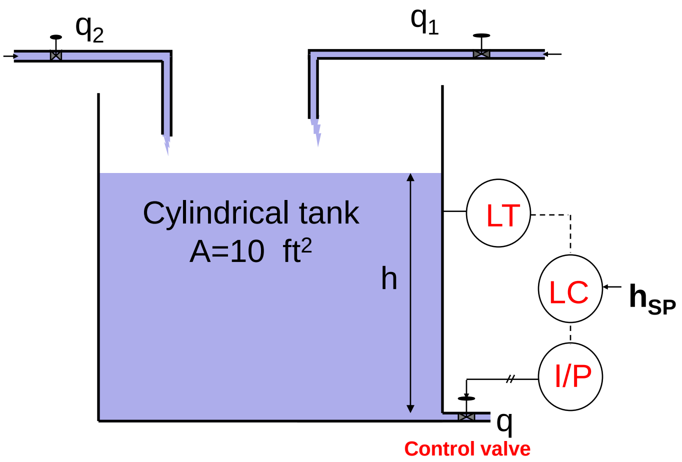
- **Example. Liquid storage tank**

o) A feed-back control system with P controller is built now in which the flow rate of the outlet stream is chosen as manipulated variable. The controller signal in electric. The control valve on the outlet stream is A/O pneumatic one with a linear trim. When the control valve is fully opened the flow rate is 24 ft^3/min. The level transmitter has a span of 8 ft.

- ➢ Draw the liquid height closed-loop feedback control system.
- ➢ Draw the corresponding transfer function block diagram.
- ➢ Derive the closed-loop transfer functions between H, Q, Q_1 and Q_2.
- ➢ Will the P controller eliminate the offset for servo and regulatory problems of step changes? Why?

Dynamic Behavior and Stability of Closed-Loop Control System

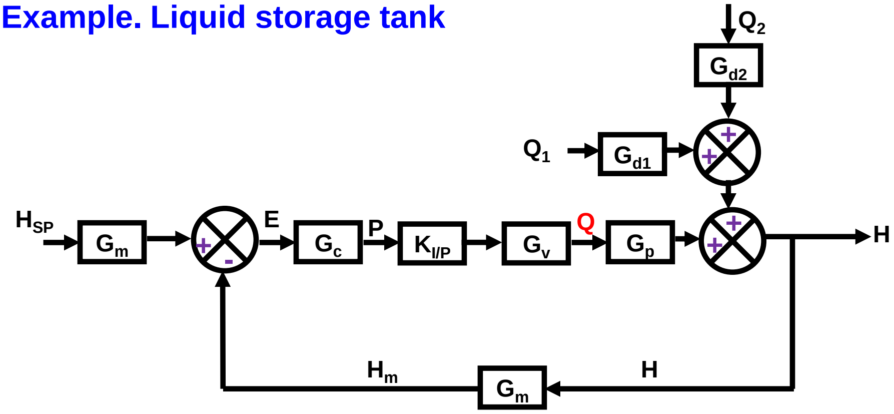
- **Example. Liquid storage tank**



"Liquid level closed-loop feedback control system"

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**



"Transfer function block diagram for liquid level closed-loop feedback control system"

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

$$H(s)=\frac{G_cG_vK_{IP}G_pG_m}{1+G_cG_vK_{IP}G_pG_m}H_{SP}(s)+\frac{G_{d1}}{1+G_cG_vK_{IP}G_pG_m}Q_1(s)+$$

$$\frac{G_{d2}}{1+G_cG_vK_{IP}G_pG_m}Q_2(s)$$

- Since the outlet flow rate is now manipulated variable, it is classified as input variable. This means that the only output variable is the controlled variable *h*. This means that the dynamic model consists now of MB only:

$$A\frac{dH}{dt}=Q_1+Q_2-Q\xrightarrow{L}H(s)=\frac{1}{As}Q_1+\frac{1}{As}Q_2-\frac{1}{As}Q$$

$$\Rightarrow G_{d1}(s)=\frac{H(s)}{Q_1(s)}=\frac{1}{As}\;;\;G_{d2}(s)=\frac{H(s)}{Q_2(s)}=\frac{1}{As};\;G_p(s)=\frac{H(s)}{Q(s)}=-\frac{1}{As}$$

→ **This is integrating system**

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

$$H(s)=\frac{-0.3K_c/s}{1-0.3K_c/s}H_{SP}(s)+\frac{0.1/s}{1-0.3K_c/s}Q_1(s)+$$

$$\frac{0.1/s}{1-0.3K_c/s}Q_2(s)$$

→ **Servo problem:** $H_{SP}(s)=a/s\;\;;\;\;Q_1(s)=Q_2(s)=0$

$$H(s)=\frac{-0.3K_c}{s-0.3K_c}\frac{a}{s}$$

$$H(t\to\infty)=\lim_{s\to0}sH(s)=\lim_{s\to0}\left[\frac{-0.3aK_c}{s-0.3K_c}\right]=a$$

$$e(t\to\infty)=H_{sp}(t\to\infty)-H(t\to\infty)=a-a=0$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

→ **Regulatory problem:** $H_{SP}(s)=0 \;\; ; \;\; Q_1(s)\, or\, Q_2(s)=b/s$

$$H(s)=\frac{0.1s}{s-0.3K_c}\,\frac{b}{s}$$

$$H(t\to\infty)=\lim_{s\to 0} sH(s)=\lim_{s\to 0}\left[\frac{0.1bs}{s-0.3K_c}\right]=0$$

$$e(t\to\infty)=H_{sp}(t\to\infty)-H(t\to\infty)=0-0=0$$

⇒ **For integrating systems, P controller eliminates the offset.**

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example. Liquid storage tank**

→ **Response of regulatory problem for step change in q_1**

$$H(s)=\frac{0.1s}{s-0.3K_c}\,\frac{b}{s}=\frac{0.1b}{s-0.3K_c}\Rightarrow H(t)=0.1be^{0.3K_c t}$$

⇒ **To have stable response, the controller must work on the direct mode (K_c <0).**

Dynamic Behavior and Stability of Closed-Loop Control System

- **General conclusions on feedback closed-loop control systems:**
  - As K_c increases: less offset, the response will be faster, more overshooting, less oscillation.
  - Integral mode eliminates the offset.
  - As τ_I decreases: the response will be faster, more overshooting, less oscillation.
  - As τ_D increases: the response will be slower, less oscillatory, less overshooting (when there is no noise).
  - Increasing τ_D gives the opportunity to increase K_c in order to enhance the speed of the response.
  - P controller eliminates the offset of integrating process
  - For integrating process, the controller must work on the direct mode (K_c <0).
  - The derivative control does not change the order of the open-loop transfer function of the closed-loop control system.

Dynamic Behavior and Stability of Closed-Loop Control System

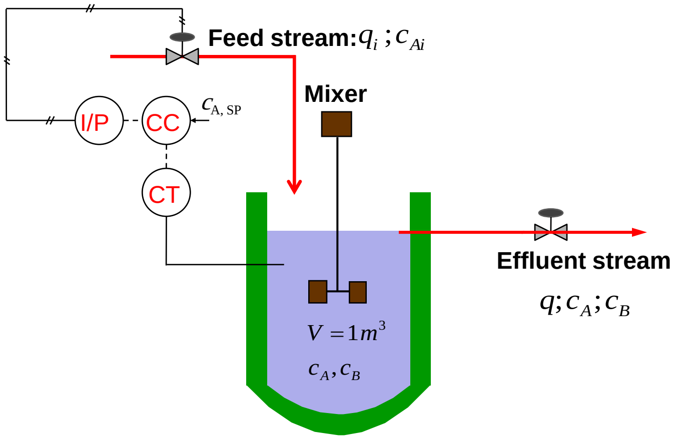
Example. A continuous stirred-tank reactor(CSTR) is used to produce a compound B according to the liquid-phase elementary reaction 2A→ B. The reactor has a constant volume of 1 m^3. The reaction rate constant is 0.2 m^3/mol. The desired steady-state conditions are: feed flow rate is 0.05 m^3/min and the feed concentration of component A is 1 mol/m^3. A feedback control system is built to control the concentration of component A inside the reactor by manipulating the feed volumetric flow. The control loop has: concentration transmitter with zero-order transfer function and gain of 0.8 mA/(mol/m^3); A/O control valve with zero-order transfer function and gain of 0.04 m^3/min/psig. If the loop has P-controller only with gain of 10 and a step change of 1 mol/m^3 occurred in the feed concentration of component A, Find:

a) The offset in concentration of component A .

b) The corresponding settling time.

Dynamic Behavior and Stability of Closed-Loop Control System

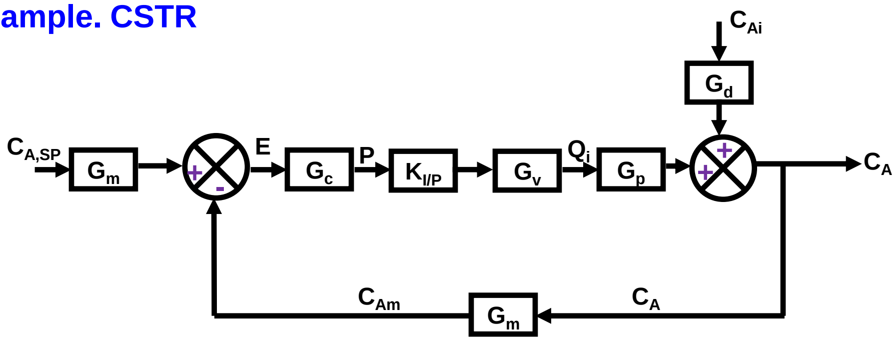
Example. CSTR



“Concentration closed-loop feedback control system”

Dynamic Behavior and Stability of Closed-Loop Control System

Example. CSTR



“Transfer function block diagram for concentration closed-loop feedback control system”

$$C_A(s)=\frac{G_cG_vK_{IP}G_pG_m}{1+G_cG_vK_{IP}G_pG_m}C_{A,SP}(s)+\frac{G_d}{1+G_cG_vK_{IP}G_pG_m}C_{Ai}(s)$$

$$C_{Ai}(s)=1/s$$

$$C_{A,SP}(s)=0$$ **“Regulatory problem”**

Dynamic Behavior and Stability of Closed-Loop Control System

Example. CSTR

- **Total MB:** $\frac{dV}{dt} = 0 = q_i - q \Rightarrow q = q_i$
- **Component A mole balance:**

$$\frac{dn_A}{dt} = \frac{dc_A V}{dt} = V\frac{dc_A}{dt} = c_{Ai}q_i - c_A q - (-r_A)V$$

Elementary reaction: $(-r_A) = kc_A{}^2$

$$\Rightarrow V\frac{dc_A}{dt} = c_{Ai}q_i - c_A q_i - kc_A{}^2 V$$

- **Linearize the nonlinear terms:**

$$f_1(c_{Ai},q_i) = c_{Ai}q_i \;\; ; f_2(c_A,q_i); \;\; f_3(c_A{}^2)$$

Dynamic Behavior and Stability of Closed-Loop Control System

Example. CSTR

$$f_1(c_{Ai},q_i) = c_{Ai}q_i \cong f_1(\bar{c}_{Ai},\bar{q}_i) + \left.\frac{\partial f_1}{\partial q_i}\right|_{\bar{c}_{Ai},\bar{q}_i}(q_i - \bar{q}_i) + \left.\frac{\partial f_1}{\partial c_{Ai}}\right|_{\bar{c}_{Ai},\bar{q}_i}(c_{Ai} - \bar{c}_{Ai})$$

$$= \bar{c}_{Ai}\bar{q}_i + \bar{c}_{Ai}(q_i - \bar{q}_i) + \bar{q}_i(c_{Ai} - \bar{c}_{Ai})$$

$$f_2(c_A,q_i) = c_A q_i \cong f_1(\bar{c}_A,\bar{q}_i) + \left.\frac{\partial f_1}{\partial q_i}\right|_{\bar{c}_A,\bar{q}_i}(q_i - \bar{q}_i) + \left.\frac{\partial f_1}{\partial c_A}\right|_{\bar{c}_A,\bar{q}_i}(c_A - \bar{c}_A)$$

$$= \bar{c}_A\bar{q}_i + \bar{c}_A(q_i - \bar{q}_i) + \bar{q}_i(c_A - \bar{c}_A)$$

$$f_3(c_A) = c_A{}^2 = f_3(\bar{c}_A) + \left.\frac{df_3}{dc_A}\right|_{\bar{c}_A}(c_A - \bar{c}_A) = \bar{c}_A{}^2 + 2\bar{c}_A(c_A - \bar{c}_A)$$

$$\Rightarrow V\frac{dc_A}{dt} = \bar{c}_{Ai}\bar{q}_i + \bar{c}_{Ai}(q_i - \bar{q}_i) + \bar{q}_i(c_{Ai} - \bar{c}_{Ai}) -$$

$$\bar{c}_A\bar{q}_i - \bar{c}_A(q_i - \bar{q}_i) - \bar{q}_i(c_A - \bar{c}_A) - kV\bar{c}_A{}^2 - 2kV\bar{c}_A(c_A - \bar{c}_A)........(1)$$

Dynamic Behavior and Stability of Closed-Loop Control System

Example. CSTR

- **Steady-state component A mole balance:**

$$\Rightarrow 0=\bar{c}_{Ai}\bar{q}_i-\bar{c}_A\bar{q}_i-kV\bar{c}_A{}^2 \quad \text{..}(2)$$

$$\bar{q}_i=0.05\,\text{m}^3/\text{min};\ \bar{c}_{Ai}=1\,\text{mol}/\text{m}^3;k=0.2\ \text{m}^3/\text{mol};\ V=1\ \text{m}^3$$

$$0=(1)(0.05)-0.05\bar{c}_A-(0.2)(1)\bar{c}_A{}^2$$

$$0.2\bar{c}_A{}^2+0.05\bar{c}_A-0.05=0$$

→ Solve to obtain the steady state concentration:

$$\bar{c}_A=0.64\ \text{mol/m}^3$$

- **Subtract Eq. 2 from (1) to obtain dynamic equation in deviation form:**

$$V\frac{dC_A}{dt}=\bar{c}_{Ai}Q_i+\bar{q}_iC_{Ai}-\bar{c}_AQ_i-\bar{q}_iC_A-2kV\bar{c}_AC_A$$

$$V\frac{dC_A}{dt}=(\bar{c}_{Ai}-\bar{c}_A)Q_i+\bar{q}_iC_{Ai}-(\bar{q}_i+2kV\bar{c}_A)C_A$$

Dynamic Behavior and Stability of Closed-Loop Control System

Example. CSTR

$$(1)\frac{dC_A}{dt}=(1-0.64)Q_i+0.05C_{Ai}-\left[0.05+(2)(0.2)(1)(0.64)\right]C_A$$

$$\Rightarrow\frac{dC_A}{dt}+0.31C_A=0.36Q_i+0.05C_{Ai}$$

- **Take Laplace Transform and rearrange:**

$$C_A(s)=\frac{0.36}{s+0.31}Q_i(s)+\frac{0.05}{s+0.31}C_{Ai}(s)$$

$$\therefore G_p(s)=\frac{C_A(s)}{Q_i(s)}=\frac{0.36}{s+0.31}$$

$$G_d(s)=\frac{C_A(s)}{C_{Ai}(s)}=\frac{0.05}{s+0.31}$$

$$G_m(s)=0.8\,\text{mA/(mol/m}^3);G_v(s)=0.04\,(\text{m}^3\text{/min})/\text{psig}\ \ ;G_c(s)=10$$

Dynamic Behavior and Stability of Closed-Loop Control System

Example. CSTR

$$C_A(s) = \frac{\dfrac{0.05}{s+0.31}}{1+(10)(0.04)(0.75)\dfrac{0.36}{s+0.31}(0.8)}\frac{1}{s} = \frac{0.05}{s(s+0.396)}$$

$$C_A(t\to\infty) = \lim_{s\to 0} sC_A(s) = 0.126\,\text{mol/m}^3$$

$$Offset = 0 - C_A(t\to\infty) = -0.126\,\text{mol/m}^3$$

- **Take Laplace Inverse of C_A(s):**

$$C_A(t) = c_A(t) - 0.64 = 0.126 - 0.126e^{-0.396t}$$

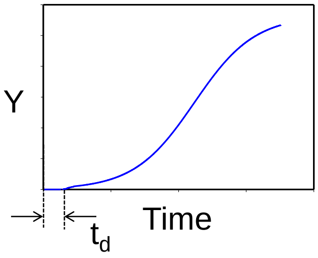
$$\Rightarrow c_A(t) = 0.766 - 0.126e^{-0.396t}$$

- **Settling time:** $c_A(t) = (0.99)(0.766) = 0.766 - 0.126e^{-0.396t_s}$

$$\Rightarrow t_s = 7.1\,\text{min}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Effect of dead time, t_d, on the closed-loop performance:**
  - In most real chemical processes, the closed-loop response has a delay of a dead time t_d.
  - In such circumstances, it is evident that the nominator of the open-loop transfer function of the real closed-loop system has the exponential term : $e^{-t_d s}$



  - This mean that the real transfer function of some closed-loop elements has this exponential term.

Ex.: $G_m(s) = \dfrac{K_m e^{-t_d s}}{\tau_m s + 1}$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Effect of dead time, $t_{d,}$ on the closed-loop response:**
  - The presence of **dead-time exponential term** in the nominator of the open-loop transfer function of the closed-loop system will be **principal source of instability** for the chemical process systems.

  Exercise. Try $G_m(s) = 2e^{-0.1s}$ in the previous example.
  - **In the absence of dead time:**
    - a closed-loop system is stable (overdamped or underdamped) if its open-loop transfer function is of first- or second-order. In this case it is NOT difficult to decide the optimum controller parameters.
    - a closed-loop system may become unstable if its open-loop transfer function is of third-order or higher. In this case, it is difficult to decide the optimum controller parameters ⇒ There is a need for **controller tuning**.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Controller tuning: deciding what values of the controller parameters (K_c, τ_I, τ_D) to be used in order to achieve stable response with convincing performance.**
  - **Remark.** Controller tuning is very important if the open-loop transfer function of the closed-loop system has dead time exponential term or if it is of third-order or higher. Otherwise, unstable response may arise.

  - There are some simple criteria used in controller tuning such as one-quarter decay ratio, minimum settling time, minimum largest error, minimum offset, and so on.
  - The most common criteria are based on minimizing the offset with one-quarter decay ratio such as:
    - Open-loop **Cohen-Coon method**.
    - Closed-loop **Ziegler-Nichols method**.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Controller tuning**
  - Other tuning criteria are based on minimizing the following integral errors:
    - Minimize Integral of the Absolute Error (**IAE**):

$$\textbf{Min}\,\mathrm{IAE} = \int_0^\infty |e(t)|\,dt$$

    - Minimize Integral of the Square Error(**ISE**):

$$\textbf{Min}\,\mathrm{ISE} = \int_0^\infty [e(t)]^2\,dt$$

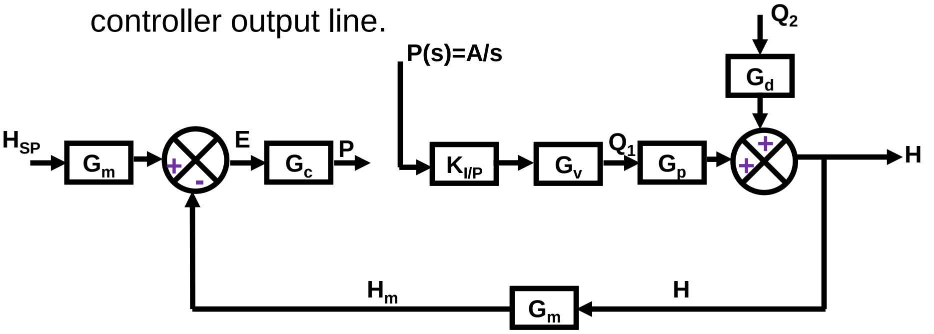
    - Minimize Integral of the Time-weighted Absolute Error (**ITAE**):

$$\textbf{Min}\,\mathrm{IAE} = \int_0^\infty t|e(t)|\,dt$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Cohen-Coon tuning method:**
  - It is developed experimentally by Cohen and Coon.
  - It is known as **process reaction curve method**.
  - **Procedure:**

    1. Open the feedback control loop by disconnecting the controller output line.



Dynamic Behavior and Stability of Closed-Loop Control System

- **Cohen-Coon Tuning Method:**

2. Introduce a step change of magnitude A in the variable P(s):

P(s)=A/s

3. Write the transfer function between $H_m(s)$ and P(s):

$$\frac{H_m(s)}{P(s)} = G_{PRC}(s) = G_v K_{IP} G_p G_m$$

$$H_m(s) = G_{PRC}(s)P(s) = G_v K_{IP} G_p G_m \frac{A}{s}$$

4. Obtain $H_m(t)$ by taking the Laplace inverse of $H_m(s)$.

$$H_m(t) = \mathbf{L}^{-1}\left[G_v K_{IP} G_p G_m \frac{A}{s}\right]$$

5. Draw $H_m(t)$ versus t. If the resulted curve has sigmoidal shape (S shape) which is called **process reaction curve**, **go to step 6**. **Otherwise, stop** and conclude that Cohen-Coon is not applicable for controller tuning.

Dynamic Behavior and Stability of Closed-Loop Control System

- **Cohen-Coon Tuning Method:**

6. Determine the following parameters from the process reaction curve:

$$G_{PRC}(s) = \frac{H_m(s)}{P(s)} \cong \frac{Ke^{-t_d s}}{\tau s+1}$$

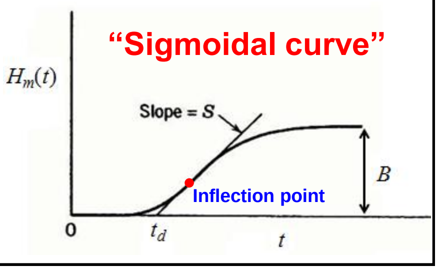
"Cohen coon relation"

$$\rightarrow K = \frac{\Delta \text{Output}}{\Delta \text{Input}} = \frac{H_m(t\rightarrow\infty) - H_m(t\rightarrow 0)}{P(t\rightarrow\infty) - P(t\rightarrow 0)} = \frac{B}{A}$$

$$\rightarrow \tau = B/S$$

Where S is the slope of the sigmoidal curve at the point of inflection.

t_d is the dead time approximated at the intersection of the tangent line(of slope S) with t-axis.



Dynamic Behavior and Stability of Closed-Loop Control System

- **Cohen-Coon Tuning Method:**

  7. Calculate the controller parameters as follows:

  - **For P controller:** $K_c = \frac{1}{K}\frac{\tau}{t_d}\left(1 + \frac{t_d}{3\tau}\right)$
  - **For PI controller:** $K_c = \frac{1}{K}\frac{\tau}{t_d}\left(0.9 + \frac{t_d}{12\tau}\right)$

    $\tau_I = t_d \frac{30 + 3t_d/\tau}{9 + 20t_d/\tau}$
  - **For PID controller:** $K_c = \frac{1}{K}\frac{\tau}{t_d}\left(\frac{4}{3} + \frac{t_d}{4\tau}\right)$; $\tau_I = t_d \frac{32 + 6t_d/\tau}{13 + 8t_d/\tau}$

    $\tau_D = t_d \frac{4}{11 + 2t_d/\tau}$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Example.** **In the previous example, suppose that the control valve transfer function is first-order with time constant of 2 min, and the level transmitter has also first-order transfer function with time constant of 1 min. Use Cohen-Coon method to tune the parameters for P, PI, or PID controller.**

$$P(s) = A/s = 1/s; G_m = \frac{2}{s+1}; G_v = \frac{2}{2s+1}; K_{IP} = 0.75; G_p = \frac{0.1}{s+0.15}$$

$$H_m(s) = G_{PRC}(s)\frac{A}{s} = K_{IP}G_mG_vG_p\frac{1}{s} = (0.75)\frac{2}{2s+1}\frac{2}{s+1}\frac{0.1}{s+0.15}$$

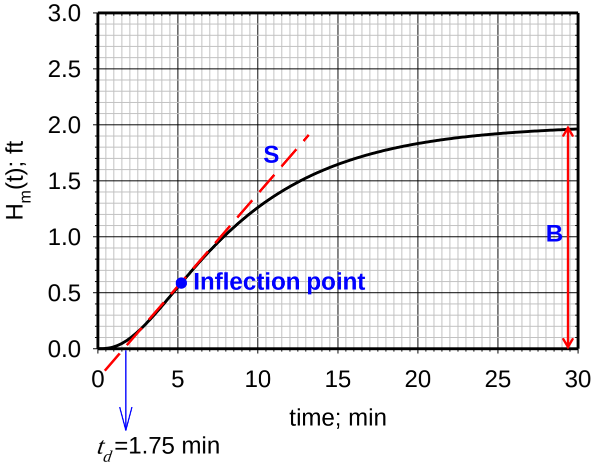
$$= \frac{0.3}{s(2s+1)(s+1)(s+0.15)}$$

$$H_m(t) = \mathbf{L}^{-1}\left[\frac{0.15}{s(s+0.5)(s+1)(s+0.15)}\right] =$$

$$= 2 + 1.7143e^{-0.5t} - 3.3613e^{-0.15t} - 0.3529e^{-t}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Cohen-Coon Tuning Method:**



→ Draw $H_m(t)$ versus t.
→ Draw the tangent line at the inflection point (red dashed line).
→ Pick up two points that coincide on the tangent line to calculate the slope S:
$(t_1,H_{m1})=(3.5,0.3)$
$(t_2,H_{m2})= (12,1.7)$

- Slope= S =(1.7-0.3)/(12/3.5)=0.165 ft/min
- B =2-0=2 ft; τ= B/S=2/0.165 =12.12 min; K=B/A=2/1=2 ft/mA

Dynamic Behavior and Stability of Closed-Loop Control System

- **Cohen-Coon Tuning Method:**

→ Calculate now the controller parameters using: K=2 ;τ= 12.12 min ; t_d=1.75 min

- **If P controller:** $K_c = \frac{1}{K}\frac{\tau}{t_d}\left(1+\frac{t_d}{3\tau}\right) = 3.63$
- **If PI controller:** $K_c = \frac{1}{K}\frac{\tau}{t_d}\left(0.9+\frac{t_d}{12\tau}\right) = 3.16$

$$\tau_I = t_d \frac{30+3t_d/\tau}{9+20t_d/\tau} = 4.48\,\text{min}$$

- **If PID controller:** $K_c = \frac{1}{K}\frac{\tau}{t_d}\left(\frac{4}{3}+\frac{t_d}{4\tau}\right) = 4.74\,;$

$$\tau_I = t_d \frac{32+6t_d/\tau}{13+8t_d/\tau} = 4.06\,\text{min} \quad ; \quad \tau_D = t_d \frac{4}{11+2t_d/\tau} = 0.62\,\text{min}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Ziegler-Nichols tuning method:**
  - It is developed by Ziegler-Nichols.
  - Unlike Cohen Coon method, it is a closed-loop tuning technique.
  - **Procedure:**
    - **1.** Bring the system to the desired steady-state.
    - **2.** Use **proportional controller only** with feedback loop closed.
    - **3.** Introduce a set point change.
    - **4.** Vary the proportional gain K_c until the system oscillates continuously. The frequency of continuous oscillation is called **crossover frequency ω_{co}**

For open-loop transfer function of the form:

$$G_{OL}(s) = \frac{Ke^{-t_d s}}{(\tau_1 s+1)((\tau_2 s+1).....(\tau_n s+1)}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Ziegler-Nichols Tuning Method:**

The crossover frequency ω_{co} can be found from:

$$-\pi = -t_d\omega_{co} + \tan^{-1}(-\tau_1\omega_{co}) + \tan^{-1}(-\tau_2\omega_{co}) + ... + \tan^{-1}(-\tau_n\omega_{co})$$

5. Find the amplitude ratio AR at that crossover frequency using:

$$\log AR = \log\left(\frac{1}{\sqrt{(\tau_1\omega_{co})^2+1}}\right) + \log\left(\frac{1}{\sqrt{(\tau_2\omega_{co})^2+1}}\right) + ... + \log\left(\frac{1}{\sqrt{(\tau_n\omega_{co})^2+1}}\right)$$

6. Compute the following two quantities:

- Ultimate gain; $K_u = 1/AR$
- Ultimate period of sustained cycling, $P_u = 2\pi/\omega_{co}$

Dynamic Behavior and Stability of Closed-Loop Control System

- **Ziegler-Nichols Tuning Method:**

7. Calculate the controller parameters as follows:

Controller	K_c	τ_I	τ_D
P	**$K_U/2$**	**-**	**-**
PI	**$K_U/2.2$**	**$P_U/1.2$**	**-**
PID	**$K_U/1.7$**	**$P_U/2$**	**$P_U/8$**

Note that for PID controller, $\tau_I = 4\ \tau_D$

Remark. According to Bode criterion the value of amplitude ratio at cross over frequency decides the stability of the closed-loop system:

If AR<1: Closed-loop response is stable

If AR>1 : Closed loop response is unstable

Dynamic Behavior and Stability of Closed-Loop Control System

- **Ziegler-Nichols Tuning Method:**

- **Example. Use Ziegler-Nichols method to tune P, PI, or PID controller parameters of the previous example.**

$$G_{OL}(s) = K_{IP}G_cG_mG_vG_p = (0.75)K_c\frac{2}{2s+1}\,\frac{2}{s+1}\,\frac{0.1}{s+0.15}$$

$$= \frac{0.3K_c}{(2s+1)(s+1)(s+0.15)} \equiv \frac{Ke^{-t_d s}}{(\tau_1 s+1)((\tau_2 s+1)(\tau_3 s+1)}$$

$$\Rightarrow \tau_1 = 2; \tau_2 = 1; \tau_3 = 1/0.15; t_d = 0; K = 2K_c$$

$$-\pi = \tan^{-1}(-\tau_1\omega_{co}) + \tan^{-1}(-\tau_2\omega_{co}) + \tan^{-1}(-\tau_3\omega_{co})$$

$$-\pi = \tan^{-1}(-2\omega_{co}) + \tan^{-1}(-1\omega_{co}) + \tan^{-1}(-\omega_{co}/0.15)$$

Solving this Eq., gives crossover frequency**:**

ω_{co} =0.852 rad/min

Dynamic Behavior and Stability of Closed-Loop Control System

- **Ziegler-Nichols Tuning Method:**

$$\log AR = \log\left(\frac{1}{\sqrt{(\tau_1\omega_{co})^2+1}}\right)+\log\left(\frac{1}{\sqrt{(\tau_2\omega_{co})^2+1}}\right)+\log\left(\frac{1}{\sqrt{(\tau_3\omega_{co})^2+1}}\right)$$

$$=\log\left(\frac{1}{\sqrt{(0.852\times1)^2+1}}\right)+\log\left(\frac{1}{\sqrt{(0.852\times2)^2+1}}\right)+$$

$$\log\left(\frac{1}{\sqrt{(0.852/0.15)^2+1}}\right)$$

$$\Rightarrow AR = 0.2133$$

→ Ultimate gain; **K_u=1/AR=4.69**

→ Ultimate period of sustained cycling:
$P_u=2\pi/\omega_{co}$ $=2\pi/0.852=7.38$ min/cycle

Dynamic Behavior and Stability of Closed-Loop Control System

- **Ziegler-Nichols Tuning Method:**

K_{cu} = 4.69; P_u =7.38 min/cycle

- **If P controller:** **$K_c = K_{cu}/2 = 2.34$**
- **If PI controller: :** **$K_c = K_{CU}/2.2=2.14$**
  $\tau_I= P_U/1.2=6.15$ min
- **If PID controller:** **$K_c = K_{CU}/1.7=2.76$**
  $\tau_I= P_U/2=3.69$ min
  $\tau_D= P_U/8=0.92$ min