Process Control

Transfer Function

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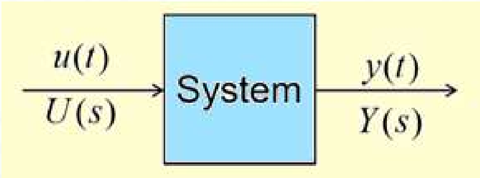
- *Definition of transfer function:*

- *It is an algebraic expression for the dynamic relationship between the input and output of the process model*



- *Let G(s) denote the transfer function between an input, u, and an output, y. Then, by definition:*

$$G(s) = \frac{Y(s)}{U(s)}$$

where:

$$Y(s) = L[y(t)]$$

$$U(s) = L[u(t)]$$

- *How to find transfer function G(s)*

1. *If the dynamic model is nonlinear, linearize it around the desired steady-state point.*
2. *Write the steady-state Eq.*
3. *Define deviation variables and write the linearized dynamic model in terms of these deviation variables.*
1. *Take Laplace transform and rearrange to have the desired transfer function.*

Example:

Find the transfer function that relates the output y with input u:

$$5\frac{dy}{dt} + 4y = u; \quad y(0) = 1$$

Solution:

$$\bar{y} = y(0) = 1$$

Steady-state Eq.:

$$0 + 4\bar{y} = \bar{u} \rightarrow \bar{u} = 4$$

Deviated variables:

$$\tilde{y} = y - \bar{y} \rightarrow y = \tilde{y} + 1$$

$$\tilde{u} = u - \bar{u} \rightarrow u = \tilde{u} + 4$$

A dynamic model in terms of deviation variables:

$$5\frac{d\tilde{y}}{dt} + 4(\tilde{y} + 1) = \tilde{u} + 4$$

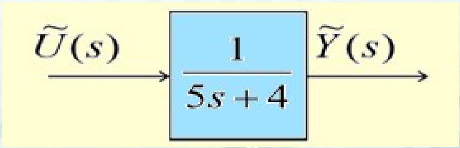
$$\therefore 5\frac{d\tilde{y}}{dt} + 4\tilde{y} = \tilde{u}; \qquad \tilde{y}(0) = \tilde{u}(0) = 0$$

- *Take Laplace transform:*

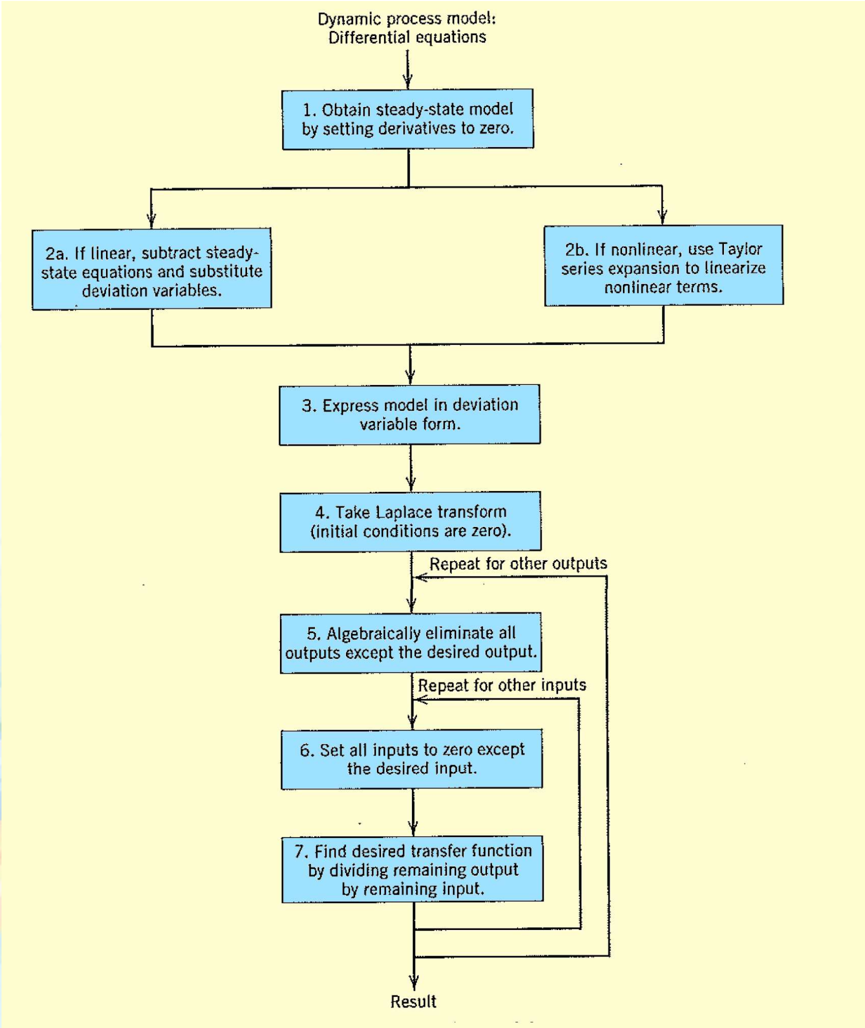
$$L\left[5\frac{d\tilde{y}}{dt}+4\tilde{y}\right]=L[\tilde{u}]$$

$$5s\tilde{Y}(s)+4\tilde{Y}(s)=\tilde{U}(s)$$

$$\rightarrow G(s)=\frac{\tilde{Y}(s)}{\tilde{U}(s)}=\frac{1}{5s+4}$$



General Procedure for Developing Transfer Function Model:



➢ Properties of Transfer Function Models

- *The order of the transfer function (TF) is defined to be the order of its denominator polynomial. Note that the order of the TF is equal to the order of the ODE.*
- *The TF of the previous example is 1st-order.*
- *Steady-state Gain (K): the ratio between ultimate changes in output and input:*

$$Gain = K = \frac{\Delta output}{\Delta input} = \frac{\big(y(\infty) - y(0)\big)}{\big(u(\infty) - u(0)\big)}$$

✓ *For a unit step change in input, the gain is the change in output*

✓ *From the final value theorem, a unit step change in input with zero initial condition gives:*

$$K = \frac{Y(\infty)}{1} = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)\frac{1}{s} = \lim_{s\to 0} G(s)$$

➢ *Properties of Transfer Function Models:*

- *Some TF models do not have steady-state gain as for integrating processes and processes with sustaining oscillation in output.*
- *Example: for the previous example find the static gain with a unit step change in input.*

$$\lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{1}{5s+4} = 0.25$$

- *Physical realizability: TF is physically unrealizable if the order of the numerator (m) is greater than that of the denominator (n). This means that the order of derivative for the input is higher than that of the output.*

Example:

$a_0 y = b_1 \frac{du}{dt} + b_0 u$ *and step change in u*

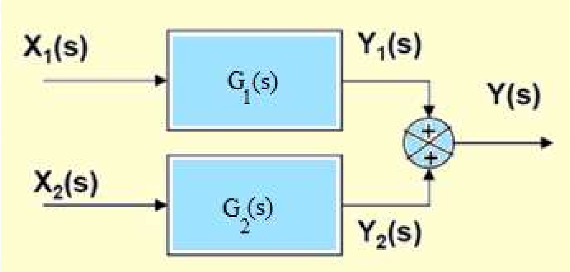
- *Physical realizability ≅ requires future input values for current output!*

➢ *Properties of Transfer Function Models:*

- *Additive property:*

$$Y(s) = Y_1(s) + Y_2(s)$$

$$Y(s) = G_1(s)X_1(s) + G_2(s)X_2(s)$$

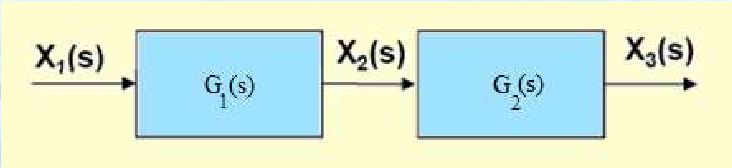


- *Multiplicative property:*

$$X_3(s) = G_2(s)X_2(s)$$

$$= G_2(s)[G_1(s)X_1(s)]$$

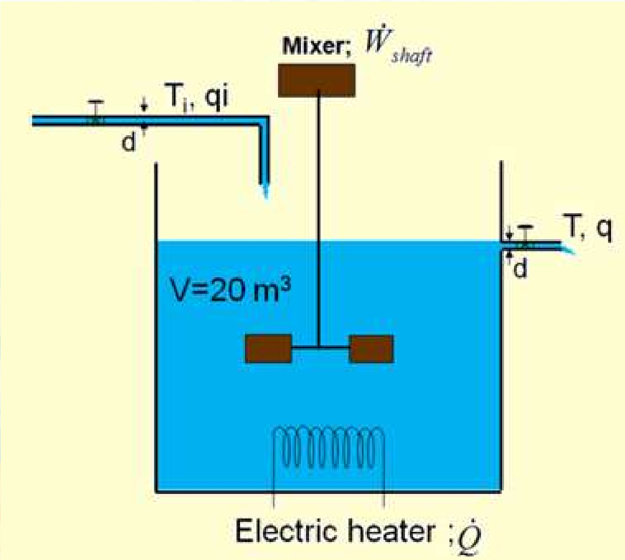
$$= G_2(s)G_1(s)X_1(s)$$



Example 1:

For the stirred-tank heating process with constant holdup, V, find the transfer function between:

Outlet temperature deviation and inlet temperature deviation.



See the dynamic model of this example in the Modelling Chapter:

$$\tau \frac{dy}{dt} = y_i - y + Ku \qquad at\ t = 0\ (old\ desired\ st.st.): y = 0$$

where

$$K = \frac{1}{\rho q C};\ \tau = \frac{V}{q}$$

and "deviation variables" are:

$$y = T - \bar{T}$$

$$y_i = T_i - \bar{T}_i$$

$$u = \dot{Q} - \bar{\dot{Q}}$$

Take the Laplace transform of the dynamic model equation:

$$L\left\{\tau \frac{dy}{dt}\right\} = L\{y_i - y + Ku\}$$

$$\rightarrow \tau sY = Y_i - Y + KU$$

Rearrange the Eq. as:

$$(\tau s + 1)Y = Y_i + KU$$

$$Y = \frac{1}{(\tau s + 1)} Y_i + \frac{K}{(\tau s + 1)} U$$

- *The transfer function between outlet temperature deviation and inlet temperature deviation is:*

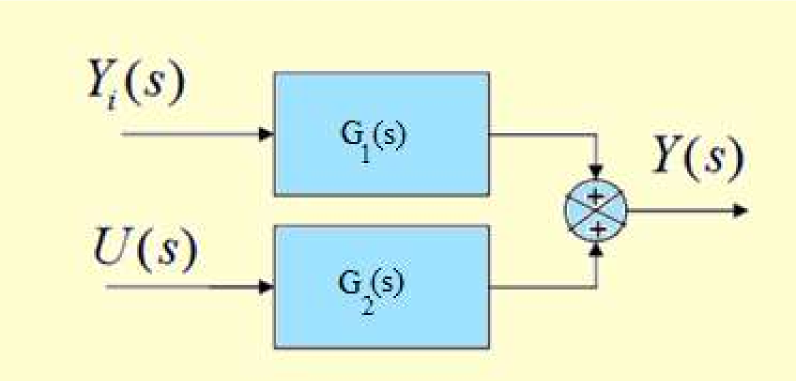
$$G_1(s) = \frac{Y(s)}{Y_i(s)} = \frac{1}{(\tau s + 1)}$$

- *The transfer function between outlet temperature deviation and inlet heating rate deviation is:*

$$G_2(s) = \frac{Y(s)}{U(s)} = \frac{K}{(\tau s + 1)}$$

$$Y = G_1(s)Y_i + G_2(s)U$$

Using the addition property, this can be represented by the following block diagram:



$$G_1(s) = \frac{Y(s)}{Y_i(s)} = \frac{1}{(\tau s + 1)}$$

$$G_2(s) = \frac{Y(s)}{U(s)} = \frac{K}{(\tau s + 1)}$$

Example 2:

In the previous example if the liquid volume in the tank is 20 m^3 and the flow rate is 10 m^3/hr., the liquid density is 1000 kg/m^3 and the liquid heat capacity is 1 cal/g.^oC. Suppose there is a step change in heating rate from 30 kcal/s to 99 kcal/s and no change in the inlet temperature. Determine the outlet deviated temperature as a function of time.

$$Y(s) = \frac{1}{(\tau s + 1)} Y_i(s) + \frac{K}{(\tau s + 1)} U(s)$$

No change in the inlet temperature:

$$Y(s) = \frac{K}{(\tau s + 1)} U(s)$$

A step change in heating rate:

$$U(s) = \frac{99 - 30}{s} = \frac{69}{s}$$

$$K = \frac{1}{C\rho q} = \frac{1}{1000 \times 1000 \times 10} = 1 \times 10^{-7}\frac{^oC.hr}{cal} = 0.36\ \frac{^oC.s}{kcal}$$

$$Y(s) = \frac{0.36}{(2s+1)}\frac{69}{s} = \frac{24.8}{s(2s+1)} = 24.8\left[\frac{1}{s} - \frac{2}{2s+1}\right]$$

Take Laplace inverse transform

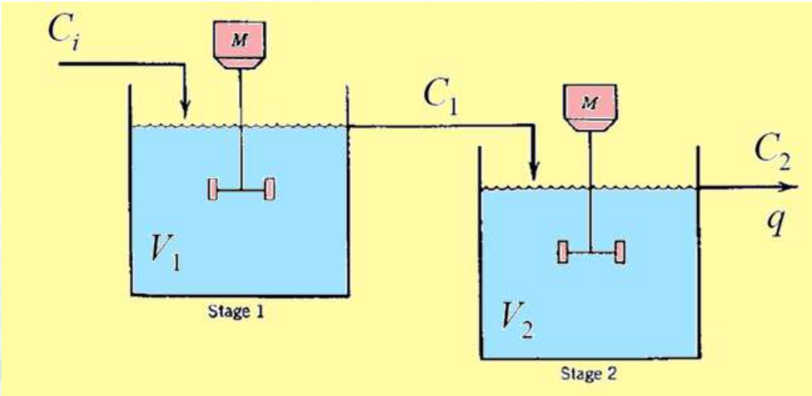
$$y(t) = 24.8\left(1 - e^{-t/2}\right)$$

If the initial steady state temperature in the tank is 50 ^oC, the new steady state value is 50+24.8 = 74.8 ^oC. And the steady-state gain is

$$Gain = K = \frac{y(\infty) - y(0)}{u(\infty) - u(0)} = \frac{24.8 - 0}{(99 - 30) - 0} = 0.36\ ^oC.s/kCal$$

Example 3:

Two stirred tanks in series for mixing of two liquid components (constant volumes, no reactions). Find the transfer function between component concentration out of 2nd tank and to the feed concentration to first tank.



Parameters: V1/q = 2 min, V2/q = 1.5 min.

Initial conditions:

$$c_1(0) = c_2(0) = 1\ kg\ mol/m^3$$

Dynamic model:

- *Apply component mole balance on tank 1:*

$$V_1 \frac{dc_1}{dt} + qc_1 = qc_i$$

- *Apply component mole balance on tank 2:*

$$V_2 \frac{dc_2}{dt} + qc_2 = qc_1$$

- *Use deviation variables, take the Laplace transforms, and rearrange to have the following transfer functions:*
  - ✓ *TF between concentration out of the 1st tank to the feed concentration to the 1st tank:*

$$\frac{\tilde{C}_1(s)}{\tilde{C}_i(s)} = \frac{1}{(V_1/q)s + 1} = \frac{1}{(2s+1)} (1^{st} - order\ TF)$$

- *TF between concentration out of the 2nd tank and the feed concentration to the 2nd tank:*

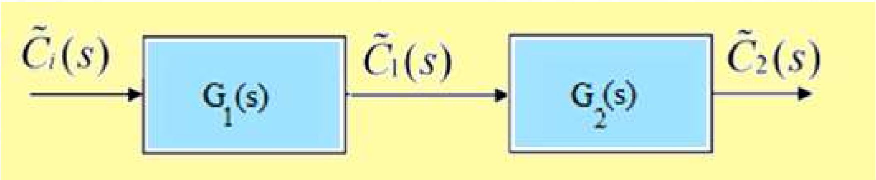
$$\frac{\tilde{C}_2(s)}{\tilde{C}_1(s)} = \frac{1}{(V_2/q)s+1} = \frac{1}{(1.5s+1)} \left(1^{st} - order\ TF\right)$$

- *TF between concentration out of the 2nd tank and the feed concentration to the 1st tank (chain rule):*

$$\frac{\tilde{C}_2(s)}{\tilde{C}_i(s)} = \frac{\tilde{C}_2(s)\tilde{C}_1(s)}{\tilde{C}_1(s)\tilde{C}_i(s)} = \frac{1}{\big((V_2/q)s+1\big)\big((V_1/q)s+1\big)}$$

$$= \frac{1}{(2s+1)(1.5s+1)} \qquad \left(2^{nd} - order\ TF\right)$$

- *Using the multiplication properly, this can be represented by the following block diagram:*

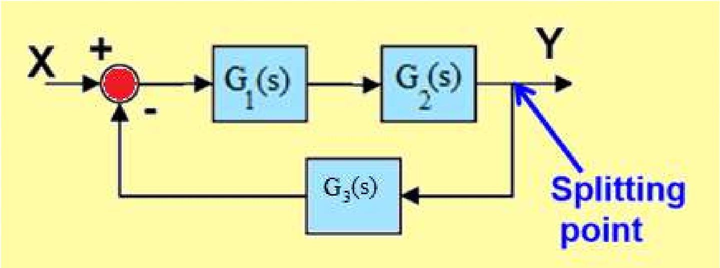


$$G_1(s) = \frac{1}{(2s+1)}$$

$$G_2(s) = \frac{1}{(1.5s+1)}$$

Example 4:

Use the TF block diagram shown below to find: Y (s) X (s)



Solution

Y_1 and Y_2 are intermediate variables:

$$Y = G_1 G_2 Y_2 = G_1 G_2 (X - Y_1)$$

$$= G_1 G_2 (X - G_3 Y)$$

$$= G_1 G_2 X - G_1 G_2 G_3 Y$$

$$\rightarrow Y + G_1 G_2 G_3 Y = G_1 G_2 X$$

$$Y(1 + G_1 G_2 G_3) = G_1 G_2 X$$

$$\therefore \frac{Y}{X} = \frac{G_1 G_2}{1 + G_1 G_2 G_3}$$