Process Control
Frequency Response

The University of Jordan

Chemical Engineering Department

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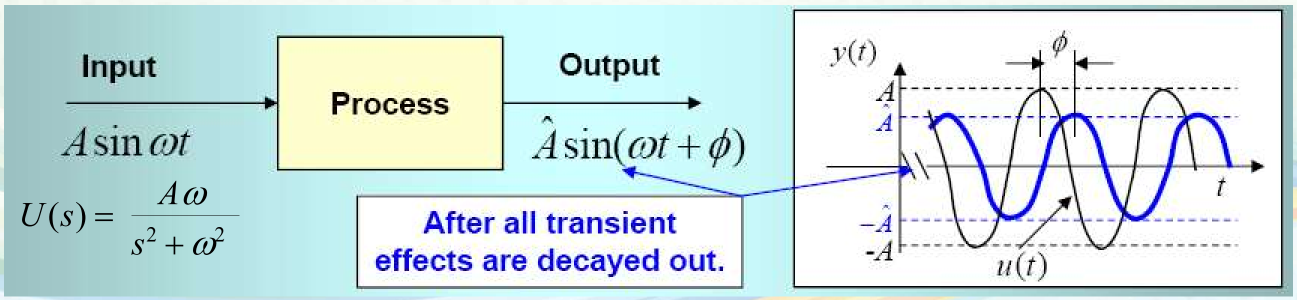
Introduction

- Frequency response concepts and techniques play an important role in
  1. Stability analysis
  2. Control system design
  3. Robustness assessment

Frequency Response

Definition of frequency response

- For a linear system: the ultimate output response of a process for a sinusoidal input of certain frequency will show amplitude change and phase shift at the same frequency depending on the process characteristics.



- Amplitude ratio (AR): attenuation of amplitude, $AR = \frac{\hat{A}}{A}$
- Phase angle (ϕ): phase shift compared to input
- These two quantities are function of frequency.

Definition of frequency response

- *Input:* $u(t) = A\sin(\omega t)$

$$L\big(u(t)\big) \rightarrow U(s) = \frac{A\omega}{s^2 + \omega^2}$$

$$y_\infty(t) = \lim_{t\to\infty} \frac{KA}{\omega^2\tau^2+1}\left(\cancelto{0}{\omega\tau e^{-t/\tau}} - \omega\tau\cos\omega t + \sin\omega t\right)$$

$$= \frac{KA}{\omega^2\tau^2+1}\left(-\omega\tau\cos\omega t + \sin\omega t\right)$$

$$= \underbrace{\frac{KA}{\sqrt{\omega^2\tau^2+1}}}_{\textbf{Amplitude}}\sin(\omega t + \underbrace{\phi}_{\textbf{Phase angle}}) \qquad (\phi = -\tan^{-1}\omega\tau)$$

- *Ultimate Output:*

$$Y(s) = G(s)\frac{A\omega}{s^2+\omega^2} \rightarrow y(t) = L^{-1}[Y(s)]$$

Ultimate Output (frequency response):

$$y(t\rightarrow\infty) = \hat{A}\sin(\omega t + \phi)$$

- $AR = \hat{A}/A$ *is the normalized amplitude ratio*
- ϕ *is the phase angle (PA) or response angle (RA)*
- AR *and* ϕ *are functions of* $\boldsymbol{\omega}$.

Getting frequency response

- Without calculating transient response Y(t), the frequency response can be obtained directly as follows:

✓ For a given transfer function G(s) let:

$$s = j\omega \qquad G(j\omega) = K_1 + K_2 j \qquad j = \sqrt{-1}$$

$$|G| = AR = \sqrt{K_1^2 + K_2^2}$$

$$\phi = \angle G = \arctan\frac{K_2}{K_1}$$

Note that unstable transfer function (TF) does not have a frequency response because a sinusoidal input produces an unstable output response.

Getting frequency response

- *For transfer function of the form:*

$$G = G_1 . G_2 . G_3$$

$$|G| = |G_1| . |G_2| . |G_3|$$

$$log|G| = log|G_1| + log|G_2| + log|G_3|$$

$$\angle G = \angle G_1 + \angle G_2 + \angle G_3$$

- *For transfer function of the form:*

$$G = \frac{G_1}{G_2}$$

$$|G| = \frac{|G_1|}{|G_2|}$$

$$log|G| = log|G_1| - log|G_2|$$

$$\angle G = \angle G_1 - \angle G_2$$

Getting frequency response

- *In general for the transfer function of the form:*

$$G(s) = \frac{G_a(s)G_b(s)G_c(s)\ \dots\dots.}{G_1(s)G_2(s)G_3(s)\ \dots\dots}$$

$$G(j\omega) = \frac{G_a(j\omega)G_b(j\omega)G_c(j\omega)\ \dots\dots.}{G_1(j\omega)G_2(j\omega)G_3(j\omega)\ \dots\dots}$$

$$|G(j\omega)| = \frac{|G_a(j\omega)||G_b(j\omega)||G_c(j\omega)|\ \dots\dots.}{|G_1(j\omega)||G_2(j\omega)||G_3(j\omega)|\ \dots\dots}$$

$$\measuredangle G(j\omega) = \measuredangle G_a(j\omega) + \measuredangle G_b\,(j\omega) + \measuredangle G_c(j\omega)\ \dots\dots. - \measuredangle G_1(j\omega) - \measuredangle G_2\,(j\omega) - \measuredangle G_3\,(j\omega)\ \dots\dots$$

Example:

First order transfer function: $G(s) = \frac{1}{\tau s+1}$

$$G(j\omega) = \frac{1}{1+\tau j\omega} \cdot \frac{1-\tau j\omega}{1-\tau j\omega}$$

$$G(j\omega) = \frac{1}{1+\omega^2\tau^2} - \frac{\tau\omega}{1+\omega^2\tau^2} j$$

$$|G| = AR = \sqrt{K_1^2 + K_2^2} \rightarrow |G| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

$$\phi = \angle G = arctan\frac{K_2}{K_1} \rightarrow \phi = -\arctan(\omega\tau)$$

$$as\ \omega \rightarrow \infty, \phi \rightarrow -90^{\circ}$$

Bode diagram

Bode plots show the frequency response, that is, the changes in magnitude and phase as a function of frequency.

- *Bode diagram is a plot of:*
  - log AR vs. log(ω) or log(τω) → *log-log plot*
  - ϕ vs. log ω or log(τω) → *semi-log plot*

- *Bode diagram is useful to*
  - ✓ *Illustrate frequency response characteristics.*
  - ✓ *Design and analyze the stability of the closed-loop system.*

Example 1

Draw Bode diagram for first-order t TF

$$G(s) = \frac{\mathrm{K}}{\tau s + 1}$$

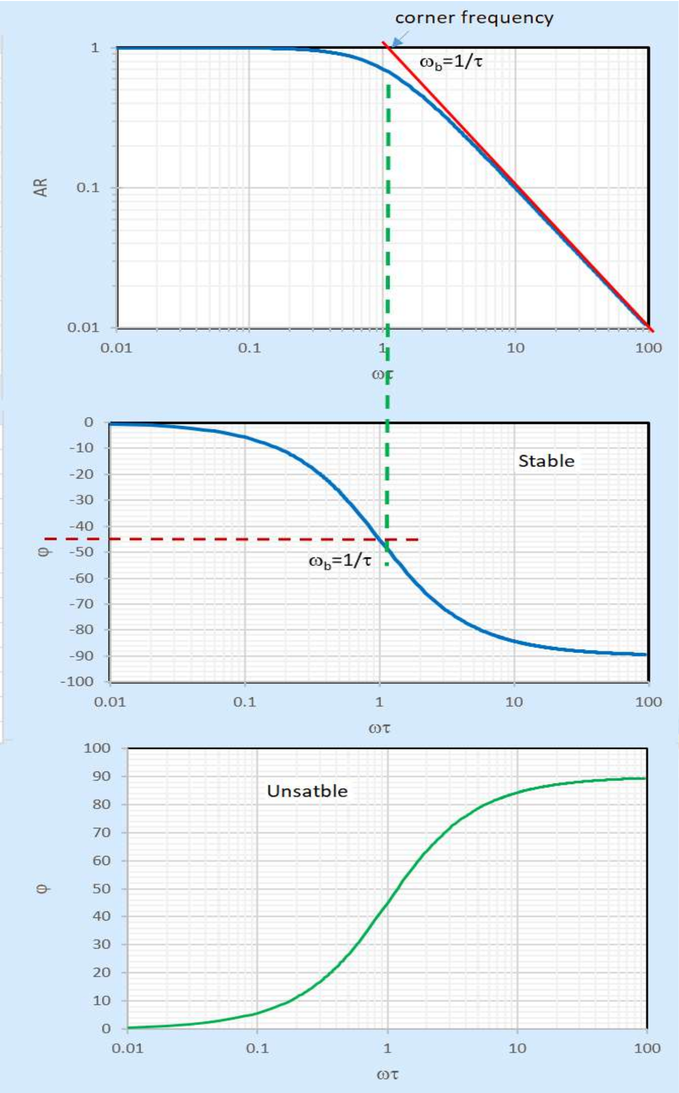
$$|G| = AR = \frac{K}{\sqrt{1 + \omega^2 \tau^2}}$$

The normalized amplitude ratio AR_N is:

$$AR_N = \frac{AR}{K} = \frac{1}{\sqrt{1 + \omega^2 \tau^2}}$$

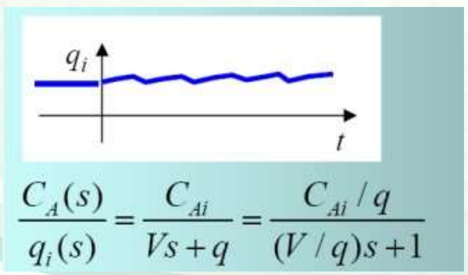
$$\phi = -\arctan(\omega\tau)$$

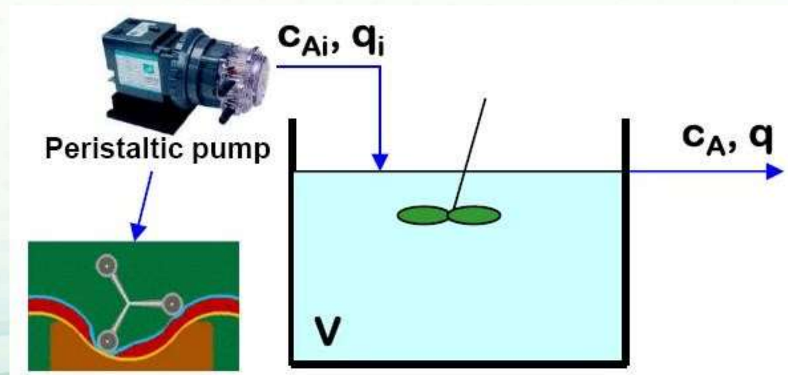
$$as\ \omega \to \infty, \phi \to -90^\circ$$



Example 2

If a feed is pumped by 3 plades peristaltic pump to a CSTR. The rpm of the peristaltic pump is 10 rpm. V= 50 cm³, the time-averaged feed flow rate is 94 cm³/min. Will ± 5% fluctuation in the feed flow appear in the output?





- *Process average-time constant*: $\tau = \frac{V}{q} = \frac{50}{94} = 0.53 min$
- *Input frequency;* $\omega = 2\pi P = 2\pi \times rpm \times 3 \leftarrow 3\, plades = 188.6\ rad/min$

$$\omega\tau = 100\, rad$$

- *From first-order transfer function Bode diagram at* $\omega\tau = 100\, rad$*:*
- $AR_N(normalized\ amplitude\ ratio) = 0.01\ ;\ \phi = -90^{\circ} = -\frac{\pi}{2}$

$$u(t) = q_i - \bar{q}_i + Asin(188.6\, t) \rightarrow U(t) = Asin(188.6\, t)$$

$$C_A(t \rightarrow \infty) = \hat{A}sin\left(188.6\, t - \frac{\pi}{t}\right);\ \ \hat{A} = (AR)A$$

$$c_A(t \rightarrow \infty) = \bar{c}_A + \hat{A}sin\left(188.6\, t - \frac{\pi}{t}\right)$$

- *For fluctuation in* q_i *of* $\mathcal{U}(t) = \pm 5\%$ *of nominal flow rate, the fluctuation in the output concentration will be about*

$$C_A = AR \times U(t) = \pm 5\% \times 0.01 = \pm 0.05\%$$

which is almost unnoticeable.

Example 3:

Bode diagram of unstable pole TF:

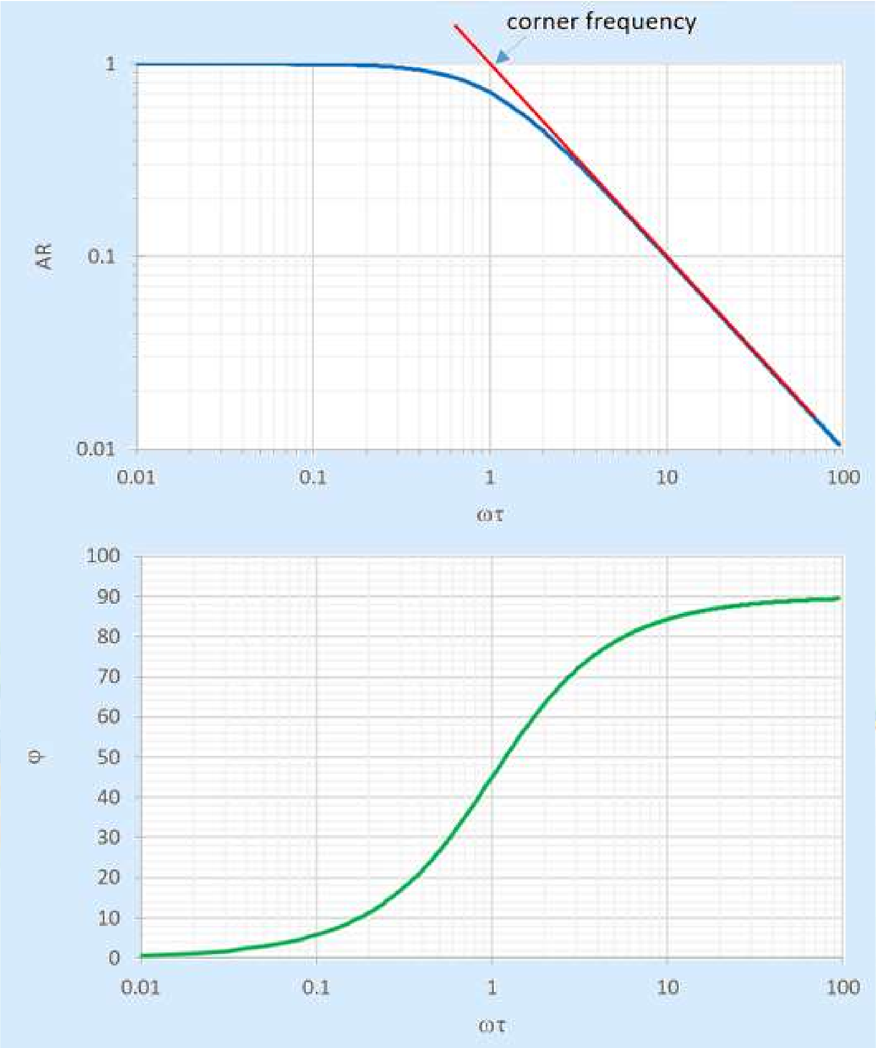
$$G(s) = \frac{1}{-\tau s + 1}$$

$$G(j\omega) = \frac{1}{1 - j\tau\omega} = \frac{1}{1 + \tau^2\omega^2}(1 + j\tau\omega)$$

$$AR = |G(j\omega)| = \frac{1}{\sqrt{1 + \tau^2\omega^2}}$$

$$\phi = \measuredangle G(j\omega) = tan^{-1}\frac{Im\big(G(j\omega)\big)}{Re\big(G(j\omega)\big)} = tan^{-1}(\omega\tau)$$

- *Note that for this unstable process, the phase angle is positive.*
- *The physical interpretation of frequency response is not valid for unstable systems, because a sinusoidal input produces an unbounded output response instead of a sinusoidal response.*



Example 4:

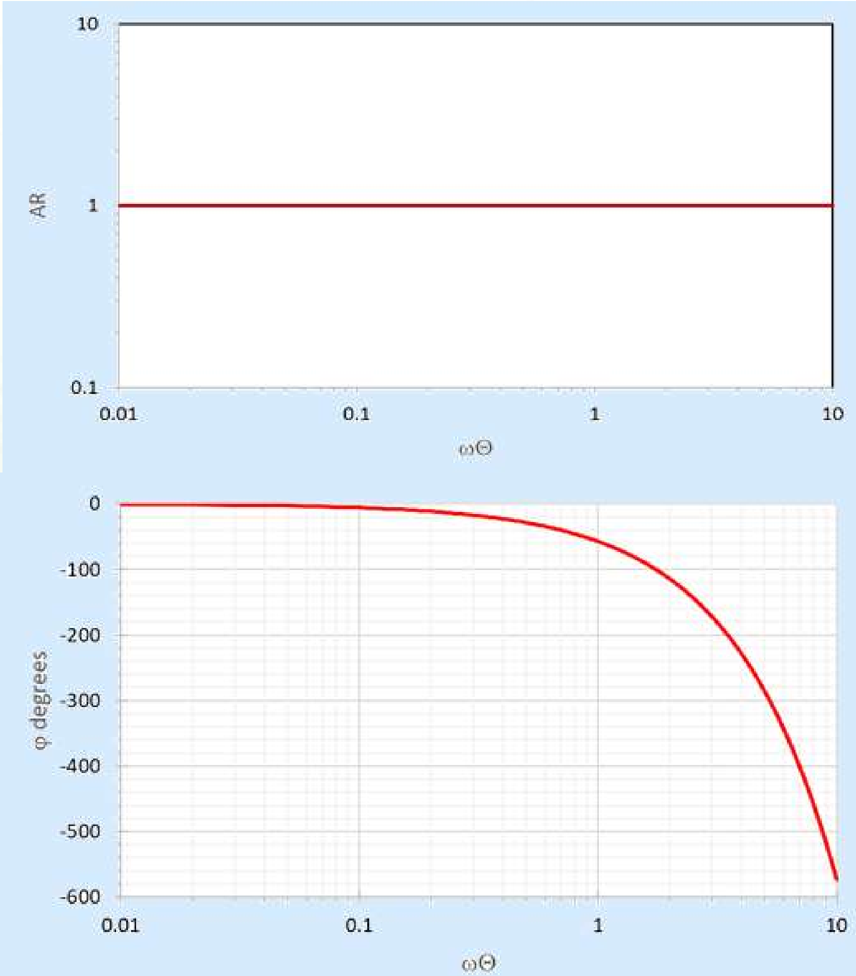
Bode diagram of pure time delay TF:

$$G(s) = e^{-\theta}$$

$$G(j\omega) = e^{-j\theta\omega} = cos(\theta\omega) - jsin(\theta\omega)$$

$$AR = |G(j\omega)| = 1$$

$$\phi = \measuredangle G(j\omega) = tan^{-1}tan(\theta\omega) = -\theta\omega$$



Example 5:

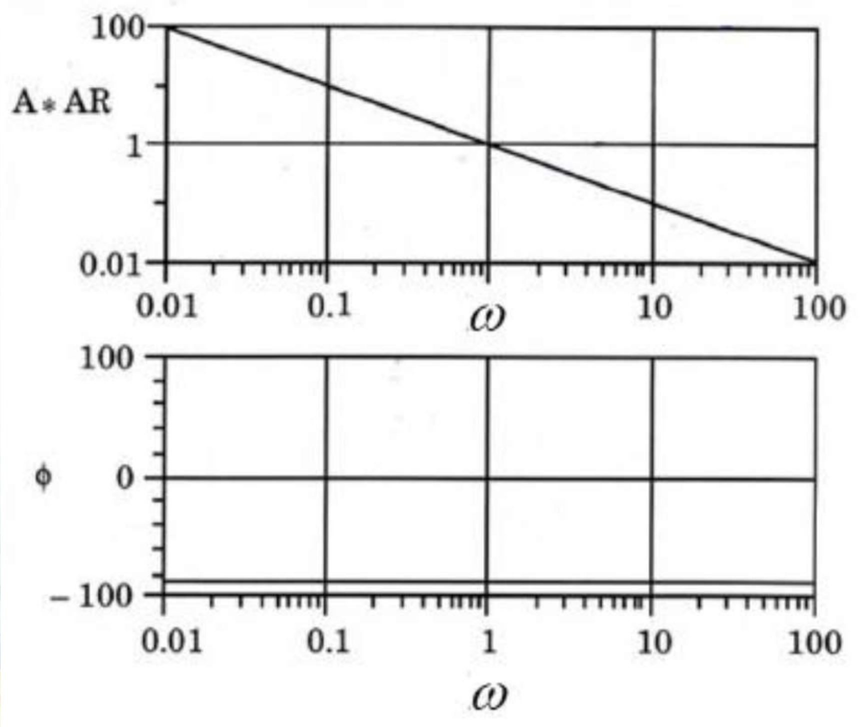
Bode diagram of integrating TF:

$$G(s) = \frac{1}{As}$$

$$G(j\omega) = \frac{1}{jA\omega} = -\frac{1}{A\omega}j$$

$$AR = |G(j\omega)| = \frac{1}{A\omega}$$

$$\phi = \measuredangle G(j\omega) = tan^{-1}\left(-\frac{1}{0.\omega}\right) = -\frac{\pi}{2}$$



Example 6:

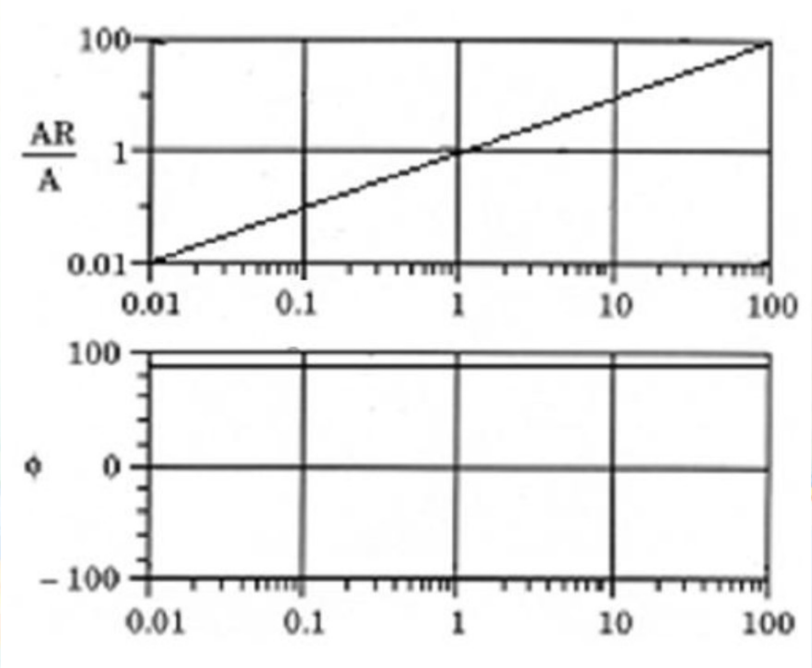
Bode diagram of differentiator TF:

$$G(s) = \mathrm{A}s$$

$$G(j\omega) = jA\omega$$

$$AR = |G(j\omega)| = A\omega$$

$$\phi = \measuredangle G(j\omega) = tan^{-1}\left(\frac{1}{0.\omega}\right) = \frac{\pi}{2}$$



Example 7:

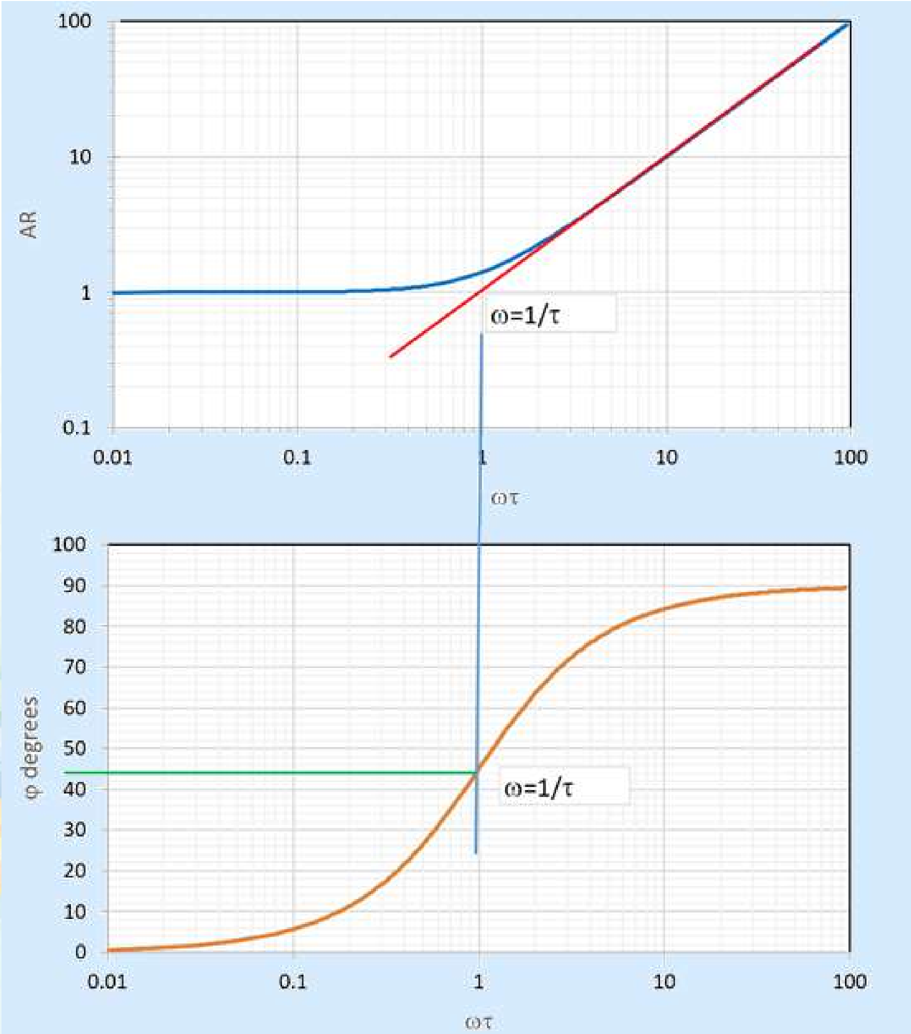
Draw Bode diagram for zero lead (lag) TF:

$$G(s) = \tau s + 1$$

$$G(j\omega) = 1 + j\omega\tau$$

$$AR = |G(j\omega)| = \sqrt{1 + \omega^2\tau^2}$$

$$\phi = \measuredangle G(j\omega) = tan^{-1}(\omega\tau)$$



Example 8:

Draw Bode diagram for second-order TF:

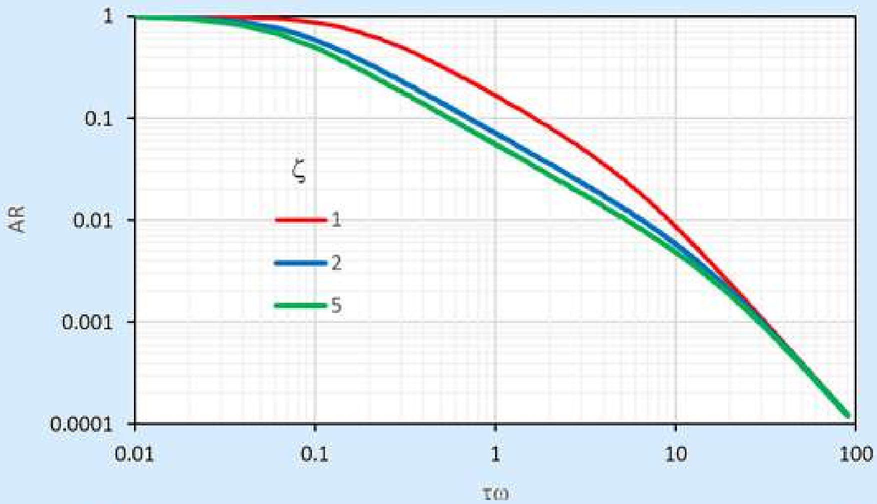
$$G(s) = \frac{K}{(\tau^2 s^2 + 2\xi\tau s \quad)}$$

$$G(j\omega) = \frac{K}{(1 - \tau^2\omega^2) + 2j\xi\tau\omega}$$

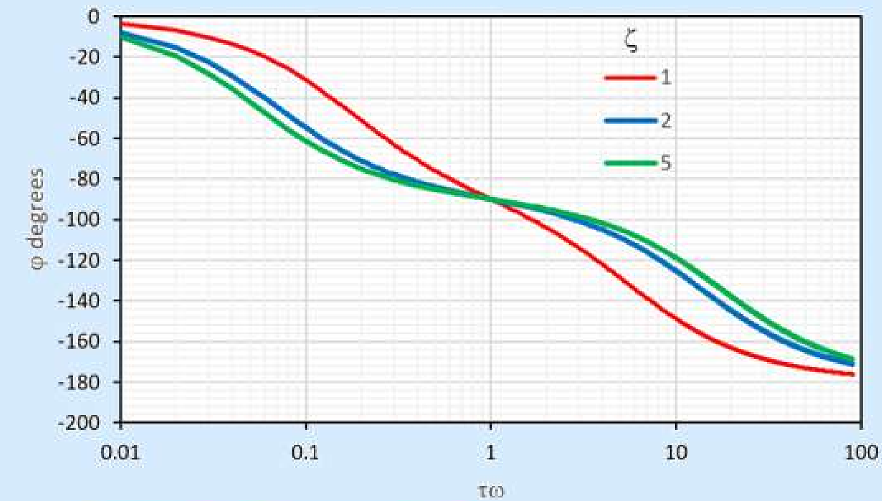
$$AR = |G(j\omega)| = \frac{K}{\sqrt{(1 - \tau^2\omega^2)^2 + (2j\xi\tau\omega)^2}}$$

$$\phi = \measuredangle G(j\omega) = tan^{-1}\frac{Im\big(G(j\omega)\big)}{Re\big(G(j\omega)\big)}$$

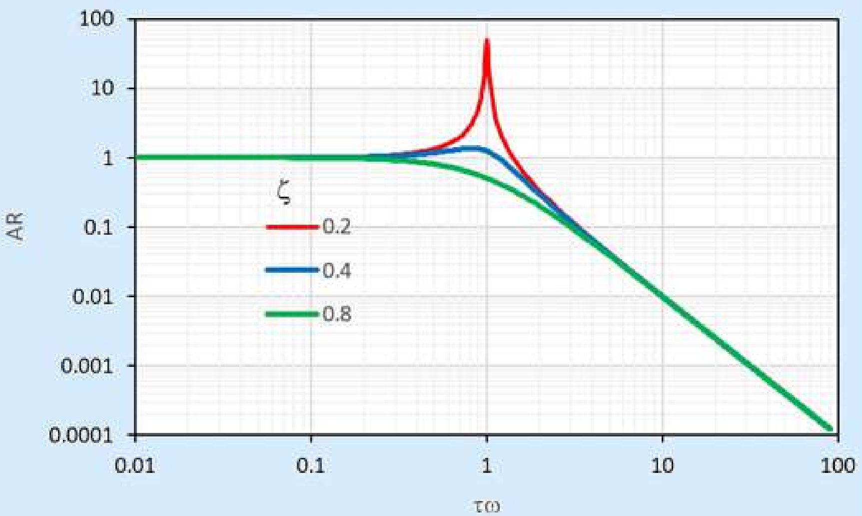
$$= tan^{-1}\frac{2\xi\tau\omega}{(1 - \tau^2\omega^2)}$$



AR
τω
ζ
1
2
5



φ degrees
τω
ζ
1
2
5



AR
τω
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0.2
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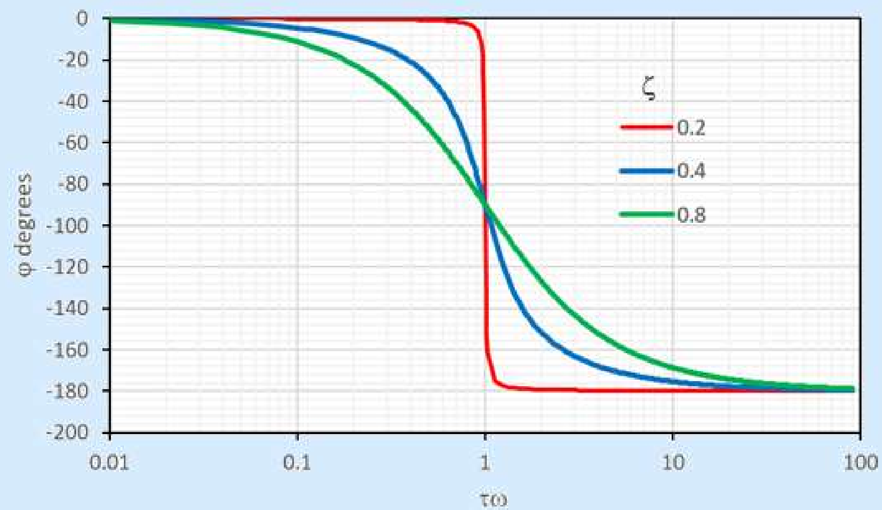
0.8



φ degrees
τω
ζ
0.2
0.4
0.8

Example 9:

Bode diagram of a process with TF:

$$G(s) = \frac{5(0.5s+1)e^{-0.5s}}{(20s+1)(4s+1)}$$

$G(s) = 5G_1(s)G_2(s)G_3(s)G_4(s)$

$G_1(s) = 0.5s + 1 \rightarrow |G_1| = \sqrt{1+0.25\omega^2}\ ; \phi_1 = tan^{-1}(0.25\omega) \rightarrow Zero\ lead\ TF$

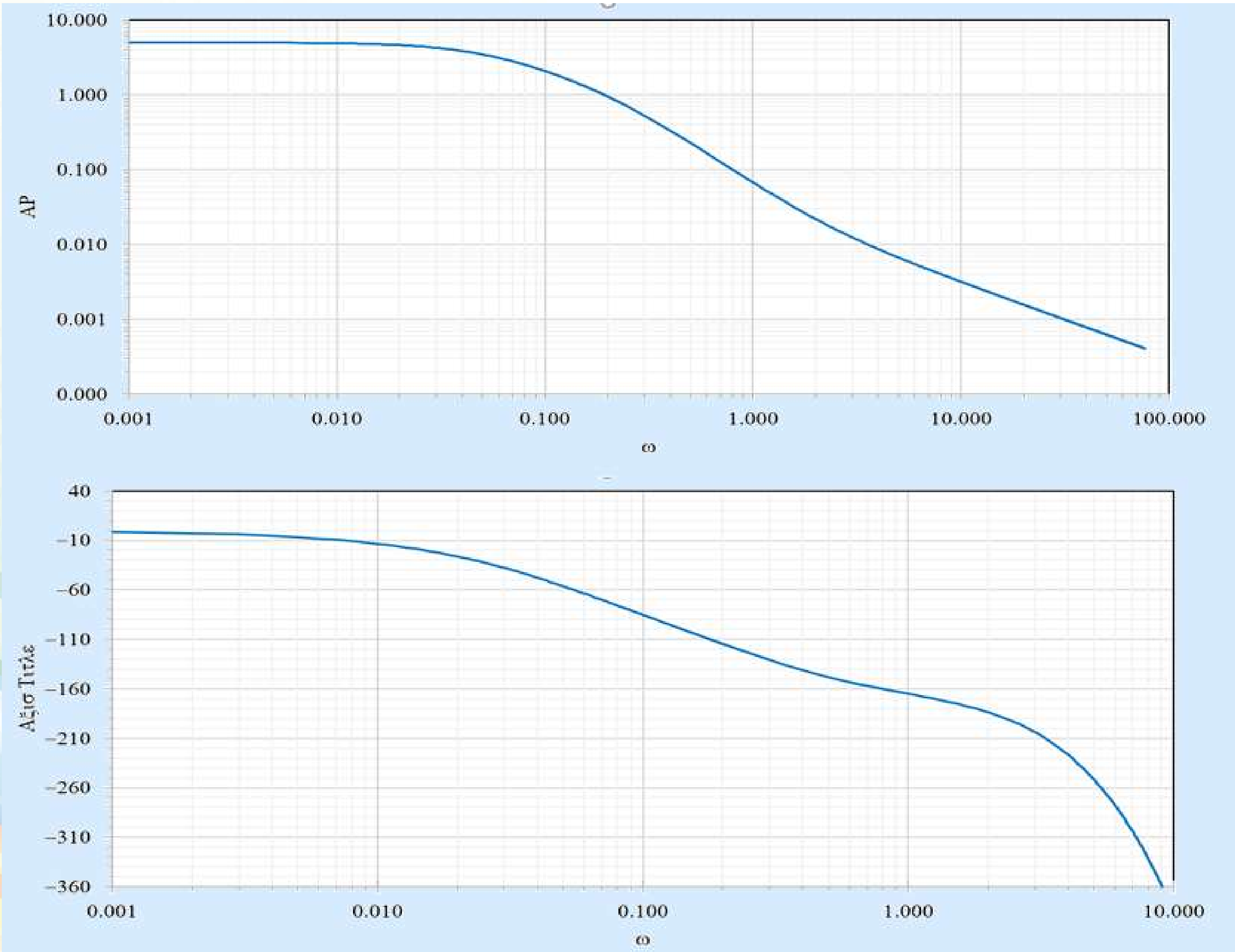
$G_2(s) = e^{-0.5s} \rightarrow |G_2| = 1\ ; \phi_2 = -0.5\omega \rightarrow Pure\ delay\ TF$

$G_3(s) = \frac{1}{20s+1} \rightarrow |G_3| = \frac{1}{\sqrt{1+400\ \ {}^2}}; \phi_3 = -tan^{-1}(20\omega) \rightarrow 1st - order\ TF$

$G_4(s) = \frac{1}{4s+1} \rightarrow |G_4| = \frac{1}{\sqrt{1+16\omega^2}}; \phi_4 = -tan^{-1}(4\omega) \rightarrow 1st - order\ TF$

$$AR = |G_1||G_2||G_3||G_4| = 5\sqrt{\frac{1+0.25\omega^2}{(1+400\omega^2)(1+16\omega^2)}}$$

$\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4 = tan^{-1}(0.25\omega) - 0.5\omega - tan^{-1}(20\omega) - tan^{-1}(4\omega)$


10.000
1.000
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0.001
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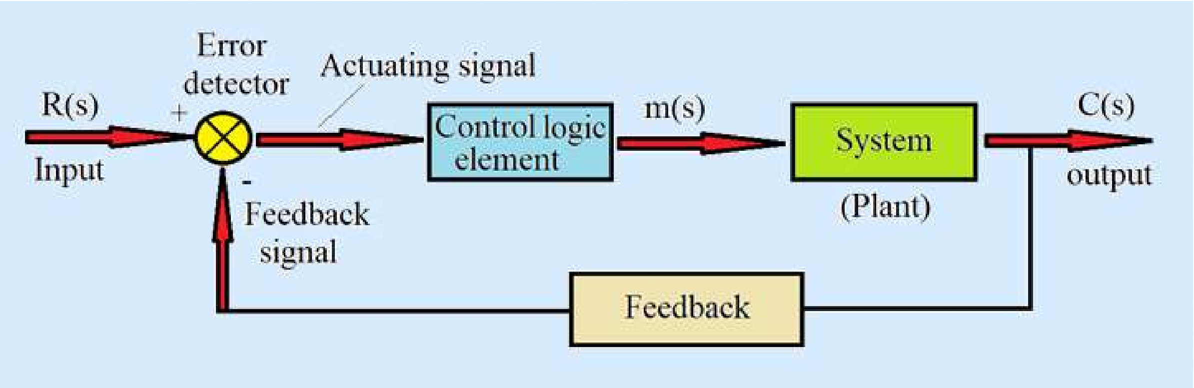
ω
40
-10
-60
-110
-160
-210
-260
-310
-360
Αξισ Τιτλε
0.001
0.010
0.100
1.000
10.000
ω

Controllers



1- Proportional, $G(s) = K_p$

2- Integral Controller, $G(s) = \frac{K_I}{s}$

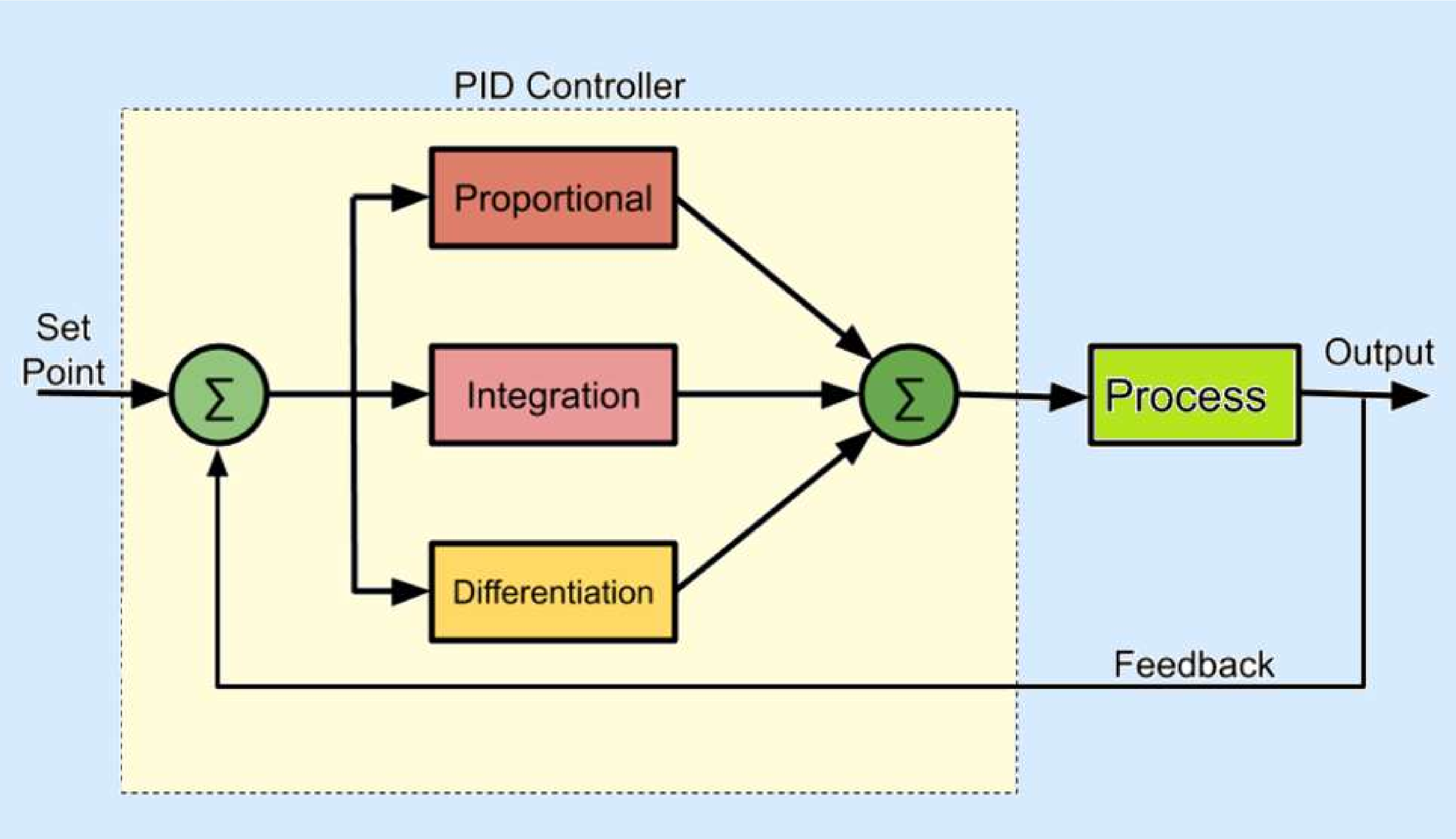
3- Proportional Integral, $G(s) = K_p + \frac{K_I}{s}$

4- Derivative Controller, $G(s) = K_D s$

5- Proportional derivative, $G(s) = K_p + K_D s$

6- Proportional Integral Derivative, $G(s) = K_p + \frac{K_I}{s} + K_D s$

Controllers



Frequency response characteristics of controllers

A. Proportional Controller

$$G_c = K_c \qquad \therefore AR = |K_c|, \qquad \phi = 0$$

B. PI Controller

$$G_c = K_c\left(1 + \frac{1}{\tau_I s}\right)$$

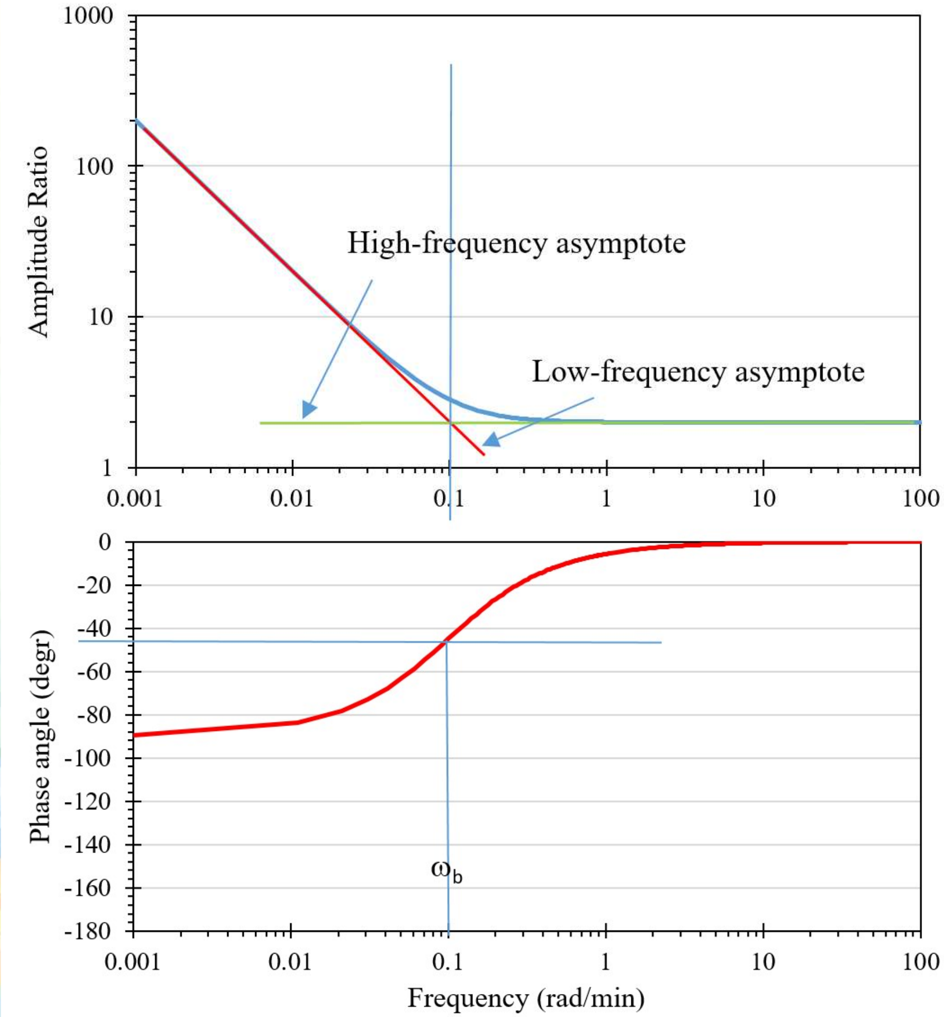
$$AR = K_c\sqrt{\frac{1}{\omega^2\tau_I^2} + 1}$$

$$\phi = tan^{-1}\left(-\frac{1}{\tau_I\omega}\right) = tan^{-1}(\tau_I\omega) - 90^{\circ}$$

The Bode plot for a PI controller is shown in next slide.

Note: $\omega_b = \dfrac{1}{\tau_b}$

$$G_c(s) = 2\left(1 + \frac{1}{10s}\right)$$



C. Ideal PD Controller: $G(s) = K_c(1 + \tau_D s)$

$AR = K_c\sqrt{(\omega\tau_D)^2 + 1}$ $\qquad \phi = tan^{-1}(\omega\tau_D)$

D. PD Controller with filter: $G(s) = K_c\left(\frac{1+\tau_D s}{1+\alpha\tau_D s}\right)$

E. Ideal (Parallel) PID Controller: $G(s) = K_c\left(1 + \frac{1}{\tau_I s} + \tau_D s\right)$

F. Actual (Series) PID Controller without filter: $G(s) = K_c\left(\frac{1+\tau_I s}{\tau_I s}\right)(\tau_D s + 1)$

G. Actual Series PID Controller (Series PID Controller) with filter:

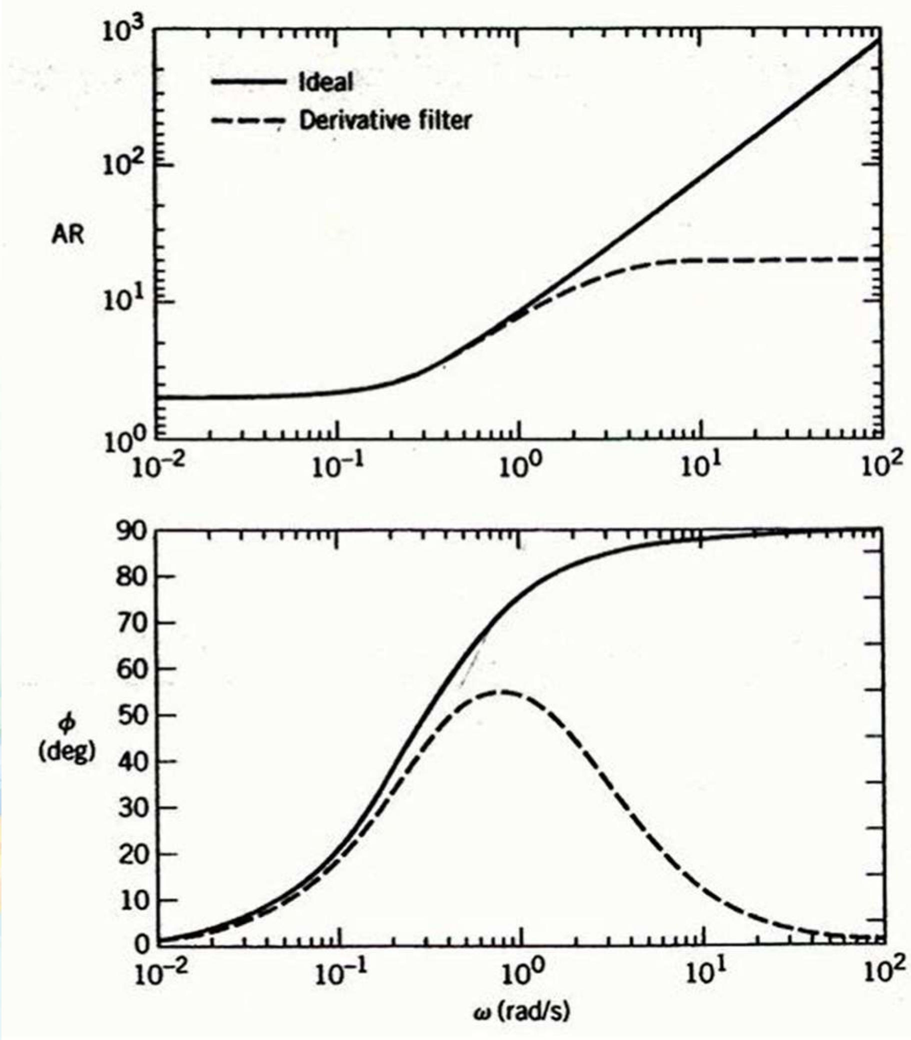
$$G(s) = K_c\left(\frac{1 + \tau_I s}{\tau_I s}\right)\left(\frac{\tau_D s + 1}{1 + \alpha\tau_D s}\right)$$

Bode plots of an ideal PD controller and a PD controller with derivative filter

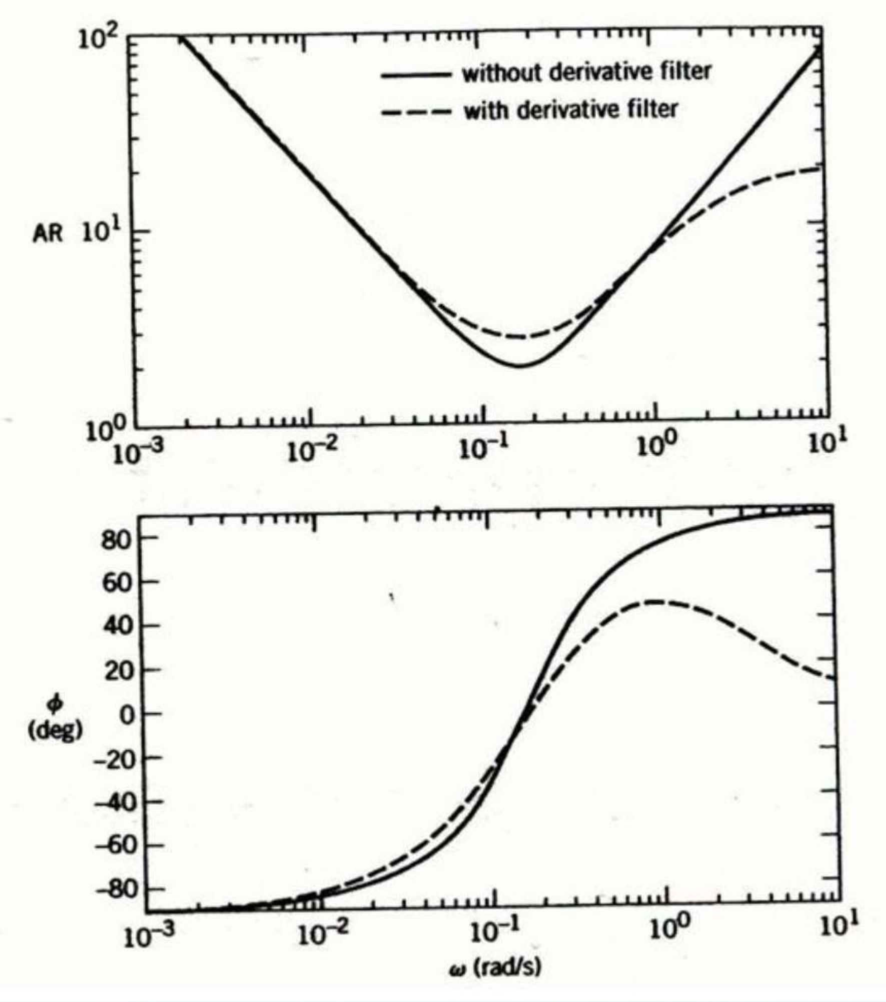
$Ideal$: $G_c(s) = 2(4s + 1)$

$With\ Derivative\ Filter$:

$$G_c(s) = 2\left(\frac{4s+1}{0.4s+1}\right)$$



Bode plots of an ideal parallel PID controller and series PID controller with derivative filter (α=1)



$$\textit{Ideal parallel:}\quad G_c(s) = 2\left(1 + \frac{1}{10s} + 4s\right)$$

Series with Derivative Filter:

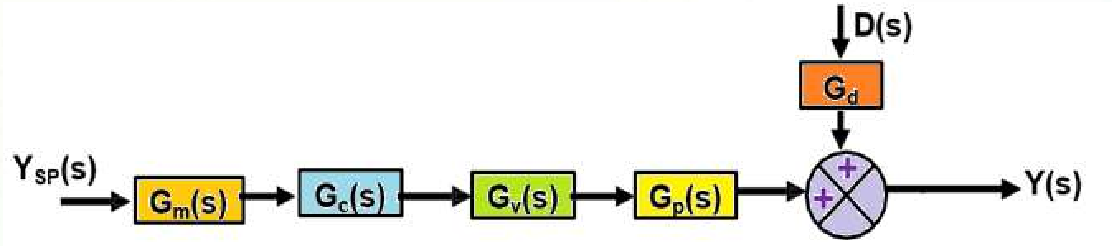
$$G_c(s) = 2\left(\frac{10s+1}{10s}\right)\left(\frac{4s+1}{0.4s+1}\right)$$

Open and Closed Control System

Types of control systems

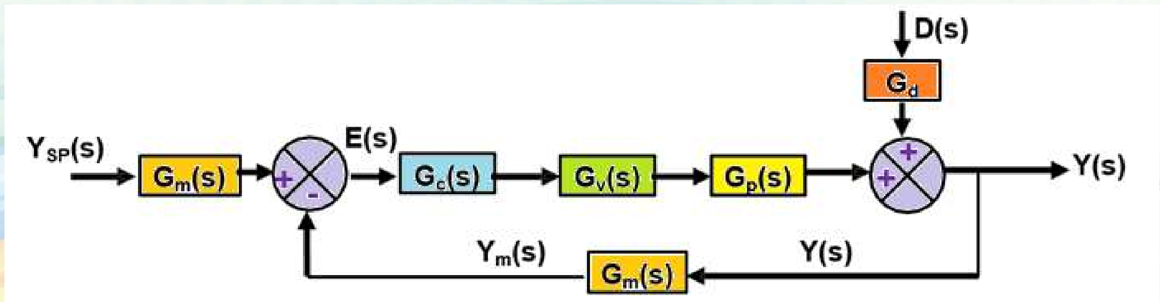
1- Open loop control system (non feedback system)

- Output is directly controlled by input.
- Does not have feedback system.



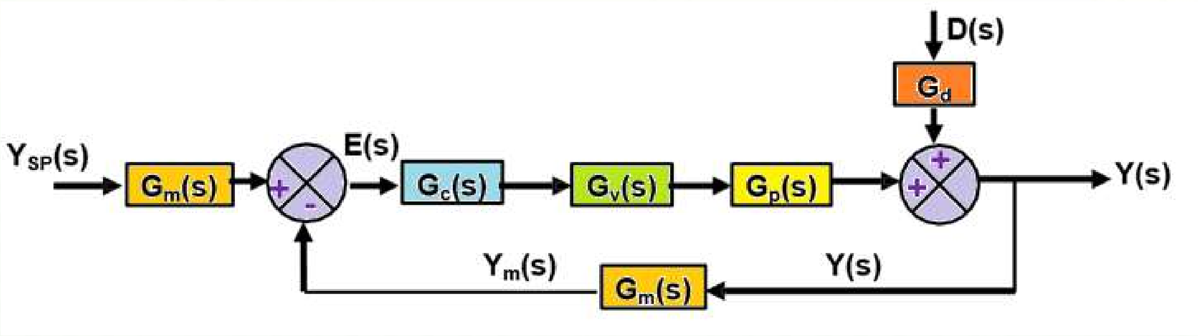
2- Closed loop control system

- Output has an effect on the control action of the input.
- Output is feedback to the input (feedback system)



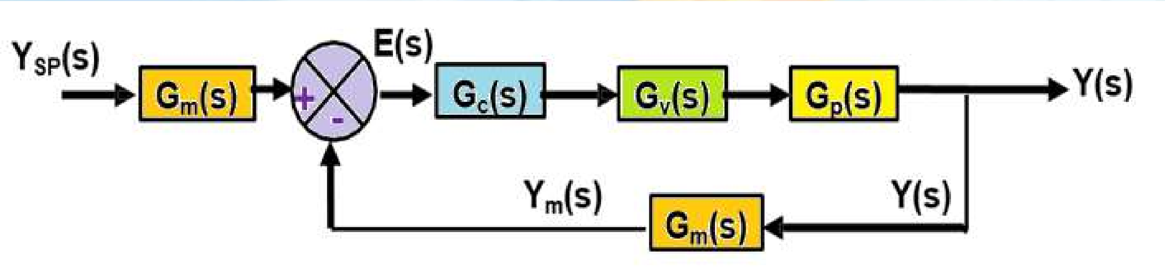
Regulator problem:

Manipulate the system input to counteract the effects of disturbances.

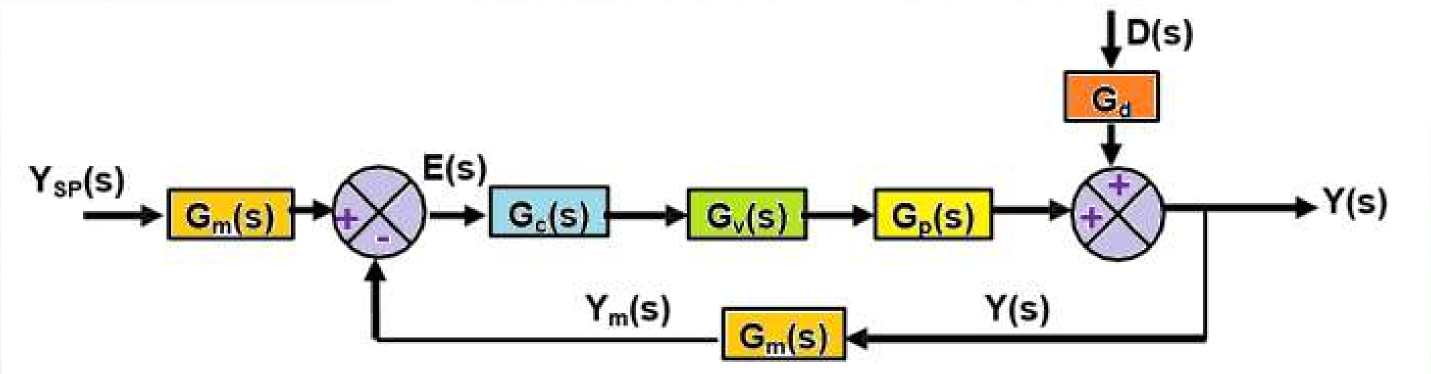


Servo Problem (i.e. tracking problem):

Manipulate the system input to keep the output close to a given reference trajectory



Stability of closed-loop frequency response (FR)



$$\frac{Y(s)}{Y_{SP}(s)} = \frac{G_{OL}}{1+G_{OL}}$$

$$\frac{Y(s)}{D(s)} = \frac{G_d}{1+G_{OL}}$$

$$G_{OL} = G_m G_c G_v G_p$$

Characteristic equation:

$$1+ G_{OL} = 0$$

$$Y_{SP}(s) = \frac{A\omega}{s^2+\omega^2}$$ **"Servo"**

or

$$D(s) = \frac{A\omega}{s^2+\omega^2}$$ **"Regulatory"**

Stability of closed-loop frequency response

- **Stability margins:** as mentioned early the roots of closed-loop characteristic equation must be negative:

$$\text{Characteristic equation: } 1 + G_{OL} = 0$$

- Thus, the margin values for stability of closed-loop system is determined from : $G_{OL}(s) = -1 + 0j$
- This means that the stability margin value for amplitude ratio of the open-loop TF G_{OL} must be:

$$(AR_{OL})_{margin} = |G_{OL}(s)| = 1$$

and the corresponding stability margin value of phase angle of G_{OL} is:

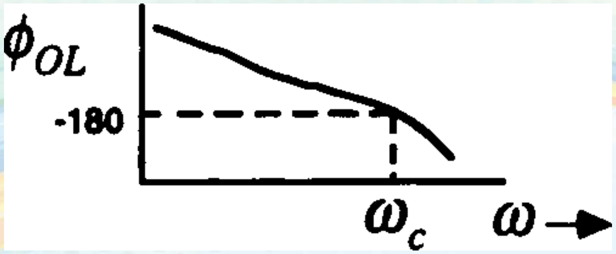
$$(\phi_{OL})_{margin} = tan^{-1}\left(\frac{0}{-1}\right) = -\pi$$

Bode stability criterion: *A closed-loop frequency response (FR) is unstable if the G_{OL} has an amplitude ratio, AR_{OL}, greater than one at the critical frequency (ω_c). Otherwise the closed-loop system is stable:*

$$AR_{OL}|_{\omega_c} < (AR_{OL})_{margin} = 1$$

$$\therefore If\, AR_{OL}|_{\omega_c} < 1 \rightarrow \text{"}stable\ FR\text{"}$$

Where the critical frequency (ω_c) is the value of ω where the open-loop phase angle is $\phi_{OL} = -\pi$



- *Bode stability criterion provides info on closed-loop stability from open-loop frequency response information.*

Example:

A process has $G_p = \frac{2}{(0.5s+1)^3}$, $G_v = 0.1$, $G_m = 10$. *All signals in the closed-loop control system are electrical and time is in minutes. A proportional controller is used, what is the ultimate controller gain,* K_{cu}, *below which the frequency response is stable?*

$$G_{OL} = G_v G_m G_c G_p = (0.1)(10)(K_c)\frac{2}{(0.5s+1)^3} = \frac{2K_c}{(0.5s+1)^3}$$

$$G_{OL} = 2K_c G_1 G_2 G_3$$

where $G_1 = G_2 = G_3 = \frac{1}{0.5s+1}$

$$|G_1| = |G_2| = |G_3| = \frac{1}{\sqrt{1+\tau^2\omega^2}} = \frac{1}{\sqrt{1+0.25\omega^2}}$$

$$\phi_1 = \phi_2 = \phi_3 = -tan^{-1}(\omega\tau) = -tan^{-1}(0.5\omega)$$

$$G(j\omega) = \frac{1}{1-j\tau\omega} = \frac{1}{1+\tau^2\omega^2}(1+j\tau\omega)$$

$$AR = |G(j\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

$$\phi = \measuredangle G(j\omega) = tan^{-1}\frac{Im\big(G(j\omega)\big)}{Re\big(G(j\omega)\big)} = tan^{-1}(\omega\tau)$$

$$AR_{OL} = 2K_c|G_1||G_2||G_3| = 2K_c\left(\frac{1}{\sqrt{1+0.25\omega^2}}\right)^3 = 2K_c(1+0.25\omega^2)^{-1.5}$$

$$\phi_{OL} = \phi_1 + \phi_2 + \phi_3 = -3tan^{-1}(0.5\omega)$$

- *Let us find the critical frequency,* ω_c*, at* $\phi_{OL} = -\pi$

$$-\pi = -3tan^{-1}(0.5\omega_c) \rightarrow \frac{\pi}{3} = tan^{-1}\,(0.5\omega_c) \Rightarrow \omega_c = 3.467\; rad/min$$

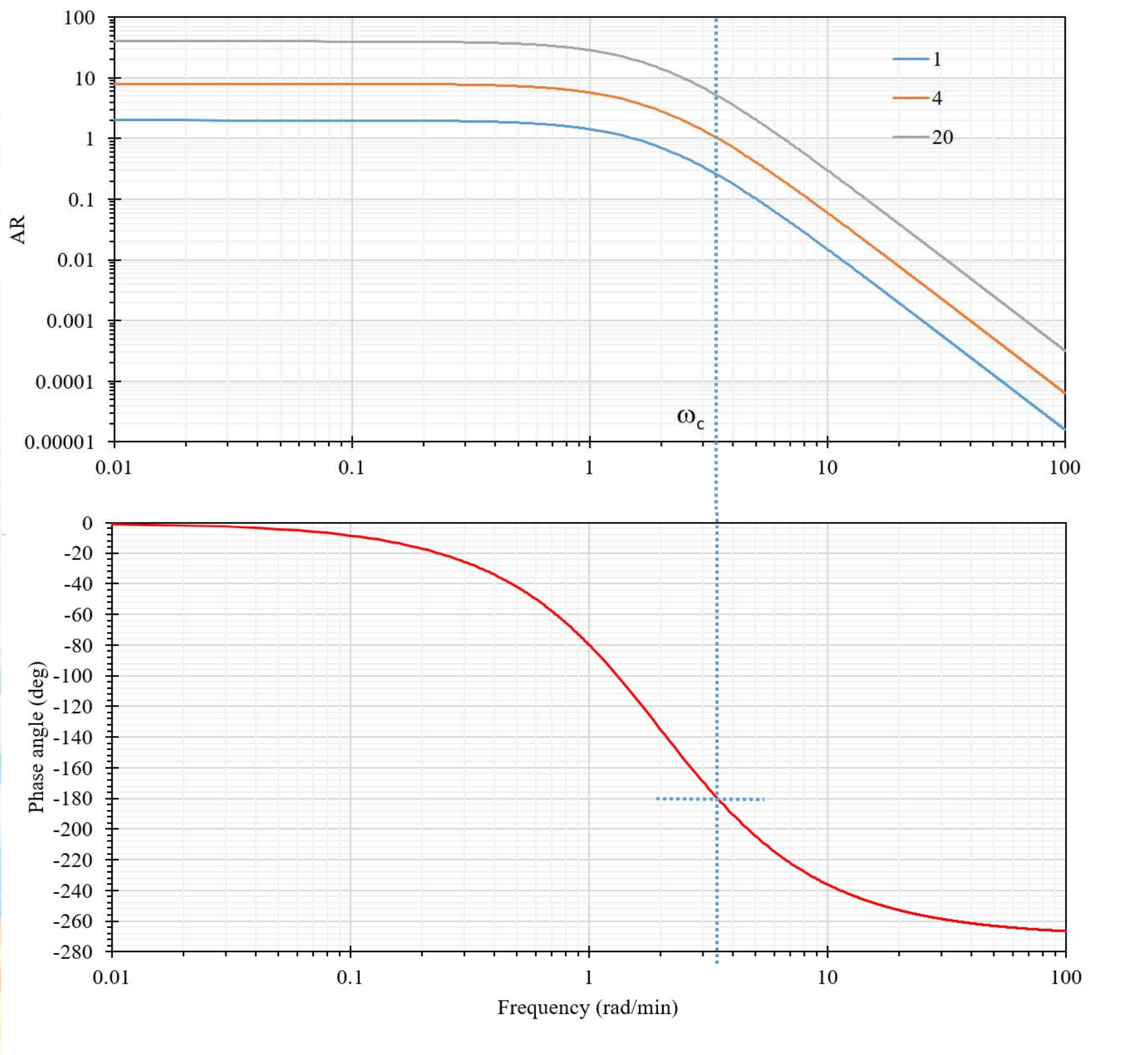
- *The open-loop amplitude ratio,* AR_{OL}*, at this critical frequency, is:*

$$AR_{OL|\omega_c} = 2K_c(1+0.25{\omega_c}^2)^{-1.5} = 0.25K_c$$

- *To achieve stable FR:*

$$AR_{OL|\omega_c} < 1 \Rightarrow 0.25K_c < 1$$

$$\therefore K_c < 4 \rightarrow \boxed{K_{cu} = 4}$$


100
10
1
0.1
0.01
0.001
0.0001
0.00001
AR
1
4
20
ωc
0.01
0.1
1
10
100
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
-200
-220
-240
-260
-280
Phase angle (deg)
0.01
0.1
1
10
100
Frequency (rad/min)

Example:

A process has $G_p = \frac{4e^{-s}}{5s+1}$, $G_v = 2$, $G_m = 0.25$, $G_c = K_c$. *All signals in the closed-loop control system are electrical and time is in minutes. Find the* K_{cu} *of P controller for stable frequency response. Find the corresponding ultimate period of oscillation.*

$$G_{OL} = G_v G_m G_c G_p = (2)(0.25)(K_c)\frac{4e^{-s}}{5s+1} = \frac{2K_c e^{-s}}{0.5s+1}$$

Let $G_{OL} = 2K_c G_1 G_2$

where $G_1 = e^{-s} \rightarrow |G_1| = 1 \;\; ; \;\; \phi_1 = -\omega \;\; \rightarrow Pure\ delay\ TF$

$$G_2 = \frac{1}{5s+1} \rightarrow |G_2| = \frac{1}{\sqrt{1+25\omega^2}} \quad ; \phi_2 = -tan^{-1}(5\omega)$$

$$AR_{OL} = 2K_c|G_1||G_2| = \frac{2K_c}{\sqrt{1+0.25\omega^2}}$$

$$\phi_{OL} = \phi_1 + \phi_2 = -\omega - tan^{-1}(5\omega)$$

- *Let us find the critical frequency,* ω_c*, at* $\phi_{OL} = -\pi$

$= -\omega - tan^{-1}(5\omega)$

- *Solve to obtain* w_c *= 1.69 rad/min*

$$AR_{OL|\omega_c} = 2K_c(1+0.25{\omega_c}^2)^{-1.5} = 0.25K_c$$

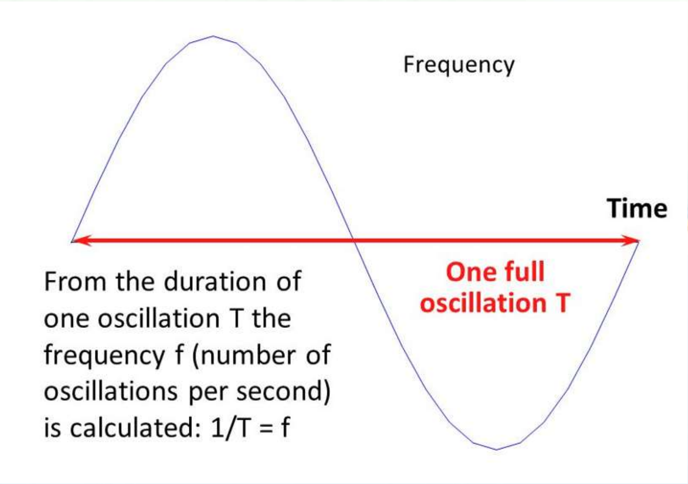
- *To achieve stable FR:*

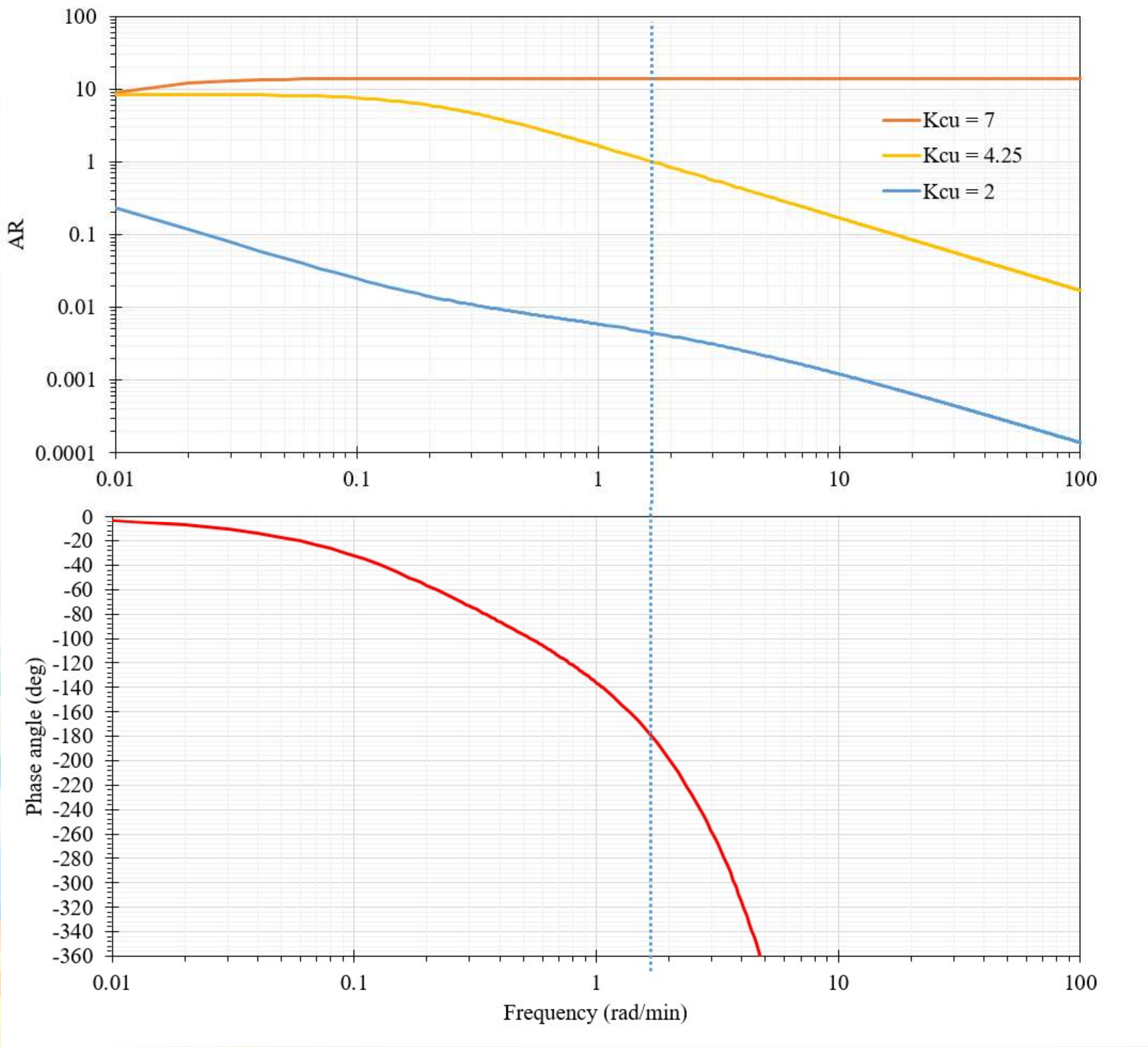
$$AR_{OL|\omega_c} = \frac{2K_c}{\sqrt{1+25\omega^2}} = \frac{2K_c}{\sqrt{1+25(1.69)^2}} = 0.235K_c$$

$$\therefore 0.235K_{cu} = 1 \rightarrow K_{cu} = 4.25$$

If $K_{cu} < 4.25$ *(stable)*

Ultimate period of oscillation: $P_{cu} = \frac{2\pi}{\omega_c} = \frac{2\pi}{1.69} = 3.72\ min$




100
10
1
0.1
0.01
0.001
0.0001
AR
Kcu = 7
Kcu = 4.25
Kcu = 2
0.01
0.1
1
10
100
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
-200
-220
-240
-260
-280
-300
-320
-340
-360
Phase angle (deg)
Frequency (rad/min)

Proportional Gain and Phase Margins

- Gain Margin (GM): According to the Bode stability criterion,

$$AR_{OL}|_{\omega_c} < 1 \rightarrow GM = \frac{1}{AR_{OL}|_{\omega_c}} > 1$$

- Phase Margin(PM): Let ω_g is the frequency at which $AR_{OL} = 1.0$ and the corresponding phase angle is $\phi_{OL|g}$.
- According to the Bode stability criterion,

$$\phi_{OL|\omega c} = -180$$

When $\phi_{OL|\omega c} > -180^{\circ} \rightarrow AR_{OL}|_{\omega_c} < 1$

$$\phi_{OL|\omega_g} + 180^{\circ} > 180^{\circ} - 180^{\circ} \rightarrow \phi_{OL|\omega_g} + 180^{\circ} > 0$$

$\phi_{OL|\omega_g} + 180^{\circ}$ Phase Margin

Thus, for stability PM > 0 and GM > 0

Proportional Gain and Phase Margins

- The greater the Gain Margin (GM), the greater the stability of the system.
- The gain margin refers to the amount of gain, which can be increased or decreased without making the system unstable. It is usually expressed as a magnitude in dB.
- The greater the Phase Margin (PM), the greater will be the stability of the system.
- The phase margin refers to the amount of phase, which can be increased or decreased without making the system unstable. It is usually expressed as a phase in degrees.

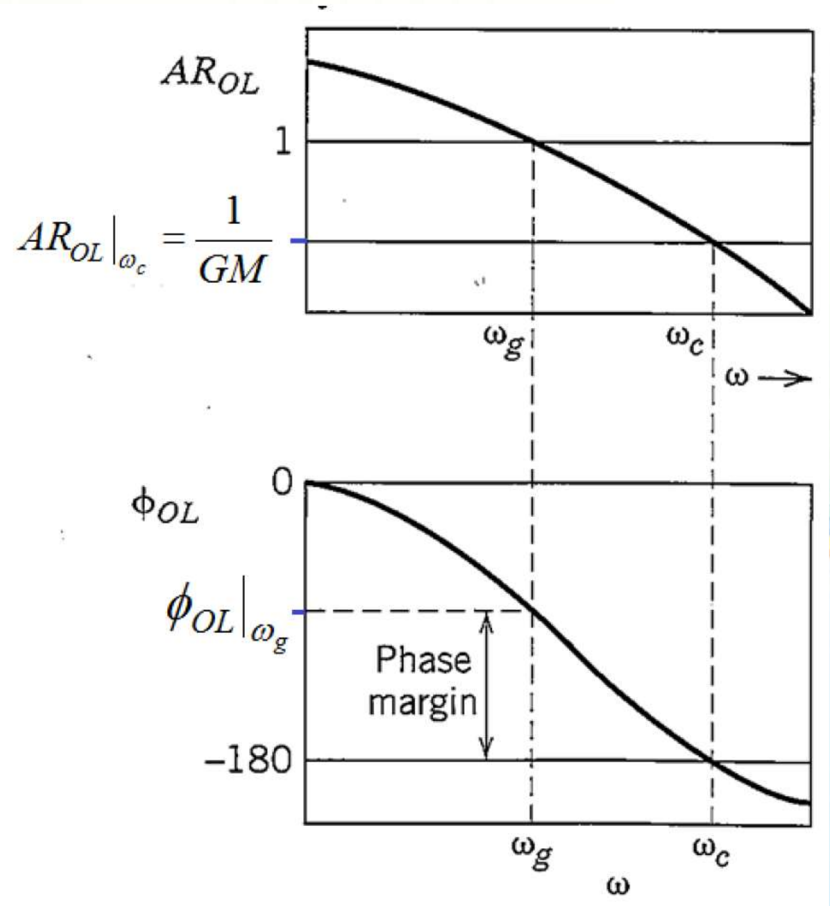
Proportional Gain and Phase Margins

Rules of thumb:

A well-designed FB control system will have:

$$1.7 \le GM \le 2.0$$

$$30° \le PM \le 45°$$



Nyquist stability criterion

- The Nyquist stability criterion is similar to the Bode criterion in that it determines closed-loop stability from the open-loop frequency response characteristics.
- The Nyquist stability criterion is based on two concepts from complex variable theory, Contour Mapping and the Principle of the Argument.
- Nyquist Stability Criterion. Consider an open-loop transfer function $G_{OL}(s)$ that is proper and has no unstable pole-zero cancellations. Let N be the number of times that the Nyquist plot for $G_{OL}(s)$ encircles the -1 point in the clockwise direction. Also let P denote the number of poles of $G_{OL}(s)$ that lie to the right of the imaginary axis. Then, $Z = N + P$ where Z is the number of roots of the characteristic equation that lie to the right of the imaginary axis (that is, its number of "zeros").
- The closed-loop system is stable if and only if $Z = 0$.

Some important properties of the Nyquist stability criterion

1. It provides a necessary and sufficient condition for closed-loop stability based on the open-loop transfer function.
2. The reason the -1 point is so important can be deduced from the characteristic equation, $1 + G_{OL}(s) = 0$. This equation can also be written as $G_{OL}(s) = -1$, which implies that $AR_{OL} = 1$ and , as noted earlier. The -1 point is referred to as the critical point.
3. Most process control problems are open-loop stable. For these situations, $P = 0$ and thus $Z = N$. Consequently, the closed-loop system is unstable if the Nyquist plot for $G_{OL}(s)$ encircles the -1 point, one or more times.
4. A negative value of N indicates that the -1 point is encircled in the opposite direction (counter-clockwise). This situation implies that each countercurrent encirclement can stabilize one unstable pole of the open-loop system 0.

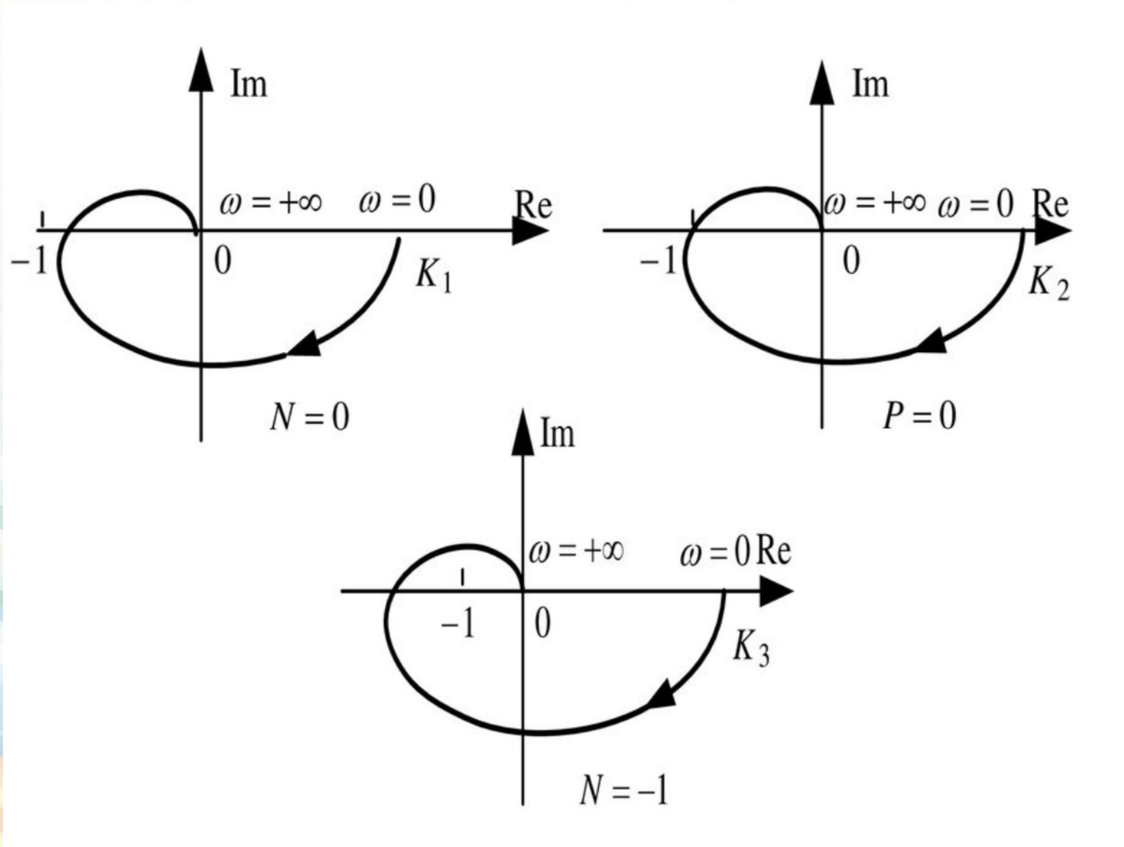
Some important properties of the Nyquist stability criterion

5. Unlike the Bode stability criterion, the Nyquist stability criterion is applicable to open-loop unstable processes.
6. Unlike the Bode stability criterion, the Nyquist stability criterion can be applied when multiple values of ω_c or ω_g occur.

Nyquist stability Criterion:

Let N be the number of times the Nyquist plot for $G_{OL}(s)$ encircles the (-1,0) point in the clockwise direction. Also let P denotes the number positive poles of $G_{OL}(s)$. Then, $Z = N+P$ is the number of positive roots of the characteristic equation ($G_{OL}(s) + 1 = 0$). Thus, the closed-loop system is stable if and only if $Z = 0$

Nyquist stability criterion:



Example:

A process has $G_p = \frac{4e^{-s}}{5s+1}$ *,* G_v*= 2,* G_m*= 0.25,* $G_c = K_c$*. all signals in the closed-loop control system are electrical and time is in minute. Draw Nyquist plots for Kc = 4, 6.38, and 50.*

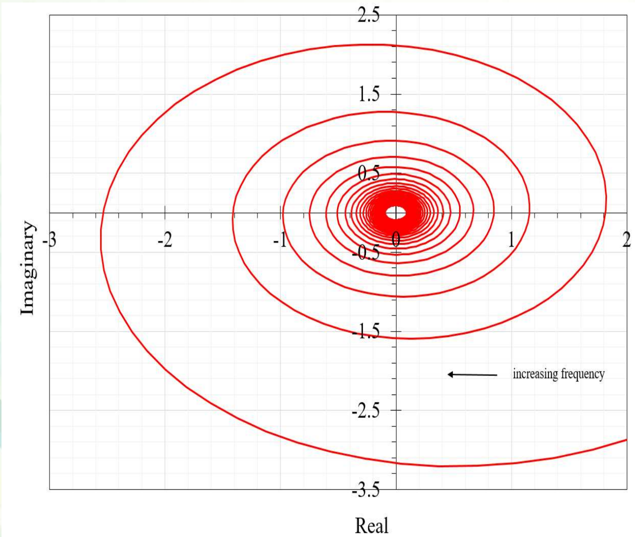
$$G_{OL} = \frac{2K_c e^{-s}}{5s+1} \Rightarrow G_{OL}(\omega j) = \frac{2K_c e^{-\omega j}}{5\omega j + 1} = \frac{2K_c(cos\omega - jsin\omega)}{5\omega j + 1} \times \frac{1 - 5\omega j}{1 - 5\omega j}$$

$$= \frac{2K_c}{25\omega^2 + 1}(cos\omega - jsin\omega)(1 - 5\omega j)$$

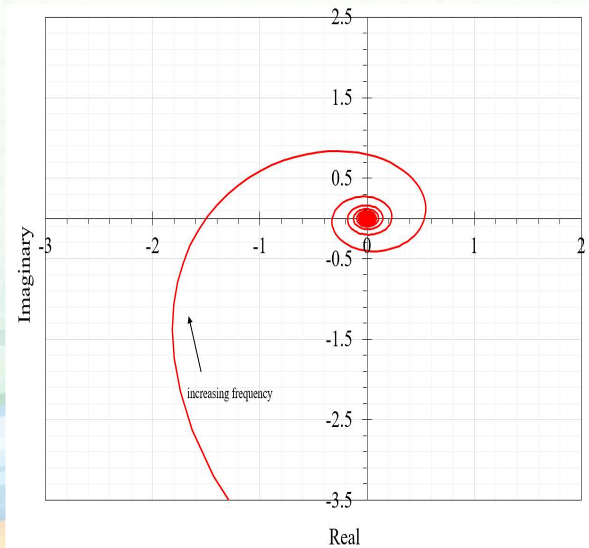
$$= \frac{2K_c}{25\omega^2 + 1}[(cos\omega - 5\omega sin\omega) - (sin\omega + 5\omega cos\omega)j]$$

$$= \frac{2K_c(cos\omega - 5\omega sin\omega)}{25\omega^2 + 1} - \frac{2K_c(sin\omega + 5\omega cos\omega)}{25\omega^2 + 1}j$$

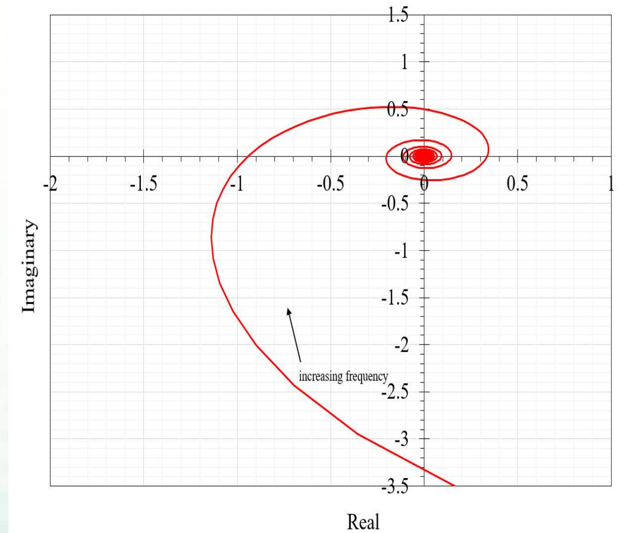
Now prepare a polar plot of $G_{OL}(\omega j)$ *for every Kc in the direction of increasing* ω *value.*



Nyquist plot for K_c= 50



Nyquist plot for K_c= 6.38



Nyquist plot for K_c = 4

Use the Nyquist stability criterion for a closed-loop system of the previous example.

$$G_{OL}(s) = \frac{2K_c e^{-s}}{5s+1}$$

Poles of G_{OL}: $5s + 1 = 0 \rightarrow s = -\frac{1}{5}$ *"one negative pole"*

- *This means that there is no positive poles of* $G_{OL} \rightarrow P = 0$
- *See previous plots to count the number of times, N, that Nyquist plot encircles the point (-1, 0):*

- *At* $K_c = 6.38$: $N = 1 \rightarrow Z = N + P = 1 + 0 = 1 \rightarrow$ *Not stable FR*
- *At* $K_c = 50$: $N = 3 \rightarrow Z = N + P = 3 + 0 = 3 \rightarrow$ *Not stable FR*
- *At* $K_c = 4$: $N = 0 \rightarrow Z = N + P = 0 + 0 = 0 \rightarrow$ *stable FR*
- *For* $K_c < 4.25$: $Z = N + P = 0 + 0 = 0 \rightarrow$ *stable FR*