Process Control

Dynamic Behavior and Stability of Closed-Loop Control System

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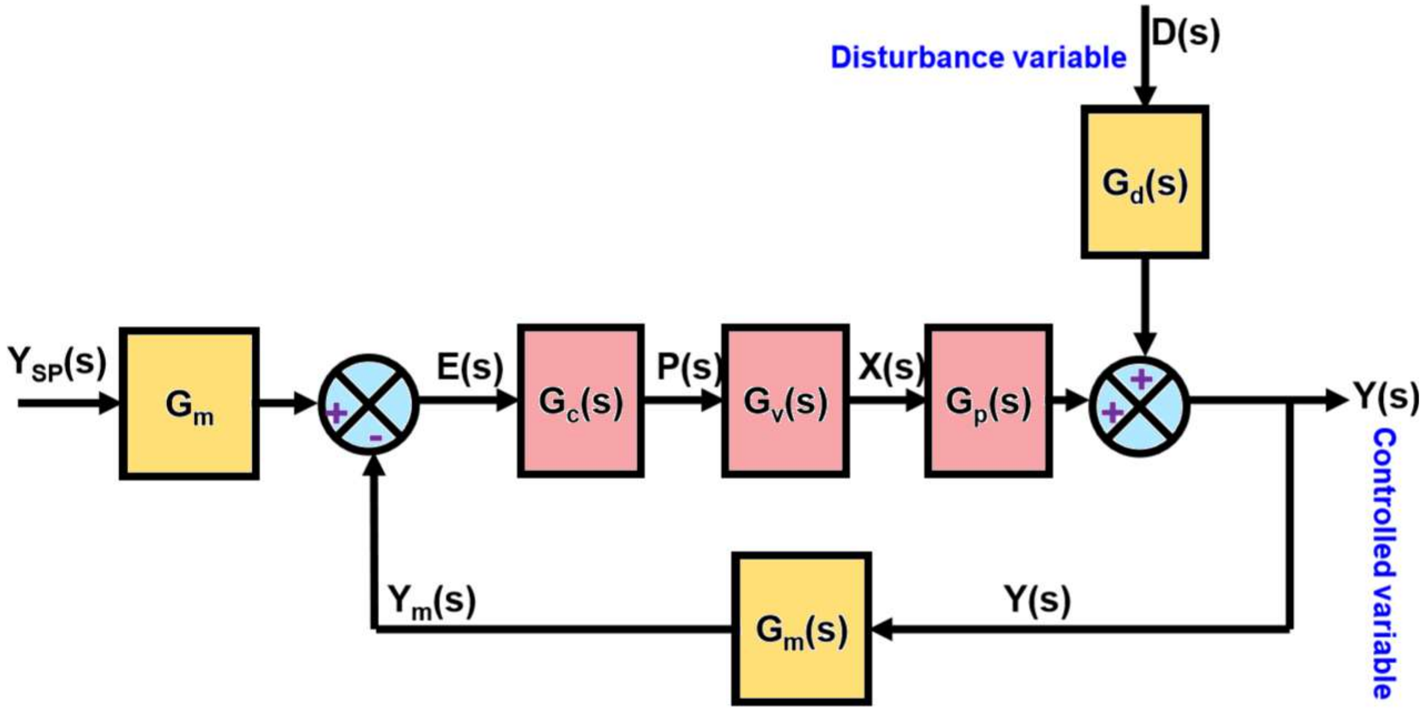
Dynamic Behavior and Stability of Closed-Loop Control System



"Standard block diagram of closed-loop feedback control system with one disturbance"

Closed-Loop Transfer Functions:

- Using additive and multiplicative properties of transfer functions, previously explained:
  - Transfer function between controlled variable and its set point (Servo problem: change in set point; no changes disturbances):

$$\frac{Y(s)}{Y_{SP}(s)} = \frac{G_m G_c G_v G_p}{1 + G_m G_c G_v G_p}$$

  - Transfer function between controlled variable and its disturbance/load (Regulatory problem: changes in set disturbance; no change in set point disturbances:

$$\frac{Y(s)}{D(s)} = \frac{G_d}{1 + G_c G_v G_p G_m}$$

Dynamic Behavior and Stability of Closed-Loop Control System

- The closed loop becomes open when the feedback path is broken. The open-loop transfer function is:

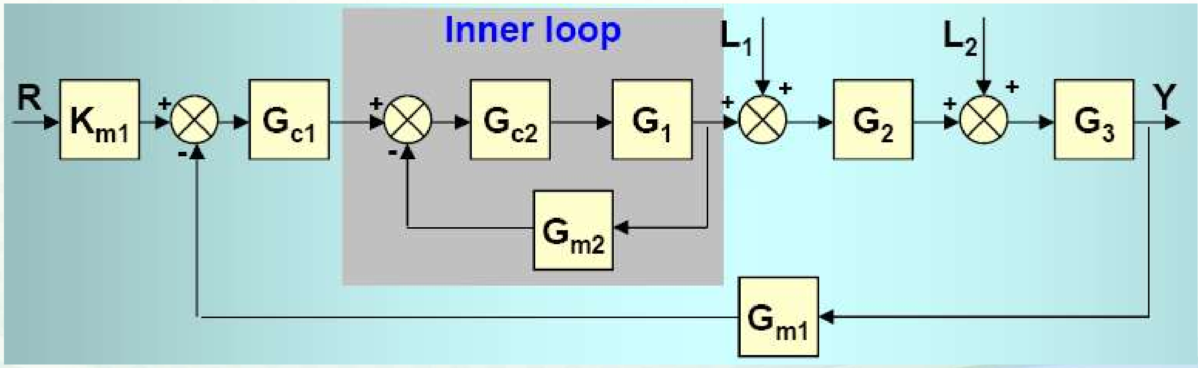
$$G_{OL} = G_m G_c G_v G_p$$

$$\frac{Y(s)}{Y_{SP}(s)} = \frac{G_{OL}}{1+G_{OL}} \qquad \frac{Y(s)}{D(s)} = \frac{G_d}{1+G_{OL}}$$

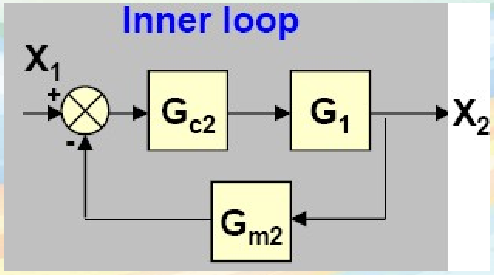
- For simultaneous changes in set point and disturbance:

$$Y(s) = \frac{G_c G_v G_p G_m}{1+G_c G_v G_p G_m} Y_{SP}(s) + \frac{G_d}{1+G_c G_v G_p G_m} D(s)$$

Example: For the control loop shown below, find the transfer functions Y/R, Y/L_1, and Y/L_2, :



Solution:



$$X_2 = \frac{G_1 G_{c2}}{1 + G_{m2} G_1 G_{c2}} X_1$$

- *Transfer function Y/R:*

$$\frac{Y}{R}=\frac{K_{m1}G_3G_2G_1G_{c2}G_{c1}}{1+G_{m2}G_1G_{c2}+G_{m1}G_3G_2G_1G_{c2}G_{c1}}$$

- *Transfer function Y/L_1:*

$$\frac{Y}{L_1}=\frac{G_3G_2(1+G_{m2}G_1G_{c2})}{1+G_{m2}G_1G_{c2}+G_{m1}G_3G_2G_1G_{c2}G_{c1}}$$

- *Transfer function Y/L_2:*

$$\frac{Y}{L_2}=\frac{G_3(1+G_{m2}G_1G_{c2})}{1+G_{m2}G_1G_{c2}+G_{m1}G_3G_2G_1G_{c2}G_{c1}}$$

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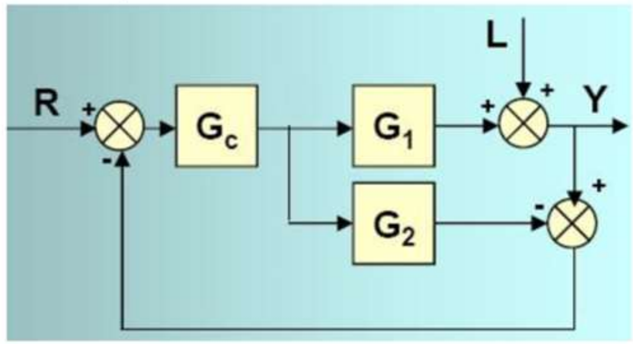
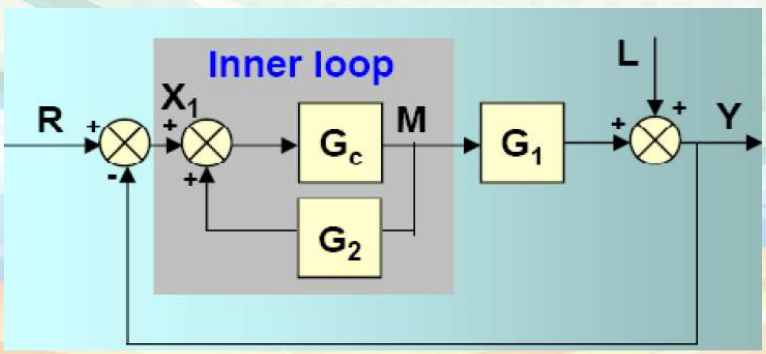
Example: For the control loop shown below, find the transfer functions Y/R and Y/L:



Solution:



$$M = \frac{G_c}{1 - G_2 G_c} X_1$$

$$\frac{Y}{R} = \frac{G_1 G_c}{1 - G_2 G_c + G_1 G_c} = \frac{G_1 G_c}{1 + (G_1 - G_2) G_c}$$

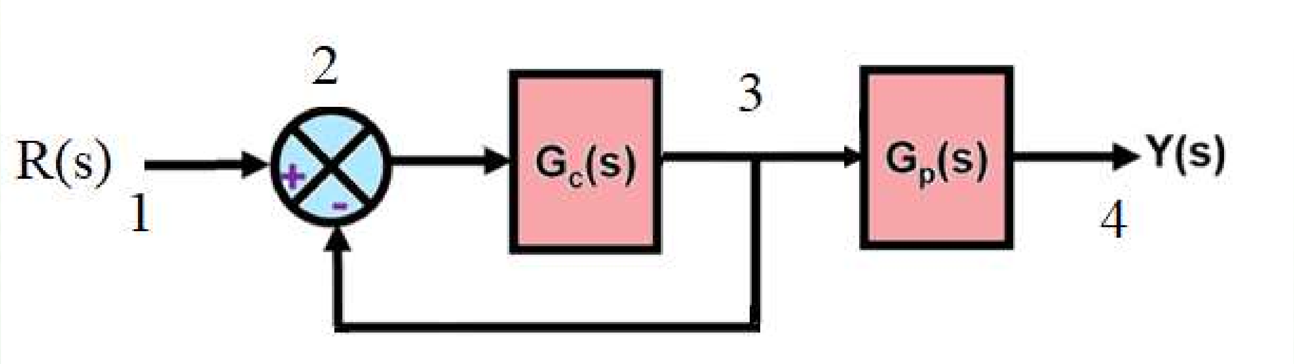
$$\frac{Y}{L} = \frac{1}{1 + \pi_f} = \frac{1 - G_2 G_c}{1 + (G_1 - G_2) G_c}$$

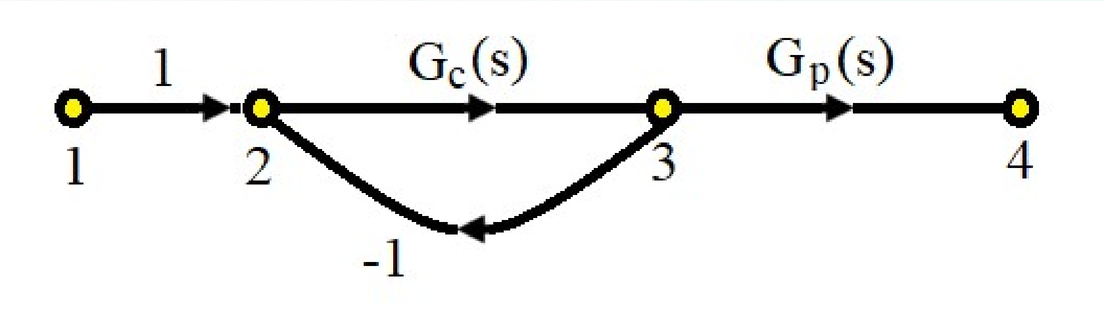
$$= \frac{1 - G_2 G_c}{1 - G_2 G_c + G_1 G_c}$$

Procedure for converting block diagram to signal flow graph

1. Assume nodes at the input, output, at summing points, at every branch point and in between cascaded blocks.
2. Draw the nodes separately as small circle and number the circle in the order 1, 2, 3,…
3. From the block diagram find the gain between each node in the main forward path and
4. Connect all the corresponding circles by straight lines and mark the gain on the nodes.
5. Draw the field forward paths between various nodes and mark the gain of field forward path along with sign.
6. Draw the feed back path between various nodes and mark the gain of the feedback paths along with sign.

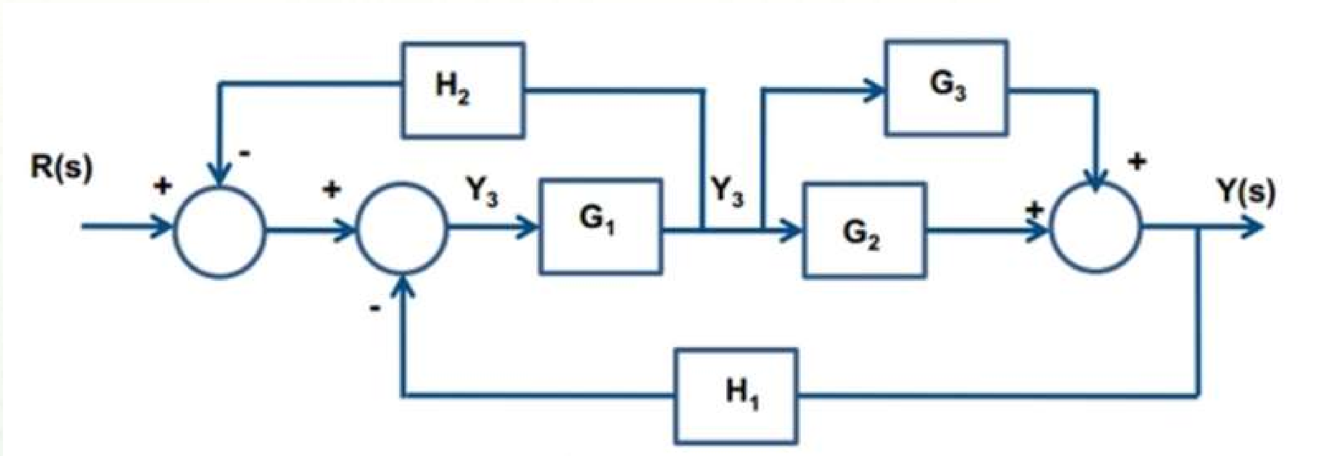
Procedure for converting block diagram to signal flow graph

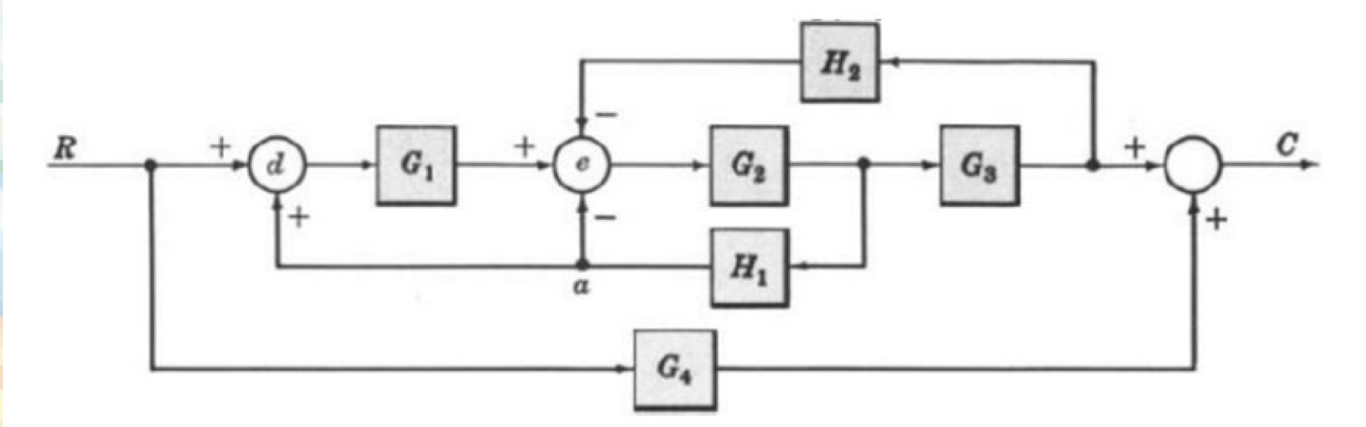




Dynamic Behavior and Stability of Closed-Loop Control System

Examples:





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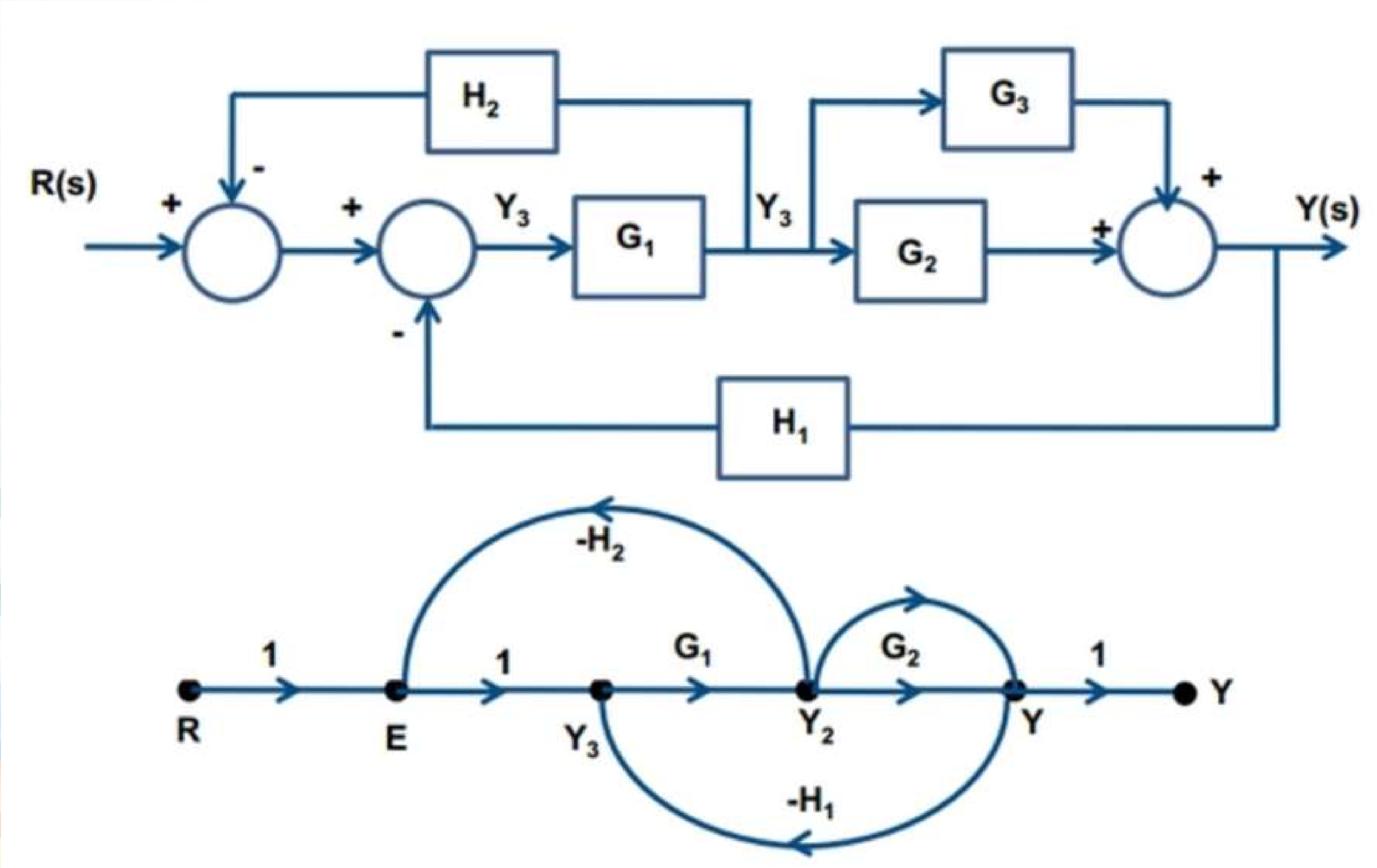

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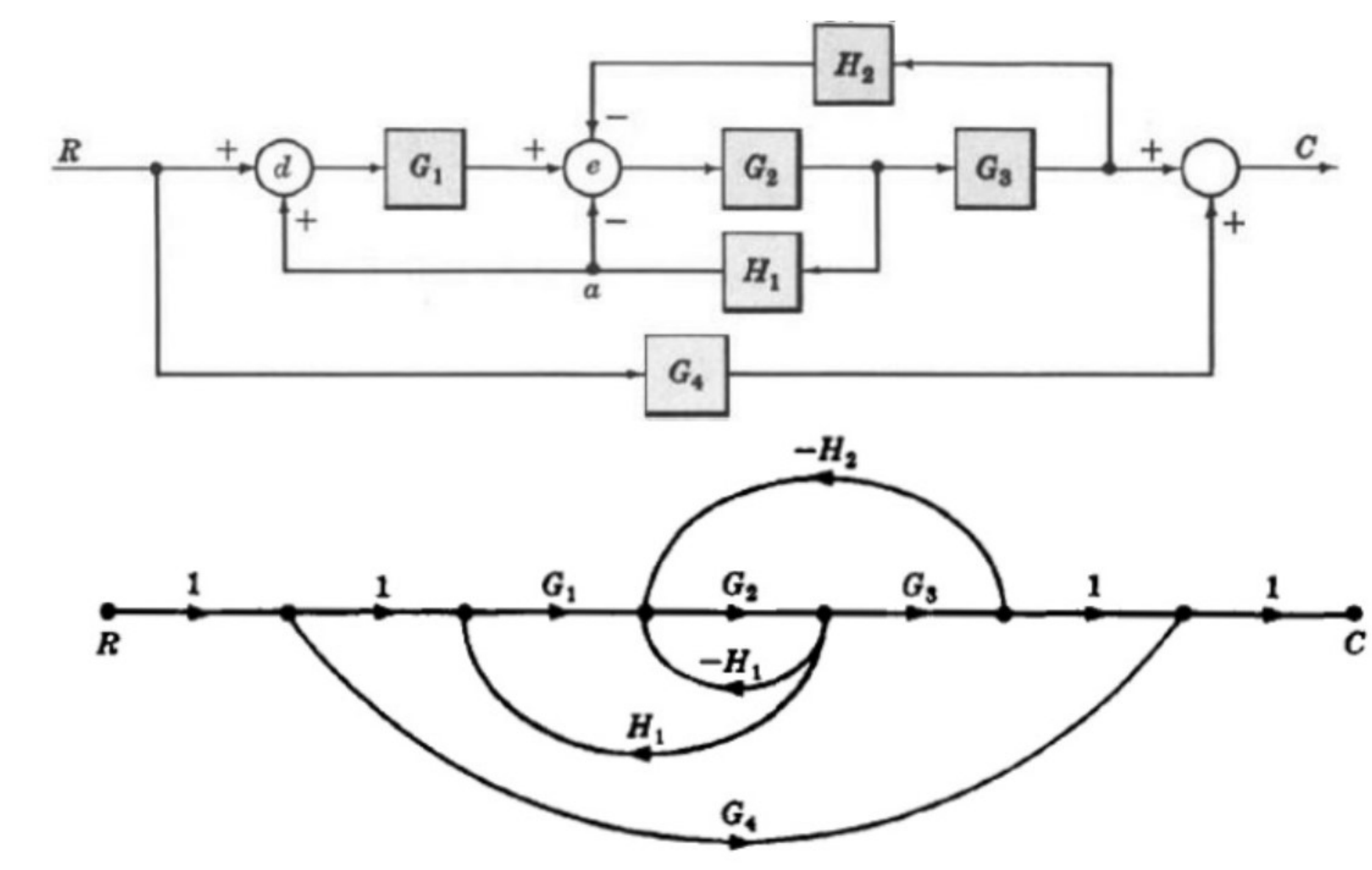


Mason's Gain Formula

Let R(s) → input of the system

Y(s) → output of the system

Transfer function of the system $T(s) = \frac{Y(s)}{R(s)}$

Overall gain $T(s) = \frac{1}{\Delta}\sum_k P_k \Delta_k$

Where: T(s) = transfer function of the system

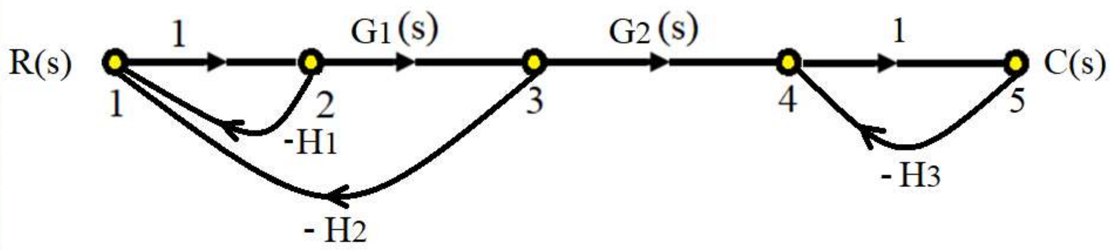
P_k = forward path gain of k^{th} forward path

Δ (Determinant) = 1 – (sum of individual loop gain) + (sum of two non touching loops) - (sum of three non touching loops) + (sum of four non touching loops) + ………

Δ_k (Associated path factor) = 1 – loop gains which are not touching to k^{th} forward

Example:

Using Mason's gain formula find the process transfer function of the process.



Transfer function of the system $G(s) = \frac{C(s)}{R(s)}$

Overall gain $G(s) = \frac{1}{\Delta}\sum_k P_k \Delta_k$

T(s) = transfer function of the system

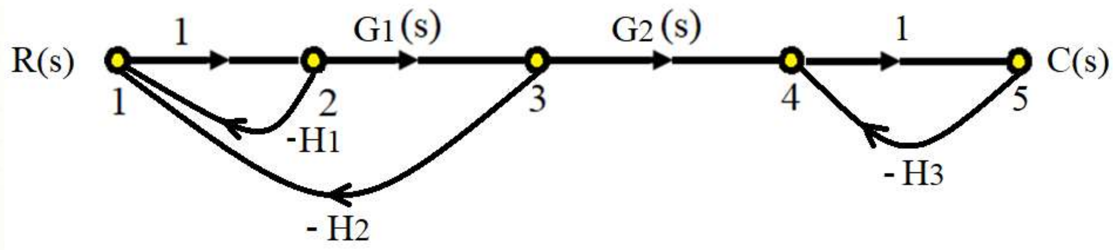
P_k *= forward path gain of* k^{th} *forward path* → $P_{15} = G_1 G_2$

Individual loop gain → $P_{21} = -H_1$ $P_{31} = -G_1 H_2$ $G_{54} = -H_3$

Two non touching loops → $P_{21,54} = H_1 H_3$ $P_{31,54} = G_1 H_2 H_3$

Three non touching loops = zero

Dynamic Behavior and Stability of Closed-Loop Control System



Δ = 1 – (sum of individual loop gain) + (sum of two non touching loops) + (sum of three non touching loops) + (sum of four non touching loops) + ...

$$\Delta = 1 - (-H_1 - G_1 H_2 - H_3) + H_1 H_3 + G_1 H_2 H_3$$

$$\Delta = 1 + H_1 + G_1 H_2 + H_3 + H_1 H_3 + G_1 H_2 H_3$$

Loop gains which are not touching to k^{th} forward = zero

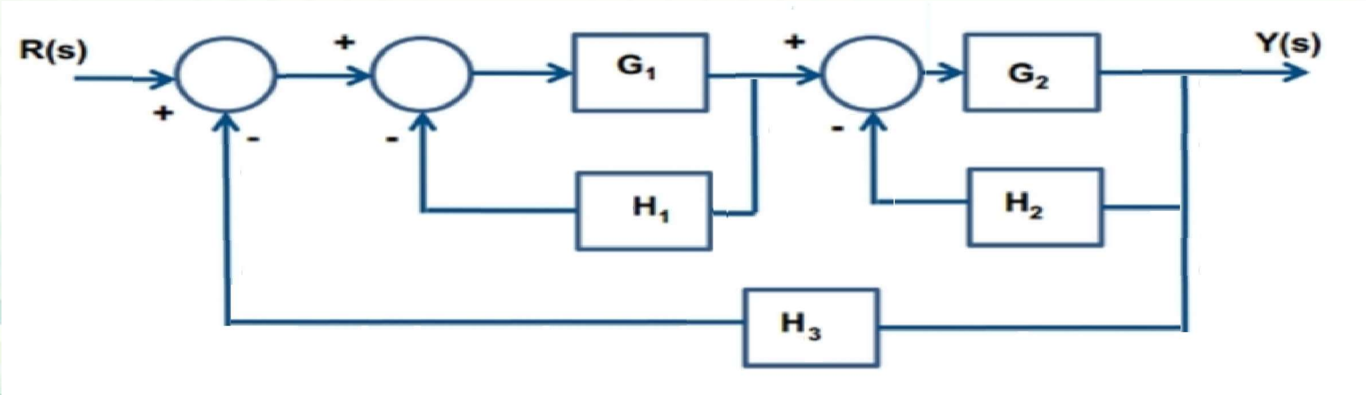
Δ_k = 1 – loop gains which are not touching to k^{th} forward = 1 -0 = 1

$$G(s) = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

$$G(s) = \frac{1}{1+H_1+G_1H_2+H_3+H_1H_3+G_1H_2H_3}(G_1G_2)$$

Exercise:

Find the transfer function of the system whose block diagram is shown below:



Solution

$$G(s) = \frac{(G_1 G_2)}{1 + G_1 H_1 + G_2 H_2 + G_1 G_2 H_3 + G_1 G_2 H_1 H_2}$$

Stability of closed-loop control system:

General stability criterion: A linear system is stable if and only if all roots (poles) of the denominator in the transfer function (TF) are negative or have negative real parts. Otherwise, the system is unstable.

- To find the roots (poles) of the denominator in TF:

Denominator of TF = 0 "Characteristic Eq."

- For standard closed-loop feedback control system, the characteristic Eq. is:

$$1 + G_m G_c G_v G_p = 0 \quad or \quad 1 + G_{OL} = 0$$

Stability of closed-loop control system:

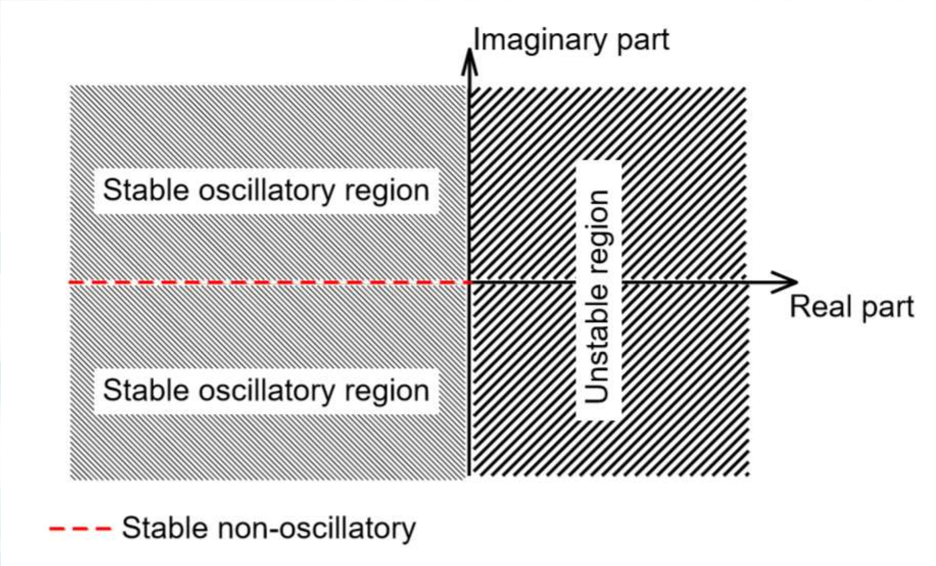
- The roots (poles) of the characteristic equation $(s - p_i)$ determine the type of response that occurs:

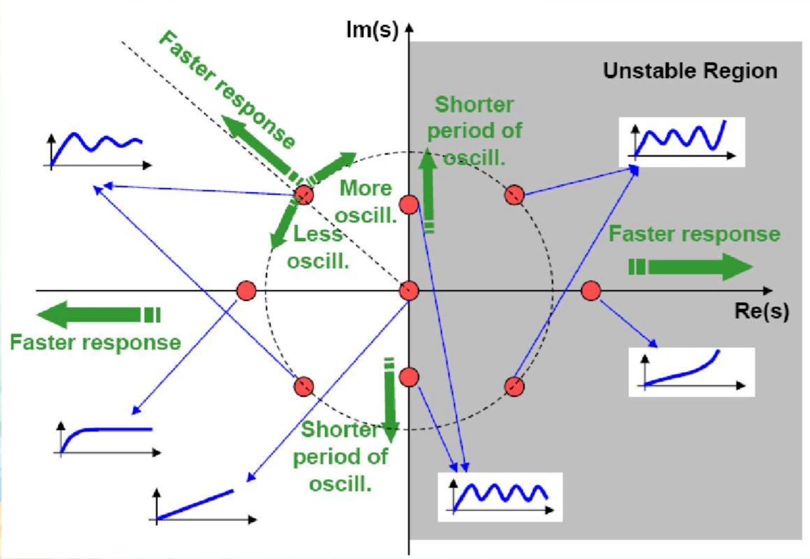
1. Real positive roots → Unstable response.
2. Real negative roots → Stable system without oscillation.
3. Complex root with negative real part → Stable Oscillatory response.
4. Complex roots with positive real parts → Unstable response.

Remark. Stability criterion help us to decide the action of controller whether reverse or direct.

Stability of closed-loop control system:

Stability regions in the complex plane for the roots of characteristic Eq.:





- If all roots in left half of complex plane → stable system

Example: Standard closed-loop feedback control system has proportional controller, A/C control valve, and transmitter. The process has first-order transfer function with positive gain and space time of 9 min. Does the controller have reverse or direct action to achieve stable response?

Characteristic Eq.: $1+G_mG_cG_vG_p=0 \Rightarrow 1+K_mK_cK_v\dfrac{K_p}{\tau_p s+1}=0$

Multiply by $\tau_p s+1$:

$$\left(\tau_p s+1\right)+K_mK_cK_vK_p=0 \Rightarrow s=-\frac{1+K_mK_cK_vK_p}{\tau_p}<0$$

$$\Rightarrow (1+K_mK_cK_vK_p)>0 \Rightarrow K_mK_cK_vK_p>-1$$

Since : $K_v < 0$ (A/C control valve); $K_m > 0$; and $K_p > 0$; the controller gain must be negative ($K_c < 0$):

→ Direct action

- K_c, K_v > 0 For A/O control valve" Reverse action"
- K_c, K_v < 0 For A/C control valve" Direct action"

Example: Study the stability of standard closed-loop feedback control system with, A/O valve:

$$G_c = K_c ; G_v = 1/(2s+1); G_m = 1; G_p = 1/(5s+1)$$

Characteristic Eq.:

$$1 + G_m G_c G_v G_p = 0 \Rightarrow 1 + (1) K_c \frac{1}{2s+1} \frac{1}{5s+1} = 0$$

$$(2s+1)(5s+1) + K_c = 0 \Rightarrow 10s^2 + 7s + K_c + 1 = 0$$

$$s = \frac{-7 \pm \sqrt{49 - 40(K_c + 1)}}{20} < 0 \Rightarrow \sqrt{49 - 40(K_c + 1)} < 7$$

$$49 - 40(K_c + 1) < 49 \Rightarrow -40(K_c + 1) < 0$$

$$K_c + 1 > 0$$

$\therefore K_c > -1$ For stability ⟹ Direct acting controller for ($K_c < 0$) which satisfies this condition but because the control valve is A/O then the controller will be reverse acting with $K_c > 0$ which also satisfies the condition.

Stability of closed-loop control system:

- Sometimes it is difficult to determine the nature of the poles of characteristic equation. In such case, root-finding techniques can be used to estimate the roots.

$$G_c = K_c; G_v = 1/(2s+1); G_m = 1/(s+1); G_p = 1/(5s+1)$$

Example: Study the stability of standard closed-loop feedback control system with:

Characteristic Eq.:

$$1 + G_m G_c G_v G_p = 0 \Rightarrow 1 + \frac{1}{s+1} K_c \frac{1}{2s+1} \frac{1}{5s+1} = 0$$

$$(s+1)(2s+1)(5s+1) + K_c = 0 \Rightarrow 10s^3 + 17s^2 + 8s + K_c + 1 = 0$$

→ Difficult to determine values of K_c such that $s < 0$.

→ Any alternative?! Yes, there are other stability criteria.

Stability of closed-loop control system:

A. Routh-Hurwitz stability criterion:

It is applicable for characteristic Eq. of the form:

$$a_n s^n + a_{n-1} s^{n-1} + ... + a_2 s^2 + a_1 s + a_0 = 0$$ *"Polynomial form"*

➡ **Construct the Routh array:**

1	a_n	a_{n-2}	a_{n-4}	...
2	a_{n-1}	a_{n-3}	a_{n-5}	...
3	b_1	b_2	b_3	...
4	c_1	c_2	...	
.	.			
.	.			
.	.			
$n+1$	z_1			

Coefficients determinations:

$$b_1 = (a_{n-1}a_{n-2} - a_n a_{n-3}) / a_{n-1}$$
$$b_2 = (a_{n-1}a_{n-4} - a_n a_{n-5}) / a_{n-1}$$
$$\vdots$$
$$c_1 = (b_1 a_{n-3} - a_{n-1} b_2) / b_1$$
$$c_2 = (b_1 a_{n-5} - a_{n-1} b_3) / b_1$$
$$\vdots$$

$$b_1 = -\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} / a_{n-1}$$

$$b_2 = -\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} / a_{n-1}$$

$$c_1 = -\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} / b_1$$

$$c_2 = -\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} / b_1$$

Dynamic Behavior and Stability of Closed-Loop Control System

Stability of closed-loop control system:

A. Routh-Hurwitz stability criterion:

- A necessary condition for stability:

⟹ all coefficients of characteristic Eq. (a_i's) are positive:

$$a_n s^n + a_{n-1} s^{n-1} + ... + a_2 s^2 + a_1 s + a_0 = 0 \;\; (a_i > 0 \;\; i = 0,...n)$$

- A necessary and sufficient condition for stability:

⟹ All of the elements in the left column of the Routh array are positive."

1	a_n	a_{n-2}	a_{n-4}	$\cdots$
2	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
3	b_1	b_2	b_3	$\cdots$
4	c_1	c_2	$\cdots$	
.	.			
.	.			
.	.			
$n+1$	d_1			

>0

Example: Use Routh-Hurwitz stability criterion to study the stability of standard closed-loop feedback control system given in previous example:

Characteristic Eq.: $10s^3 + 17s^2 + 8s + K_c + 1 = 0$

– Necessary condition:

$$a_3 = 10 > 0$$
$$a_2 = 17 > 0$$
$$a_1 = 8 > 0$$
$$a_0 = K_c + 1 > 0 \Rightarrow K_c > -1$$

"For stability"

- If any coefficient is not positive, stop and conclude the system is unstable.

- *Necessary and sufficient condition:*

Routh array:

1	a_3	a_1	→	1	10	8
2	a_2	a_0		2	17	K_c+1
3	b_1	b_2		3	$7.41-0.588K_c$	0
4	c_1			4	$1+K_c$	

$$b_1 = \frac{17(8)-10(1+K_c)}{17} = 7.41 - 0.588K_c \qquad b_2 = \frac{17(0)-10(0)}{17} = 0$$

$$c_1 = \frac{b_1(1+K_c)-17(0)}{b_1} = 1+K_c$$

Stable region:

$$K_c + 1 > 0 \Rightarrow K_c > -1$$

$$7.41 - 0.588K_c > 0 \Rightarrow K_c < 12.6$$

$$\therefore -1 < K_c < 12.6$$

"For stability without oscillation"

Stability of closed-loop control system:

B. Direct Substitution Stability Criterion:

- This stability criterion is based on the fact that the imaginary axis is the dividing line between stable and unstable systems.
- Procedure:

1. Substitute $s = j\omega$ into characteristic equation.
2. Obtain two equations: one for real part and the another for imaginary part,
3. Solve the two equations to obtain values of K_{cm} and ω. Where K_{cm} the maximum controller gain at which the roots of characteristic equation crosses the imaginary axis.
4. Determine the stable region by trying test values of K_c in the characteristic Eq.

Example: Use direct substitution stability criterion to study the stability of standard closed-loop feedback control system given in previous example:

Characteristic Eq.: $10s^3 + 17s^2 + 8s + K_c + 1 = 0$

- Set $s = j\omega$

$$-10j\omega^3 - 17\omega^2 + 8j\omega + 1 + K_{cm} = (1 + K_{cm} - 17\omega^2) + j\omega(8 - 10\omega^2) = 0$$

Real part Eq. : $(1 + K_{cm} - 17\omega^2) = 0$

Imaginary part Eq. : $\omega(8 - 10\omega^2) = 0$

Solve to obtain:

$$\omega = 0 \text{ or } \omega^2 = 0.8$$
$$\Rightarrow K_{cm} = -1 \text{ or}$$
$$K_{cm} = 12.6$$

- Try a test point such as: $K_c = 0$

$10s^3 + 17s^2 + 8s + 1 = 0$ ⟶ Stable: All +ve coefficients:

- Thus, the stable /non-oscillation region is: $-1 < K_c < 12.6$

Example: Use direct substitution stability criterion to study the stability of the system with the following characteristic Eq.: $1+5s+2K_c e^{-s}=0$

set $s=j\omega$: $1+5j\omega+2K_{cm}e^{-j\omega}=0$

But, $e^{-j\omega}=\cos\omega-j\sin\omega$

$$\Rightarrow 1+5j\omega+2K_{cm}(\cos\omega-j\sin\omega)=0$$

Real part Eq.: $1+2K_{cm}\cos\omega=0 \Rightarrow 2K_{cm}=-\frac{1}{\cos\omega}$

Imaginary part Eq.: $5\omega-2K_{cm}\sin\omega=0 \Rightarrow 5\omega+\tan\omega=0$

Solve to obtain: $\omega=1.688683$

$K_{cm}=4.25$

→ Try a test point such as: $K_c = 0$: $1+5s=0$

→ Stable: All + ve coefficients: $K_c<4.25$

Example: Use Routh-Hurwitz stability criterion to study the stability of the system with the following characteristic Eq.:

$$1 + 5s + 2K_c e^{-s} = 0$$

This characteristic Eq. does NOT have polynomial form to use Routh-Hurwitz stability. It can be rewritten in a polynomial form using 1/1 Pade Approximation:

$$e^{-\theta s} \approx \frac{1 - \frac{\theta}{2}s}{1 + \frac{\theta}{2}s}$$

$$\Rightarrow 1 + 5s + 2K_c \frac{1 - 0.5s}{1 + 0.5s} = 0$$

$$\Rightarrow (1 + 0.5s)(1 + 5s) + 2K_c(1 - 0.5s) = 0$$

$$\Rightarrow 2.5s^2 + (5.5 - K_c)s + (1 + 2K_c) = 0$$

→ *Necessary condition:*

$$a_1 = 5.5 - K_c > 0 \Rightarrow K_c < 5.5$$
$$a_0 = 1 + 2K_c > 0 \Rightarrow K_c > -0.5$$
$$\therefore -0.5 < K_c < 5.5$$

"For stability without oscillation"

→ *Necessary and sufficient condition:*

Routh array:

1	a_2	a_0
2	a_1	0
3	b_1	

→

1	2.5	$1+2K_c$
2	$5.5-K_c$	0
3	$1+2K_c$	0

$$5.5 - K_c > 0 \Rightarrow K_c < 5.5$$
$$\Rightarrow 1 + 2K_c > 0 \Rightarrow K_c > -0.5$$
$$\therefore -0.5 < K_c < 5.5$$

Remark: *In this example, Routh array does not add additional information but it confirms the stable region.*

Dynamic Behavior and Stability of Closed-Loop Control System

Stability of closed-loop control system:

- Routh-Hurwitz stability criterion with 1/1 Pade approximation of the exponential term gives a maximum controller gain of K_{cm} = 5.5. The exact value resulting from the direct substitution criterion is K_{cm} = 4.25. The percent relative error is around 28%.

Exercise: Resolve the previous example using 2/2 Pade approximation (more accurate than 1/1) given by:

$$e^{-\theta s} \approx \left[1-\frac{\theta}{2}s+\frac{\theta^2}{12}s^2\right] \Big/ \left[1+\frac{\theta}{2}s+\frac{\theta^2}{12}s^2\right]$$

- Routh-Hurwitz stability criterion with 2/2 Pade approximation of the exponential term gives maximum controller gain of K_{cm} = 4.29. The percent relative error is around 1%.

Variable	Value	f(x)	Ini Guess
x	4.2898188	-1.693E-12	1

NLE Report (safenewt)

Nonlinear equations

[1] f(x) = (5.5-x)*(2.583333+0.16666*x)-0.416666*(1+2*x) = 0

C. Root Locus Diagram

- A root locus diagram is a plot that shows how the eigenvalues of a linear (or linearized) system change as a function of a single parameter (usually the loop gain).
- Complex plane diagram shows the location of closed-loop poles (roots of characteristic equation) depending on the parameter value such as controller gain K_c. (single parametric study).
- It can be built by finding the roots at a different values of the parameter under investigation such as K_c.

Example

Consider a feedback control system with open loop transfer function:

$$G_{OL}(s) = \frac{4K_c}{(s+1)(s+2)(s+3)}$$

Plot the root locus diagram for $0 \le K_c \le 20$.

Characteristic Equation:

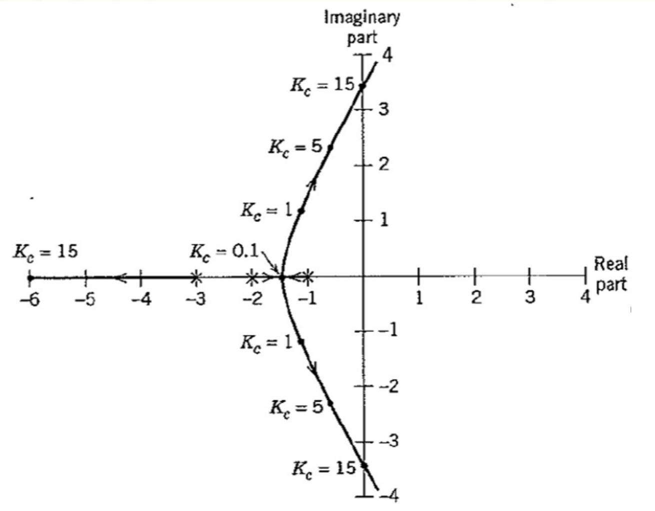
$$1 + G_{OL}(s) = 0 \rightarrow (s+1)(s+2)(s+3) + 4K_c = 0$$

- ✓ At Kc = 0 (no controller; open loop): roots = -1, -2, -3
- ✓ At Kc = 0.1: roots = -3.1597, −1.4202+0.0932i , −1.4202−0.0932i
- ✓ At Kc = 1; roots = -3.7963, −1.1018+1.1917i , −1.1018−1.1917i
- ✓ At Kc = 5; roots = -4.8371387, −0.5814+2.2443i , −0.5814−2.2443i
- ✓ At Kc = 15; roots = -6, 3.3166i, -3.3166i
- ✓ At Kc = 20; roots = −6.3862, 0.1931+3.6646i, 0.1931−3.6646i

- Localize these roots at each K_c on the complex plane to plot the root locus diagram.

Root locus diagram

- It is clear from root locus diagram that:



- ✓ The closed loop system is unstable for $K_c > 15$.
- ✓ The closed loop response will be stable for $0.1 < K_c < 15$.