Chapter 3

3.1

a) $$\mathcal{L}\left[e^{-bt}\sin\omega t\right] = \int_0^\infty e^{-bt}\sin\omega t\; e^{-st}dt = \int_0^\infty \sin\omega t\; e^{-(s+b)t}dt$$

$$= \left[e^{-(s+b)t}\,\frac{\left[-(s+b)\sin\omega t - \omega\cos\omega t\right]}{(s+b)^2+\omega^2}\right]_0^\infty$$

$$= \frac{\omega}{(s+b)^2+\omega^2}$$

b) $$\mathcal{L}\left[e^{-bt}\cos\omega t\right] = \int_0^\infty e^{-bt}\cos\omega t\; e^{-st}dt = \int_0^\infty \cos\omega t\; e^{-(s+b)t}dt$$

$$= \left[e^{-(s+b)t}\,\frac{\left[-(s+b)\cos\omega t + \omega\sin\omega t\right]}{(s+b)^2+\omega^2}\right]_0^\infty$$

$$= \frac{s+b}{(s+b)^2+\omega^2}$$

3.2

a) The Laplace transform provided is

$$Y(s) = \frac{4}{s^4+3s^3+4s^2+6s+4}$$

We also know that only sin ωt is an input, where $\omega = \sqrt{2}$. Then

$$X(s) = \frac{\omega}{s^2+\omega^2} = \frac{\sqrt{2}}{s^2+\left(\sqrt{2}\right)^2} = \frac{\sqrt{2}}{s^2+2}$$

Since $Y(s) = D^{-1}(s)\,X(s)$ where $D(s)$ is the characteristic polynomial (when all initial conditions are zero),

$$Y(s)=\frac{2\sqrt{2}}{(s^2+3s+2)}\ \frac{\sqrt{2}}{(s^2+2)}$$

and the original ode was

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=2\sqrt{2}\sin\sqrt{2}t \qquad \text{with} \quad y'(0)=y(0)=0$$

b) This is a unique result.

c) The solution arguments can be found from

$$Y(s)=\frac{2\sqrt{2}\sqrt{2}}{(s+1)(s+2)+(s^2+2)}$$

which in partial fraction form is

$$Y(s)=\frac{\alpha_1}{s+1}+\frac{\alpha_2}{s+2}+\frac{a_1s+a_2}{s^2+2}$$

Thus the solution will contain four functions of time

$$e^{-t}\quad , \quad e^{-2t}\quad , \quad \sin\sqrt{2}\,t\quad , \quad \cos\sqrt{2}\,t$$

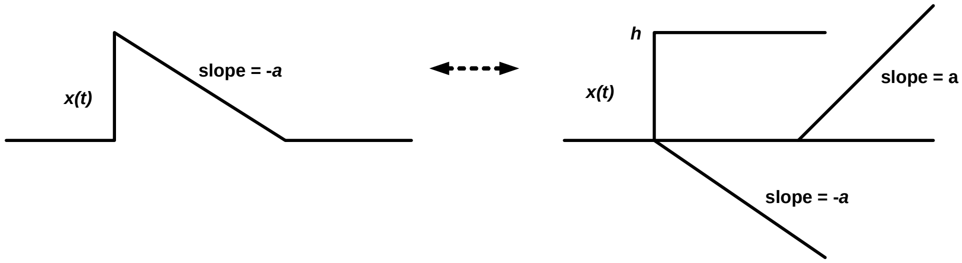
3.3

a) Pulse width is obtained when $x(t) = 0$

Since $x(t) = h - at$

$$t_\omega : \ h - at_\omega = 0 \qquad \text{or} \qquad t_\omega = h/a$$

b)



$x(t) = hS(t) - atS(t) + a(t - t_\omega)\, S(t - t_\omega)$

c) $X(s) = \frac{h}{s} - \frac{a}{s^2} + \frac{ae^{-st_\omega}}{s^2} = \frac{h}{s} + \frac{e^{-st_\omega} - 1}{s^2}$

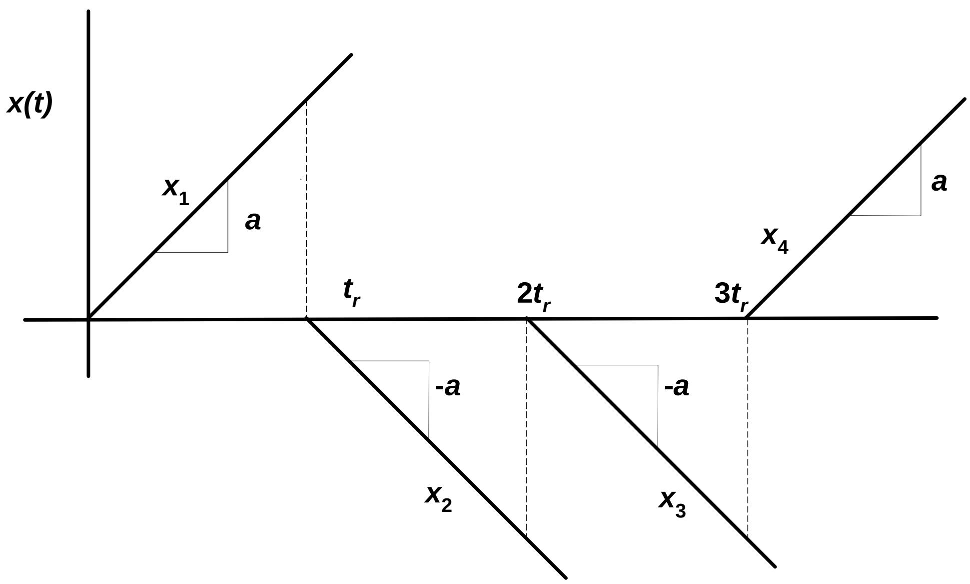
d) Area under pulse = $h\, t_\omega/2$

3.4

a) $f(t) = 5\, S(t) - 4\, S(t\text{-}2) - S(t\text{-}6)$

$F(s) = = \frac{1}{s}\left(5 - 4e^{-2s} - e^{-6s}\right)$

b)



$x(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t)$

$= at - a(t - t_r)S(t - t_r) - a(t - 2t_r)S(t - 2t_r) + a(t - 3t_r)S(t - 3t_r)$

following Eq. 3-101. Thus

$X(s) = \frac{a}{s^2}\left[1 - e^{-t_r s} - e^{-2t_r s} + e^{-3t_r s}\right]$

by utilizing the Real Translation Theorem Eq. 3-104.

3.5

$$T(t) = 20\,S(t) + \frac{55}{30}t\,S(t) - \frac{55}{30}(t\text{-}30)\,S(t\text{-}30)$$

$$T(s) = \frac{20}{s} + \frac{55}{30}\frac{1}{s^2} - \frac{55}{30}\frac{1}{s^2}e^{-30s} = \frac{20}{s} + \frac{55}{30}\frac{1}{s^2}\left(1 - e^{-30s}\right)$$

3.6

a) $$X(s) = \frac{s(s+1)}{(s+2)(s+3)(s+4)} = \frac{\alpha_1}{s+2} + \frac{\alpha_2}{s+3} + \frac{\alpha_3}{s+4}$$

$$\alpha_1 = \left.\frac{s(s+1)}{(s+3)(s+4)}\right|_{s=-2} = 1$$

$$\alpha_2 = \left.\frac{s(s+1)}{(s+2)(s+4)}\right|_{s=-3} = -6$$

$$\alpha_3 = \left.\frac{s(s+1)}{(s+2)(s+3)}\right|_{s=-4} = 6$$

$$X(s) = \frac{1}{s+2} - \frac{6}{s+3} + \frac{6}{s+4} \quad \text{and} \quad x(t) = e^{-2t} - 6e^{-3t} + 6e^{-4t}$$

b) $$X(s) = \frac{s+1}{(s+2)(s+3)(s^2+4)} = \frac{s+1}{(s+2)(s+3)(s+2j)(s-2j)}$$

$$X(s) = \frac{\alpha_1}{s+2} + \frac{\alpha_2}{s+3} + \frac{\alpha_3 + j\beta_3}{s+2j} + \frac{\alpha_3 - j\beta_3}{s+2j}$$

$$\alpha_1 = \left.\frac{s+1}{(s+3)(s^2+4)}\right|_{s=-2} = -\frac{1}{8}$$

$$\alpha_2 = \left.\frac{s+1}{(s+2)(s^2+4)}\right|_{s=-3} = \frac{2}{13}$$

$$\alpha_3 + j\beta_3 = \left.\frac{s+1}{(s+2)(s+3)(s-2j)}\right|_{s=-2j} = \frac{1-2j}{-40-8j} = \frac{-3+11j}{208}$$

$$x(t) = -\frac{1}{8}e^{-2t} + \frac{2}{13}e^{-3t} + 2\left(\frac{-3}{208}\right)\cos 2t + 2\left(\frac{11}{208}\right)\sin 2t$$

$$= -\frac{1}{8}e^{-2t} + \frac{2}{13}e^{-3t} - \frac{3}{104}\cos 2t + \frac{11}{104}\sin 2t$$

c) $$X(s) = \frac{s+4}{(s+1)^2} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{(s+1)^2} \qquad (1)$$

$$\alpha_2 = (s+4)\big|_{s=-1} = 3$$

In Eq. 1, substitute any $s \neq -1$ to determine α_1. Arbitrarily using $s=0$, Eq. 1 gives

$$\frac{4}{1^2} = \frac{\alpha_1}{1} + \frac{3}{1^2} \qquad \text{or} \qquad \alpha_1 = 1$$

$$X(s) = \frac{1}{s+1} + \frac{3}{(s+1)^2} \qquad \text{and} \qquad x(t) = e^{-t} + 3te^{-t}$$

d) $$X(s) = \frac{1}{s^2+s+1} = \frac{1}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{1}{(s+b)^2 + \omega^2}$$

where $b = \frac{1}{2}$ and $\omega = \frac{\sqrt{3}}{2}$

$$x(t) = \frac{1}{\omega}e^{-bt}\sin\omega t = \frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t$$

e) $$X(s) = \frac{s+1}{s(s+2)(s+3)}e^{-0.5s}$$

To invert, we first ignore the time delay term. Using the Heaviside expansion with the partial fraction expansion,

$$\hat{X}(s) = \frac{s+1}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

Multiply by s and let $s \to 0$

$$A = \frac{1}{(2)(3)} = \frac{1}{6}$$

Multiply by (s+2) and let $s \to -2$

$$B = \frac{-2+1}{(-2)(-2+3)} = \frac{-1}{(-2)(1)} = \frac{1}{2}$$

Multiply by (s+3) and let $s \to$-3

$$C = \frac{-3+1}{(-3)(-3+2)} = \frac{-2}{(-3)(-1)} = -\frac{2}{3}$$

Then

$$\hat{X}(s) = \frac{1/6}{s} + \frac{1/2}{s+2} + \frac{-2/3}{s+3}$$

$$\hat{x}(t) = \frac{1}{6} + \frac{1}{2}e^{-2t} - \frac{2}{3}e^{-3t}$$

Imposing shift theorem

$$x(t) = \hat{x}(t-0.5) = \frac{1}{6} + \frac{1}{2}e^{-2(t-0.5)} - \frac{2}{3}e^{-3(t-0.5)}$$

for $t \geq 0.5$

3.7

a) $$Y(s) = \frac{6(s+1)}{s^2(s+1)} = \frac{6}{s^2} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2}$$

$$\alpha_2 = s^2 \frac{6}{s^2}\bigg|_{s=0} = 6 \qquad \alpha_1 = 0$$

$$Y(s) = \frac{6}{s^2}$$

b) $$Y(s) = \frac{12(s+2)}{s\ (s^2+9)} = \frac{\alpha_1}{s} + \frac{\alpha_2 s + \alpha_3}{s^2+9}$$

Multiplying both sides by $s(s^2+9)$

$$12(s+2)=\alpha_1(s^2+9)+(\alpha_2 s+\alpha_3)(s) \qquad \text{or}$$
$$12s+24=(\alpha_1+\alpha_2)s^2+\alpha_3 s+9\alpha_1$$

Equating coefficients of like powers of s,

s^2: $\alpha_1 + \alpha_2 = 0$
s^1: $\alpha_3 = 12$
s^0: $9\alpha_1 = 24$

Solving simultaneously,

$$\alpha_1=\frac{8}{3} \quad , \quad \alpha_2=\frac{-8}{3} \quad , \quad \alpha_3=12$$

$$Y(s)=\frac{8}{3}\frac{1}{s}+\frac{\left(-\frac{8}{3}s+12\right)}{s^2+9}$$

c) $$Y(s)=\frac{(s+2)(s+3)}{(s+4)(s+5)(s+6)}=\frac{\alpha_1}{s+4}+\frac{\alpha_2}{s+5}+\frac{\alpha_3}{s+6}$$

$$\alpha_1=\left.\frac{(s+2)(s+3)}{(s+5)(s+6)}\right|_{s=-4}=1$$

$$\alpha_2=\left.\frac{(s+2)(s+3)}{(s+4)(s+6)}\right|_{s=-5}=-6$$

$$\alpha_3=\left.\frac{(s+2)(s+3)}{(s+4)(s+5)}\right|_{s=-6}=6$$

$$Y(s)=\frac{1}{s+4}-\frac{6}{s+5}+\frac{6}{s+6}$$

d) $$Y(s)=\frac{1}{\left[(s+1)^2+1\right]^2(s+2)}=\frac{1}{(s^2+2s+2)^2(s+2)}$$

$$=\frac{\alpha_1 s+\alpha_2}{s^2+2s+2}+\frac{\alpha_3 s+\alpha_4}{(s^2+2s+2)^2}+\frac{\alpha_5}{s+2}$$

Multiplying both sides by $(s^2+2s+2)^2(s+2)$ gives

$$1 = \alpha_1 s^4 + 4\alpha_1 s^3 + 6\alpha_1 s^2 + 4\alpha_1 s + \alpha_2 s^3 + 4\alpha_2 s^2 + 6\alpha_2 s + 4\alpha_2 + \alpha_3 s^2 + 2\alpha_3 s + \alpha_4 s + 2\alpha_4 + \alpha_5 s^4 + 4\alpha_5 s^3 + 8\alpha_5 s^2 + 8\alpha_5 s + 4\alpha_5$$

Equating coefficients of like power of *s*,

s^4 : $\alpha_1 + \alpha_5 = 0$

s^3 : $4\alpha_1 + \alpha_2 + 4\alpha_5 = 0$

s^2 : $6\alpha_1 + 4\alpha_2 + \alpha_3 + 8\alpha_5 = 0$

s^1 : $4\alpha_1 + 6\alpha_2 + 2\alpha_3 + \alpha_4 + 8\alpha_5 = 0$

s^0 : $4\alpha_2 + 2\alpha_4 + 4\alpha_5 = 1$

Solving simultaneously:

$\alpha_1 = -1/4$ $\qquad \alpha_2 = 0$ $\qquad \alpha_3 = -1/2$ $\qquad \alpha_4 = 0$ $\qquad \alpha_5 = 1/4$

$$Y(s) = \frac{-1/4s}{s^2+2s+2} + \frac{-1/2s}{(s^2+2s+2)^2} + \frac{1/4}{s+2}$$

3.8

a) From Eq. 3-100

$$\mathfrak{L}\left[\int_0^t f(t^*)dt^*\right] = \frac{1}{s}F(s)$$

we know that $\mathfrak{L}\left[\int_0^t e^{-\tau}d\tau\right] = \frac{1}{s}\ \mathfrak{L}\left[e^{-\tau}\right] = \frac{1}{s(s+1)}$

$\therefore$ Laplace transforming yields

$$s^2X(s) + 3X(s) + 2X(s) = \frac{2}{s(s+1)}$$

or $\quad (s^2 + 3s + 1)\,X(s) = \dfrac{2}{s(s+1)}$

$$X(s) = \frac{2}{s(s+1)^2(s+2)}$$

and $\quad x(t) = 1 - 2te^{-t} - e^{-2t}$

b) Applying the final Value Theorem

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{2}{(s+1)^2(s+2)} = 2$$

[Note that Final Value Theorem is applicable here]

3.9

a) $$X(s) = \frac{6(s+2)}{(s^2+9s+20)(s+4)} = \frac{6(s+2)}{(s\ \ +4)(s+5)(s+4)}$$

$$x(0) = \lim_{s\to\infty}\left[\frac{6s(s+2)}{(s+5)(s+4)^2}\right] = 0$$

$$x(\infty) = \lim_{s\to 0}\left[\frac{6s(s+2)}{(s+5)(s+4)^2}\right] = 0$$

$x(t)$ is converging (or bounded) because $[sX(s)]$ does not have a limit at $s = -4$, and $s = -5$ only, i.e., it has a limit for all real values of $s \geq 0$.

$x(t)$ is smooth because the denominator of $[sX(s)]$ is a product of real factors only. See Fig. S3.9a.

b) $$X(s) = \frac{10s^2-3}{(s^2-6s+10)(s+2)} = \frac{10s^2-3}{(s-3+2j)\ \ (s-3-2j)(s+2)}$$

$$x(0) = \lim_{s\to\infty}\left[\frac{10s^3-3s}{(s^2-6s+10)(s+2)}\right] = 10$$

Application of final value theorem is not valid because $[sX(s)]$ does not have a limit for some real $s \geq 0$, i.e., at $s = 3\pm 2j$. For the same reason, $x(t)$ is diverging (unbounded).

$x(t)$ is oscillatory because the denominator of $[sX(s)]$ includes complex factors. See Fig. S3.9b.

c) $$X(s) = \frac{16s+5}{(s^2+9)} = \frac{16s+5}{(s+3j)\ (s-3j)}$$

$$x(0) = \lim_{s\to\infty}\left[\frac{16s^2+5s}{(s^2+9)}\right] = 16$$

Application of final value theorem is not valid because [$sX(s)$] does not have a limit for real s = 0. This implies that $x(t)$ is not diverging, since divergence occurs only if [$sX(s)$] does not have a limit for some real value of s>0.

$x(t)$ is oscillatory because the denominator of [$sX(s)$] is a product of complex factors. Since $x(t)$ is oscillatory, it is not converging either. See Fig. S3.9c

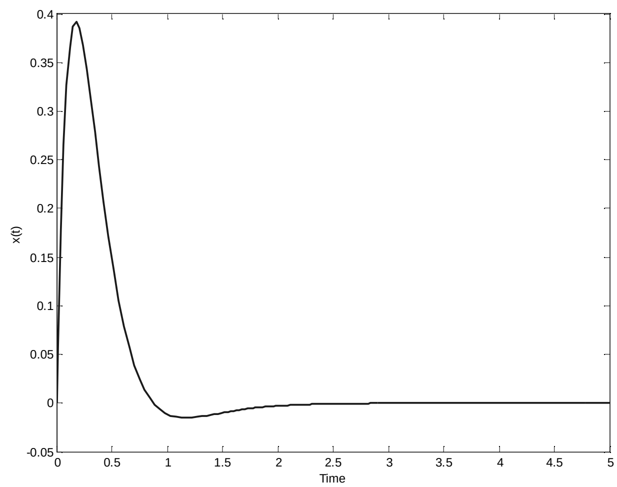


Figure S3.9a. *Simulation of X(s) for case a)*

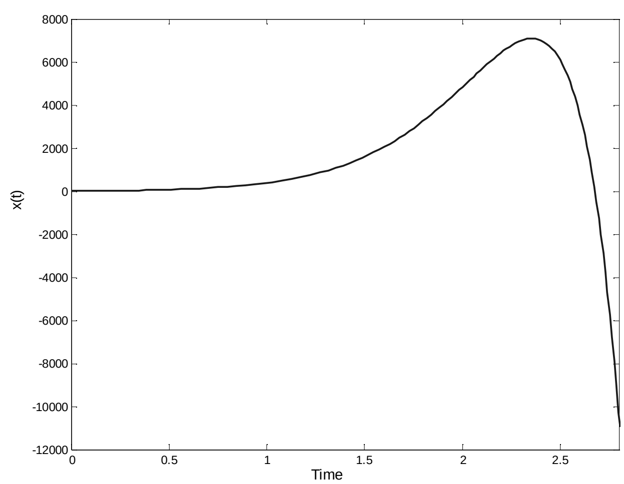


Figure S3.9b. *Simulation of X(s) for case b)*

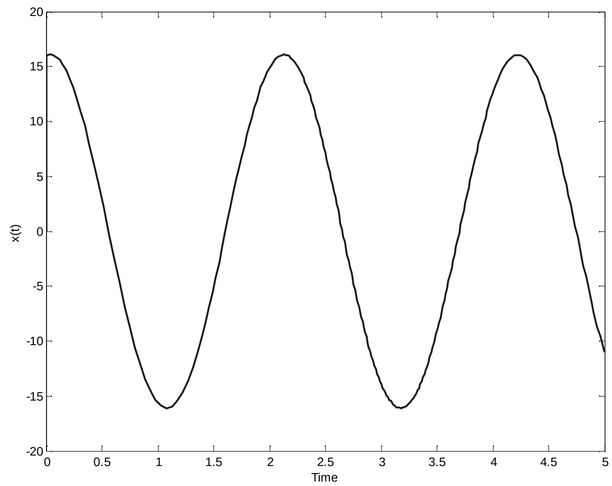


Figure S3.9c. *Simulation of X(s) for case c)*

The Simulink block diagram is shown below. An impulse input should be used to obtain the function's behavior. In this case note that the impulse input is simulated by a rectangular pulse input of very short duration. (At time t = 0 and t =0.001 with changes of magnitude 1000 and –1000 respectively). The MATLAB command *impulse* might also be used.

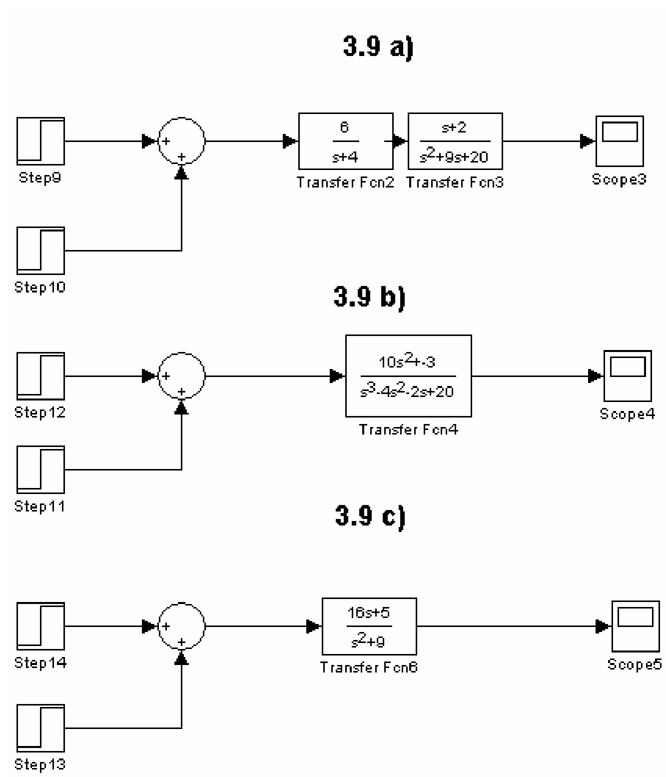


Figure S3.9d. *Simulink block diagram for cases a), b) and c).*

3.10

a)

i) $Y(s) = \dfrac{2}{s(s^2+4s)} = \dfrac{2}{s^2(s+4)} = \dfrac{A}{s^2} + \dfrac{B}{s} + \dfrac{C}{s+4}$

$\therefore$ $y(t)$ will contain terms of form: constant, t, e^{-4t}

ii) $Y(s) = \dfrac{2}{s(s^2+4s+3)} = \dfrac{2}{s(s+1)(s+3)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+3}$

$\therefore$ $y(t)$ will contain terms of form: constant, e^{-t}, e^{-3t}

iii) $Y(s) = \dfrac{2}{s(s^2+4s+4)} = \dfrac{2}{s(s+2)^2} = \dfrac{A}{s} + \dfrac{B}{(s+2)^2} + \dfrac{C}{s+2}$

$\therefore$ $y(t)$ will contain terms of form: constant, e^{-2t} , te^{-2t}

iv) $Y(s) = \dfrac{2}{s(s^2+4s+8)}$

$$s^2+4s+8 = (s^2+4s+4)+(8-4) = (s+2)^2+2^2$$
$$Y(s) = \frac{2}{s[(s+2)^2+2^2]}$$

$\therefore$ $y(t)$ will contain terms of form: constant, $e^{-2t}\sin 2t$, $e^{-2t}\cos 2t$

b) $Y(s) = \dfrac{2(s+1)}{s(s^2+4)} = \dfrac{2(s+1)}{s(s^2+2^2)} = \dfrac{A}{s} + \dfrac{Bs}{s^2+2^2} + \dfrac{C}{s^2+2^2}$

$$A = \lim_{s\to 0} \frac{2(s+1)}{(s^2+4)} = \frac{1}{2}$$

$2(s+1) = A(s^2+4) + Bs(s) + Cs$
$2s+2 = As^2 + 4A + Bs^2 + Cs$

Equating coefficients on like powers of s

s^2: $0 = A + B$ $\rightarrow$ $B = -A = -\dfrac{1}{2}$

s^1: $2 = C$ $\rightarrow$ $C = 2$

s^0: $2 = 4A$ $\rightarrow$ $A = \dfrac{1}{2}$

$$\therefore \qquad Y(s)=\frac{1/2}{s}+\frac{-(1/2)s}{s^2+2^2}+\frac{2}{s^2+2^2}$$

$$y(t) = \frac{1}{2}-\frac{1}{2}\cos 2t+\frac{2}{2}\sin 2t$$

$$y(t) = \frac{1}{2}(1-\cos 2t)+\sin 2t$$

3.11

Since convergent and oscillatory behavior does not depend on initial conditions, assume $\frac{dx^2(0)}{dt^2}=\frac{dx(0)}{dt}=x(0)=0$

a) Laplace transform of the equation gives

$$s^3X(s)+2s^2X(s)+2sX(s)+X(s)=\frac{3}{s}$$

$$X(s)=\frac{3}{s(s^3+2s^2+2s+1)}=\frac{3}{s(s+1)(s+\frac{1}{2}+\frac{\sqrt{3}}{2}j)(s+\frac{1}{2}-\frac{\sqrt{3}}{2}j)}$$

Denominator of [$sX(s)$] contains complex factors so that $x(t)$ is oscillatory, and denominator vanishes at real values of $s= -1$ and -½ which are all <0 so that $x(t)$ is convergent. See Fig. S3.11a.

b) $$s^2X(s)-X(s)=\frac{2}{s-1}$$

$$X(s)=\frac{2}{(s-1)(s^2-1)}=\frac{2}{(s-1)^2(s+1)}$$

The denominator contains no complex factors; $x(t)$ is not oscillatory.
The denominator vanishes at s=1 ≥0; $x(t)$ is divergent. See Fig. S3.11b.

c) $$s^3X(s)+X(s)=\frac{1}{s^2+1}$$

$$X(s)=\frac{1}{(s^2+1)(s^3+1)}=\frac{1}{(s+j)(s-j)(s+1)(s-\frac{1}{2}+\frac{\sqrt{3}}{2}j)(s-\frac{1}{2}-\frac{\sqrt{3}}{2}j)}$$

The denominator contains complex factors; $x(t)$ is oscillatory.
The denominator vanishes at real s = 0, ½; $x(t)$ is not convergent. See Fig. S3.11c.

d) $s^2X(s) + sX(s) = \frac{4}{s}$

$$X(s) = \frac{4}{s(s^2+s)} = \frac{4}{s^2(s+1)}$$

The denominator of [$sX(s)$] contains no complex factors; $x(t)$ is not oscillatory.
The denominator of [$sX(s)$] vanishes at $s = 0$; $x(t)$ is not convergent. See Fig. S3.11d.

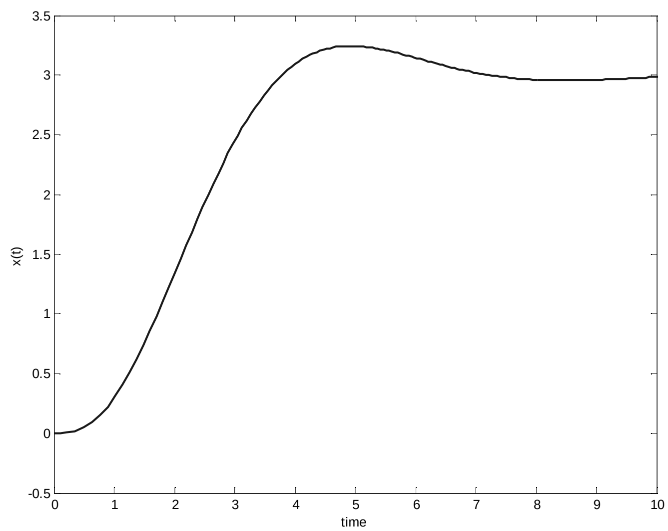


Figure S3.11a. *Simulation of X(s) for case a)*

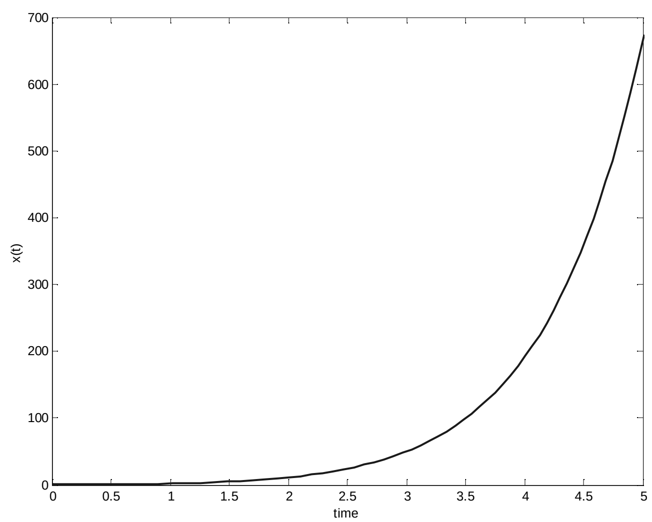


Figure S3.11b. *Simulation of X(s) for case b)*

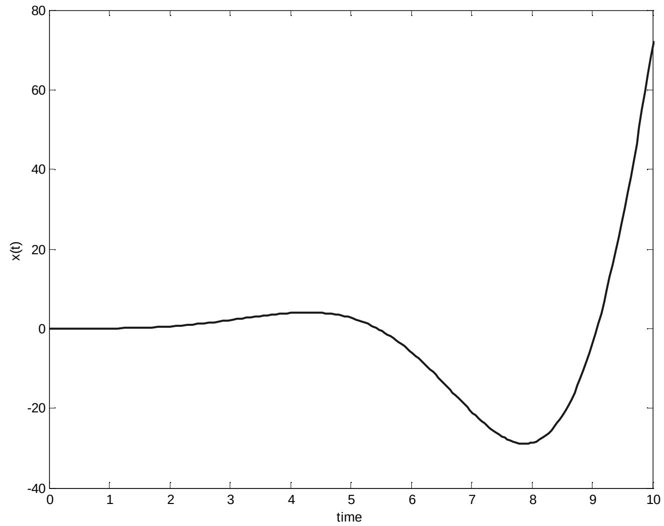


Figure S3.11c. *Simulation of X(s) for case c)*

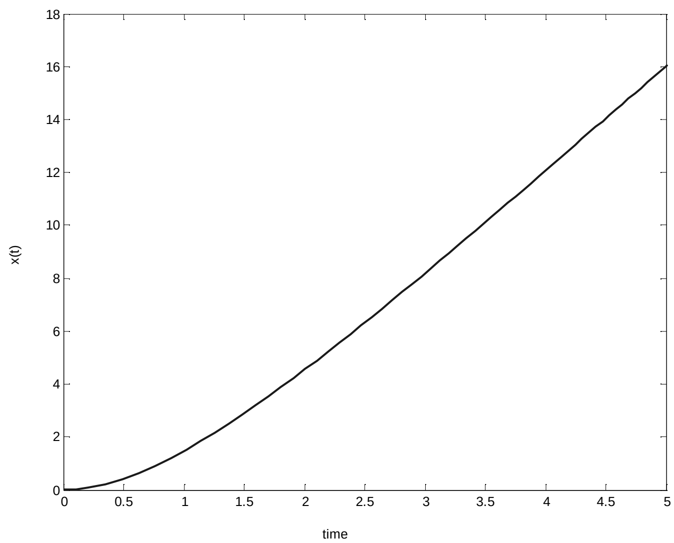


Figure S3.11d. *Simulation of X(s) for case d)*

3.12

Since the time function in the solution is not a function of initial conditions, we Laplace Transform with

$$x(0) = \frac{dx(0)}{dt} = 0$$

$$\tau_1 \tau_2 s^2 X(s) + (\tau_1 + \tau_2) s X(s) + X(s) = KU(s)$$

$$X(s) = \frac{K}{\tau_1\tau_2 s^2 + (\tau_1 + \tau_2)s + 1}U(s)$$

Factoring denominator

$$X(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}U(s)$$

a) If $u(t) = a\,S(t)$ then $U(s) = \frac{a}{s}$

$$X_a(s) = \frac{Ka}{s(\tau_1 s + 1)(\tau_2 s + 1)} \qquad \tau_1 \neq \tau_2$$

$$x_a(t) = f_a(S(t),\ \ e^{-t/\tau_1},\ \ e^{-t/\tau_2})$$

b) If $u(t) = be^{-t/\tau}$ then $U(s) = \frac{b\tau}{\tau_s + 1}$

$$X_b(s) = \frac{Kb\tau}{(\tau s + 1)(\tau_1 s + 1)(\tau_2 s + 1)} \qquad \tau \neq \tau_1 \neq \tau_2$$

$$x_b(t) = f_b(e^{-t/\tau},\ \ e^{-t/\tau_1},\ \ e^{-t/\tau_2})$$

c) If $u(t) = ce^{-t/\tau}$ where $\tau = \tau_1$, then $U(s) = \frac{\tau c}{\tau_1 s + 1}$

$$X_c(s) = \frac{Kc\tau}{(\tau_1 s + 1)^2(\tau_2 s + 1)}$$

$$x_c(t) = f_c(e^{-t/\tau_1},\ \ t\,e^{-t/\tau_1},\ \ e^{-t/\tau_2})$$

d) If $u(t) = d \sin \omega t$ then $U(s) = \frac{d\omega}{s^2 + \omega^2}$

$$X_d(s) = \frac{Kd}{(s^2 + \omega^2)(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$x_d(t) = f_d(e^{-t/\tau_1},\ \ e^{-t/\tau_2}, \sin \omega t, \cos \omega t)$$

3.13

a) $\frac{dx^3}{dt^3} + 4x = e^t$ with $\frac{d^2x(0)}{dt^2} = \frac{dx(0)}{dt} = x(0) = 0$

Laplace transform of the equation,

$$s^3 X(s) + 4X(s) = \frac{1}{s-1}$$

$$X(s) = \frac{1}{(s-1)(s^3+4)} = \frac{1}{(s-1)(s+1.59)(s-0.79+1.37j)(s-0.79-1.37j)}$$

$$= \frac{\alpha_1}{s-1} + \frac{\alpha_2}{s+1.59} + \frac{\alpha_3 + j\beta_3}{s-0.79+1.37j} + \frac{\alpha_3 - j\beta_3}{s-0.79-1.37j}$$

$$\alpha_1 = \left.\frac{1}{(s^3+4)}\right|_{s=1} = \frac{1}{5}$$

$$\alpha_2 = \left.\frac{1}{(s-1)(s-0.79+1.37j)(s-0.79-1.37j)}\right|_{s=-1.59} = -\frac{1}{19.6}$$

$$\alpha_3 + j\beta_3 = \left.\frac{1}{(s-1)(s+1.59)(s-0.79-1.37j)}\right|_{s=0.79-1.37j} = -0.74 - 0.59j$$

$$X(s) = \frac{\frac{1}{5}}{s-1} + \frac{-\frac{1}{19.6}}{s+1.59} + \frac{-0.074-0.059j}{s-0.79+1.37j} + \frac{-0.074+0.059j}{s-0.79-1.37j}$$

$$x(t) = \frac{1}{5}e^t - \frac{1}{19.6}e^{-1.59t} - 2e^{0.79t}(0.074\cos 1.37t + 0.059\sin 1.37t)$$

b) $\frac{dx}{dt} - 12x = \sin 3t$ with $x(0) = 0$

$$sX(s) - 12X(s) = \frac{3}{s^2+9}$$

$$X(s) = \frac{3}{(s^2+9)(s-12)} = \frac{3}{(s+3j)(s-3j)(s-12)}$$

$$= \frac{\alpha_1 + j\beta_1}{s+3j} + \frac{\alpha_1 - j\beta_1}{s-3j} + \frac{\alpha_3}{s-12}$$

$$\alpha_1 + j\beta_1 = \left.\frac{3}{(s-3j)(s-12)}\right|_{s=-3j} = \frac{3}{-18+72j} = -\frac{1}{102} - \frac{4}{102}j$$

$$\alpha_3 = \left.\frac{3}{(s^2+9)}\right|_{s=12} = \frac{1}{51}$$

$$X(s) = \frac{-\frac{1}{102} - \frac{4}{102}j}{s+3j} + \frac{-\frac{1}{102} + \frac{4}{102}j}{s-3j} + \frac{\frac{1}{51}}{s-12}$$

$$x(t) = -\frac{1}{51}(\cos 3t + 4\sin 3t) + \frac{1}{51}e^{12t}$$

c) $$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = e^{-t} \qquad \text{with} \qquad \frac{dx(0)}{dt} = x(0) = 0$$

$$s^2X(s) + 6sX(s) + 25X(s) + X(s) = \frac{1}{s+1} \quad \text{or} \quad X(s) = \frac{1}{(s+1)(s^2+6s+25)}$$

$$X(s) = \frac{1}{(s+1)(s+3+4j)(s+3-4j)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2 + \beta_2 j}{s+3+4j} + \frac{\alpha_2 - \beta_2 j}{s+3-4j}$$

$$\alpha_1 = \left.\frac{1}{(s^2+6s+25)}\right|_{s=-1} = \frac{1}{20}$$

$$\alpha_2 + j\beta_2 = \left.\frac{1}{(s+1)(s+3-4j)}\right|_{s=-3-4j} = -\frac{1}{40} - \frac{1}{80}j$$

$$X(s) = \frac{\frac{1}{20}}{s+1} + \frac{-\frac{1}{40} - \frac{1}{80}j}{s+3+4j} + \frac{-\frac{1}{40} - \frac{1}{80}j}{s+3-4j}$$

$$x(t) = \frac{1}{20}e^{-t} - e^{-3t}\left(\frac{1}{20}\cos 4t + \frac{1}{40}\sin 4t\right)$$

d) Laplace transforming (assuming initial conditions = 0, since they do not affect results)

$$sY_1(s) + Y_2(s) = X_1(s) \qquad (1)$$

$$sY_2(s) - 2Y_1(s) + 3Y_2(s) = X_2(s) \qquad (2)$$

From (2),

$$(s+3)\, Y_2(s) = X_2(s) + 2Y_1(s)$$

$$Y_2(s) = \frac{1}{s+3} X_2(s) + \frac{2}{s+3} Y_1(s)$$

Substitute in Eq.1

$$sY_1(\mathrm{s}) \; + \; \frac{1}{s+3} X_2(s) + \frac{2}{s+3} Y_1(s) = X_1(\mathrm{s})$$

We neglect $X_2(s)$ since it is equal to zero.

$$[s(s+3)+2]Y_1(s) = (s+3)X_1(s)$$

$$(s^2+3s+2)Y_1(s) = (s+3)X_1(s)$$

$$Y_1(s) = \frac{s+3}{s^2+3s+2} X_1(s) = \frac{s+3}{(s+1)(s+2)} X_1(s)$$

Now if $x_1(t) = \mathrm{e}^{-t}$ then $X_1(s) = \dfrac{1}{s+1}$

$$\therefore \qquad Y_1(s) = \frac{s+3}{(s+1)^2(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

so that $y_1(t)$ will contain $e^{-t/\tau}$, $te^{-t/\tau}$, e^{-2t} functions of time.

For $Y_2(s)$

$$Y_2(s) = \frac{2}{(s+1)^2(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

so that $y_2(t)$ will contain the same functions of time as $y_1(t)$ (although different coefficients).

3.14

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = 4\frac{d(x-2)}{dt} - x(t-2)$$

Taking the Laplace transform and assuming zero initial conditions,

$s^2Y(s) + 3sY(s) + Y(s) = 4\,e^{-2s}sX(s) - e^{-2s}X(s)$

Rearranging,

$$\frac{Y(s)}{X(s)} = G(s) = \frac{-(1-4s)e^{-2s}}{s^2+3s+1} \qquad (1)$$

a) The standard form of the denominator is : $\tau^2 s^2 + 2\zeta\tau s + 1$

From (1) , $\tau = 1$, $\zeta = 1.5$

Thus the system will exhibit overdamped and non-oscillatory response.

b) Steady-state gain

$$K = \lim_{s \to 0} G(s) = -1 \qquad \text{(from (1))}$$

c) For a step change in x

$X(s) = \dfrac{1.5}{s}$ and $Y(s) = \dfrac{-(1-4s)e^{-2s}}{(s^2+3s+1)}\dfrac{1.5}{s}$

Therefore $\hat{y}(t) = -1.5 + 1.5e^{-1.5t}\cosh(1.11t) + 7.38e^{-1.5t}\sinh(1.11t)$
Using MATLAB-Simulink, $y(t) = \hat{y}(t-2)$ is shown in Fig. S3.14

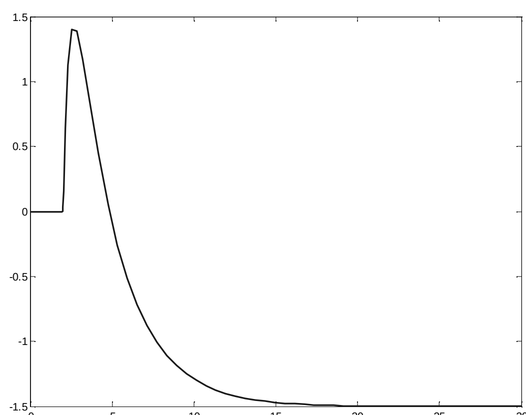


Figure S3.14. *Output variable for a step change in x of magnitude 1.5*

3.15

$$f(t) = hS(t) - hS(t - 1/h)$$

$$\frac{dx}{dt} + 4x = h[S(t) - S(t-1/h)] \qquad , \quad x(0)=0$$

Take Laplace transform,

$$sX(s) + 4X(s) = h\left(\frac{1}{s} - \frac{e^{-s/h}}{s}\right)$$

$$X(s) = h(1-e^{-s/h})\frac{1}{s(s+4)} = h(1-e^{-s/h})\left[\frac{\alpha_1}{s} + \frac{\alpha_2}{s+4}\right]$$

$$\alpha_1 = \left.\frac{1}{s+4}\right|_{s=0} = \frac{1}{4} \quad , \quad \alpha_2 = \left.\frac{1}{s}\right|_{s=-4} = -\frac{1}{4}$$

$$X(s) = \frac{h}{4}(1-e^{-s/h})\left[\frac{1}{s} - \frac{1}{s+4}\right]$$

$$= \frac{h}{4}\left[\frac{1}{s} - \frac{e^{-s/h}}{s} - \frac{1}{s+4} + \frac{e^{-s/h}}{s+4}\right]$$

$$x(t) = \begin{cases} 0 & t<0 \\ \frac{h}{4}(1-e^{-4t}) & 0<t<1/h \\ \frac{h}{4}\left[e^{-4(t-1/h)} - e^{-4t}\right] & t>1/h \end{cases}$$

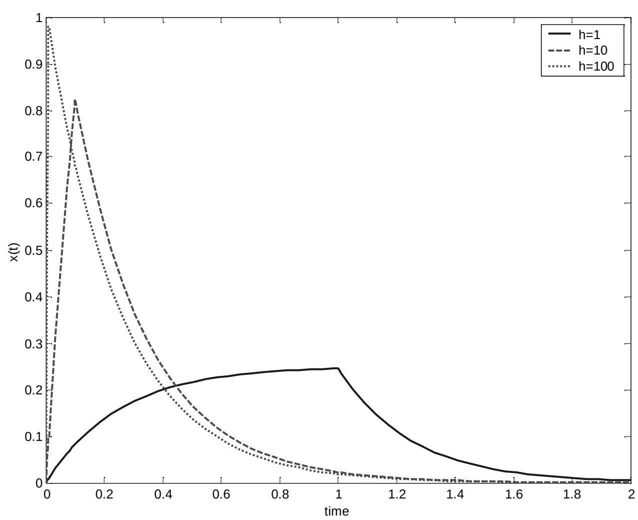


Figure S3.15. *Solution for values h= 1, 10 and 100*

3.16

a) Laplace transforming

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 6\left[sY(s) - y(0)\right] + 9Y(s) = \frac{s}{s^2+1}$$

$$(s^2 + 6s + 9)Y(s) - s(1) - 2 - (6)(1) = \frac{s}{s^2+1}$$

$$(s^2 + 6s + 9)Y(s) = \frac{s}{s^2+1} + s + 8$$

$$(s^2 + 6s + 9)Y(s) = \frac{s + s^3 + s + 8s^2 + 8}{s^2+1}$$

$$Y(s) = \frac{s^3 + 8s^2 + 2s + 8}{(s+3)^2(s^2+1)}$$

To find $y(t)$ we have to expand $Y(s)$ into its partial fractions

$$Y(s) = \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{Cs}{s^2+1} + \frac{D}{s^2+1}$$

$$y(t) = Ate^{-3t} + Be^{-3t} + C\cos t + D\sin t$$

b) $Y(s) = \dfrac{s+1}{s(s^2+4s+8)}$

Since $\dfrac{4^2}{4} < 8$ we know we will have complex factors.

$\therefore$ complete square in denominator

$s^2 + 4s + 8 = s^2 + 4s + 4 + 8 - 4$

$= s^2 + 4s + 4 + 4 = (s+2)^2 + (2)^2$ $\{ b = 2 \ , \ \omega = 2\}$

$\therefore$ Partial fraction expansion gives

$$Y(s) = \frac{A}{s} + \frac{B(s+2)}{s^2+4s+8} + \frac{C}{s^2+4s+8} = \frac{s+1}{s(s^2+4s+8)}$$

Multiply by s and let $s \rightarrow 0$

$A = 1/8$

Multiply by $s(s^2+4s+8)$

$A(s^2+4s+8) + B(s+2)s + Cs = s + 1$

$As^2 + 4As + 8A + Bs^2 + 2Bs + Cs = s + 1$

s^2: $\quad A + B = 0 \quad \rightarrow \quad B = -A = -\frac{1}{8}$

s^1: $\quad 4A + 2B + C = 1 \quad \rightarrow \quad C = 1 + 2\left(\frac{1}{8}\right) - 4\left(\frac{1}{8}\right) = \frac{3}{4}$

s^0: $\quad 8A = 1 \quad \rightarrow A = \frac{1}{8}$ $\quad$ (This checks with above result)

$$Y(s) = \frac{1/8}{s} + \frac{(-1/8)(s+2)}{(s+2)^2+2^2} + \frac{3/4}{(s+2)^2+2^2}$$

$$y(t) = \left(\frac{1}{8}\right) - \left(\frac{1}{8}\right)e^{-2t}\cos 2t + \left(\frac{3}{8}\right)e^{-2t}\sin 2t$$

3.17

$$V\frac{dC}{dt} + qC = qC_i$$

Since V and q are constant, we can Laplace Transform

$sVC(s) + qC(s) = q\,C_i(s)$

Note that $c(t = 0) = 0$

Also, $\quad c_i(t) = 0 \quad , \quad t \leq 0$

$\quad\quad\quad c_i(t) = \bar{c}_i \quad , \quad t > 0$

Laplace transforming the input function, a constant,

$$C_i(s) = \frac{\bar{c}_i}{s}$$

so that

$$sVC(s) + qC(s) = q\frac{\bar{c}_i}{s} \qquad \text{or} \qquad C(s) = \frac{q\bar{c}_i}{(sV+q)s}$$

Dividing numerator and denominator by q

$$C(s) = \frac{\bar{c}_i}{\left(\frac{V}{q}s+1\right)s}$$

Use Transform pair #3 in Table 3.1 to invert ($\tau = V/q$)

$$c(t) = \bar{c}_i\left(1 - e^{-\frac{V}{q}t}\right)$$

Using MATLAB, the concentration response is shown in Fig. S3.17. (Consider V = 2 m^3, C_i=50 Kg/m^3 and q = 0.4 m^3/min)

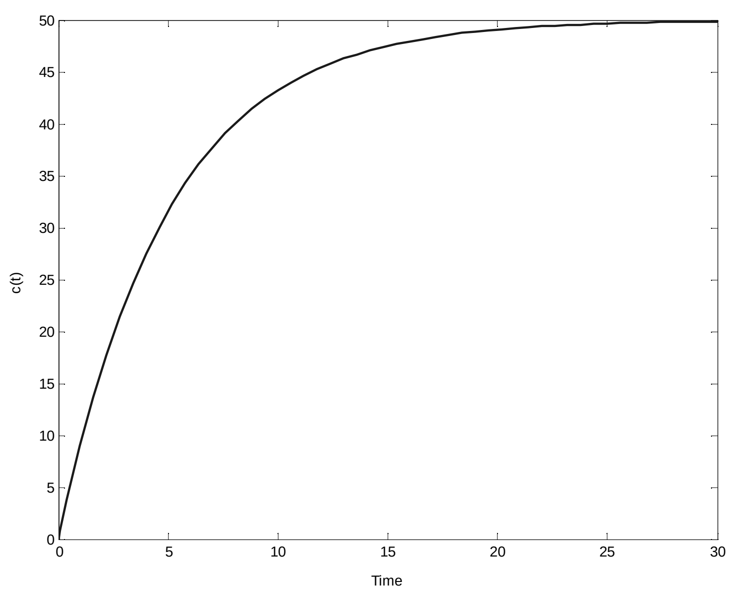


Figure S3.17. *Concentration response of the reactor effluent stream.*

3.18

a) If $Y(s) = \dfrac{KA\omega}{s(s^2+\omega^2)}$

and input $U(s) = \dfrac{A\omega}{(s^2+\omega^2)} = \mathfrak{L}\{A \sin \omega t\}$

then the differential equation had to be

$$\frac{dy}{dt} = Ku(t) \qquad \text{with} \qquad y(0) = 0$$

b) $Y(s) = \dfrac{KA\omega}{s(s^2+\omega^2)} = \dfrac{\alpha_1}{s} + \dfrac{\alpha_2 s}{s^2+\omega^2} + \dfrac{\alpha_3\omega}{s^2+\omega^2}$

$$\alpha_1 = \left.\frac{KA\omega}{s^2+\omega^2}\right|_{s\to 0} = \frac{KA}{\omega}$$

Find α_2 and α_3 by equating coefficients

$$KA\omega = \alpha_1(s^2+\omega^2) + \alpha_2 s^2 + \alpha_3\omega s$$

$$KA\omega = \alpha_1 s^2 + \alpha_1\omega^2 + \alpha_2 s^2 + \alpha_3\omega s$$

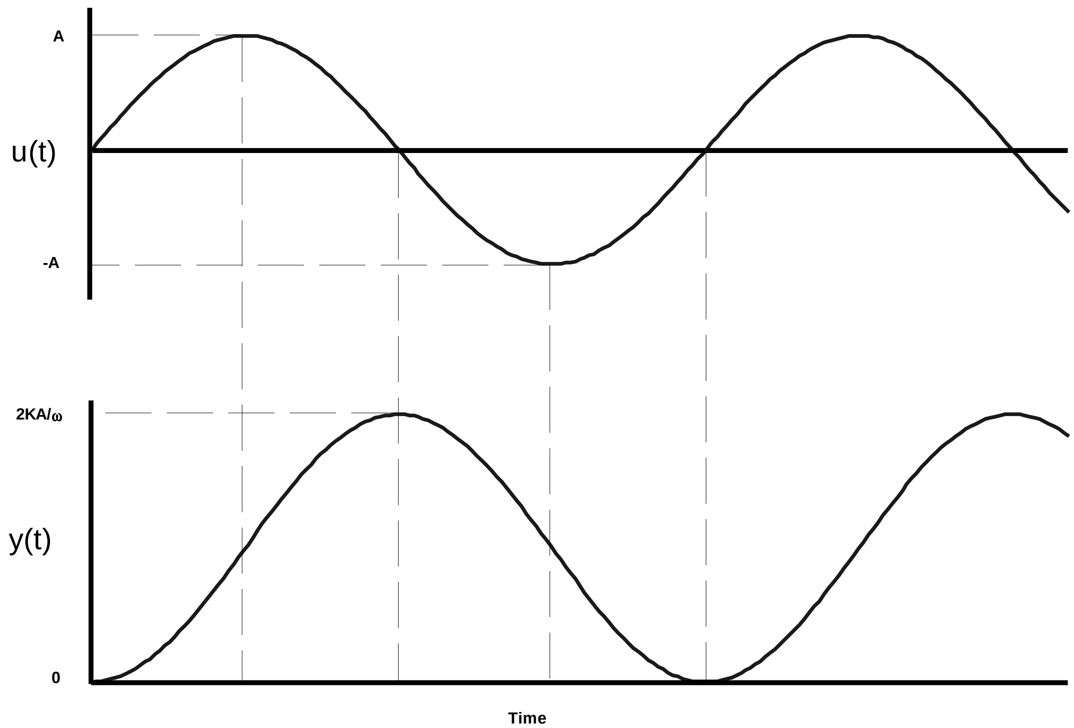
$$s^2: \quad 0 = \alpha_1 + \alpha_2 \quad \rightarrow \alpha_2 = -\alpha_1 = \frac{-KA}{\omega}$$

$$s: \quad 0 = \alpha_3\,\omega \quad \rightarrow \alpha_3 = 0$$

$$\therefore \; Y(s) = \frac{KA\omega}{s(s^2+\omega^2)} = \frac{KA/\omega}{s} - \frac{(KA/\omega)s}{s^2+\omega^2}$$

$$y(t) = \frac{KA}{\omega}(1-\cos\omega t)$$

c)



i) We see that $y(t)$ follows behind $u(t)$ by 1/4 cycle = $2\pi/4 = \pi/2$ rad. which is constant for all ω

ii) The amplitudes of the two sinusoidal quantities are:

y : KA/ω
u: A

Thus their ratio is K/ω, which is a function of frequency.