Chapter 4

4.1

a) iii
b) iii
c) v
d) v

4.2

a) 5

b) 10

c) $Y(s) = \dfrac{10}{s(10s+1)}$

From the Final Value Theorem, $y(t) = 10$ when $t \to \infty$

d) $y(t) = 10(1-e^{-t/10})$, then $y(10) = 6.32 = 63.2\%$ of the final value.

e) $Y(s) = \dfrac{5}{(10s+1)} \dfrac{(1-e^{-s})}{s}$

From the Final Value Theorem, $y(t) = 0$ when $t \to \infty$

f) $Y(s) = \dfrac{5}{(10s+1)} 1$

From the Final Value Theorem, $y(t) = 0$ when $t \to \infty$

g) $Y(s) = \dfrac{5}{(10s+1)} \dfrac{6}{(s^2+9)}$ then

$y(t) = 0.33e^{-0.1t} - 0.33\cos(3t) + 0.011\sin(3t)$

The sinusoidal input produces a sinusoidal output and $y(t)$ does not have a limit when $t \to \infty$.

By using Simulink-MATLAB, above solutions can be verified:

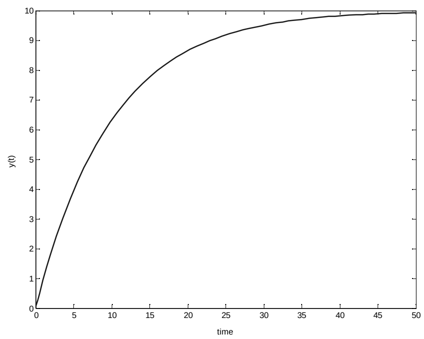

Fig S4.2a. *Output for part c) and d)*

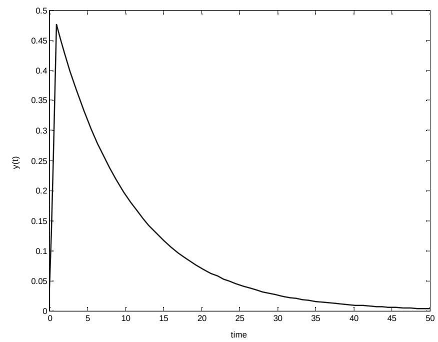

Fig S4.2b. *Output for part e)*

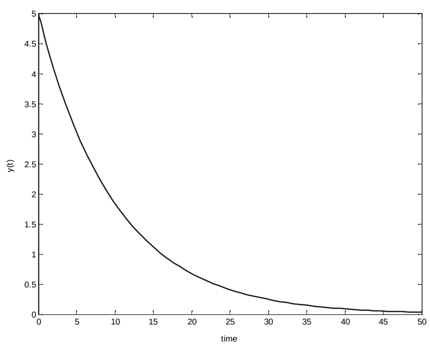

Fig S4.2c. *Output for part f)*

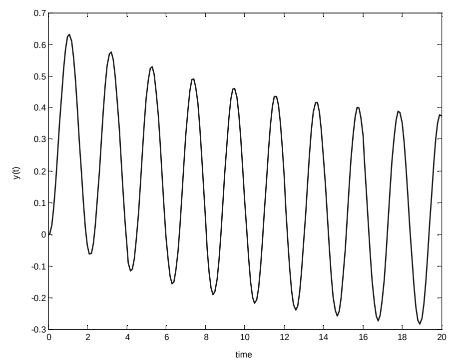

Fig S4.2d. *Output for part g)*

4.3

a) The dynamic model of the system is given by

$$\frac{dV}{dt} = \frac{1}{\rho}(w_i - w) \tag{2-45}$$

$$\frac{dT}{dt} = \frac{w_i}{V\rho}(T_i - T) + \frac{Q}{V\rho C} \tag{2-46}$$

Let the right-hand side of Eq. 2-46 be f(w_i,V,T),

$$\frac{dT}{dt} = f(w_i, V, T) = \left(\frac{\partial f}{\partial w_i}\right)_s w_i' + \left(\frac{\partial f}{\partial V}\right)_s V' + \left(\frac{\partial f}{\partial T}\right)_s T' \tag{1}$$

$$\left(\frac{\partial f}{\partial w_i}\right)_s = \frac{1}{\bar{V}\rho}(T_i - \bar{T})$$

$$\left(\frac{\partial f}{\partial V}\right)_s = -\frac{\bar{w}_i}{\bar{V}^2\rho}(T_i - \bar{T}) - \frac{Q}{\bar{V}^2\rho C} = -\frac{1}{\bar{V}}\left(\frac{dT}{dt}\right)_s = 0$$

$$\left(\frac{\partial f}{\partial T}\right)_s = -\frac{\bar{w}_i}{\bar{V}\rho}$$

$$\frac{dT}{dt} = \frac{1}{\bar{V}\rho}(T_i - \bar{T})w_i' - \frac{\bar{w}_i}{\bar{V}\rho}T' \quad , \qquad \frac{dT}{dt} = \frac{dT'}{dt}$$

Taking Laplace transform and rearranging

$$\frac{T'(s)}{W_i'(s)} = \frac{(T_i - \bar{T})/\bar{w}_i}{\left(\frac{\bar{V}\rho}{\bar{w}_i}\right)s + 1} \tag{2}$$

Laplace transform of Eq. 2-45 gives $V'(s) = \dfrac{W_i'(s)}{\rho s}$ (3)

If $\left(\dfrac{\partial f}{\partial V}\right)_s$ were not zero, then using (3)

$$\frac{T'(s)}{W_i'(s)} = \frac{\left[\frac{(T_i - \bar{T})}{\bar{w}_i} + \frac{\bar{V}}{\bar{w}_i}\left(\frac{\partial f}{\partial V}\right)_s \frac{1}{s}\right]}{\left(\frac{\bar{V}\rho}{\bar{w}_i}\right)s + 1} \tag{4}$$

Appelpolscher guessed the incorrect form (4) instead of the correct form (2) because he forgot that $\left(\dfrac{\partial f}{\partial V}\right)_s$ would vanish.

b) From Eq. 3,

$$\frac{V'(s)}{W_i'(s)} = \frac{1}{\rho s}$$

4.4

$$Y(s)=G(s)X(s)=\frac{K}{s(\tau s+1)}$$

$G(s)$	Interpretation of $G(s)$	$U(s)$	Interpretation of $u(t)$
$\frac{K}{s(\tau s+1)}$	*2nd order process* *	1	$\delta(0)$ [Delta function]
$\frac{K}{\tau s+1}$	*1st order process*	$\frac{1}{s}$	$S(0)$ [Unit step function]
$\frac{K}{s}$	*Integrator*	$\frac{K}{\tau s+1}$	$\frac{1}{\tau}e^{-t/\tau}$ [Exponential input]
K	*Simple gain (i.e no dynamics)*	$\frac{1}{s(\tau s+1)}$	$1-e^{-t/\tau}$ [Step + exponential input]

* *2nd order or combination of integrator and 1st order process*

4.5

a) $$2\frac{dy_1}{dt} = -2y_1 - 3y_2 + 2u_1 \qquad (1)$$

$$\frac{dy_2}{dt} = 4y_1 - 6y_2 + 2u_1 + 4u_2 \qquad (2)$$

Taking Laplace transform of the above equations and rearranging,

$$(2s+2)Y_1(s) + 3Y_2(s) = 2U_1(s) \qquad (3)$$

$$-4\,Y_1(s) + (s+6)Y_2(s)=2U_1(s) + 4U_2(s) \qquad (4)$$

Solving Eqs. 3 and 4 simultaneously for $Y_1(s)$ and $Y_2(s)$,

$$Y_1(s) = \frac{(2s+6)U_1(s) - 12U_2(s)}{2s^2 + 14s + 24} = \frac{2(s+3)U_1(s) - 12U_2(s)}{2(s+3)(s+4)}$$

$$Y_2(s) = \frac{(4s+12)U_1(s) - (8s+8)U_2(s)}{2s^2 + 14s + 24} = \frac{4(s+3)U_1(s) + 8(s+1)U_2(s)}{2(s+3)(s+4)}$$

Therefore,

$$\frac{Y_1(s)}{U_1(s)} = \frac{1}{s+4} \quad , \quad \frac{Y_1(s)}{U_2(s)} = \frac{-6}{(s+3)(s+4)}$$

$$\frac{Y_2(s)}{U_1(s)} = \frac{2}{s+4} \quad , \quad \frac{Y_2(s)}{U_2(s)} = \frac{4(s+1)}{(s+3)(s+4)}$$

4.6

The physical model of the CSTR is (Section 2.4.6)

$$V\frac{dc_A}{dt} = q(c_{Ai} - c_A) - Vkc_A \qquad (2\text{-}66)$$

$$V\rho C\frac{dT}{dt} = wC(T_i - T) + (-\Delta H)Vkc_A + UA(T_c - T) \qquad (2\text{-}68)$$

where: $k = k_o\, e^{-E/RT}$ (2-63)

These equations can be written as,

$$\frac{dc_A}{dt} = f_1(c_A, T) \qquad (1)$$

$$\frac{dT}{dt} = f_2(c_A, T, T_c) \qquad (2)$$

Because both equations are nonlinear, linearization is required. After linearization and introduction of deviation variables, we could get an expression for $c'_A(s)/\,T'(s)$.

But it is not possible to get an expression for $T'(s)/T_c'(s)$ from (2) due to the presence of c_A in (2). Thus the proposed approach is <u>not feasible</u> because the CSTR is an interacting system.

Better approach:

After linearization etc., solve for $T'(s)$ from (1) and substitute into the linearized version of (2). Then rearrange to obtain the desired, $C_A'(s)/T_c'(s)$ (See Section 4.3)

4.7

a) The assumption that H is constant is redundant. For equimolal overflow,

$$L_0 = L_1 = L \quad , \quad V_1 = V_2 = V$$

$$\frac{dH}{dt} = L_0 + V_2 - L_1 - V_1 = 0 \quad \text{, i.e., } H \text{ is constant.}$$

The simplified stage concentration model becomes

$$H\frac{dx_1}{dt} = L(x_0 - x_1) + V(y_2 - y_1) \tag{1}$$

$$y_1 = a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 \tag{2}$$

b) Let the right-hand side of Eq. 1 be f(L, x_0, x_1, V, y_1, y_2)

$$H\frac{dx_1}{dt} = f(L, x_0, x_1, V, y_1, y_2) = \left(\frac{\partial f}{\partial L}\right)_s L' + \left(\frac{\partial f}{\partial x_0}\right)_s x_0' + \left(\frac{\partial f}{\partial x_1}\right)_s x_1'$$

$$+\left(\frac{\partial f}{\partial V}\right)_s V' + \left(\frac{\partial f}{\partial y_1}\right)_s y_1' + \left(\frac{\partial f}{\partial y_2}\right)_s y_2'$$

Substituting for the partial derivatives and noting that $\dfrac{dx_1}{dt} = \dfrac{dx_1'}{dt}$

$$H\frac{dx_1'}{dt} = (\bar{x}_0 - \bar{x}_1)L' + \bar{L}x_0' - \bar{L}x_1' + (\bar{y}_2 - \bar{y}_1)V' + \bar{V}y_2' - \bar{V}y_1' \tag{3}$$

Similarly,

$$y_1' = g(x_1) = \left(\frac{\partial g}{\partial x_1}\right)_s x_1' = (a_1 + 2a_2\bar{x}_1 + 3a_3\bar{x}_1^{\,2})x_1' \tag{4}$$

c) For constant liquid and vapor flow rates, $L' = V' = 0$

Taking Laplace transform of Eqs. 3 and 4,

$$HsX_1'(s) = \bar{L}X_0'(s) - \bar{L}X_1'(s) + \bar{V}Y_2'(s) - \bar{V}Y_1'(s) \tag{5}$$

$$Y_1'(s) = (a_1 + 2a_2\bar{x}_1 + 3a_3\bar{x}_1^{\,2})X_1'(s) \tag{6}$$

From Eqs. 5 and 6, the desired transfer functions are

$$\frac{X_1'(s)}{X_0'(s)} = \frac{\frac{\bar{L}}{H}\tau}{\tau s + 1} \quad , \quad \frac{X_1'(s)}{Y_2'(s)} = \frac{\frac{\bar{V}}{H}\tau}{\tau s + 1}$$

$$\frac{Y_1'(s)}{X_0'(s)} = \frac{(a_1 + 2a_2\bar{x}_1 + 3a_3\bar{x}_1^{\,2})\frac{\bar{L}}{H}\tau}{\tau s + 1}$$

$$\frac{Y_1'(s)}{Y_2'(s)} = \frac{(a_1 + 2a_2\bar{x}_1 + 3a_3\bar{x}_1^{\,2})\frac{\bar{V}}{H}\tau}{\tau s + 1}$$

where

$$\tau = \frac{H}{\bar{L} + \bar{V}(a_1 + 2a_2\bar{x}_1 + 3a_3\bar{x}_1^{\,2})}$$

4.8

From material balance,

$$\frac{d(\rho A h)}{dt} = w_i - Rh^{1.5}$$

$$\frac{dh}{dt} = \frac{1}{\rho A}w_i - \frac{R}{\rho A}h^{1.5}$$

We need to use a Taylor series expansion to linearize

$$\frac{dh}{dt} = \left[\frac{1}{\rho A}\overline{w}_i - \frac{R}{\rho A}\overline{h}^{1.5}\right] + \frac{1}{\rho A}(w_i - \overline{w}_i) - \frac{1.5R\overline{h}^{0.5}}{\rho A}(h - \overline{h})$$

Since the bracketed term is identically zero at steady state,

$$\frac{dh'}{dt} = \frac{1}{\rho A}w_i' - \frac{1.5R\overline{h}^{0.5}}{\rho A}h'$$

Rearranging

$$\frac{\rho A}{1.5R\overline{h}^{0.5}}\frac{dh'}{dt} + h' = \frac{1}{1.5R\overline{h}^{0.5}}w_i'$$

Hence $\dfrac{H'(s)}{W_i'(s)} = \dfrac{K}{\tau s + 1}$

where

$$K = \frac{1}{1.5R\overline{h}^{0.5}} = \frac{\overline{h}}{1.5R\overline{h}^{1.5}} = \frac{\overline{h}}{1.5\overline{w}} = \frac{[height]}{[flowrate]}$$

$$\tau = \frac{\rho A}{1.5R\overline{h}^{0.5}} = \frac{\rho A\overline{h}}{1.5R\overline{h}^{1.5}} = \frac{\rho \overline{V}}{1.5\overline{w}} = \frac{[mass]}{[mass/time]} = [time]$$

4.9

a) The model for the system is given by

$$mC\frac{dT}{dt} = wC(T_i - T) + h_pA_p(T_w - T) \quad \text{(2-51)}$$

$$m_wC_w\frac{dT_w}{dt} = h_sA_s(T_s - T_w) - h_pA_p(T_w - T) \quad \text{(2-52)}$$

Assume that m, m_w, C, C_w, h_p, h_s, A_p, A_s, and w are constant. Rewriting the above equations in terms of deviation variables, and noting that

$$\frac{dT}{dt} = \frac{dT'}{dt} \qquad \frac{dT_w}{dt} = \frac{dT'_w}{dt}$$

$$mC\frac{dT'}{dt} = wC(T'_i - T') + h_p A_p (T'_w - T')$$
$$m_w C_w \frac{dT'_w}{dt} = h_s A_s (0 - T'_w) - h_p A_p (T'_w - T')$$

Taking Laplace transforms and rearranging,

$$(mCs + wC + h_p A_p)T'(s) = wCT'_i(s) + h_p A_p T'_w(s) \qquad (1)$$

$$(m_w C_w s + h_s A_s + h_p A_p)T'_w(s) = h_p A_p T'(s) \qquad (2)$$

Substituting in Eq. 1 for $T'_w(s)$ from Eq. 2,

$$(mCs + wC + h_p A_p)T'(s) = wCT'_i(s) + h_p A_p \frac{h_p A_p}{(m_w C_w s + h_s A_s + h_p A_p)} T'(s)$$

Therefore,

$$\frac{T'(s)}{T'_i(s)} = \frac{wC(m_w C_w s + h_s A_s + h_p A_p)}{(mCs + wC + h_p A_p)(m_w C_w s + h_s A_s + h_p A_p) - (h_p A_p)^2}$$

b) The gain is $\left[\frac{T'(s)}{T'_i(s)}\right]_{s=0} = \frac{wC(h_s A_s + h_p A_p)}{wC(h_s A_s + h_p A_p) + h_s A_s h_p A_p}$

c) No, the gain would be expected to be 1 only if the tank were insulated so that $h_p A_p = 0$. For heated tank the gain is not 1 because heat input changes as T changes.

4.10

Additional assumptions

1) perfect mixing in the tank
2) constant density, ρ, and specific heat, C.
3) T_i is constant.

Energy balance for the tank,

$$\rho VC\frac{dT}{dt} = wC(T_i - T) + Q - (\overline{U} + bv)A(T - T_a)$$

Let the right-hand side be f(T,v),

$$\rho VC\frac{dT}{dt} = f(T,v) = \left(\frac{\partial f}{\partial T}\right)_s T' + \left(\frac{\partial f}{\partial v}\right)_s v' \qquad (1)$$

$$\left(\frac{\partial f}{\partial T}\right)_s = -wC\ (\overline{U} + b\overline{v})A$$

$$\left(\frac{\partial f}{\partial v}\right)_s = -\,bA(\overline{T} - T_a)$$

Substituting for the partial derivatives in Eq. 1 and noting that $\dfrac{dT}{dt} = \dfrac{dT'}{dt}$

$$\rho VC\frac{dT'}{dt} = -\left[wC + (\overline{U} + b\overline{v})A\right]T' - bA(\overline{T} - T_a)v'$$

Taking the Laplace transform and rearranging

$$\left[\rho VCs + wC + (\overline{U} + b\overline{v})A\right]T'(s) = -\,bA(\overline{T} - T_a)v'(s)$$

$$\frac{T'(s)}{v'(s)} = \frac{\left[\dfrac{-\,bA(\overline{T} - T_a)}{wC + (\overline{U} + b\overline{v})A}\right]}{\left[\dfrac{\rho VC}{wC + (\overline{U} + b\overline{v})A}\right]s + 1}$$

4.11

a) Mass balances on surge tanks

$$\frac{dm_1}{dt} = w_1 - w_2 \qquad (1)$$

$$\frac{dm_2}{dt} = w_2 - w_3 \qquad (2)$$

Ideal gas law

$$P_1 V_1 = \frac{m_1}{M} RT \tag{3}$$

$$P_2 V_2 = \frac{m_2}{M} RT \tag{4}$$

Flows (Ohm's law is $I = \dfrac{E}{R} = \dfrac{\text{Driving Force}}{\text{Resistance}}$)

$$w_1 = \frac{1}{R_1}(P_c - P_1) \tag{5}$$

$$w_2 = \frac{1}{R_2}(P_1 - P_2) \tag{6}$$

$$w_3 = \frac{1}{R_3}(P_2 - P_h) \tag{7}$$

Degrees of freedom:

number of parameters : 8 (V_1, V_2, M, R, T, R_1, R_2, R_3)

number of variables : 9 (m_1, m_2, w_1, w_2, w_3, P_1, P_2, P_c, P_h)

number of equations : 7

$\therefore$ number of degrees of freedom that must be eliminated = 9 − 7 = 2

Since P_c and P_h are known functions of time (i.e., inputs), $N_F = 0$.

b) Development of model

Substitute (3) into (1) : $\dfrac{MV_1}{RT}\dfrac{dP_1}{dt} = w_1 - w_2$ (8)

Substitute (4) into (2) : $\dfrac{MV_2}{RT}\dfrac{dP_2}{dt} = w_2 - w_3$ (9)

Substitute (5) and (6) into (8) :

$$\frac{MV_1}{RT}\frac{dP_1}{dt} = \frac{1}{R_1}(P_c - P_1) - \frac{1}{R_2}(P_1 - P_2)$$

$$\frac{MV_1}{RT}\frac{dP_1}{dt} = \frac{1}{R_1}P_c(t) - \left(\frac{1}{R_1} + \frac{1}{R_2}\right)P_1 + \frac{1}{R_2}P_2 \tag{10}$$

Substitute (6) and (7) into (9):

$$\frac{MV_2}{RT}\frac{dP_2}{dt} = \frac{1}{R_2}(P_1 - P_2) - \frac{1}{R_3}(P_2 - P_h)$$

$$\frac{MV_2}{RT}\frac{dP_2}{dt} = \frac{1}{R_2}P_1 - (\frac{1}{R_2} + \frac{1}{R_3})P_2 + \frac{1}{R_3}P_h(t) \qquad (11)$$

Note that $\frac{dP_1}{dt} = f_1(P_1, P_2)$ from Eq. 10

$\frac{dP_2}{dt} = f_2(P_1, P_2)$ from Eq. 11

This is exactly the same situation depicted in Figure 6.13, therefore the two tanks interact. This system has the following characteristics:

i) Interacting (Eqs. 10 and 11 interact with each other)
ii) 2nd-order denominator (2 differential equations)
iii) Zero-order numerator (See example 4.4 in text)
iv) No integrating elements
v) Gain of $\frac{W_3'(s)}{P_c'(s)}$ is not equal to unity. (Cannot be because the units on the two variables are different).

4.12

a) $A\frac{dh}{dt} = q_i - C_v h^{1/2}$

Let $f = q_i - C_v h^{1/2}$

Then $f \approx \bar{q}_i - C_v \bar{h}^{1/2} + q_i - \bar{q}_i - \frac{1}{2}C_v \bar{h}^{-1/2}(h - \bar{h})$

so $A\frac{dh'}{dt} = q_i' - \frac{C_v}{2\bar{h}^{1/2}}h'$ because $\bar{q}_i - C_v \bar{h}^{1/2} \equiv 0$

$$\left[sA + \frac{C_v}{2\bar{h}^{1/2}}\right]H'(s) = Q_i'(s)$$

$$\frac{H'(s)}{Q_i'(s)} = \frac{1}{sA + \dfrac{C_v}{2\bar{h}^{1/2}}}$$

Note: Not a standard form

$$\frac{H'(s)}{Q_i'(s)} = \frac{2\bar{h}^{1/2}/C_v}{\dfrac{2A\bar{h}^{1/2}}{C_v}s + 1}$$

where $K = \dfrac{2\bar{h}^{1/2}}{C_v}$ and $\tau = \dfrac{2A\bar{h}^{1/2}}{C_v}$

b) Because $q = C_v h^{1/2}$

$$q' = C_v \frac{1}{2}\bar{h}^{-1/2}h' = \frac{C_v}{2\bar{h}^{1/2}}h' = \frac{1}{K}h'$$

$$\therefore \qquad \frac{Q'(s)}{H'(s)} = \frac{1}{K}\ , \quad \frac{Q'(s)}{H'(s)}\frac{H'(s)}{Q_i'(s)} = \frac{1}{K}\frac{K}{\tau s + 1}$$

and $\dfrac{Q'(s)}{Q_i'(s)} = \dfrac{1}{\tau s + 1}$

c) For a linear outlflow relation

$$A\frac{dh}{dt} = q_i - C_v^{\,*}h$$

Note that $C_v^{\,*} \neq C_v$

$$A\frac{dh'}{dt} = q_i' - C_v^{\,*}h'$$

$$A\frac{dh'}{dt} + C_v^{\,*}h' = q_i' \qquad \text{or} \qquad \frac{A}{C_v^{\,*}}\frac{dh'}{dt} + h' = \frac{1}{C_v^{\,*}}q_i'$$

Multiplying numerator and denominator by $\bar{h}$ on each side yields

$$\frac{A\bar{h}}{C_v^{\,*}\bar{h}}\frac{dh'}{dt} + h' = \frac{\bar{h}}{C_v^{\,*}\bar{h}}q_i'$$

or $\dfrac{\bar{V}}{\bar{q}_i}\dfrac{dh'}{dt} + h' = \dfrac{\bar{h}}{\bar{q}_i} q_i'$

$\tau^* = \dfrac{\bar{V}}{\bar{q}i}$ $\qquad K^* = \dfrac{\bar{h}}{\bar{q}_i}$ $\qquad$ q.e.d

To put τ and K in comparable terms for the square root outflow form of the transfer function, multiply numerator and denominator of each by $\bar{h}^{1/2}$.

$$K = \frac{2\bar{h}^{1/2}}{C_v}\frac{\bar{h}^{1/2}}{\bar{h}^{1/2}} = \frac{2\bar{h}}{C_v \bar{h}^{1/2}} = \frac{2\bar{h}}{\bar{q}_i} = 2K^*$$

$$\tau = \frac{2A\bar{h}^{1/2}}{C_v}\frac{\bar{h}^{1/2}}{\bar{h}^{1/2}} = \frac{2A\bar{h}}{C_v \bar{h}^{1/2}} = \frac{2\bar{V}}{\bar{q}_i} = 2\tau^*$$

Thus level in the square root outflow TF is twice as sensitive to changes in q_i and reacts only ½ as fast (two times more slowly) since $\tau = 2\tau^*$.

4.13

a) The nonlinear dynamic model for the tank is:

$$\frac{dh}{dt} = \frac{1}{\pi(D-h)h}\left(q_i - C_v\sqrt{h}\right) \qquad (1)$$

(corrected nonlinear ODE; model in first printing of book is incorrect)

To linearize Eq. 1 about the operating point $(h = \bar{h}\ ,\ q_i = \bar{q}_i)$, let

$$f = \frac{q_i - C_v\sqrt{h}}{\pi(D-h)h}$$

Then,

$$f(h, q_i) \approx \left(\frac{\partial f}{\partial h}\right)_s h' + \left(\frac{\partial f}{\partial q_i}\right)_s q_i'$$

where

$$\left(\frac{\partial f}{\partial q_i}\right)_s = \frac{1}{\pi(D-\bar{h})\bar{h}}$$

$$\left(\frac{\partial f}{\partial h}\right)_s = -\frac{1}{2}\frac{C_v}{\sqrt{\bar{h}}}\frac{1}{\pi(D-\bar{h})\bar{h}} + \left(\bar{q}_i - C_v\sqrt{\bar{h}}\right)\left[\frac{-\pi D + 2\pi h}{\left(\pi(D-\bar{h})\bar{h}\right)^2}\right]$$

Notice that the second term of last partial derivative is zero from the steady-state relation, and the term $\pi(D-\bar{h})\bar{h}$ is finite for all $0 < h < D$. Consequently, the linearized model of the process, after substitution of deviation variables is,

$$\frac{dh'}{dt} = \left[-\frac{1}{2}\frac{C_v}{\sqrt{\bar{h}}}\frac{1}{\pi(D-\bar{h})\bar{h}}\right]h' + \left[\frac{1}{\pi(D-\bar{h})\bar{h}}\right]q_i'$$

Since $\bar{q}_i = C_v\sqrt{\bar{h}}$

$$\frac{dh'}{dt} = \left[-\frac{1}{2}\frac{\bar{q}_i}{\bar{h}}\frac{1}{\pi(D-\bar{h})\bar{h}}\right]h' + \left[\frac{1}{\pi(D-\bar{h})\bar{h}}\right]q_i'$$

or $$\frac{dh'}{dt} = ah' + bq_i'$$

where

$$a = \left[-\frac{1}{2}\frac{\bar{q}_i}{\bar{h}}\frac{1}{\pi(D-\bar{h})\bar{h}}\right] = -\frac{\bar{q}_i}{\bar{V}_o} \quad , \quad b = \left[\frac{1}{\pi(D-\bar{h})\bar{h}}\right]$$

$\bar{V}_o$ = volume at the initial steady state

b) Taking Laplace transform and rearranging

$$s\,h'(s) = ah'(s) + bq_i'(s)$$

Therefore

$$\frac{h'(s)}{q_i'(s)} = \frac{b}{(s-a)} \quad \text{or} \quad \frac{h'(s)}{q_i'(s)} = \frac{(-b/a)}{(-1/a)s+1}$$

Notice that the time constant is equal to the residence time at the initial steady state.

4.14

Assumptions

1) Perfectly mixed reactor
2) Constant fluid properties and heat of reaction.

a) Component balance for A,

$$V\frac{dc_A}{dt} = q(c_{A_i} - c_A) - Vk(T)c_A \qquad (1)$$

Energy balance for the tank,

$$\rho VC\frac{dT}{dt} = \rho qC(T_i - T) + (-\Delta H)Vk(T)c_A \qquad (2)$$

Since a transfer function with respect to c_{Ai} is desired, assume the other inputs, namely q and T_i, to be constant.

Linearize (1) and (2) and not that $\frac{dc_A}{dt} = \frac{dc'_A}{dt}$, $\frac{dT}{dt} = \frac{dT'}{dt}$,

$$V\frac{dc'_A}{dt} = qc'_{Ai} - (q + Vk(\bar{T}))c'_A - V\bar{c}_A k(\bar{T})\frac{20000}{\bar{T}^2}T' \qquad (3)$$

$$\rho VC\frac{dT'}{dt} = -\left(\rho qC + \Delta HV\bar{c}_A k(\bar{T})\frac{20000}{\bar{T}^2}\right)T' + (-\Delta H)Vk(\bar{T})c'_A \qquad (4)$$

Taking the Laplace transforms and rearranging

$$\left[Vs + q + Vk(\bar{T})\right]C'_A(s) = qC'_{Ai}(s) - V\bar{c}_A k(\bar{T})\frac{20000}{\bar{T}^2}T'(s) \qquad (5)$$

$$\left[\rho VCs + \rho qC - (-\Delta H)V\bar{c}_A k(\bar{T})\frac{20000}{\bar{T}^2}\right]T'(s) = (-\Delta H)Vk(\bar{T})C'_A(s) \qquad (6)$$

Substituting for $C'_A(s)$ from Eq. 5 into Eq. 6 and rearranging,

$$\frac{T'(s)}{C'_{Ai}(s)}=\frac{-\Delta HVk(\bar{T})q}{\left[Vs+q+Vk(\bar{T})\right]\left[\rho VCs+\rho qC-(-\Delta H)V\bar{c}_A k(\bar{T})\frac{20000}{\bar{T}^2}\right]+(-\Delta H)\bar{c}_A V^2 k^2(\bar{T})\frac{20000}{\bar{T}^2}} \quad (7)$$

$\bar{c}_A$ is obtained from Eq. 1 at steady state,

$$\bar{c}_A=\frac{q\bar{c}_{Ai}}{q+Vk(\bar{T})}=0.0011546 \text{ mol/cu.ft.}$$

Substituting the numerical values of $\bar{T}$, ρ, C, $(-\Delta H)$, q, V, $\bar{c}_A$ into Eq. 7 and simplifying,

$$\frac{T'(s)}{C'_{Ai}(s)}=\frac{11.38}{(0.0722s+1)(50s+1)}$$

b) The gain K of the above transfer function is $\left[\frac{T'(s)}{C'_{Ai}(s)}\right]_{s=0}$,

$$K=\frac{0.15766\ \bar{q}}{\left(\frac{\bar{q}}{1000}-3.153\times10^6\frac{\bar{c}_A}{\bar{T}^2}\right)\left(\frac{\bar{q}}{1000}+13.84\right)+4.364.10^7\frac{\bar{c}_A}{\bar{T}^2}} \quad (8)$$

obtained by putting s=0 in Eq. 7 and substituting numerical values for ρ, C, $(-\Delta H)$, V. Evaluating sensitivities,

$$\frac{dK}{d\bar{q}}=\frac{K}{\bar{q}}-\frac{K^2}{0.15766\bar{q}}\left[2\frac{\bar{q}}{10^6}+0.01384-3153\frac{\bar{c}_A}{\bar{T}^2}\right]=-6.50\times10^{-4}$$

$$\frac{dK}{d\bar{T}}=-\frac{K^2}{3.153}\left[\left(\frac{\bar{q}}{1000}+13.84\right)\left(\frac{3.153\times10^6\bar{c}_A\times2}{\bar{T}^3}\right)-\frac{2\times4.364\times10^7\bar{c}_A}{\bar{T}^3}\right]$$

$$=-2.57\times10^{-5}$$

$$\frac{dK}{d\bar{c}_{Ai}}=\frac{dK}{d\bar{c}_A}\times\frac{d\bar{c}_A}{d\bar{c}_{Ai}}$$

$$=\frac{-K^2}{0.15766\bar{q}}\left[-\left(\frac{\bar{q}}{1000}+13.84\right)\left(\frac{3.153\times10^6}{\bar{T}^2}\right)+\frac{4.364\times10^7}{\bar{T}^2}\right]\left(\frac{\bar{q}}{\bar{q}+13840}\right)$$

$$=8.87\times10^{-3}$$

4.15

From Example 4.4, system equations are:

$$A_1 \frac{dh_1'}{dt} = q_i' - \frac{1}{R_1} h_1' \qquad , \qquad q_1' = \frac{1}{R_1} h_1'$$

$$A_2 \frac{dh_2'}{dt} = \frac{1}{R_1} h_1' - \frac{1}{R_2} h_2' \qquad , \qquad q_2' = \frac{1}{R_2} h_2'$$

Using state space representation,

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $x = \begin{bmatrix} h_1' \\ h_2' \end{bmatrix}$, $u = q_i$ and $y = q_2'$

then,

$$\begin{bmatrix} \dfrac{dh_1'}{dt} \\ \dfrac{dh_2'}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{R_1 A_1} & 0 \\ \dfrac{1}{R_1 A_1} & -\dfrac{1}{R_2 A_2} \end{bmatrix} \begin{bmatrix} h_1' \\ h_2' \end{bmatrix} + \begin{bmatrix} \dfrac{1}{A_1} \\ 0 \end{bmatrix} q_i'$$

$$q_2' = \begin{bmatrix} 0 & \dfrac{1}{R_2} \end{bmatrix} \begin{bmatrix} h_1' \\ h_2' \end{bmatrix}$$

Therefore,

$$A = \begin{bmatrix} -\dfrac{1}{R_1 A_1} & 0 \\ \dfrac{1}{R_1 A_1} & -\dfrac{1}{R_2 A_2} \end{bmatrix} \quad , \quad B = \begin{bmatrix} \dfrac{1}{A_1} \\ 0 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 0 & \dfrac{1}{R_2} \end{bmatrix} \quad , \quad E = 0$$

4.16

Applying numerical values, equations for the three-stage absorber are:

$$\frac{dx_1}{dt} = 0.881y_f - 1.173x_1 + 0.539x_2$$

$$\frac{dx_2}{dt} = 0.634x_1 - 1.173x_2 + 0.539x_3$$

$$\frac{dx_3}{dt} = 0.634x_2 - 1.173x_3 + 0.539x_f$$

$$y_i = 0.72x_i$$

Transforming into a state-space representation form:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} -1.173 & 0.539 & 0 \\ 0.634 & -1.173 & 0.539 \\ 0 & 0.634 & -1.173 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0.881 \\ 0 \\ 0 \end{bmatrix} y_f$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.72 & 0 & 0 \\ 0 & 0.72 & 0 \\ 0 & 0 & 0.72 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} y_f$$

Therefore, because x_f can be neglected in obtaining the desired transfer functions,

$$A = \begin{bmatrix} -1.173 & 0.539 & 0 \\ 0.634 & -1.173 & 0.539 \\ 0 & 0.634 & -1.173 \end{bmatrix} \quad B = \begin{bmatrix} 0.881 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.72 & 0 & 0 \\ 0 & 0.72 & 0 \\ 0 & 0 & 0.72 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying the MATLAB function *ss2tf* , the transfer functions are:

$$\frac{Y_1'(s)}{Y_f'(s)} = \frac{0.6343s^2 + 1.4881s + 0.6560}{s^3 + 3.5190s^2 + 3.443s + 0.8123}$$

$$\frac{Y_2'(s)}{Y_f'(s)} = \frac{0.4022s + 0.4717}{s^3 + 3.5190s^2 + 3.443s + 0.8123}$$

$$\frac{Y_2'(s)}{Y_f'(s)} = \frac{0.2550}{s^3 + 3.5190s^2 + 3.443s + 0.8123}$$

4.17

Dynamic model:

$$\frac{dX}{dt} = \mu(S)X - DX$$

$$\frac{dS}{dt} = -\mu(S)X / Y_{X/S} - D(S_f - S)$$

Linearization of non-linear terms: (reference point = steady state point)

1. $f_1(S,X) = \mu(S)X = \dfrac{\mu_m S}{K_s + S} X$

$$f_1(S,X) \approx f_1(\bar{S},\bar{X}) + \left.\frac{\partial f_1}{\partial S}\right|_{\bar{S},\bar{X}} (S - \bar{S}) + \left.\frac{\partial f_1}{\partial X}\right|_{\bar{S},\bar{X}} (X - \bar{X})$$

Putting into deviation form,

$$f_1(S',X') \approx \left.\frac{\partial f_1}{\partial S}\right|_{\bar{S},\bar{X}} S' + \left.\frac{\partial f_1}{\partial X}\right|_{\bar{S},\bar{X}} X' = \left(\frac{\mu_m (K_s + \bar{S}) - \mu_m \bar{S}}{(K_s + \bar{S})^2} \bar{X}\right) S' + \left(\frac{\mu_m \bar{S}}{K_s + \bar{S}}\right) X'$$

Substituting the numerical values for $\mu_m, K_s, \bar{S}$ and $\bar{X}$ then:

$$f_1(S', X') \approx 0.113S' + 0.1X'$$

2. $f_2(D,S,S_f) = D(S_f - S)$

$$f_2(D',S',S_f') \approx \left.\frac{\partial f_2}{\partial D}\right|_{\bar{D},\bar{S},\bar{S}_f} D' + \left.\frac{\partial f_2}{\partial S}\right|_{\bar{D},\bar{S},\bar{S}_f} S' + \left.\frac{\partial f_2}{\partial S_f}\right|_{\bar{D},\bar{S},\bar{S}_f} S_f'$$

$$f_2(D',S',S_f') \approx (\bar{S}_f - \bar{S})D' - \bar{D}S' + \bar{D}S_f'$$

$$f_2(D',S',S_f') \approx 9D' - 0.1S' + 0.1S_f'$$

3. $f_3(D,X) = DX$

$$f_3(D',X') \approx D'\bar{X} + X'\bar{D} = 2.25D' + 0.1X'$$

Returning to the dynamic equation: putting them into deviation form by including the linearized terms:

$$\frac{dX'}{dt} = 0.113S' + 0.1X' - 2.25D' - 0.1X'$$

$$\frac{dS'}{dt} = \frac{-0.113}{0.5}S' - \frac{0.1}{0.5}X' - 9D' + 0.1S' - 0.1S_f'$$

Rearranging:

$$\frac{dX'}{dt} = 0.113S' - 2.25D'$$

$$\frac{dS'}{dt} = -0.126S' - 0.2X' - 9D' - 0.1S_f'$$

Laplace Transforming:

$$sX'(s) = 0.113S'(s) - 2.25D'(s)$$

$$sS'(s) = -0.126S'(s) - 0.2X'(s) - 9D'(s) - 0.1S_f'(s)$$

Then,

$$X'(s) = \frac{0.113}{s} S'(s) \; - \; \frac{2.25}{s} D'(s)$$

$$S'(s) = \frac{-0.2}{s+0.126} X'(s) - \frac{9}{s+0.126} D'(s) \; - \frac{0.1}{s+0.126} S'_f(s)$$

or

$$X'(s)\left[1 + \frac{0.0226}{s(s+0.126)}\right] =$$

$$= -\frac{1.017}{s+0.126} D'(s) \; - \frac{0.0113}{s+0.126} S'_f(s) - \frac{2.25}{s} D'(s)$$

Therefore,

$$\frac{X'(s)}{D'(s)} = \frac{-1.3005 - 2.25s}{s^2 + 0.126s + 0.0226}$$