Chapter 5

5.1

a) $x_{DP}(t) = hS(t) - 2hS(t\text{-}t_w) + hS(t\text{-}2t_w)$

$x_{DP}(s) = \dfrac{h}{s}(1 - 2e^{-t_w s} + e^{-2t_w s})$

b) Response of a first-order process,

$$Y(s) = \left(\frac{K}{\tau s + 1}\right)\frac{h}{s}(1 - 2e^{-t_w s} + e^{-2t_w s})$$

or $$Y(s) = (1 - 2e^{-t_w s} + e^{-2t_w s})\left[\frac{\alpha_1}{s} + \frac{\alpha_2}{\tau s + 1}\right]$$

$$\alpha_1 = \left.\frac{Kh}{\tau s + 1}\right|_{s=0} = Kh \qquad \alpha_2 = \left.\frac{Kh}{s}\right|_{s=-\frac{1}{\tau}} = -Kh\tau$$

$$Y(s) = (1 - 2e^{-t_w s} + e^{-2t_w s})\left[\frac{Kh}{s} - \frac{Kh\tau}{\tau s + 1}\right]$$

$$y(t) = \begin{cases} Kh(1 - e^{-t/\tau}) & , \quad 0 < t < t_w \\ Kh(-1 - e^{-t/\tau} + 2e^{-(t-t_w)/\tau}) & , \quad t_w < t < 2t_w \\ Kh(-e^{-t/\tau} + 2e^{-(t-t_w)/\tau} - e^{-(t-2t_w)/\tau}) & , \quad 2t_w < t \end{cases}$$

Response of an integrating element,

$$Y(s) = \frac{K}{s}\frac{h}{s}(1 - 2e^{-t_w s} + e^{-2t_w s})$$

$$y(t) = \begin{cases} Kht & , \quad 0 < t < t_w \\ Kh(-t + 2\,t_w) & , \quad t_w < t < 2t_w \\ 0 & , \quad 2t_w < t \end{cases}$$

c) This input gives a response, for an integrating element, which is zero after a finite time.

5.2

a) For a step change in input of magnitude *M*

$y(t) = KM(1 - e^{-t/\tau}) + y(0)$

We note that $KM = y(\infty) - y(0) = 280 - 80 = 200°C$

Then $K = \dfrac{200^\circ C}{0.5Kw}$ = 400 °C/Kw

At time $t = 4$, $y(4) = 230$ °C

Thus $\dfrac{230-80}{280-80} = 1 - e^{-4/\tau}$ or $\tau = 2.89$ min

$$\therefore \qquad \frac{T'(s)}{P'(s)} = \frac{400}{2.89s+1} \; [°C/Kw]$$

a) For an input ramp change with slope $a = 0.5$ Kw/min

$Ka = 400 \times 0.5 = 200$ °C/min

This maximum rate of change will occur as soon as the transient has died out, i.e., after

5×2.89 min ≈ 15 min have elapsed.

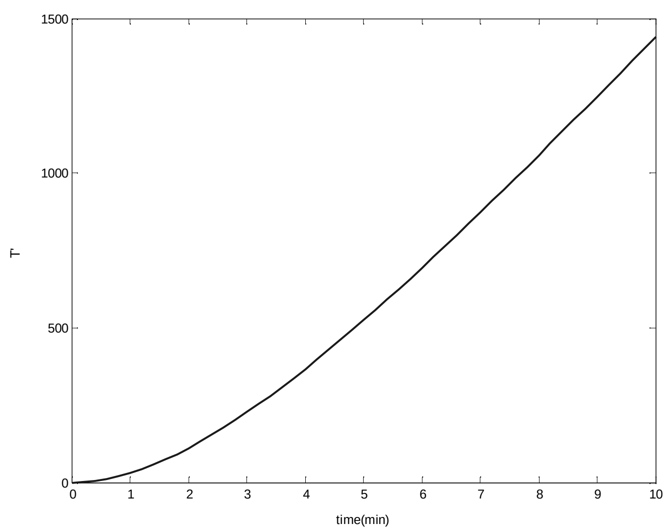


Fig S5.2. *Temperature response for a ramp input of magnitude 0.5 Kw/min.*

5.3

The contaminant concentration c increases according to this expression:

$$c(t) = 5 + 0.2t$$

Using deviation variables and Laplace transforming,

$$c'(t) = 0.2t \qquad \text{or} \qquad C'(s) = \frac{0.2}{s^2}$$

Hence

$$C'_m(s) = \frac{1}{10s+1} \cdot \frac{0.2}{s^2}$$

and applying Eq. 5-21

$$c'_m(t) = 2(e^{-t/10} - 1) + 0.2t$$

As soon as $c'_m(t) \geq 2$ ppm the alarm sounds. Therefore,

$$\Delta t = 18.4 \text{ s} \qquad \text{(starting from the beginning of the ramp input)}$$

The time at which the actual concentration exceeds the limit (t = 10 s) is subtracted from the previous result to obtain the requested Δt .

$$\Delta t = 18.4 - 10.0 = 8.4 \text{ s}$$

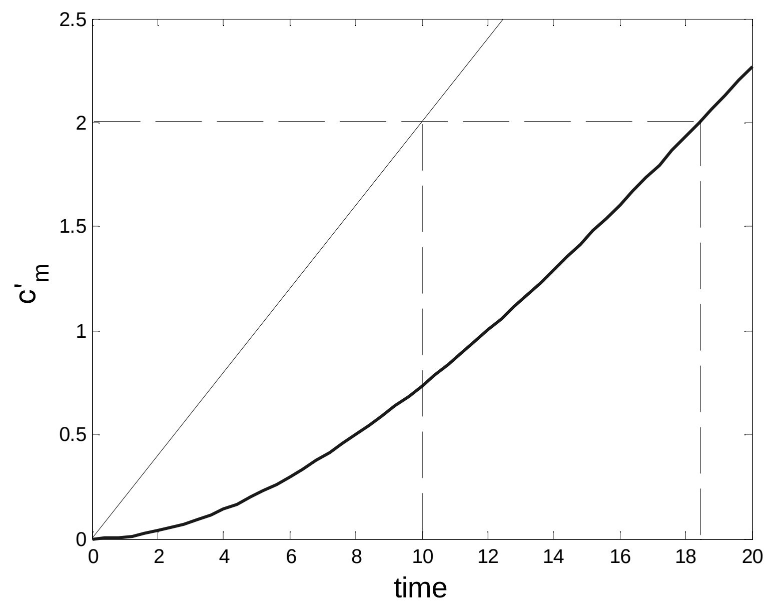


Fig S5.3. *Concentration response for a ramp input of magnitude 0.2 Kw/min.*

5.4

a) Using deviation variables, the rectangular pulse is

$$c'_F = \begin{cases} 0 & t < 0 \\ 2 & 0 \le t < 2 \\ 0 & 2 \le t \le \infty \end{cases}$$

Laplace transforming this input yields

$$c'_F(s) = \frac{2}{s}\left(1 - e^{-2s}\right)$$

The input is then given by

$$c'(s) = \frac{8}{s(2s+1)} - \frac{8e^{-2s}}{s(2s+1)}$$

and from Table 3.1 the time domain function is

$$c'(t) = 8(1 - e^{-t/2}) - 8(1 - e^{-(t-2)/2})\,\text{S}\,(t-2) \qquad (1)$$

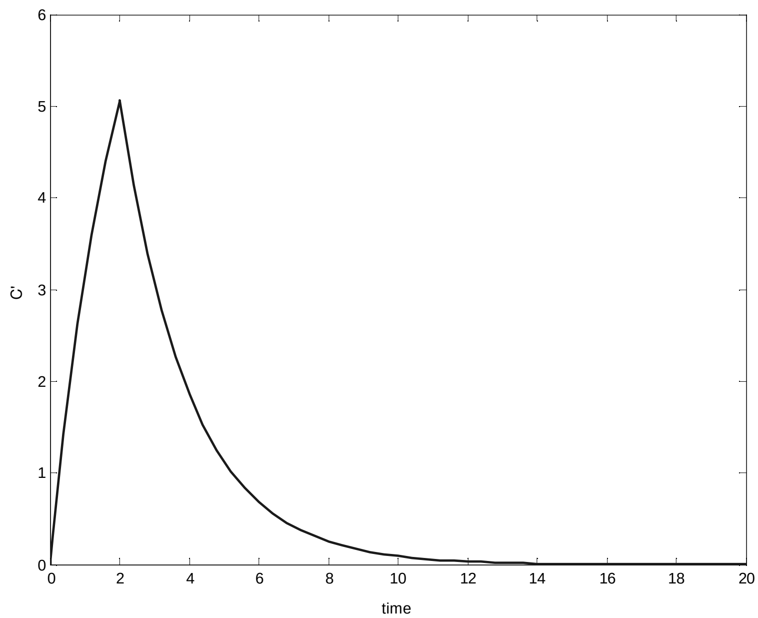


Fig S5.4. *Exit concentration response for a rectangular input.*

b) By inspection of Eq. 1, the time at which this function will reach its maximum value is 2, so maximum value of the output is given by

$$c'(2) = 8(1 - e^{-1}) - 8(1 - e^{-0/2})\,S(0) \qquad (2)$$

and since the second term is zero, $c'(2) = 5.057$

c) By inspection, the steady state value of $c'(t)$ will be zero, since this is a first-order system with no integrating poles and the input returns to zero. To obtain $c'(\infty)$, simplify the function derived in a) for all time greater than 2, yielding

$$c'(t) = 8(e^{-(t-2)/2} - e^{-t/2}) \qquad (3)$$

which will obviously converge to zero.

Substituting $c'(t) = 0.05$ in the previous equation and solving for t gives

$$t = 9.233$$

5.5

a) Energy balance for the thermocouple,

$$mC\frac{dT}{dt} = hA(T_s - T) \qquad (1)$$

where m is mass of thermocouple
C is heat capacity of thermocouple
h is heat transfer coefficient
A is surface area of thermocouple
t is time in sec

Substituting numerical values in (1) and noting that

$$\overline{T}_s = \overline{T} \quad \text{and} \quad \frac{dT}{dt} = \frac{dT'}{dt},$$

$$15\frac{dT'}{dt} = T_s' - T'$$

Taking Laplace transform, $\dfrac{T'(s)}{T_s'(s)} = \dfrac{1}{15s+1}$

b) $T_s(t) = 23 + (80 - 23)\, S(t)$

$\overline{T}_s = \overline{T} = 23$

From $t = 0$ to $t = 20$,

$$T_s'(t) = 57\, S(t) \quad , \quad T_s'(s) = \frac{57}{s}$$

$$T'(s) = \frac{1}{15s+1} T_s'(s) = \frac{57}{s(15s+1)}$$

Applying inverse Laplace Transform,

$$T'(t) = 57(1 - e^{-t/15})$$

Then

$$T(t) = T'(t) + \overline{T} = 23 + 57(1 - e^{-t/15})$$

Since $T(t)$ increases monotonically with time, maximum $T = T(20)$.

Maximum $T(t) = T(20) = 23 + 57\,(1 - e^{-20/15}) = 64.97$ °C

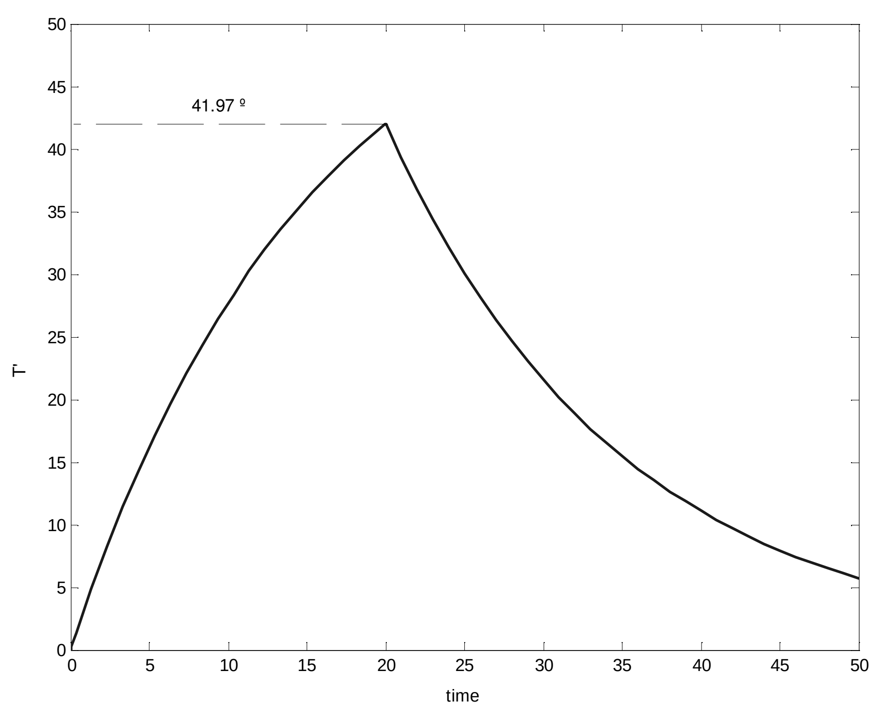


Fig S5.5. *Thermocouple output for part b)*

5.6

a) The overall gain of G is $G\big|_{s=0}$

$$= \frac{K_1}{\tau_1 \times 0 + 1} \cdot \frac{K_2}{\tau_2 \times 0 + 1} = K_1 K_2$$

b) If the equivalent time constant is equal to $\tau_1 + \tau_2 = 5 + 3 = 8$, then

$y(t = 8)/KM = 0.632$ for a first-order process.

$$y(t = 8)/KM = 1 - \frac{5e^{-8/5} - 3e^{-8/3}}{5 - 3} = 0.599 \neq 0.632$$

Therefore, the equivalent time constant is not equal to $\tau_1 + \tau_2$

c) The roots of the denominator of G are

$-1/\tau_1$ and $-1/\tau_2$

which are negative real numbers. Therefore the process transfer function G cannot exhibit oscillations when the input is a step function.

5.7

Assume that at steady state the temperature indicated by the sensor T_m is equal to the actual temperature at the measurement point T. Then,

$$\frac{T'_m(s)}{T'(s)} = \frac{K}{\tau s + 1} = \frac{1}{1.5s + 1}$$

$$\overline{T}_m = \overline{T} = 350^\circ C$$

$$T'_m(t) = 15 \sin \omega t$$

where $\omega = 2\pi \times 0.1$ rad/min = 0.628 rad/min

At large times when $t/\tau >> 1$, Eq. 5-26 shows that the amplitude of the sensor signal is

$$A_m = \frac{A}{\sqrt{\omega^2\tau^2 + 1}}$$

where A is the amplitude of the actual temperature at the measurement point.

Therefore $\quad A = 15\sqrt{(0.628)^2(1.5)^2 + 1} = 20.6°\text{C}$

Maximum $T = \overline{T} + A$ =350 + 20.6 = 370.6

Maximum T_{center} = 3 (max T) – 2 T_{wall}

= (3×370.6)–(2×200) = 711.8°C

Therefore, the catalyst will not sinter instantaneously, but will sinter if operated for several hours.

5.8

a) Assume that q is constant. Material balance over the tank,

$$A\frac{dh}{dt} = q_1 + q_2 - q$$

Writing in deviation variables and taking Laplace transform

$$AsH'(s) = Q_1'(s) + Q_2'(s)$$

$$\frac{H'(s)}{Q_1'(s)} = \frac{1}{As}$$

b) $q_1'(t) = 5\, S(t) - 5S(t\text{-}12)$

$$Q_1'(s) = \frac{5}{s} - \frac{5}{s}e^{-12s}$$

$$H'(s) = \frac{1}{As}Q_1'(s) = \frac{5/A}{s^2} - \frac{5/A}{s^2}e^{-12s}$$

$$h'(t) = \frac{5}{A} t\, S(t) - \frac{5}{A}(t-12)\, S(t\text{-}12)$$

$$h(t) = \begin{cases} 4 + \dfrac{5}{A}t = 4 + 0.177t & 0 \le t \le 12 \\[2ex] 4 + \left(\dfrac{5}{A} \times 12\right) = 6.122 & 12 < t \end{cases}$$

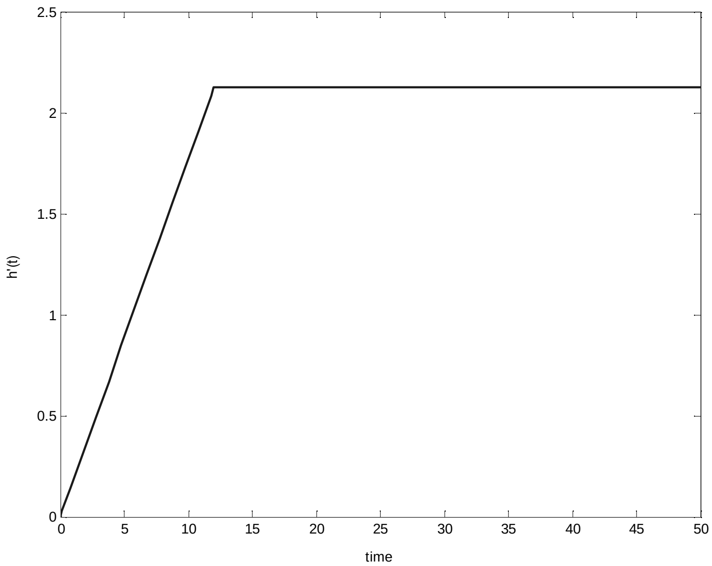


Fig S5.8a. *Liquid level response for part b)*

c) $\bar{h} = 6.122$ ft at the new steady state $t \ge 12$

d) $q_1'(t) = 5\,S(t) - 10S(t\text{-}12) + 5S(t\text{-}24)$; $t_w = 12$

$$Q_1'(s) = \frac{5}{s}\left(1 - 2e^{-12s} + e^{-24s}\right)$$

$$H'(s) = \frac{5/A}{s^2} - \frac{10/A}{s^2}e^{-12s} + \frac{5/A}{s^2}e^{-24s}$$

$h(t) = 4 + 0.177tS(t) - 0.354(t\text{-}12)S(t\text{-}12) + 0.177(t\text{-}24)S(t\text{-}24)$

For $t \ge 24$

$$\bar{h} = 4 + 0.177t - 0.354(t-12) + 0.177(t-24) = 4 \text{ ft } at\ t \ge 24$$

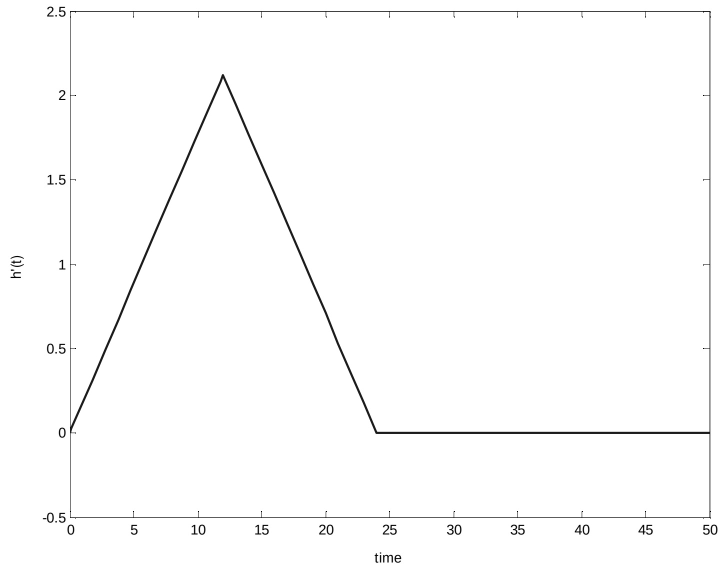


Fig S5.8b. *Liquid level response for part d)*

5.9

a) Material balance over tank 1.

$$A\frac{dh}{dt} = C(q_i - 8.33h)$$

where $A = \pi \times (4)^2/4 = 12.6 \text{ ft}^2$

$$C = 0.1337 \quad \frac{\text{ft}^3/\text{min}}{\text{USGPM}}$$

$$AsH'(s) = CQ_i'(s) - (C \times 8.33)H'(s)$$

$$\frac{H'(s)}{Q_i'(s)} = \frac{0.12}{11.28s + 1}$$

For tank 2,

$$A\frac{dh}{dt} = C(q_i - q)$$

$$AsH'(s) = CQ_i'(s) \quad , \quad \frac{H'(s)}{Q_i'(s)} = \frac{0.011}{s}$$

b) $Q_i'(s) = 20/s$

For tank 1, $H'(s) = \dfrac{2.4}{s(11.28s+1)} = \dfrac{2.4}{s} - \dfrac{27.1}{11.28s+1}$

$h(t) = 6 + 2.4(1 - e^{-t/11.28})$

For tank 2, $H'(s) = 0.22/s^2$

$h(t) = 6 + 0.22t$

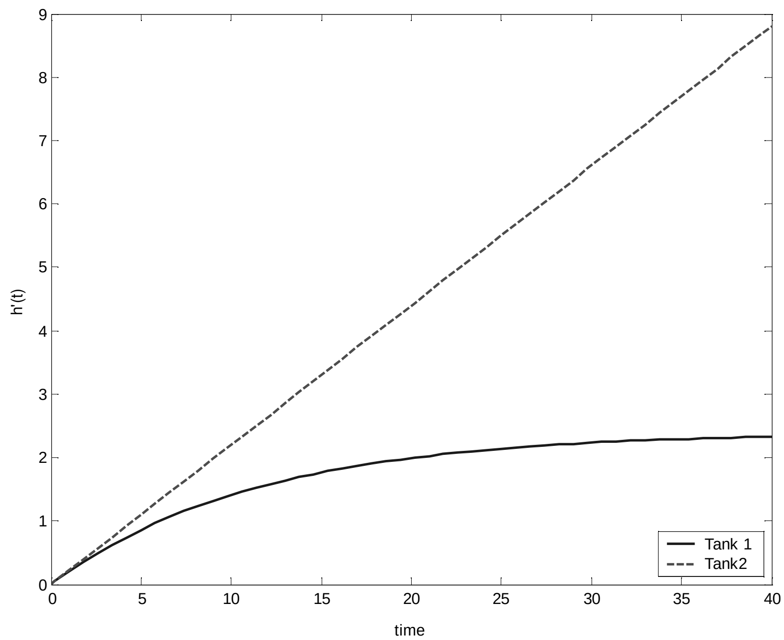


Fig S5.9. *Transient response in tanks 1 and 2 for a step input.*

c) For tank 1, $h(\infty) = 6 + 2.4 - 0 = 8.4$ ft

For tank 2, $h(\infty) = 6 + (0.22 \times \infty) = \infty$ ft

d) For tank 1, $8 = 6 + 2.4(1 - e^{-t/11.28})$
$h = 8$ ft at $t = 20.1$ min
For tank 2, $8 = 6 + 0.22t$
$h = 8$ ft at $t = 9.4$ min

Tank 2 overflows first, at 9.4 min.

5.10

a) The dynamic behavior of the liquid level is given by

$$\frac{d^2h'}{dt} + A\frac{dh'}{dt} + Bh' = C\,p'(t)$$

where

$$A = \frac{6\mu}{R^2\rho} \qquad B = \frac{3g}{2L} \qquad and \;\; C = \frac{3}{4\rho L}$$

Taking the Laplace Transform and assuming initial values = 0

$$s^2H'(s) + AsH'(s) + BH'(s) = C\,P'(s)$$

or $$H'(s) = \frac{C/B}{\frac{1}{B}s^2 + \frac{A}{B}s + 1}P'(s)$$

We want the previous equation to have the form

$$H'(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}P'(s)$$

Hence $K = C/B = \dfrac{1}{2\rho g}$

$$\tau^2 = \frac{1}{B} \qquad \text{then} \quad \tau = \sqrt{1/B} = \left(\frac{2L}{3g}\right)^{1/2}$$

$$2\zeta\tau = \frac{A}{B} \qquad \text{then} \quad \zeta = \frac{3\mu}{R^2\rho}\left(\frac{2L}{3g}\right)^{1/2}$$

b) The manometer response oscillates as long as $0 < \zeta < 1$ or

$$0 < \frac{3\mu}{R^2\rho}\left(\frac{2L}{3g}\right)^{1/2} < 1$$

b) If ρ is larger , then ζ is smaller and the response would be more oscillatory.

If μ is larger, then ζ is larger and the response would be less oscillatory.

5.11

$$Y(s) = \frac{KM}{s^2(\tau s+1)} = \frac{K_1}{s^2} + \frac{K_2}{s(\tau s+1)}$$

$$K_1\tau s + K_1 + K_2 s = KM$$

$$K_1 = KM$$
$$K_2 = -K_1\tau = -KM\tau$$

Hence

$$Y(s) = \frac{KM}{s^2} - \frac{KM\tau}{s(\tau s+1)}$$

or

$$y(t) = KMt - KM\tau\,(1-e^{-t/\tau})$$

After a long enough time, we can simplify to

$$y(t) \approx KMt - KM\tau \qquad \text{(linear)}$$

slope = KM
intercept = $-KM\tau$

That way we can get K and τ

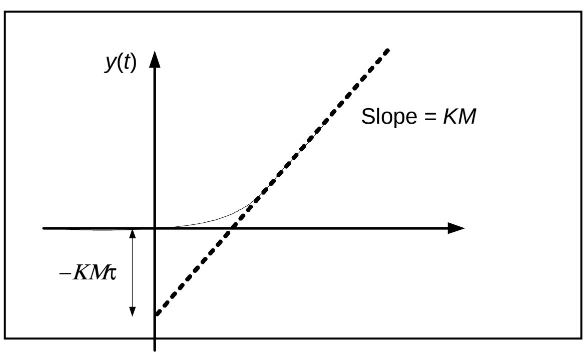


Figure S5.11. *Time domain response and parameter evaluation*

5.12

a) $\ddot{y} + K\dot{y} + 4y = x$

Assuming y(0) = $\dot{y}(0) = 0$

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2 + Ks + 4} = \frac{0.25}{0.25s^2 + 0.25Ks + 1}$$

b) Characteristic equation is

$$s^2 + Ks + 4 = 0$$

The roots are $s = \dfrac{-K \pm \sqrt{K^2 - 16}}{2}$

$-10 \le K < -4$ Roots : positive real, distinct
Response : $A + Be^{t/\tau_1} + C\ e^{t/\tau_2}$

$K = -4$ Roots : positive real, repeated
Response : $A + Be^{t/\tau} + C\ e^{t/\tau}$

$-4 < K < 0$ Roots: complex with positive real part.
Response: $A + e^{\zeta t/\tau}(B \cos\sqrt{1-\zeta^2}\ \frac{t}{\tau} + C \sin\sqrt{1-\zeta^2}\ \frac{t}{\tau})$

$K = 0$ Roots: imaginary, zero real part.
Response: $A + B \cos t/\tau + C \sin t/\tau$

$0 < K < 4$ Roots: complex with negative real part.
Response: $A + e^{-\zeta t/\tau}(B \cos\sqrt{1-\zeta^2}\ \frac{t}{\tau} + C \sin\sqrt{1-\zeta^2}\ \frac{t}{\tau})$

$K = 4$ Roots: negative real, repeated.
Response: $A + Be^{-t/\tau} + C\ t\ e^{-t/\tau}$

$4 < K \le 10$ Roots: negative real, distinct
Response: $A + Be^{-t/\tau_1} + C\ e^{-t/\tau_2}$

Response will converge in region $0 < K \le 10$, and will not converge in region $-10 \le K \le 0$

5.13

a) The solution of a critically-damped second-order process to a step change of magnitude *M* is given by Eq. 5-50 in text.

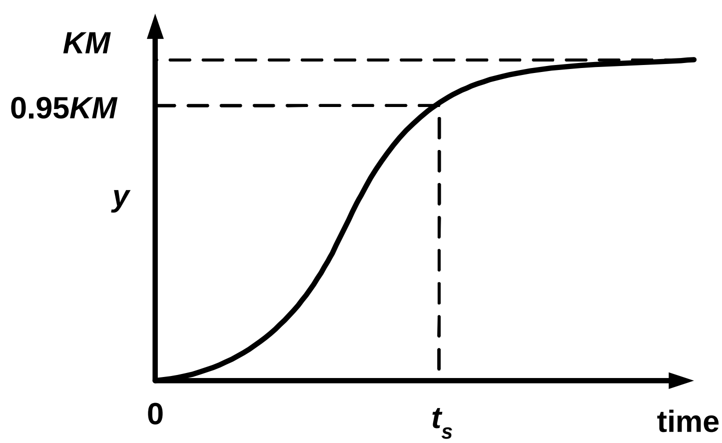
$$y(t) = KM\left[1-\left(1+\frac{t}{\tau}\right)e^{-t/\tau}\right]$$

Rearranging

$$\frac{y}{KM} = 1-\left(1+\frac{t}{\tau}\right)e^{-t/\tau}$$

$$\left(1+\frac{t}{\tau}\right)e^{-t/\tau} = 1-\frac{y}{KM}$$

When *y/KM* = 0.95, the response is 0.05 *KM* below the steady-state value.



$$\left(1+\frac{t_s}{\tau}\right)e^{-t/\tau} = 1-0.95 = 0.05$$

$$\ln\left(1+\frac{t_s}{\tau}\right)-\frac{t_s}{\tau} = \ln(0.05) = -3.00$$

$$\text{Let} \quad E = \ln\left(1+\frac{t_s}{\tau}\right)-\frac{t_s}{\tau}+3$$

and find value of $\frac{t_s}{\tau}$ that makes $E \approx 0$ by trial-and-error.

t_s/τ	E
4	0.6094
5	-0.2082
4.5	0.2047
4.75	-0.0008

$\therefore$ a value of $t = 4.75\tau$ is t_s, the settling time.

b) $$Y(s) = \frac{Ka}{s^2(\tau s+1)^2} = \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{\tau s+1} + \frac{a_4}{(\tau s+1)^2}$$

We know that the a_3 and a_4 terms are exponentials that go to zero for large values of time, leaving a linear response.

$$a_2 = \lim_{s\to 0} \frac{Ka}{(\tau s+1)^2} = Ka$$

Define $Q(s) = \frac{Ka}{(\tau s+1)^2}$

$$\frac{dQ}{ds} = \frac{-2Ka\tau}{(\tau s+1)^3}$$

Then $a_1 = \frac{1}{1!} \lim_{s\to 0}\left[\frac{-2Ka\tau}{(\tau s+1)^3}\right]$

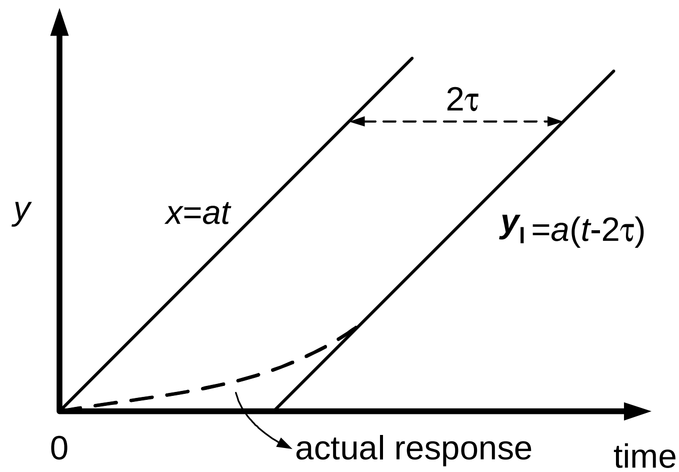
(from Eq. 3-62)

$$a_1 = -2\,Ka\tau$$

$\therefore$ the long-time response (after transients have died out) is

$$\begin{aligned} y_\ell(t) &= Kat - 2Ka\tau = Ka(t-2\tau) \\ &= a(t-2\tau) \qquad \text{for } K = 1 \end{aligned}$$

and we see that the output lags the input by a time equal to 2τ.



5.14

a) Gain = $\dfrac{11.2\text{mm} - 8\text{mm}}{31\text{psi} - 15\text{psi}} = 0.20\text{mm/psi}$

Overshoot = $\dfrac{12.7\text{mm} - 11.2\text{mm}}{11.2\text{mm} - 8\text{mm}} = 0.47$

Overshoot = $\exp\left(\dfrac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 0.47$, $\zeta = 0.234$

Period = $\left(\dfrac{2\pi\tau}{\sqrt{1-\zeta^2}}\right) = 2.3 \text{ sec}$

$$\tau = 2.3 \text{ sec} \times \frac{\sqrt{1-0.234^2}}{2\pi} = 0.356 \text{ sec}$$

$$\frac{R'(s)}{P'(s)} = \frac{0.2}{0.127s^2 + 0.167s + 1} \qquad (1)$$

b) From Eq. 1, taking the inverse Laplace transform,

$$0.127\,\ddot{R}' + 0.167\,\dot{R}' + R' = 0.2\,P'$$

$$\ddot{R}' = \ddot{R} \qquad \dot{R}' = \dot{R} \qquad R' = R\text{-}8 \qquad P' = P\text{-}15$$

$$0.127\,\ddot{R} + 0.167\,\dot{R} + R = 0.2\,P + 5$$

$$\ddot{R} + 1.31\,\dot{R} + 7.88\,R = 1.57\,P + 39.5$$

5.15

$$\frac{P'(s)}{T'(s)} = \frac{3}{(3)^2 s^2 + 2(0.7)(3)s + 1} \qquad [°C/kW]$$

Note that the input change $p'(t) = 26 - 20 = 6$ kw

Since K is 3 °C/kW, the output change in going to the new steady state will be

$$\underset{t \to \infty}{T'} = (3^\circ C/\text{kW})(6\text{ kW}) = 18^\circ \text{C}$$

a) Therefore the expression for $T(t)$ is Eq. 5-51

$$T(t) = 70^\circ + 18^\circ \left\{ 1 - e^{-\frac{0.7t}{3}} \left(\cos\left(\frac{\sqrt{1-(0.7)^2}}{3} t \right) + \frac{0.7}{\sqrt{1-(0.7)^2}} \sin\left(\frac{\sqrt{1-(0.7)^2}}{\tau} t \right) \right) \right\}$$

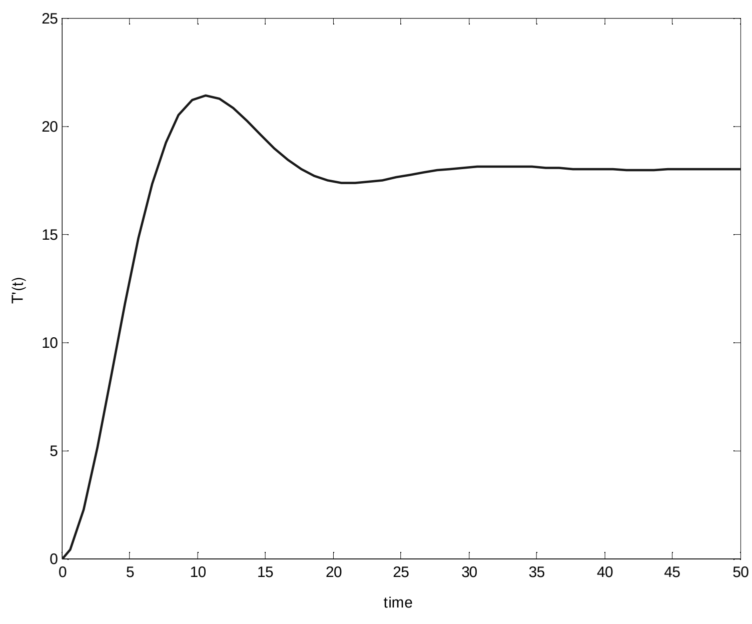


Fig S5.15. *Process temperature response for a step input*

b) The overshoot can obtained from Eq. 5-52 or Fig. 5.11. From Figure 5.11 we see that OS ≈ 0.05 for ζ=0.7. This means that maximum temperature is

$$T_{max} \approx 70° + (18)(1.05) = 70 + 18.9 = 88.9°$$

From Fig S5.15 we obtain a more accurate value.

The time at which this maximum occurs can be calculated by taking derivative of Eq. 5-51 or by inspection of Fig. 5.8. From the figure we see that $t / \tau = 3.8$ at the point where an (interpolated) ζ=0.7 line would be.

$\therefore \quad t_{max} \approx 3.8$ (3 min) = 11.4 minutes

5.16

For underdamped responses,

$$y(t) = KM\left\{1 - e^{-\zeta t/\tau}\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right\} \qquad (5\text{-}51)$$

a) At the response peaks,

$$\frac{dy}{dt} = KM\left\{\frac{\zeta}{\tau}e^{-\zeta t/\tau}\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right.$$

$$\left. - e^{-\zeta t/\tau}\left[-\frac{\sqrt{1-\zeta^2}}{\tau}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \frac{\zeta}{\tau}\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right\} = 0$$

Since $KM \neq 0$ and $e^{-\zeta t/\tau} \neq 0$

$$0 = \left(\frac{\zeta}{\tau} - \frac{\zeta}{\tau}\right)\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \left(\frac{\zeta^2}{\tau\sqrt{1-\zeta^2}} + \frac{\sqrt{1-\zeta^2}}{\tau}\right)\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)$$

$$0 = \sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) = \sin n\pi \quad , \quad t = n\frac{\pi\tau}{\sqrt{1-\zeta^2}}$$

where n is the number of the peak.

Time to the first peak, $t_p = \dfrac{\pi\tau}{\sqrt{1-\zeta^2}}$

b) Overshoot, OS = $\dfrac{y(t_p) - KM}{KM}$

$$\text{OS} = -\exp\left(\frac{-\zeta t}{\tau}\right)\left[\cos(\pi) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\pi)\right]$$

$$= \exp\left[\frac{-\zeta\tau\pi}{\tau\sqrt{1-\zeta^2}}\right] = \exp\left[\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right]$$

c) Decay ratio, DR = $\dfrac{y(t_{3p}) - KM}{y(t_p) - KM}$

where $y(t_{3p}) = \dfrac{3\pi\tau}{\sqrt{1-\zeta^2}}$ is the time to the third peak.

$$\text{DR} = \frac{KM\ e^{-\zeta t_{3p}/\tau}}{KM\ e^{-\zeta t_p/\tau}} = \exp\left[-\frac{\zeta}{\tau}(t_{3p} - t_p)\right] = \exp\left[-\frac{\zeta}{\tau}\left(\frac{2\pi\tau}{\sqrt{1-\zeta^2}}\right)\right]$$

$$= \exp\left[\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right] = (\text{OS})^2$$

d) Consider the trigonometric identity

sin (A+B) = sin A cos B + cos A sin B

Let B = $\left(\dfrac{\sqrt{1-\zeta^2}}{\tau}t\right)$, sin A = $\sqrt{1-\zeta^2}$, cos A = ζ

$$y(t) = KM\left\{1 - e^{-\zeta t/\tau}\frac{1}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos B + \zeta\sin B\right]\right\}$$

$$= KM\left\{1 - \frac{e^{-\zeta t/\tau}}{\sqrt{1-\zeta^2}}\sin(A+B)\right\}$$

Hence for $t \geq t_s$, the settling time,

$$\left|\frac{e^{-\zeta t/\tau}}{\sqrt{1-\zeta^2}}\right| \leq 0.05 \quad \text{, or} \quad t \geq -\ln\left(0.05\sqrt{1-\zeta^2}\right)\frac{\tau}{\zeta}$$

Therefore, $t_s \geq \dfrac{\tau}{\zeta}\ln\left(\dfrac{20}{\sqrt{1-\zeta^2}}\right)$

5.17

a) Assume underdamped second-order model (exhibits overshoot)

$$K = \frac{\Delta \text{output}}{\Delta \text{input}} = \frac{10-6 \text{ ft}}{140-120 \text{ gal/min}} = 0.2 \frac{\text{ft}}{\text{gal/min}}$$

$$\text{Fraction overshoot} = \frac{11-10}{10-6} = \frac{1}{4} = 0.25$$

From Fig 5.11, this corresponds (approx) to $\zeta = 0.4$

From Fig. 5.8 , $\zeta = 0.4$, we note that $t_p/\tau \approx 3.5$

Since t_p = 4 minutes (from problem statement), $\tau = 1.14$ min

$$\therefore \qquad G_p(s) = \frac{0.2}{(1.14)^2 s^2 + 2(0.4)(1.14)s + 1} = \frac{0.2}{1.31s^2 + 0.91s + 1}$$

b) In Chapter 6 we see that a 2nd-order overdamped process model with a numerator term can exhibit overshoot. But if the process is underdamped, it is unique.

5.18

a) Assuming constant volume and density,

Overall material balances yield: $q_2 = q_1 = q$ (1)

Component material balances:

$$V_1 \frac{dc_1}{dt} = q(c_i - c_1) \qquad (2)$$

$$V_2 \frac{dc_2}{dt} = q(c_1 - c_2) \qquad (3)$$

b) Degrees of freedom analysis

3 Parameters : V_1, V_2, q

3 Variables : c_i, c_1, c_2

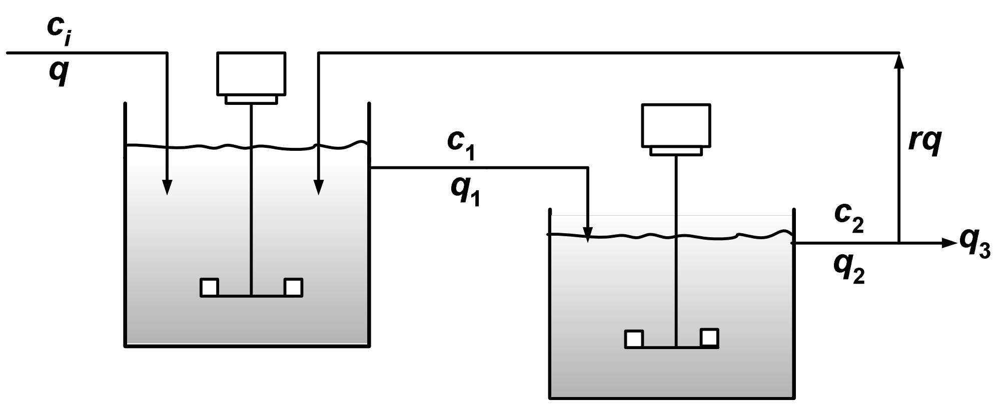
2 Equations: (2) and (3)

$N_F = N_V - N_E = 3 - 2 = 1$

Hence one input must be a specified function of time.

2 Outputs = c_1, c_2
1 Input = c_i

c) If a recycle stream is used



Overall material balances:

$$q_1 = (1+r)q \tag{4}$$

$$q_2 = q_1 = (1+r)q \tag{5}$$

$$q_3 = q_2 - rq = (1+r)q - rq = q \tag{6}$$

Component material balances:

$$V_1 \frac{dc_1}{dt} = qc_i + rqc_2 - (1+r)qc_1 \tag{7}$$

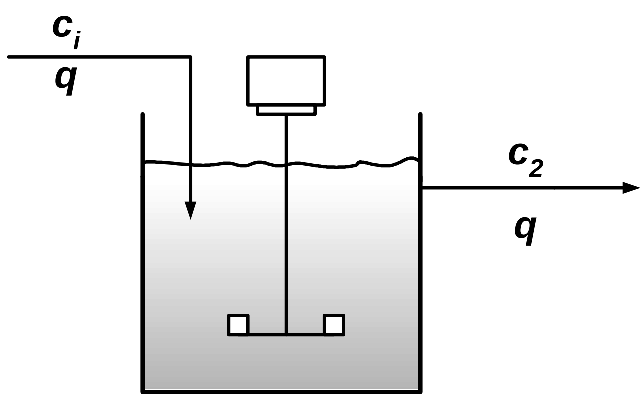
$$V_2 \frac{dc_2}{dt} = (1+r)qc_1 - (1+r)qc_2 \tag{8}$$

Degrees of freedom analysis is the same except now we have

4 parameters : V_1, V_2, q, r

d) If $r \to \infty$, there will a large amount of mixing between the two tanks as a result of the very high internal circulation.

Thus the process acts like



Total Volume = $V_1 + V_2$

Model :

$$(V_1 + V_2)\frac{dc_2}{dt} = q(c_i - c_2) \tag{9}$$

$$c_1 = c_2 \text{ (complete internal mixing)} \tag{10}$$

Degrees of freedom analysis is same as part b)

5.19

a) For the original system,

$$A_1\frac{dh_1}{dt} = Cq_i - \frac{h_1}{R_1}$$

$$A_2\frac{dh_2}{dt} = \frac{h_1}{R_1} - \frac{h_2}{R_2}$$

where $A_1 = A_2 = \pi(3)^2/4 = 7.07 \text{ ft}^2$

$$C = 0.1337\ \frac{\text{ft}^3/\text{min}}{\text{gpm}}$$

$$R_1 = R_2 = \frac{\bar{h}_1}{C\bar{q}_i} = \frac{2.5}{0.1337 \times 100} = 0.187\frac{\text{ft}}{\text{ft}^3/\text{min}}$$

Using deviation variables and taking Laplace transforms,

$$\frac{H_1'(s)}{Q_i'(s)} = \frac{C}{A_1 s + \frac{1}{R_1}} = \frac{CR_1}{A_1 R_1 s + 1} = \frac{0.025}{1.32s + 1}$$

$$\frac{H_2'(s)}{H_1'(s)} = \frac{1/R_1}{A_2 s + \frac{1}{R_2}} = \frac{R_2 / R_1}{A_2 R_2 s + 1} = \frac{1}{1.32s + 1}$$

$$\frac{H_2'(s)}{Q_i'(s)} = \frac{0.025}{(1.32s + 1)^2}$$

For step change in q_i of magnitude M,

$$h'_{1\max} = 0.025M$$

$$h'_{2\max} = 0.025M$$ since the second-order transfer function

$$\frac{0.025}{(1.32s + 1)^2}$$ is critically damped (ζ=1), not underdamped

Hence $M_{max} = \dfrac{2.5 \text{ ft}}{0.025 \text{ ft/gpm}} = 100$ gpm

For the modified system,

$$A\frac{dh}{dt} = Cq_i - \frac{h}{R}$$

$$A = \pi(4)^2 / 4 = 12.6 \text{ ft}^2$$

$V = V_1 + V_2 = 2 \times 7.07\text{ft}^2 \times 5\text{ft} = 70.7\text{ft}^3$

$h_{max} = V/A = 5.62$ ft

$$R = \frac{\bar{h}}{C\bar{q}_i} = \frac{0.5 \times 5.62}{0.1337 \times 100} = 0.21 \frac{\text{ft}}{\text{ft}^3\text{/min}}$$

$$\frac{H'(s)}{Q_i'(s)} = \frac{C}{As + \frac{1}{R}} = \frac{CR}{ARs + 1} = \frac{0.0281}{2.64s + 1}$$

$$h'_{\max} = 0.0281M$$

$$M_{max} = \frac{2.81 \text{ ft}}{0.0281 \text{ ft/gpm}} = 100 \text{ gpm}$$

Hence, both systems can handle the same maximum step disturbance in q_i.

b) For step change of magnitude M, $Q_i'(s) = \dfrac{M}{s}$

For original system,

$$Q_2'(s) = \frac{1}{R_2} H_2'(s) = \frac{1}{0.187} \frac{0.025}{(1.32s+1)^2} \frac{M}{s}$$

$$= 0.134M \left[\frac{1}{s} - \frac{1.32}{(1.32s+1)} - \frac{1.32}{(1.32s+1)^2} \right]$$

$$q_2'(t) = 0.134M \left[1 - \left(1 + \frac{t}{1.32} \right) e^{-t/1.32} \right]$$

For modified system,

$$Q'(s) = \frac{1}{R} H'(s) = \frac{1}{0.21} \frac{0.0281}{(2.64s+1)} \frac{M}{s} = 0.134M \left[\frac{1}{s} - \frac{2.64}{2.64s+1} \right]$$

$$q'(t) = 0.134M \left[1 - e^{-t/2.64} \right]$$

Original system provides better damping since $q_2'(t) < q'(t)$ for $t < 3.4$.

5.20

a) Caustic balance for the tank,

$$\rho V \frac{dC}{dt} = w_1 c_1 + w_2 c_2 - wc$$

Since V is constant, $w = w_1 + w_2 = 10$ lb/min

For constant flows,

$$\rho V s C'(s) = w_1 C_1'(s) + w_2 C_2'(s) - w C'(s)$$

$$\frac{C'(s)}{C_1'(s)} = \frac{w_1}{\rho V s + w} = \frac{5}{(70)(7)s + 10} = \frac{0.5}{49s+1}$$

$$\frac{C'_m(s)}{C'(s)} = \frac{K}{\tau s + 1} \quad , \quad K = (3\text{-}0)/3 = 1 \quad , \quad \tau \approx 6 \text{ sec} = 0.1 \text{ min}$$

(from the graph)

$$\frac{C'_m(s)}{C'_1(s)} = \frac{1}{(0.1s+1)} \frac{0.5}{(49s+1)} = \frac{0.5}{(0.1s+1)(49s+1)}$$

b) $$C'_1(s) = \frac{3}{s}$$

$$C'_m(s) = \frac{1.5}{s(0.1s+1)(49s+1)}$$

$$c'_m(t) = 1.5\left[1 + \frac{1}{(49-0.1)}(0.1e^{-t/0.1} - 49e^{-t/49})\right]$$

c) $$C'_m(s) = \frac{0.5}{(49s+1)} \frac{3}{s} = \frac{1.5}{s(49s+1)}$$

$$c'_m(t) = 1.5\left(1 - e^{-t/49}\right)$$

d) The responses in b) and c) are nearly the same. Hence the dynamics of the conductivity cell are negligible.

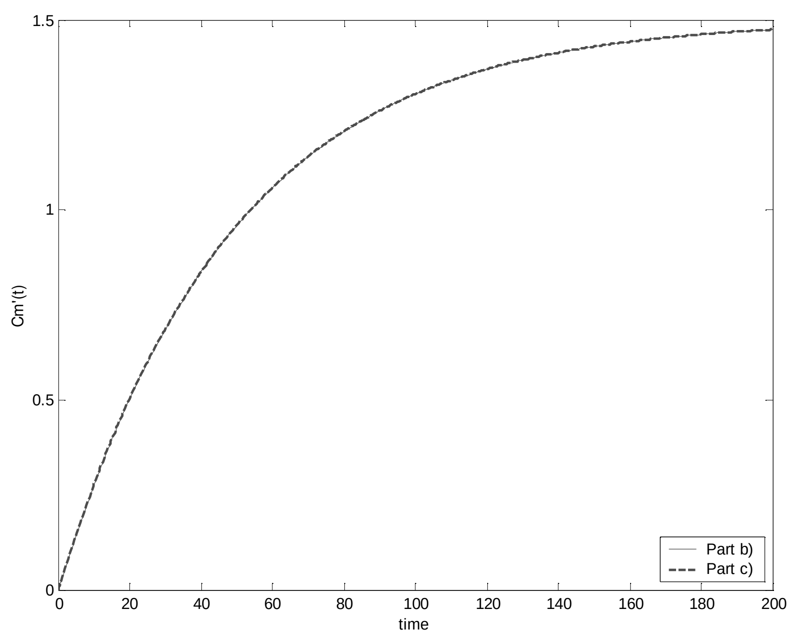


Fig S5.20. *Step responses for parts b) and c)*

5.21

Assumptions: 1) Perfectly mixed reactor
2) Constant fluid properties and heat of reaction

a) Component balance for A,

$$V\frac{dc_A}{dt} = q(c_{Ai} - c_A) - Vk(T)c_A \qquad (1)$$

Energy balance for the tank,

$$\rho VC\frac{dT}{dt} = \rho qC(T_i - T) + (-\Delta H_R)Vk(T)c_A \qquad (2)$$

Since a transfer function with respect to c_{Ai} is desired, assume the other inputs, namely q and T_i, are constant. Linearize (1) and (2) and note that

$$\frac{dc_A}{dt} = \frac{dc'_A}{dt} \quad , \quad \frac{dT}{dt} = \frac{dT'}{dt},$$

$$V\frac{dc'_A}{dt} = qc'_{Ai} - (q + Vk(\bar{T}))c'_A - V\bar{c}_A k(\bar{T})\frac{20000}{\bar{T}^2}T' \qquad (3)$$

$$\rho VC\frac{dT'}{dt} = -\left(\rho qC + \Delta H_R V\bar{c}_A k(\bar{T})\frac{20000}{\bar{T}^2}\right)T' - \Delta H_R Vk(\bar{T})c'_A \qquad (4)$$

Taking Laplace transforms and rearranging

$$\left[Vs + q + Vk(\bar{T})\right]C'_A(s) = qC'_{Ai}(s) - V\bar{c}_A k(\bar{T})\frac{20000}{\bar{T}^2}T'(s) \qquad (5)$$

$$\left[\rho VCs + \rho qC - (-\Delta H_R)V\bar{c}_A k(\bar{T})\frac{20000}{\bar{T}^2}\right]T'(s) = (-\Delta H_R)Vk(\bar{T})C'_A(s) \qquad (6)$$

Substituting $C'_A(s)$ from Eq. 5 into Eq. 6 and rearranging,

$$\frac{T'(s)}{C'_A(s)} = \frac{(-\Delta H_R)Vk(\bar{T})q}{\left[Vs + q + Vk(\bar{T})\right]\left[\rho VCs + \rho qC - (-\Delta H_R)V\bar{c}_A k(\bar{T})\frac{20000}{\bar{T}^2}\right] + (-\Delta H_R)V^2\bar{c}_A k^2(\bar{T})\frac{20000}{\bar{T}^2}} \qquad (7)$$

$\bar{c}_A$ is obtained from Eq. 1 at steady state,

$$\bar{c}_A = \frac{q\bar{c}_{Ai}}{q + Vk(\bar{T})} = 0.001155 \text{ lb mol/cu.ft.}$$

Substituting the numerical values of $\bar{T}$, ρ, C, $-\Delta H_R$, q, V, $\bar{c}_A$ into Eq. 7 and simplifying,

$$\frac{T'(s)}{C'_{Ai}(s)} = \frac{11.38}{(0.0722s+1)(50s+1)}$$

For step response, $C'_{Ai}(s) = 1/s$

$$T'(s) = \frac{11.38}{s(0.0722s+1)(50s+1)}$$

$$T'(t) = 11.38\left[1 + \frac{1}{(50-0.0722)}(0.0722e^{-t/0.0722} - 50e^{-t/50})\right]$$

A first-order approximation of the transfer function is

$$\frac{T'(s)}{C'_{Ai}(s)} = \frac{11.38}{50s+1}$$

For step response, $T'(s) = \dfrac{11.38}{s(50s+1)}$ or $T'(t) = 11.38\left[1 - e^{-t/50}\right]$

The two step responses are very close to each other hence the approximation is valid.

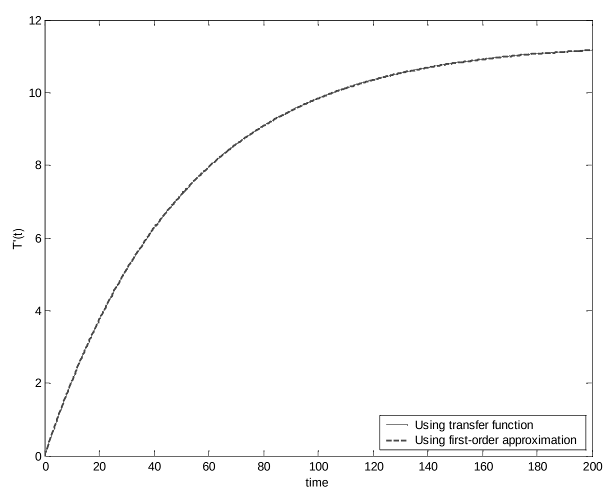


Fig S5.21. *Step responses for the 2nd order t.f and 1st order approx.*

5.22

$(\tau_a s+1)Y_1(s) = K_1U_1(s) + K_b\, Y_2(s)$ (1)
$(\tau_b s+1)Y_2(s) = K_2U_2(s) + Y_1(s)$ (2)

a) Since the only transfer functions requested involve $U_1(s)$, we can let $U_2(s)$ be zero. Then, substituting for $Y_1(s)$ from (2)

$Y_1(s) = (\tau_b s+1)Y_2(s)$ (3)

$(\tau_a s+1)(\tau_b s+1)Y_2(s) = K_1U_1(s) + K_bY_2(s)$ (4)

Rearranging (4)

$[(\tau_a s+1)(\tau_b s+1) - K_b]Y_2(s) = K_1U_1(s)$

$$\therefore \quad \frac{Y_2(s)}{U_1(s)} = \frac{K_1}{(\tau_a s+1)(\tau_b s+1)-K_b} \tag{5}$$

Also, since

$$\frac{Y_1(s)}{Y_2(s)} = \tau_b s+1 \tag{6}$$

From (5) and (6)

$$\frac{Y_1(s)}{U_1(s)} = \frac{Y_2(s)}{U_1(s)}\times\frac{Y_1(s)}{Y_2(s)} = \frac{K_1(\tau_b s+1)}{(\tau_a s+1)(\tau_b s+1)-K_b} \tag{7}$$

b) The gain is the change in y_1(or y_2) for a unit step change in u_1. Using the FVT with $U_1(s)$ = 1/s.

$$y_2(t\to\infty) = \lim_{s\to0}\left[s\frac{K_1}{(\tau_a s+1)(\tau_b s+1)-K_b}\frac{1}{s}\right] = \frac{K_1}{1-K_b}$$

This is the gain of TF $Y_2(s)/U_1(s)$.

Alternatively,

$$K = \lim_{s\to0}\left[\frac{Y_2(s)}{U_1(s)}\right] = \lim_{s\to0}\left[\frac{K_1}{(\tau_a s+1)(\tau_b s+1)-K_b}\right] = \frac{K_1}{1-K_b}$$

For $Y_1(s)/U_1(s)$

$$y_1(t \to \infty) = \lim_{s \to 0}\left[s \frac{K_1(\tau_b s+1)}{(\tau_a s+1)(\tau_b s+1) - K_b} \frac{1}{s} \right] = \frac{K_1}{1-K_b}$$

In other words, the gain of each transfer function is $\dfrac{K_1}{1-K_b}$

c) $$\frac{Y_2(s)}{U_1(s)} = \frac{K_1}{(\tau_a s+1)(\tau_b s+1) - K_b} \tag{5}$$

Second-order process but the denominator is not in standard form, i.e., $\tau^2 s^2 + 2\zeta\tau s + 1$

Put it in that form

$$\frac{Y_2(s)}{U_1(s)} = \frac{K_1}{\tau_a \tau_b s^2 + (\tau_a + \tau_b)s + 1 - K_b} \tag{8}$$

Dividing through by 1- K_b

$$\frac{Y_2(s)}{U_1(s)} = \frac{K_1/(1-K_b)}{\dfrac{\tau_a\tau_b}{1-K_b}s^2 + \dfrac{(\tau_a+\tau_b)}{1-K_b}s + 1} \tag{9}$$

Now we see that the gain $K = K_1/(1-K_b)$, as before

$$\tau^2 = \frac{\tau_a \tau_b}{1-K_b} \qquad \tau = \sqrt{\frac{\tau_a\tau_b}{1-K_b}} \tag{10}$$

$2\zeta\tau = \dfrac{\tau_a+\tau_b}{1-K_b}$, then

$$\zeta = \frac{1}{2}\frac{\tau_a+\tau_b}{1-K_b}\sqrt{\frac{1-K_b}{\tau_a\tau_b}} = \left[\frac{1}{2}\frac{\tau_a+\tau_b}{\sqrt{\tau_a\tau_b}}\right]\frac{1}{\sqrt{1-K_b}} \tag{11}$$

Investigating Eq. 11 we see that the quantity in brackets is the same as ζ for an overdamped 2nd-order system (ζ_{OD}) [from Eq. 5-43 in text].

$$\zeta = \frac{\zeta_{OD}}{\sqrt{1-K_b}} \tag{12}$$

where $\zeta_{OD} = \dfrac{1}{2}\dfrac{\tau_a+\tau_b}{\sqrt{\tau_a+\tau_b}}$

Since $\zeta_{OD} > 1$,

$\zeta > 1$, for all $0 < K_b < 1$.

In other words, since the quantity in brackets is the value of ζ for an overdamped system (i.e. for $\tau_a \neq \tau_b$ is >1) and $\sqrt{1 - K_b}$ <1 for any positive K_b, we can say that this process will be more overdamped (larger ζ) if K_b is positive and <1.

For negative K_b we can find the value of K_b that makes $\zeta = 1$, i.e., yields a critically-damped 2nd-order system.

$$\zeta = 1 = \frac{\zeta_{OD}}{\sqrt{1 - K_{b1}}} \tag{13}$$

or $$1 = \frac{\zeta_{OD}^{\,2}}{1 - K_{b1}}$$

$$1 - K_{b1} = \zeta_{OD}^{\,2}$$
$$K_{b1} = 1 - \zeta_{OD}^{\,2} \tag{14}$$

where

$K_{b1} < 0$ is the value of K_b that yields a critically-damped process.

Summarizing, the system is overdamped for $1 - \zeta_{OD}^{\,2} < K_b < 1$.

Regarding the integrator form, note that

$$\frac{Y_2(s)}{U_1(s)} = \frac{K_1}{\tau_a \tau_b s^2 + (\tau_a + \tau_b)s + 1 - K_b} \tag{8}$$

For $K_b = 1$

$$\frac{Y_2(s)}{U_1(s)} = \frac{K_1}{\tau_a \tau_b s^2 + (\tau_a + \tau_b)s} = \frac{K_1}{s[\tau_a \tau_b s + (\tau_a + \tau_b)]}$$

$$= \frac{K_1/(\tau_a + \tau_b)}{s\left[\dfrac{\tau_a \tau_b}{\tau_a + \tau_b} s + 1\right]}$$

which has the form $= \dfrac{K_1'}{s(\tau' s + 1)}$ (s indicates presence of integrator)

d) Return to Eq. 8

System A:

$$\frac{Y_2(s)}{U_1(s)} = \frac{K_1}{(2)(1)s^2 + (2+1)s + 1 - 0.5} = \frac{2K_1}{4s^2 + 6s + 1} = \frac{1}{4s^2 + 6s + 1}$$

$\tau^2 = 4 \quad \rightarrow \quad \tau = 2$

$2\zeta\tau = 6 \quad \rightarrow \quad \zeta = 1.5$

System B:

For system $\dfrac{1}{(2s+1)(s+1)} = \dfrac{1}{2s^2 + 3s + 1}$

$\tau_2^2 = 2 \quad \rightarrow \quad \tau_2 = \sqrt{2}$

$2\zeta_2\tau_2 = 3 \quad \rightarrow \quad \zeta_2 = \dfrac{3}{2\sqrt{2}} = \dfrac{1.5}{\sqrt{2}} \approx 1.05$

Since system A has larger τ (2 vs. $\sqrt{2}$) and larger ζ (1.5 vs 1.05), it will respond slower. These results correspond to our earlier analysis.