Chapter 6

6.1

a) By using MATLAB, the poles and zeros are:

Zeros: (-1 +1i) , (-1 -1i)
Poles: -4.3446
(-1.0834 +0.5853i)
(-1.0834 –0.5853i)
(+0.7557 +0.5830i)
(+0.7557 –0.5830i)

These results are shown in Fig E6.1

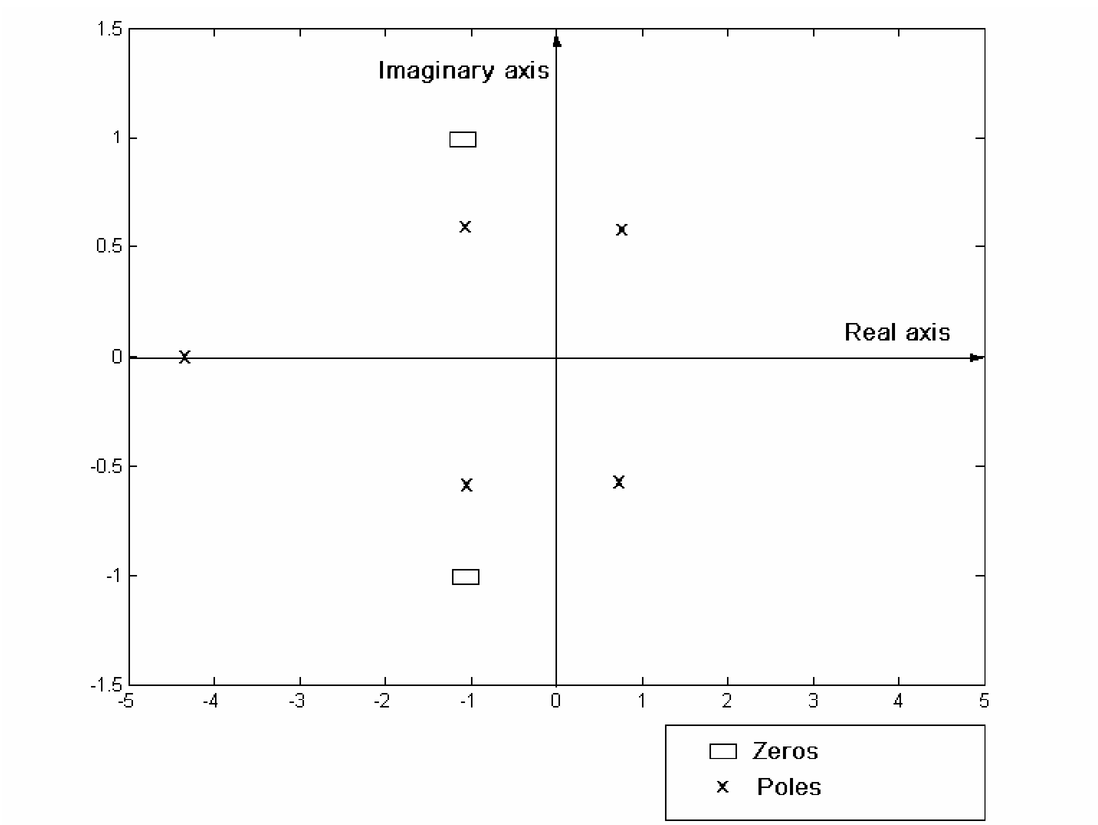


Figure S6.1. *Poles and zeros of G(s) plotted in the complex s plane.*

b) Process output will be unbounded because some poles lie in the right half plane.

c) By using Simulink-MATLAB

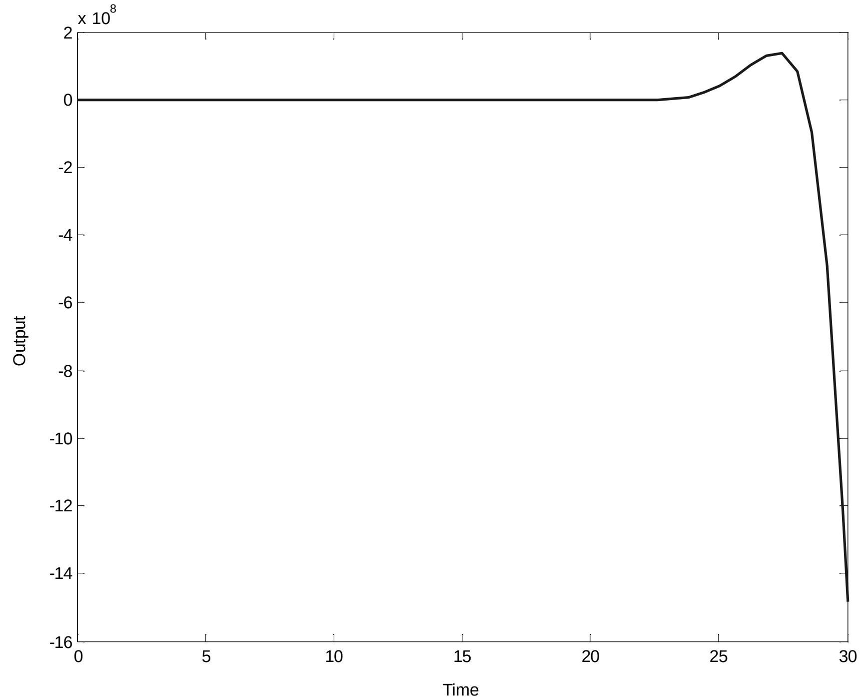


Figure E6.1b. *Response of the output of this process to a unit step input.*

As shown in Fig. S6.1b, the right half plane pole pair makes the process unstable.

6.2

a) Standard form = $\dfrac{K(\tau_a s+1)}{(\tau_1 s+1)(\tau_2 s+1)}$

b)

Hence $G(s)=\dfrac{0.5(2s+1)e^{-5s}}{(0.5s+1)(2s+1)}$

Applying zero-pole cancellation:

$$G(s)=\frac{0.5e^{-5s}}{(0.5s+1)}$$

c) Gain = 0.5
Pole = –2
Zeros = No zeros due to the zero-pole cancellation.

d) 1/1 Pade approximation: $e^{-5s} = \dfrac{(1-5/2s)}{(1+5/2s)}$

The transfer function is now

$$G(s) = \frac{0.5}{0.5s+1} \times \frac{(1-5/2s)}{(1+5/2s)}$$

Gain = 0.5
Poles = –2, –2/5
Zeros = + 2/5

6.3

$$\frac{Y(s)}{X(s)} = \frac{K(\tau_a s+1)}{(\tau_1 s+1)} \quad , \quad X(s) = \frac{M}{s}$$

From Eq. 6-13

$$y(t) = KM\left[1-\left(1-\frac{\tau_a}{\tau_1}\right)e^{-t/\tau_1}\right] = KM\left[1+\frac{\tau_a-\tau_1}{\tau_1}e^{-t/\tau_1}\right]$$

a) $$y(0^+) = KM\left[1+\frac{\tau_a-\tau_1}{\tau_1}\right] = \frac{\tau_a}{\tau_1}KM$$

b) Overshoot $\rightarrow$ $y(t) > KM$

$$KM\left[1+\frac{\tau_a-\tau_1}{\tau_1}e^{-t/\tau_1}\right] > KM$$

or $\tau_a - \tau_1 > 0$, that is, $\tau_a > \tau_1$

$$\dot{y} = -KM\frac{(\tau_a-\tau_1)}{\tau_1^{\,2}}e^{-t/\tau_1} < 0 \qquad for \quad KM > 0$$

c) Inverse response $\rightarrow$ $y(t) < 0$

$$KM\left[1+\frac{\tau_a-\tau_1}{\tau_1}e^{-t/\tau_1}\right] < 0$$

$$\frac{\tau_a-\tau_1}{\tau_1} < -e^{+t/\tau_1} \quad \text{or} \quad \frac{\tau_a}{\tau_1} < 1-e^{+t/\tau_1} < 0 \quad \text{at } t = 0.$$

Therefore $\tau_a < 0$.

6.4

$$\frac{Y(s)}{X(s)} = \frac{K(\tau_a s+1)}{(\tau_1 s+1)(\tau_2 s+1)} \quad , \ \tau_1>\tau_2, \qquad X(s) = M/s$$

From Eq. 6-15

$$y(t) = KM\left[1+\frac{\tau_a-\tau_1}{\tau_1-\tau_2}e^{-t/\tau_1} - \frac{\tau_a-\tau_2}{\tau_1-\tau_2}e^{-t/\tau_2}\right]$$

a) Extremum $\rightarrow \quad \dot{y}(t) = 0$

$$KM\left[0-\frac{1}{\tau_1}\left(\frac{\tau_a-\tau_1}{\tau_1-\tau_2}\right)e^{-t/\tau_1} + \frac{1}{\tau_2}\left(\frac{\tau_a-\tau_2}{\tau_1-\tau_2}\right)e^{-t/\tau_2}\right] = 0$$

$$\frac{1-\tau_a/\tau_2}{1-\tau_a/\tau_1} = e^{-t\left(\frac{1}{\tau_1}-\frac{1}{\tau_2}\right)} \geq 1 \quad \text{since} \quad \tau_1>\tau_2$$

b) Overshoot $\rightarrow \quad y(t) > KM$

$$KM\left[1+\frac{\tau_a-\tau_1}{\tau_1-\tau_2}e^{-t/\tau_1} - \frac{\tau_a-\tau_2}{\tau_1-\tau_2}e^{-t/\tau_2}\right] > KM$$

$$\frac{\tau_a-\tau_1}{\tau_a-\tau_2} > e^{-t\left(\frac{1}{\tau_2}-\frac{1}{\tau_1}\right)} > 0, \quad \text{therefore } \tau_a>\tau_1$$

c) Inverse response $\rightarrow \quad \dot{y}(t) < 0$ at $t = 0^+$

$$KM\left[0-\frac{1}{\tau_1}\left(\frac{\tau_a-\tau_1}{\tau_1-\tau_2}\right)e^{-t/\tau_1} + \frac{1}{\tau_2}\left(\frac{\tau_a-\tau_2}{\tau_1-\tau_2}\right)e^{-t/\tau_2}\right] < 0 \quad \text{at } t = 0^+$$

$$-\frac{1}{\tau_1}\left(\frac{\tau_a-\tau_1}{\tau_1-\tau_2}\right)+\frac{1}{\tau_2}\left(\frac{\tau_a-\tau_2}{\tau_1-\tau_2}\right) < 0$$

$$\frac{\tau_a\left(\frac{1}{\tau_2}-\frac{1}{\tau_1}\right)}{\tau_1-\tau_2} < 0$$

Since $\tau_1 > \tau_2$, $\tau_a < 0$.

d) If an extremum in y exists, then from (a)

$$e^{-t\left(\frac{1}{\tau_1}-\frac{1}{\tau_2}\right)} = \left(\frac{1-\tau_a/\tau_2}{1-\tau_a/\tau_1}\right)$$

$$t = \frac{\tau_1\tau_2}{\tau_1-\tau_2}\ln\left(\frac{1-\tau_a/\tau_2}{1-\tau_a/\tau_1}\right)$$

6.5

Substituting the numerical values into Eq. 6-15

Case (i) : $y(t) = 1\,(1 + 1.25e^{-t/10} - 2.25e^{-t/2})$

Case (ii(a)) : $y(t) = 1\,(1 - 0.75e^{-t/10} - 0.25e^{-t/2})$

Case (ii(b)) : $y(t) = 1\,(1 - 1.125e^{-t/10} + 0.125e^{-t/2})$

Case (iii) : $y(t) = 1\,(1 - 1.5e^{-t/10} + 0.5e^{-t/2})$

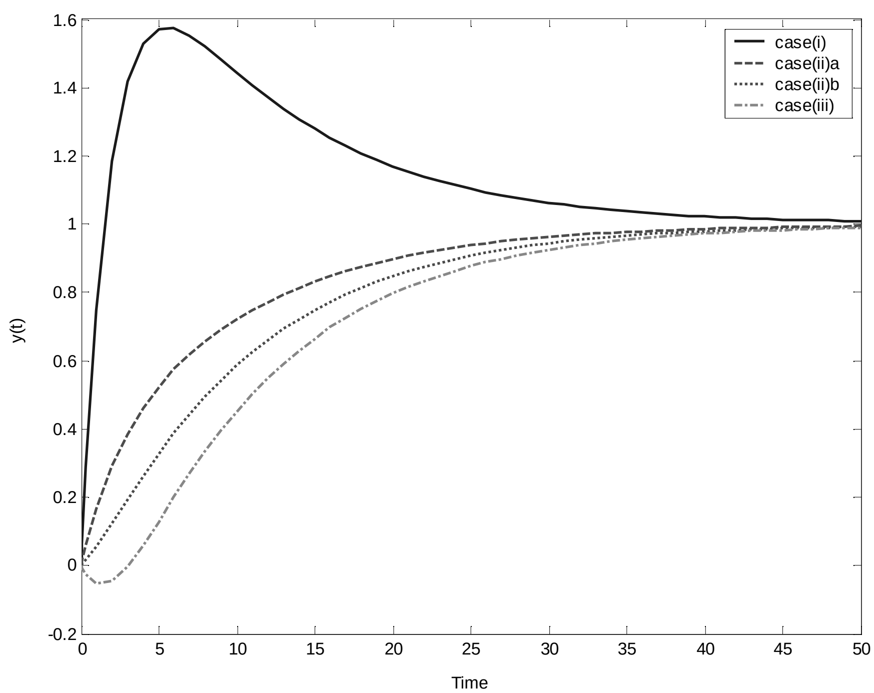


Figure S6.5. *Step response of a second-order system with a single zero.*

Conclusions:

$\tau_a > \tau_1$ gives overshoot.

$0 < \tau_a < \tau_1$ gives response similar to ordinary first-order process response.

$\tau_a < 0$ gives inverse response.

6.6

$$Y(s) = \frac{K_1}{s}U(s) + \frac{K_2}{\tau s+1}U(s) = \left[\frac{K_1}{s} + \frac{K_2}{\tau s+1}\right]U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{K_1\tau s + K_1 + K_2 s}{s(\tau s+1)} = \frac{(K_1\tau + K_2)s + K_1}{s(\tau s+1)}$$

Put in standard K/τ form for analysis:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K_1\left[\left(\tau + \frac{K_2}{K_1}\right)s + 1\right]}{s(\tau s+1)}$$

a) Order of $G(s)$ is 2 (maximum exponent on s in denominator is 2)

b) Gain of $G(s)$ is K_1. Gain is negative if $K_1 < 0$.

c) Poles of $G(s)$ are: $s_1 = 0$ and $s_2 = -1/\tau$

s_1 is on imaginary axis; s_2 is in LHP.

d) Zero of $G(s)$ is:

$$s_a = \frac{-1}{\left(\tau + \frac{K_2}{K_1}\right)} = \frac{-K_1}{K_1\tau + K_2}$$

If $\frac{K_1}{K_1\tau + K_2} < 0$, the zero is in RHP.

Two possibilities: 1. $K_1<0$ and $K_1\tau + K_2 >0$

2. $K_1 > 0$ and $K_1\tau + K_2 < 0$

e) Gain is negative if $K_1 < 0$

Then zero is RHP if $K_1\tau + K_2 > 0$

This is the only possibility.

f) Constant term and $e^{-t/\tau}$ term.

g) If input is *M/s*, the output will contain a *t* term, that is, it is not bounded.

6.7

a)

$$p'(t) = (4-2)S(t) \qquad , \; P'(s) = \frac{2}{s}$$

$$Q'(s) = \frac{-3}{20s+1}P'(s) = \frac{-3}{20s+1}\,\frac{2}{s}$$

$$Q'(t) = -6(1-e^{-t/20})$$

b)

$$R'(s) + Q'(s) = P_m'(s)$$

$$r'(t) + q'(t) = p_m'(t) = p_m(t) - p_m(0)$$

$$r'(t) = p_m(t) - 12 + 6(1-e^{-t/20})$$

$$K = \frac{r'(t=\infty)}{p(t=\infty) - p(t=0)} = \frac{18-12+6(1-0)}{4-2} = 6$$

Overshoot,

$$\text{OS} = \frac{r'(t=15) - r'(t=\infty)}{r'(t=\infty)} = \frac{27-12+6(1-e^{-15/20})-12}{12} = 0.514$$

$$OS = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 0.514 \qquad , \qquad \zeta = 0.2$$

Period, *T*, for $r'(t)$ is equal to the period for $p_m(t)$ since $e^{-t/20}$ decreases monotonically.

Thus, $T = 50 - 15 = 35$

and $$\tau = \frac{T}{2\pi}\sqrt{1-\zeta^2} = 5.46$$

c) $$\frac{P_m'(s)}{P'(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} + \frac{K'}{\tau' s + 1}$$

$$= \frac{(K'\tau^2)s^2 + (K\tau' + 2K'\zeta\tau)s + (K + K')}{(\tau^2 s^2 + 2\zeta\tau s + 1)(\tau' s + 1)}$$

d) Overall process gain = $\left.\frac{P_m'(s)}{P'(s)}\right|_{s=0} = K + K' = 6 - 3 = 3\frac{\%}{\text{psi}}$

6.8

a) Transfer Function for blending tank:

$$G_{bt}(s) = \frac{K_{bt}}{\tau_{bt} s + 1} \qquad \text{where} \quad K_{bt} = \frac{q_{in}}{\sum q_i} \neq 1$$

$$\tau_{bt} = \frac{2\text{m}^3}{1\text{m}^3/\text{min}} = 2\,\text{min}$$

Transfer Function for transfer line

$$G_{tl}(s) = \frac{K_{tl}}{(\tau_{tl} s + 1)^5} \qquad \text{where} \quad K_{tl} = 1$$

$$\tau_{tl} = \frac{0.1\text{m}^3}{5\times 1\text{m}^3/\text{min}} = 0.02\,\text{min}$$

$$\therefore \qquad \frac{C'_{out}(s)}{C'_{in}(s)} = \frac{K_{bt}}{(2s+1)(0.02s+1)^5}$$

a 6th-order transfer function.

b) Since $\tau_{bt} >> \tau_{tl}$ [2 >> 0.02] we can approximate $\dfrac{1}{(0.02s+1)^5}$ by $e^{-\theta s}$

where $\theta = \sum_{i=1}^{5}(0.02) = 0.1$

$$\therefore \qquad \frac{C'_{out}(s)}{C'_{in}(s)} \approx \frac{K_{bt}e^{-0.1s}}{2s+1}$$

c) Since $\tau_{bt} \approx 100\ \tau_{tl}$, we can imagine that this approximate TF will yield results very close to those from the original TF (part (a)). We also note that this approximate TF is exactly the same as would have been obtained using a plug flow assumption for the transfer line. Thus we conclude that investing a lot of effort into obtaining an accurate dynamic model for the transfer line is not worthwhile in this case.

[Note that, if $\tau_{bt} \approx \tau_{tl}$, this conclusion would not be valid]

d) By using Simulink-MATLAB,

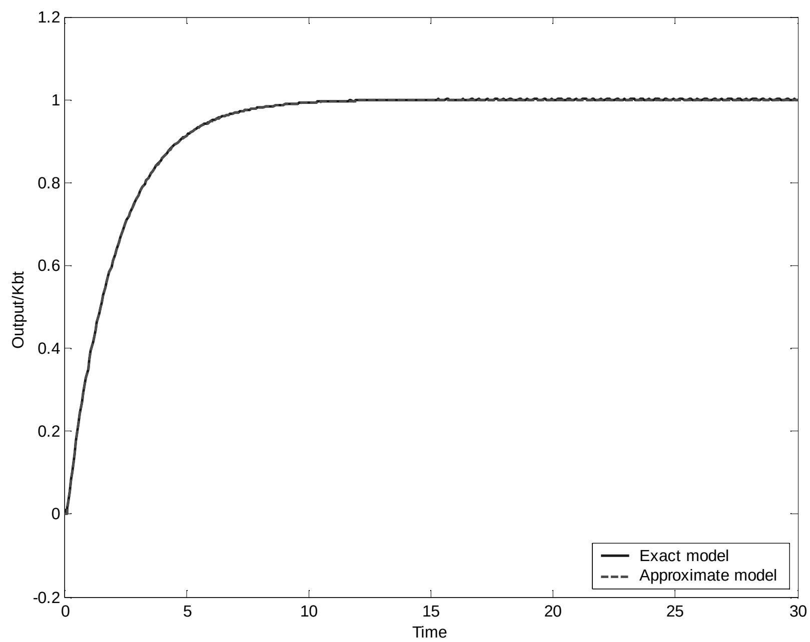


Fig S6.8. *Unit step responses for exact and approximate model.*

6.9

a), b) Represent processes that are (approximately) critically damped. A step response or frequency response in each case can be fit graphically or numerically.

c) $\theta = 2$, $\tau = 10$

d) Exhibits strong overshoot. Can't approximate it well.

e) $\theta = 0.5$, $\tau = 10$

f) $\theta = 1$, $\tau = 10$

g) Underdamped (oscillatory). Can't approximate it well.

h) $\theta = 2$, $\tau = 0$

By using Simulink-MATLAB, models for parts c), e), f) and h) are compared: (Suppose $K = 1$)

Part c)

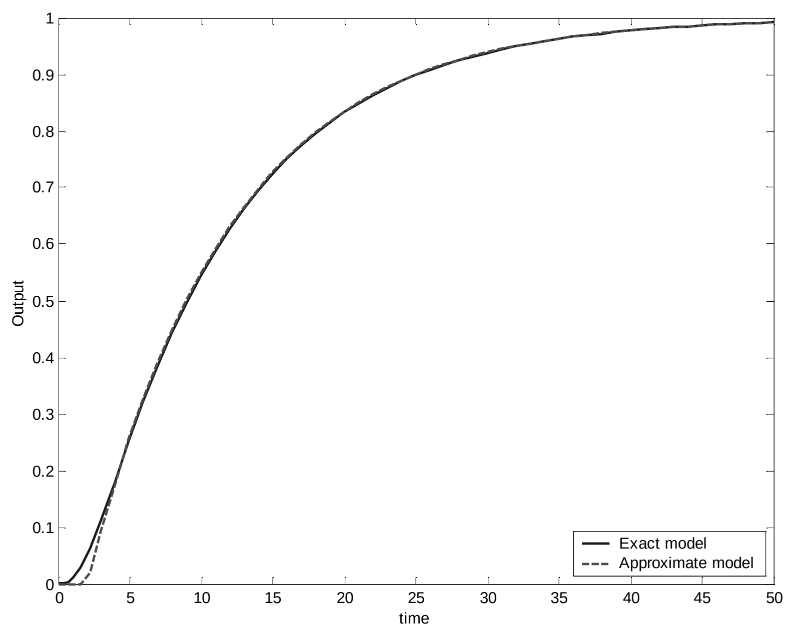


Figure S6.9a. *Unit step responses for exact and approximate model in part c)*

Part e)

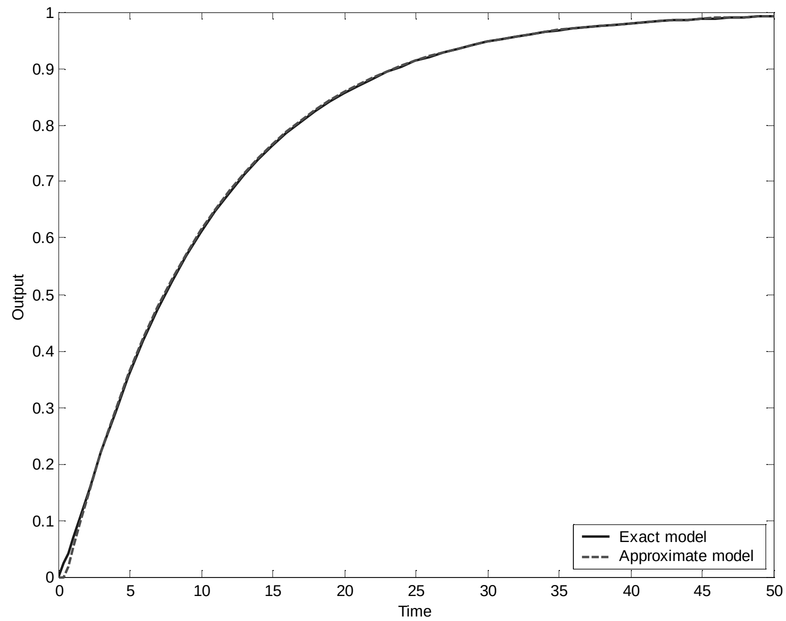


Figure S6.9b. *Unit step responses for exact and approximate model in part e)*

Part f)

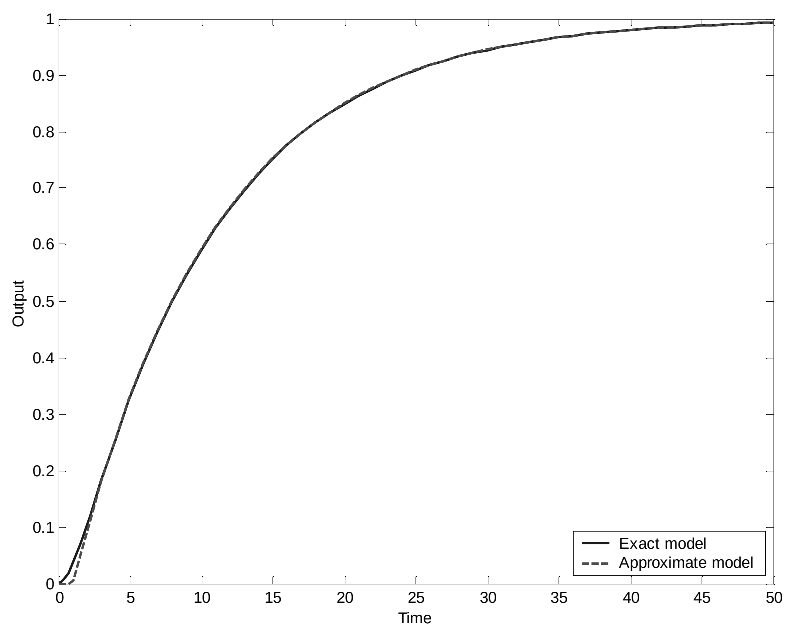


Figure S6.9c. *Unit step responses for exact and approximate model in part f)*

Part h)

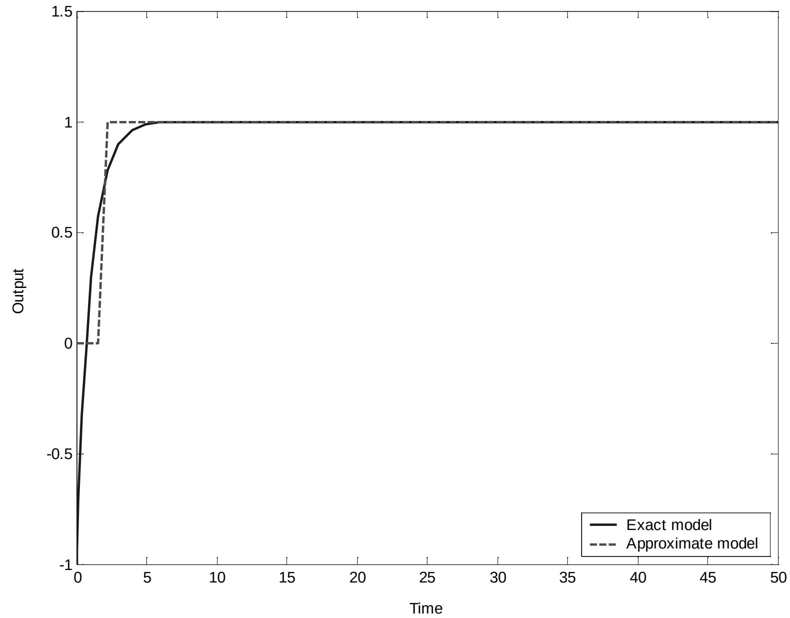


Figure S6.9d. *Unit step responses for exact and approximate model in part h)*

6.10

a) The transfer function for each tank is

$$\frac{C_i'(s)}{C_{i-1}'(s)} = \frac{1}{\left(\frac{V}{q}\right)s+1} \quad , \qquad i = 1, 2, \ldots, 5$$

where i represents the i^{th} tank.

c_o is the inlet concentration to tank 1.

V is the volume of each tank.

q is the volumetric flow rate.

$$\frac{C_5'(s)}{C_0'(s)} = \prod_{i=1}^{5}\left[\frac{C_i'(s)}{C_{i-1}'(s)}\right] = \left(\frac{1}{6s+1}\right)^5 ,$$

Then, by partial fraction expansion,

$$c_5(t) = 0.60 - 0.15\left[1 - e^{-t/6}\left\{1 + \frac{t}{6} + \frac{1}{2!}\left(\frac{t}{6}\right)^2 + \frac{1}{3!}\left(\frac{t}{6}\right)^3 + \frac{1}{4!}\left(\frac{t}{6}\right)^4\right\}\right]$$

b) Using Simulink,

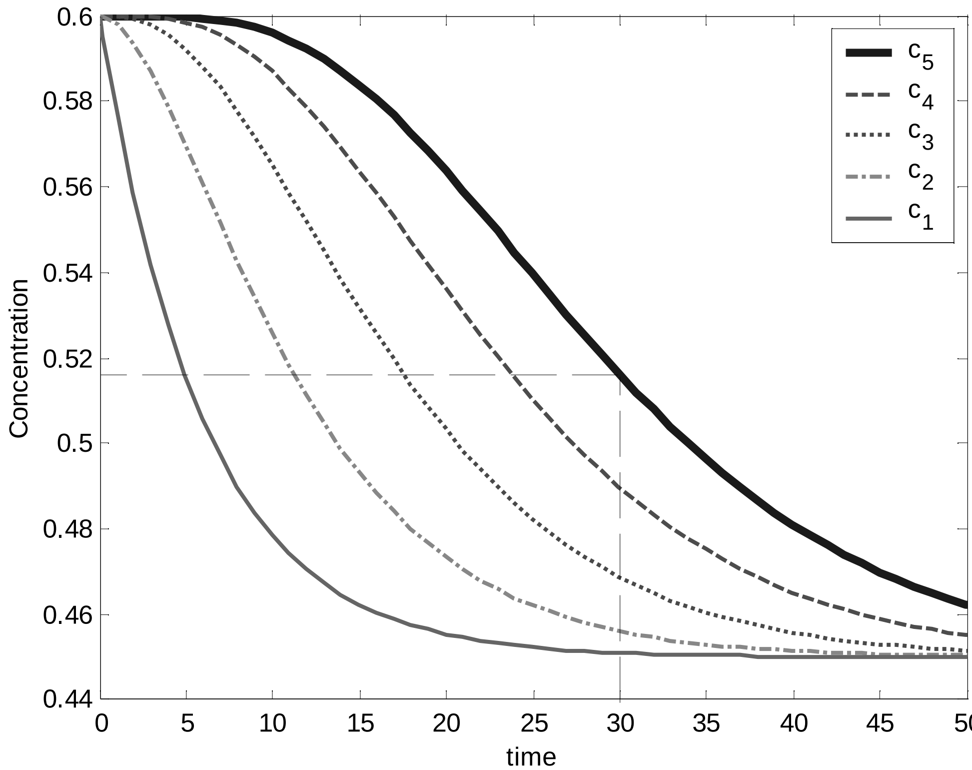


Figure S6.10. *Concentration step responses of the stirred tank.*

The value of the expression for $c_5(t)$ verifies the simulation results above:

$$c_5(30) = 0.60 - 0.15\left[1 - e^{-5}\left\{1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!}\right\}\right] = 0.5161$$

6.11

a) $$Y(s) = \frac{-\tau_a s + 1}{\tau_1 s + 1}\frac{E}{s^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{\tau_1 s + 1}$$

We only need to calculate the coefficients A and B because $Ce^{-t/\tau_1} \to 0$ for t >> τ_1. However, there is a repeated pole at zero.

$$B = \lim_{s \to 0} \left[\frac{E(-\tau_a s + 1)}{\tau_1 s + 1} \right] = E$$

Now look at

$$E(-\tau_a s + 1) = As(\tau_1 s + 1) + B(\tau_1 s + 1) + Cs^2$$

$$-E\tau_a s + E = A\tau_1 s^2 + As + B\tau_1 s + B + Cs^2$$

Equate coefficients on s:

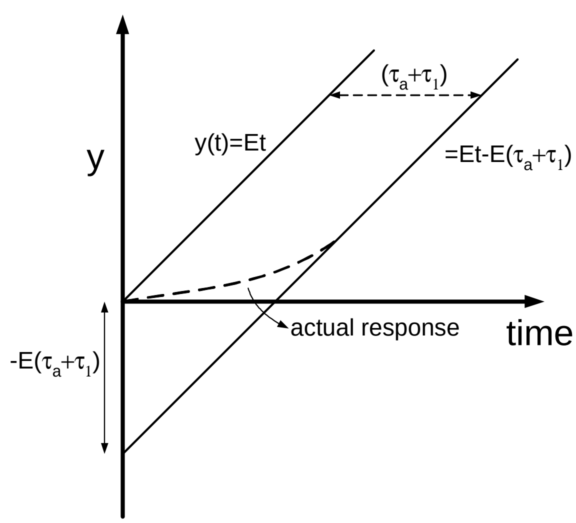
$$-E\tau_a = A + B\tau_1$$

$$A = -E(\tau_a + \tau_1)$$

Then the long-time solution is

$$y(t) \approx Et - E(\tau_a + \tau_1)$$

Plotting



b) For a LHP zero, the apparent lag would be $\tau_1 - \tau_a$

c) For no zero, the apparent lag would be τ_1

6.12

a) Using Skogestad's method

$$G(s)_{approx} = \frac{5e^{-(0.5+0.2)s}}{(10s+1)\left((4+0.5)s+1\right)} = \frac{5e^{-0.7s}}{(10s+1)(4.5s+1)}$$

b) By using Simulink-MATLAB

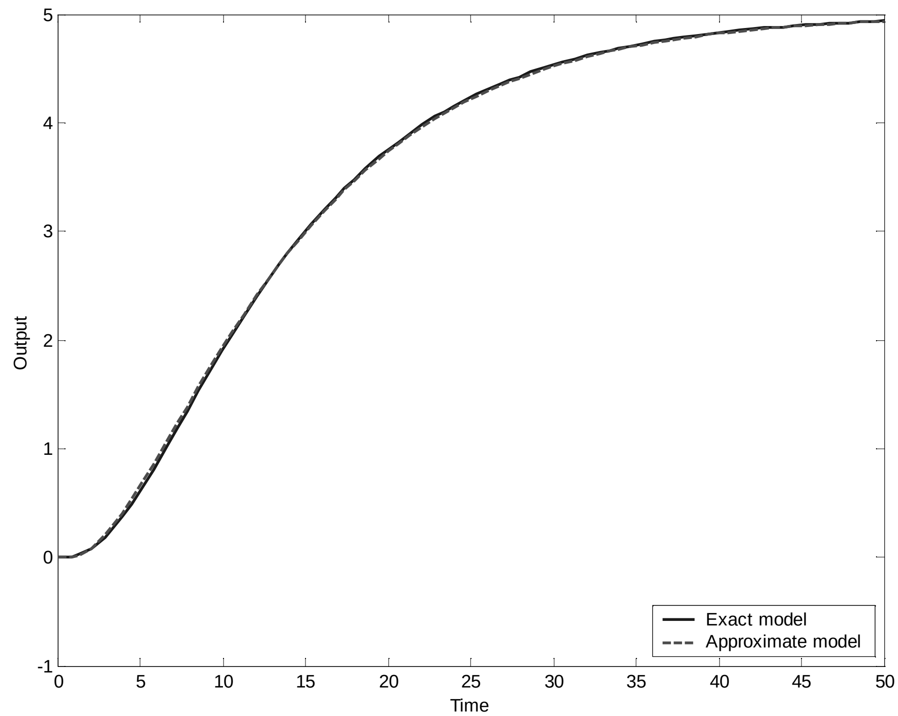


Figure S6.12a. *Unit step responses for exact and approximate model.*

c) Using MATLAB and saving output data on vectors, the maximum error is

Maximum error = 0.0521 at = 5.07 s

This maximum error is graphically shown in Fig. S6.12b

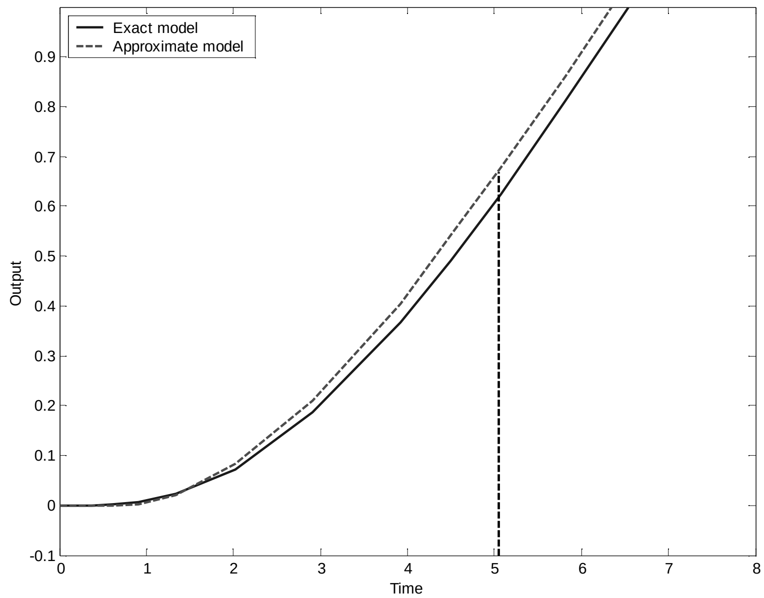


Figure S6.12b. *Maximum error between responses for exact and approximate model.*

6.13

From the solution to Problem 2-5 (a) , the dynamic model for isothermal operation is

$$\frac{V_1 M}{RT_1}\frac{dP_1}{dt} = \frac{P_d - P_1}{R_a} - \frac{P_1 - P_2}{R_b} \qquad (1)$$

$$\frac{V_2 M}{RT_2}\frac{dP_2}{dt} = \frac{P_1 - P_2}{R_b} - \frac{P_2 - P_f}{R_c} \qquad (2)$$

Taking Laplace transforms, and noting that $P_f'(s) = 0$ since P_f is constant,

$$P_1'(s) = \frac{K_b P_d'(s) + K_a P_2'(s)}{\tau_1 s + 1} \qquad (3)$$

$$P_2'(s) = \frac{K_c P_1'(s)}{\tau_2 s + 1} \qquad (4)$$

where

$$K_a = R_a/(R_a + R_b)$$
$$K_b = R_b/(R_a + R_b)$$

$$K_c = R_c/(R_b + R_c)$$

$$\tau_1 = \frac{V_1 M}{RT_1}\frac{R_a R_b}{(R_a + R_b)}$$

$$\tau_2 = \frac{V_2 M}{RT_2}\frac{R_b R_c}{(R_b + R_c)}$$

Substituting for $P_1'(s)$ from Eq. 3 into 4,

$$\frac{P_2'(s)}{P_d'(s)} = \frac{K_b K_c}{(\tau_1 s + 1)(\tau_2 s + 1) - K_a K_c} = \frac{\left(\dfrac{K_b K_c}{1 - K_a K_c}\right)}{\left(\dfrac{\tau_1 \tau_2}{1 - K_a K_c}\right)s^2 + \left(\dfrac{\tau_1 + \tau_2}{1 - K_a K_c}\right)s + 1} \quad (5)$$

Substituting for $P_2'(s)$ from Eq. 5 into 4,

$$\frac{P_1'(s)}{P_d'(s)} = \frac{\left(\dfrac{K_b}{1 - K_a K_c}\right)(\tau_2 s + 1)}{\left(\dfrac{\tau_1 \tau_2}{1 - K_a K_c}\right)s^2 + \left(\dfrac{\tau_1 + \tau_2}{1 - K_a K_c}\right)s + 1} \quad (6)$$

To determine whether the system is over- or underdamped, consider the denominator of transfer functions in Eqs. 5 and 6.

$$\tau^2 = \left(\frac{\tau_1 \tau_2}{1 - K_a K_c}\right) \quad , \quad 2\zeta\tau = \frac{\tau_1 + \tau_2}{1 - K_a K_c}$$

Therefore,

$$\zeta = \frac{1}{2}\frac{(\tau_1 + \tau_2)}{(1 - K_a K_c)}\frac{\sqrt{(1 - K_a K_c)}}{\sqrt{\tau_1 \tau_2}} = \frac{1}{2}\left(\sqrt{\frac{\tau_1}{\tau_2}} + \sqrt{\frac{\tau_2}{\tau_1}}\right)\frac{1}{\sqrt{(1 - K_a K_c)}}$$

Since $x + 1/x \geq 2$ for all positive x,

$$\zeta \geq \frac{1}{\sqrt{(1 - K_a K_c)}}$$

Since $K_a K_c \geq 0$,

$$\zeta \geq 1$$

Hence the system is overdamped.

6.14

a) For $X(s) = \dfrac{M}{s}$

$$Y(s) = \frac{KM}{s(1-s)(\tau s+1)} = \frac{A}{s} + \frac{B}{1-s} + \frac{C}{\tau s+1}$$

$$A = \lim_{s\to 0} = \frac{KM}{(1-s)(\tau s+1)} = KM$$

$$B = \lim_{s\to 1} = \frac{KM}{s(\tau s+1)} = \frac{KM}{\tau+1}$$

$$C = \lim_{s\to -1/\tau}\left[\frac{KM}{s(1-s)}\right] = \frac{KM}{\left(-\frac{1}{\tau}\right)\left(1+\frac{1}{\tau}\right)} = \frac{-KM\tau^2}{\tau+1}$$

Then,

$$y_1(t) = KM\left[1 - \frac{e^t}{\tau+1} - \frac{\tau}{\tau+1}e^{-t/\tau}\right]$$

For $M = 2$, $K = 3$, and $\tau = 3$, the Simulink response is shown:

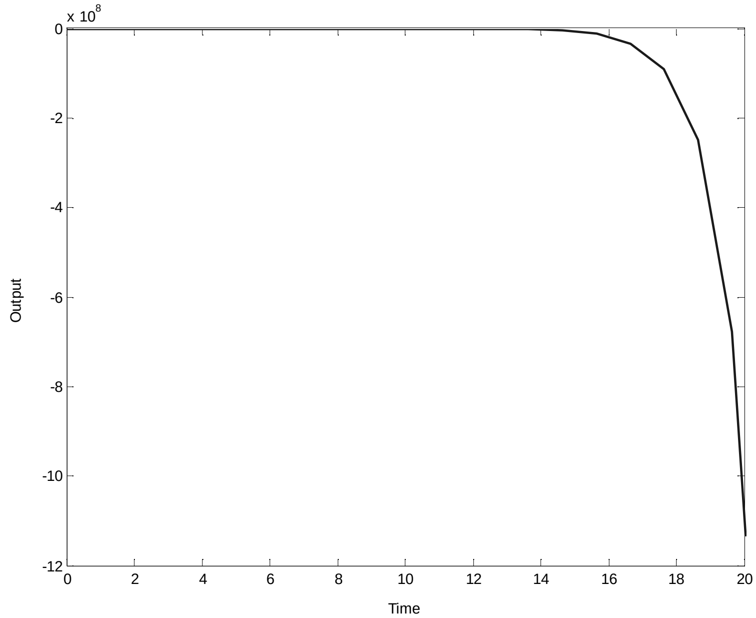


Figure S6.14a. *Unit step response for part a).*

b) If $G_2(s) = \dfrac{Ke^{-2s}}{(1-s)(\tau s+1)}$ then,

$$y_2(t) = KM\left[1 - \frac{e^{t-2}}{\tau+1} - \frac{\tau}{\tau+1}e^{-(t-2)/\tau}\right]S(t-2)$$

Note presence of positive exponential term.

c) Approximating $G_2(s)$ using a Padé function

$$G_2(s) = \frac{K(1-s)}{(s+1)(\tau s+1)(1-s)} = \frac{K}{(s+1)(\tau s+1)}$$

Note that the two remaining poles are in the LHP.

d) For $X(s) = \dfrac{M}{s}$

$$Y(s) = \frac{KM}{s(s+1)(\tau s+1)}$$

Using Table 3.1

$$\tau_1 = 1 \quad , \quad \tau_2 = \tau$$

$$y_3(t) = KM\left[1 + \frac{1}{\tau - 1}(e^{-t} - \tau e^{-t/\tau})\right]$$

Note that no positive exponential term is present.

e) Instability may be hidden by a pole-zero cancellation.

f) By using Simulink-MATLAB, unit step responses for parts b) and c) are shown below: (M = 2 , K = 3 , τ = 3)

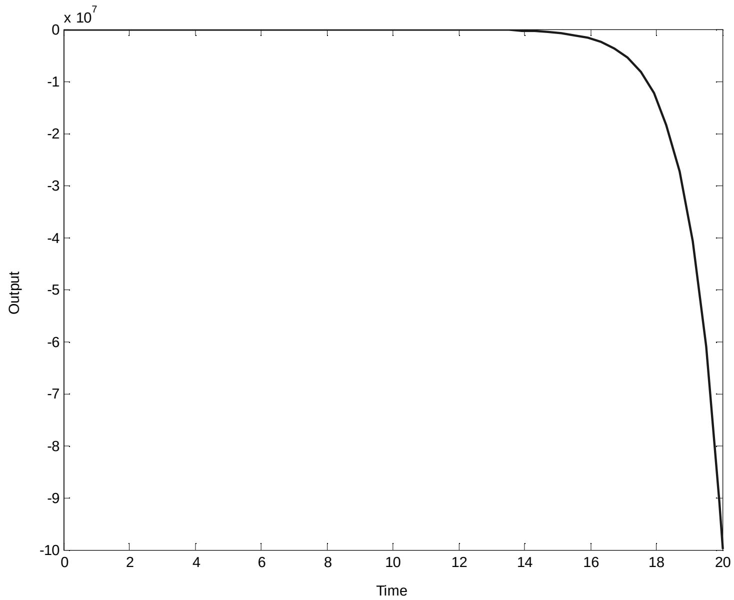


Figure S6.14b. *Unit step response for part b).*

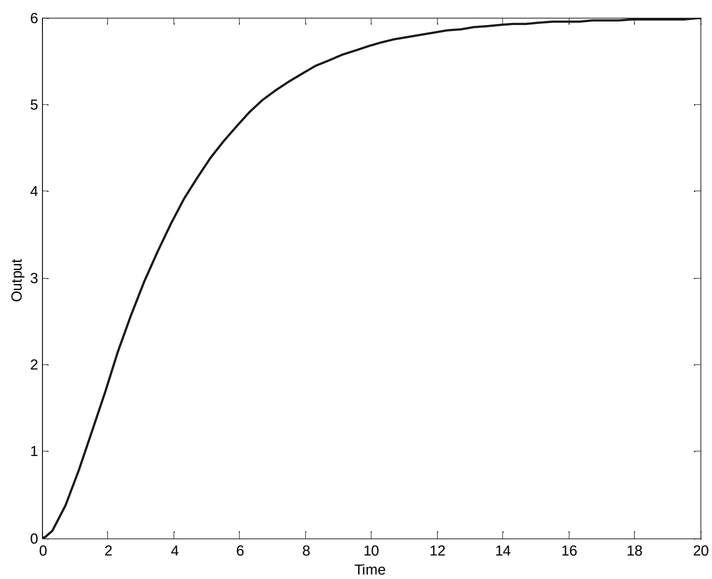


Figure S6.14c. *Unit step response for part c) .*

6.15

From Eq. 6-71 and 6-72,

$$\zeta = \frac{R_2 A_2 + R_1 A_1 + R_2 A_1}{2\sqrt{R_1 R_2 A_1 A_2}} = \frac{1}{2}\left(\sqrt{\frac{R_1 A_1}{R_2 A_2}} + \sqrt{\frac{R_2 A_2}{R_1 A_1}}\right) + \frac{1}{2}\sqrt{\frac{R_2 A_1}{R_1 A_2}}$$

Since $x + \frac{1}{x} \geq 2$ for all positive x and since R_1, R_2, A_1, A_2 are positive

$$\zeta \geq \frac{1}{2}(2) + \frac{1}{2}\sqrt{\frac{R_2 A_1}{R_1 A_2}} \geq 1$$

6.16

a) If $w_1 = 0$ and ρ = constant

$$\rho A_2 \frac{dh_2}{dt} = w_0 - w_2$$

$$w_2 = \frac{1}{R_2} h_2$$

[Note: could also define R_2 by $q_2 = \frac{1}{R_2} h_2 \rightarrow w_2 = \rho q_2 = \frac{\rho}{R_2} h_2$]

Substituting,

$$\rho A_2 \frac{dh_2}{dt} = w_0 - \frac{1}{R_2} h_2$$

or $$\rho A_2 R_2 \frac{dh_2}{dt} = R_2 w_0 - h_2$$

Taking deviation variables and Laplace transforming

$$\rho A_2 R_2 s H_2'(s) + H_2'(s) = R_2 W_0'(s)$$

$$\frac{H_2'(s)}{W_0'(s)} = \frac{R_2}{\rho A_2 R_2 s + 1}$$

Since $W_2'(s) = \frac{1}{R_2} H_2'(s)$

$$\frac{W_2'(s)}{W_0'(s)} = \frac{1}{R_2} \frac{R_2}{\rho A_2 R_2 s + 1} = \frac{1}{\rho A_2 R_2 s + 1}$$

Let $\tau_2 = \rho A_2 R_2$

$$\frac{W_2'(s)}{W_0'(s)} = \frac{1}{\tau_2 s + 1}$$

b) ρ = constant

$$\rho A_1 \frac{dh_1}{dt} = -w_1 \qquad \rho A_2 \frac{dh_2}{dt} = w_0 + w_1 - w_2$$

$$w_1 = \frac{1}{R_1}(h_1 - h_2) \qquad w_2 = \frac{1}{R_2} h_2$$

c) Since this clearly is an interacting system, there will be a single zero. Also, we know the gain must be equal to one.

$$\therefore \quad \frac{W_1'(s)}{W_0'(s)} = \frac{\tau_a s + 1}{\tau^2 s^2 + 2\zeta\tau s + 1} \qquad \frac{W_2'(s)}{W_0'(s)} = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

or

$$\frac{W_1'(s)}{W_0'(s)} = \frac{\tau_a s + 1}{(\tau_1' s + 1)(\tau_2' s + 1)} \qquad \frac{W_2'(s)}{W_0'(s)} = \frac{1}{(\tau_1' s + 1)(\tau_2' s + 1)}$$

where τ_1' and τ_2' are functions of the resistances and areas and can only be obtained by factoring.

f) Case b will be slower since the interacting system is 2^{nd}-order, "including" the 1^{st}-order system of Case a as a component.

6.17

The input is $T_i'(t) = 12 \sin \omega t$

where $\omega = \dfrac{2\pi \text{ radians}}{24 \text{ hours}} = 0.262 \text{ hr}^{-1}$

The Laplace transform of the input is from Table 3.1,

$$T_i'(s) = \frac{12\omega}{s^2 + \omega^2}$$

Multiplying the transfer function by the input transform yields

$$T_i'(s) = \frac{(-72 + 36s)\omega}{(10s+1)(5s+1)(s^2 + \omega^2)}$$

To invert, either (1) make a partial fraction expansion manually, or (2) use the Matlab residue function. The first method requires solution of a system of algebraic equations to obtain the coefficients of the four partial fractions. The second method requires that the numerator and denominator be defined as coefficients of descending powers of *s* prior to calling the Matlab residue function:

Matlab Commands

```
>> b = [ 36*0.262  –72*0.262]

b =

   9.4320  –18.8640

>> a = conv([10 1], conv([5 1], [1 0 0.262^2]))

b =

   50.0000      15.0000          4.4322           1.0297           0.0686

>> [r,p,k] = residue(b,a)

r =
          6.0865 – 4.9668i
          6.0865 + 4.9668i
          38.1989
        –50.3718
```

p =

–0.0000 – 0.2620i
–0.0000 + 0.2620i
–0.2000
–0.1000

k =

[]

Note: the residue function recomputes all the poles (listed under p). These are, in reverse order: p_1 = 0.1($\tau_1 = 10$), p_2 = 0.2($\tau_2 = 5$), and the two purely imaginary poles corresponding to the sine and cosine functions. The residues (listed under r) are exactly the coefficients of the corresponding poles, in other words, the coefficients that would have been obtained via a manual partial fraction expansion. In this case, we are not interested in the real poles since both of them yield exponential functions that go to 0 as $t \to \infty$.

The complex poles are interpreted as the sine/cosine terms using Eqs. 3-69 and 3-74. From (3-69) we have:

α_1= 6.0865, β_1 = 4.9668, b = 0, and ω=0.262.

Eq. 3-74 provides the coefficients of the periodic terms:

$$y(t) = 2\alpha_1 e^{-bt}\cos\omega t + 2\beta_1 e^{-bt}\sin\omega t + ...$$

Substituting coefficients (because b= 0, the exponential terms = 1)

$$y(t) = 2(6.068)\cos\omega t + 2(4.9668)\sin\omega t + ...$$

or $$y(t) = 12.136\cos\omega t + 9.9336\sin\omega t + ...$$

The amplitude of the composite output sinusoidal signal, for large values, of t is given by

$$A = \sqrt{(12.136)^2 + (9.9336)^2} = 15.7$$

Thus the amplitude of the output is 15.7° for the specified 12° amplitude input.

6.18

a) Taking the Laplace transform of the dynamic model in (2-7)

$$[(\gamma Vs + (q+q_R)]C'_{T1}(s) = qC'_{Ti}(s) + q_R C'_T(s) \tag{1}$$

$$[(1-\gamma)Vs + (q+q_R)]C'_T(s) = (q+q_R)C'_{T1}(s) \tag{2}$$

Substituting for $C'_T(s)$ from (2) into (1),

$$\frac{C'_{T1}(s)}{C'_{Ti}(s)} = \frac{q[(1-\gamma)Vs + (q+q_R)]}{[\gamma Vs + (q+q_R)][(1-\gamma)Vs + (q+q_R)] - q_R(q+q_R)}$$

$$= \frac{\left[\dfrac{(1-\gamma)V}{(q+q_R)}\right]s+1}{\left[\dfrac{\gamma(1-\gamma)V^2}{q(q+q_R)}\right]s^2 + \left[\dfrac{V}{q}\right]s+1} \tag{3}$$

Substituting for $C'_{T1}(s)$ from (3) into (2),

$$\frac{C'_T(s)}{C'_{Ti}(s)} = = \frac{1}{\left[\dfrac{\gamma(1-\gamma)V^2}{q(q+q_R)}\right]s^2 + \left[\dfrac{V}{q}\right]s+1} \tag{4}$$

b) Case (i), $\gamma \to 0$

$$\frac{C'_T(s)}{C'_{Ti}(s)} = = \frac{1}{\left[\dfrac{V}{q}\right]s+1} \qquad \frac{C'_{T1}(s)}{C'_{Ti}(s)} = = \frac{\left[\dfrac{V}{q+q_R}\right]s+1}{\left[\dfrac{V}{q}\right]s+1}$$

Case (ii), $\gamma \to 1$

$$\frac{C'_T(s)}{C'_{Ti}(s)} = \frac{1}{\left[\dfrac{V}{q}\right]s+1} = \frac{C'_{T1}(s)}{C'_{Ti}(s)}$$

Case (iii), $q_R \to 0$

$$\frac{C'_T(s)}{C'_{Ti}(s)} = = \frac{1}{\left[\frac{\gamma(1-\gamma)V^2}{q^2}\right]s^2 + \left[\frac{V}{q}\right]s + 1} \quad , \quad \frac{C'_{T1}(s)}{C'_{Ti}(s)} = = \frac{1}{\left[\frac{\gamma V}{q}\right]s + 1}$$

Case (iv), $q_R \to \infty$

$$\frac{C'_T(s)}{C'_{Ti}(s)} = \frac{1}{\left[\frac{V}{q}\right]s + 1} = \frac{C'_{T1}(s)}{C'_{Ti}(s)}$$

c) Case (i), $\gamma \to 0$

This corresponds to the physical situation with no top tank. Thus the dynamics for C_T are the same as for a single tank, and $C'_{T1} \approx C'_{Ti}$ for small q_R.

Case (ii), $\gamma \to 1$

Physical situation with no bottom tank. Thus the dynamics for C_{T1} are the same as for a single tank, and $C_T = C_{T1}$ at all times.

Case (iii), $q_R \to 0$

Physical situation with two separate non-interacting tanks. Thus, top tank dynamics, C_{T1}, are first order, and bottom tank, C_T, is second order.

Case (iv), $q_R \to \infty$

Physical situation of a single perfectly mixed tank. Thus, $C_T = C_{T1}$, and both exhibit dynamics that are the same as for a single tank.

d) In Eq.(3),

$$\left[\frac{(1-\gamma)V}{(q+q_R)}\right] \geq 0$$

Hence the system cannot exhibit an inverse response. From the denominator of the transfer functions in Eq.(3) and (4),

$$\zeta = \frac{1}{2}\frac{V}{q}\left[\frac{\gamma(1-\gamma)V^2}{q(q+q_R)}\right]^{-\frac{1}{2}} = \left[\frac{(q+q_R)}{4q\gamma(1-\gamma)}\right]^{\frac{1}{2}}$$

Since $\gamma(1-\gamma) \le (0.5)(1-0.5)$ for $0 \le \gamma \le 1$,

$$\zeta = \left[\frac{(q+q_R)}{q}\right]^{\frac{1}{2}} \ge 1$$

Hence, the system is overdamped and cannot exhibit overshoot.

e) Since $\zeta \ge 1$, the denominator of transfer function in Eq.(3) and (4) can be written as $(\tau_1 s+1)(\tau_2 s+1)$ where, using Eq. 5-45 and 5-46,

$$\tau_1 = \frac{\left[\frac{\gamma(1-\gamma)V^2}{q(q+q_R)}\right]^{\frac{1}{2}}}{\left[\frac{(q+q_R)}{4q\gamma(1-\gamma)}\right]^{\frac{1}{2}} - \left[\frac{(q+q_R)}{4q\gamma(1-\gamma)} - 1\right]^{\frac{1}{2}}}$$

$$\tau_2 = \frac{\left[\frac{\gamma(1-\gamma)V^2}{q(q+q_R)}\right]^{\frac{1}{2}}}{\left[\frac{(q+q_R)}{4q\gamma(1-\gamma)}\right]^{\frac{1}{2}} + \left[\frac{(q+q_R)}{4q\gamma(1-\gamma)} - 1\right]^{\frac{1}{2}}}$$

It is given that

$$C'_{Ti}(s) = \frac{h}{s}\left[1-e^{-t_w s}\right] = \frac{h}{s} - \frac{h}{s}e^{-t_w s}$$

Then using Eq. 5-48 and (4)

$$c_T(t) = S(t)h\left[1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2}\right]$$

$$- S(t-t_w)h\left[1 - \frac{\tau_1 e^{-(t-t_w)/\tau_1} - \tau_2 e^{-(t-t_w)/\tau_2}}{\tau_1 - \tau_2}\right]$$

And using Eq. 6-15 and (3)

$$C_{T1}(t) = S(t)h\left[1 + \frac{\tau_a - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{\tau_a - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}\right]$$

$$- S(t - t_w)h\left[1 + \frac{\tau_a - \tau_1}{\tau_1 - \tau_2} e^{-(t-t_w)/\tau_1} + \frac{\tau_a - \tau_2}{\tau_2 - \tau_1} e^{-(t-t_w)/\tau_2}\right]$$

where

$$\tau_a = \left[\frac{(1-\gamma)V}{(q + q_R)}\right]$$

The pulse response can be approximated reasonably well by the impulse response in the limit as $t_w \rightarrow 0$, keeping ht_w constant.

6.19

Let V_R = volume of each tank
$A_1 = \rho_1 C_{p1} V_R$
$A_2 = \rho_2 C_{p2} V_R$
$B_1 = w_1 C_{p1}$
$B_2 = w_2 C_{p2}$
$K = UA$

Then energy balances over the six tanks give

$$A_2 \frac{dT_8}{dt} = B_2(T_6 - T_8) + K(T_3 - T_8) \qquad (1)$$

$$A_2 \frac{dT_6}{dt} = B_2(T_4 - T_6) + K(T_5 - T_6) \qquad (2)$$

$$A_2 \frac{dT_4}{dt} = B_2(T_2 - T_4) + K(T_7 - T_4) \qquad (3)$$

$$A_1 \frac{dT_7}{dt} = B_1(T_5 - T_7) + K(T_4 - T_7) \qquad (4)$$

$$A_1 \frac{dT_5}{dt} = B_1(T_3 - T_5) + K(T_6 - T_5) \qquad (5)$$

$$A_1 \frac{dT_3}{dt} = B_1\left(T_1 - T_3\right) + K\left(T_8 - T_3\right) \qquad (6)$$

Define vectors

$$\underline{T}'(s) = \left[T_8'(s), T_7'(s), T_6'(s), T_5'(s), T_4'(s), T_3'(s)\right]^T$$

$$\underline{T}^*(s) = \begin{bmatrix} T_2'(s) \\ T_1'(s) \end{bmatrix}$$

Using deviation variables, and taking the Laplace transform of Eqs.1 to 6, we obtain an equation set that can be represented in matrix notation as

$$s\,\underline{\underline{I}}\,\underline{T}'(s) = \underline{\underline{A}}\,\underline{T}'(s) + \underline{\underline{B}}\,\underline{T}^*(s) \qquad (7)$$

where $\underline{\underline{I}}$ is the 6×6 identity matrix

$$\underline{\underline{A}} = \begin{bmatrix} \frac{-K-B_2}{A_2} & 0 & \frac{B_2}{A_2} & 0 & 0 & \frac{K}{A_2} \\ 0 & \frac{-K-B_1}{A_1} & 0 & \frac{B_1}{A_1} & \frac{K}{A_1} & 0 \\ 0 & 0 & \frac{-K-B_2}{A_2} & \frac{K}{A_2} & \frac{B_2}{A_2} & 0 \\ 0 & 0 & \frac{K}{A_1} & \frac{-K-B_1}{A_1} & 0 & \frac{B_1}{A_1} \\ 0 & \frac{K}{A_2} & 0 & 0 & \frac{-K-B_2}{A_2} & 0 \\ \frac{K}{A_1} & 0 & 0 & 0 & 0 & \frac{-K-B_1}{A_1} \end{bmatrix}$$

$$\underline{\underline{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{B_2}{A_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{B_1}{A_1} \end{bmatrix}$$

From Eq. 7,

$$\underline{T}'(s) = (s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\,\underline{\underline{B}}\,\underline{T}^*(s)$$

Then

$$\begin{bmatrix} T_8'(s) \\ T_7'(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} (s\underline{\underline{I}} - \underline{\underline{A}})^{-1} \underline{\underline{B}}\, \underline{T}^*(s)$$

6.20

The dynamic model for the process is given by Eqs. 2-45 and 2-46, which can be written as

$$\frac{dh}{dt} = \frac{1}{\rho A}(w_i - w) \tag{1}$$

$$\frac{dT}{dt} = \frac{w_i}{\rho A h}(T_i - T) + \frac{Q}{\rho A h C} \tag{2}$$

where h is the liquid-level
A is the constant cross-sectional area

System outputs: h , T
System inputs : w, Q

Hence assume that w_i and T_i are constant. In Eq. 2, note that the nonlinear term $\left(h\frac{dT}{dt}\right)$ can be linearized as

$$\bar{h}\frac{dT'}{dt} + \frac{d\bar{T}}{dt}h'$$

or $\bar{h}\frac{dT'}{dt}$ since $\frac{d\bar{T}}{dt} = 0$

Then the linearized deviation variable form of (1) and (2) is

$$\frac{dh'}{dt} = -\frac{1}{\rho A}w'$$

$$\frac{dT'}{dt} = \frac{-w_i}{\rho A\bar{h}}T' + \frac{1}{\rho A\bar{h}C}Q'$$

Taking Laplace transforms and rearranging,

$$\frac{H'(s)}{W'(s)} = \frac{K_1}{s} \quad , \quad \frac{H'(s)}{Q'(s)} = 0 \quad , \quad \frac{T'(s)}{W'(s)} = 0 \quad , \quad \frac{T'(s)}{Q'(s)} = \frac{K_2}{\tau_2 s + 1}$$

where $K_1 = -\frac{1}{\rho A}$; and $K_2 = \frac{1}{w_i C}$, $\tau_2 = \frac{\rho A \bar{h}}{w_i}$

Unit-step change in Q: $h(t) = \bar{h}$, $T(t) = \bar{T} + K_2(1 - e^{-t/\tau_2})$

Unit step change in w: $h(t) = \bar{h} + K_1 t$, $T(t) = \bar{T}$

6.21

Additional assumptions:

(i) The density, ρ, and the specific heat, C, of the process liquid are constant.

(ii) The temperature of steam, T_s, is uniform over the entire heat transfer area.

(iii) The feed temperature T_F is constant (not needed in the solution).

Mass balance for the tank is

$$\frac{dV}{dt} = q_F - q \tag{1}$$

Energy balance for the tank is

$$\rho C \frac{d[V(T - T_{ref})]}{dt} = q_F \rho C(T_F - T_{ref}) - q\rho C(T - T_{ref}) + UA(T_s - T) \tag{2}$$

where T_{ref} is a constant reference temperature
A is the heat transfer area

Eq. 2 is simplified by substituting for $\frac{dV}{dt}$ from Eq. 1. Also, replace V by $A_T h$ (where A_T is the tank area) and replace A by $p_T h$ (where p_T is the perimeter of the tank). Then,

$$A_T \frac{dh}{dt} = q_F - q \tag{3}$$

$$\rho C A_T h \frac{dT}{dt} = q_F \rho C (T_F - T) + U p_T h (T_s - T) \tag{4}$$

Then, Eqs. 3 and 4 constitute the dynamic model for the system.

a) Making Taylor series expansion of nonlinear terms in (4) and introducing deviation variables, Eqs. 3 and 4 become:

$$A_T \frac{dh'}{dt} = q'_F - q' \tag{5}$$

$$\rho C A_T \bar{h} \frac{dT'}{dt} = \rho C (T_F - \bar{T}) q'_F - (\rho C \bar{q}_F + U p_T \bar{h}) T' + U p_T \bar{h} T'_s + U p_T (\bar{T}_s - \bar{T}) h' \tag{6}$$

Taking Laplace transforms,

$$H'(s) = \frac{1}{A_T s} Q'_F(s) - \frac{1}{A_T s} Q'(s) \tag{7}$$

$$\left[\left(\frac{\rho C A_T \bar{h}}{\rho C \bar{q}_F + U p_T \bar{h}}\right)s + 1\right] T'(s) = \left[\frac{\rho C (T_F - \bar{T})}{\rho C \bar{q}_F + U p_T \bar{h}}\right] Q'_F(s) + \left[\frac{U p_T \bar{h}}{\rho C \bar{q}_F + U p_T \bar{h}}\right] T'_s(s) + \left[\frac{U p_T (\bar{T}_s - \bar{T})}{\rho C \bar{q}_F + U p_T \bar{h}}\right] H'(s) \tag{8}$$

Substituting for $H'(s)$ from (7) into (8) and rearranging gives

$$[A_T s]\left[\left(\frac{\rho C A_T \bar{h}}{\rho C \bar{q}_F + U p_T \bar{h}}\right)s + 1\right] T'(s) = \left[\frac{\rho C (T_F - \bar{T})}{\rho C \bar{q}_F + U p_T \bar{h}} A_T s\right] Q'_F(s) + \left[\frac{U p_T \bar{h} A_T s}{\rho C \bar{q}_F + U p_T \bar{h}}\right] T'_s(s) + \left[\frac{U p_T (\bar{T}_s - \bar{T})}{\rho C \bar{q}_F + U p_T \bar{h}}\right] [Q'_F(s) - Q'(s)] \tag{9}$$

Let $\tau = \dfrac{\rho C A_T \bar{h}}{\rho C \bar{q}_F + U p_T \bar{h}}$

Then from Eq. 7

$$\frac{H'(s)}{Q'_F(s)} = \frac{1}{A_T s} \quad , \quad \frac{H'(s)}{Q'(s)} = -\frac{1}{A_T s} \quad , \quad \frac{H'(s)}{T_s'(s)} = 0$$

And from Eq. 9

$$\frac{T'(s)}{Q'_F(s)} = \frac{\left[\dfrac{Up_T(\bar{T}_s - \bar{T})}{\rho C\bar{q}_F + Up_T\bar{h}}\right]\left[\left(\dfrac{\rho C(T_F - \bar{T})A_T}{Up_T(\bar{T}_s - \bar{T})}\right)s + 1\right]}{(A_T s)(\tau s + 1)}$$

$$\frac{T'(s)}{Q'(s)} = \frac{-\left[\dfrac{Up_T(\bar{T}_s - \bar{T})}{\rho C\bar{q}_F + Up_T\bar{h}}\right]}{(A_T s)(\tau s + 1)}$$

$$\frac{T'(s)}{T_s'(s)} = \frac{\left[\dfrac{Up_T\bar{h}}{\rho C\bar{q}_F + Up_T\bar{h}}\right]}{\tau s + 1}$$

Note:

$\tau_2 = \dfrac{\rho C(T_F - \bar{T})A_T}{Up_T(\bar{T}_s - \bar{T})}$ is the time constant in the numerator.

Because $T_F - \bar{T} < 0$ (heating) and $\bar{T}_s - T > 0$, τ_2 is negative.

We can show this property by using Eq. 2 at steady state:

$$\rho C\bar{q}_F(T_F - \bar{T}) = -Up_T\bar{h}(\bar{T}_s - \bar{T})$$

or $\rho C(T_F - \bar{T}) = \dfrac{-Up_T\bar{h}(\bar{T}_s - \bar{T})}{\bar{q}_F}$

Substituting

$$\tau_2 = -\frac{\bar{h}A_T}{\bar{q}_F}$$

Let $\bar{V} = \bar{h}A_T$ so that $\tau_2 = -\dfrac{\bar{V}}{\bar{q}_F} = -$(initial residence time of tank)

For $\dfrac{T'(s)}{Q'_F(s)}$ and $\dfrac{T'(s)}{Q'(s)}$ the “gain” in each transfer function is

$$K = \left[\frac{Up_T(\bar{T}_s - \bar{T})}{A_T\left(\rho C\bar{q}_F + Up_T\bar{h}\right)} \right]$$

and must have the units temp/volume .

(The integrator s has units of t^{-1}).

To simplify the transfer function gain we can substitute

$$Up_T(\bar{T}_s - \bar{T}) = -\frac{\rho C\bar{q}_F(\bar{T}_F - \bar{T})}{\bar{h}}$$

from the steady-state relation. Then

$$K = \frac{-\rho C\bar{q}_F(\bar{T}_F - \bar{T})}{\bar{h}A_T\left(\rho C\bar{q}_F + Up_T\bar{h}\right)}$$

or $$K = \frac{\bar{T} - \bar{T}_F}{\bar{V}\left(1 + \dfrac{Up_T\bar{h}}{\rho C\bar{q}_F}\right)}$$

and we see that the gain is positive since $\bar{T} - \bar{T}_F > 0$.

Further, it has dimensions of temp/volume.

(The ratio $\dfrac{Up_T\bar{h}}{\rho C\bar{q}_F}$ is dimensionless).

b) $h - q_F$ transfer function is an integrator with a positive gain. Liquid level accumulates any changes in q_F , increasing for positive changes and vice-versa.

$h - q$ transfer function is an integrator with a negative gain. h accumulates changes in q, in opposite direction, decreasing as q increases and vice versa.

$h - T_s$ transfer function is zero. Liquid level is independent of T_s, and of the steam pressure P_s .

$T - q$ transfer function is second-order due to the interaction with liquid level; it is the product of an integrator and a first-order process.

$T - q_F$ transfer function is second-order due to the interaction with liquid level and has numerator dynamics since q_F affects T directly as well if $T_F \neq \overline{T}$.

$T - T_s$ transfer function is simple first-order because there is no interaction with liquid level.

c) $h - q_F$: h increases continuously at a constant rate.

$h - q$: h decreases continuously at a constant rate.

$h - T_s$: h stays constant.

$T - q_F$: for $T_F < \overline{T}$, T decreases initially (inverse response) and then increases. After long times, T increases like a ramp function.

$T - q$: T decreases, eventually at a constant rate.

$T - T_s$: T increases with a first-order response and attains a new steady state.

6.22

a) The two-tank process is described by the following equations in deviation variables:

$$\frac{dh_1'}{dt} = \frac{1}{\rho A_1}\left[w_1' - \frac{1}{R}(h_1' - h_2')\right] \quad (1)$$

$$\frac{dh_2'}{dt} = \frac{1}{\rho A_2}\left[\frac{1}{R}(h_1' - h_2')\right] \quad (2)$$

Laplace transforming

$$\rho A_1 R s H_1'(s) = R W_i'(s) - H_1'(s) + H_2'(s) \quad (3)$$

$$\rho A_2 R s H_2'(s) = H_1'(s) - H_2'(s) \quad (4)$$

From (4)

$$(\rho A_2 Rs+1)H_2^{'}(s)=H_1^{'}(s) \tag{5}$$

or

$$\frac{H_2^{'}(s)}{H_1^{'}(s)}=\frac{1}{\rho A_2 Rs+1}=\frac{1}{\tau_2 s+1} \tag{6}$$

where $\tau_2=\rho A_2 R$

Returning to (3)

$$(\rho A_1 Rs+1)H_1^{'}(s)-H_2^{'}(s)=RW_i^{'}(s) \tag{7}$$

Substituting (6) with $\tau_1=\rho A_1 R$

$$\left[(\tau_1 s+1)-\frac{1}{\tau_2 s+1}\right]H_1^{'}(s)=RW_i^{'}(s) \tag{8}$$

or

$$\left[(\tau_1\tau_2)s^2+(\tau_1\tau_2)s\right]H_1^{'}(s)=R(\tau_2 s+1)W_i^{'}(s) \tag{9}$$

$$\frac{H_1^{'}(s)}{W_1^{'}(s)}=\frac{R(\tau_2 s+1)}{s\left[\tau_1\tau_2 s+(\tau_1+\tau_2)\right]} \tag{10}$$

Dividing numerator and denominator by $(\tau_1+\tau_2)$ to put into standard form

$$\frac{H_1^{'}(s)}{W_1^{'}(s)}=\frac{[R/(\tau_1+\tau_2)](\tau_2 s+1)}{s\left[\dfrac{\tau_1\tau_2}{\tau_1+\tau_2}s+1\right]} \tag{11}$$

Note that

$$K=\frac{R}{\tau_1+\tau_2}=\frac{R}{\rho A_1 R+\rho A_2 R}=\frac{1}{\rho(A_1+A_2)}=\frac{1}{\rho A} \tag{12}$$

since $A=A_1+A_2$

Also, let

$$\tau_s = \frac{\tau_1\tau_2}{\tau_1+\tau_2} = \frac{\rho^2 R^2 A_1 A_2}{\rho R(A_1+A_2)} = \frac{\rho R A_1 A_2}{A} \qquad (13)$$

so that

$$\frac{H_1'(s)}{W_i'(s)} = \frac{K(\tau_2 s+1)}{s(\tau_3 s+1)} \qquad (14)$$

and

$$\frac{H_2'(s)}{W_i'(s)} = \frac{H_2'(s)}{H_1'(s)}\frac{H_1'(s)}{W_i'(s)} = \frac{1}{(\tau_2 s+1)}\frac{K(\tau_2 s+1)}{s(\tau_3 s+1)}$$
$$= \frac{K}{s(\tau_3 s+1)} \qquad (15)$$

Transfer functions (6), (14) and (15) define the operation of the two-tank process.

The single-tank process is described by the following equation in deviation variables:

$$\frac{dh'}{dt} = \frac{1}{\rho A} w_i' \qquad (16)$$

Note that $\bar{\omega}$, which is constant, subtracts out.

Laplace transforming and rearranging:

$$\frac{H'(s)}{W_i'(s)} = \frac{1/\rho A}{s} \qquad (17)$$

Again

$$K = \frac{1}{\rho A}$$
$$\frac{H'(s)}{W_i'(s)} = \frac{K}{s} \qquad (18)$$

which is the expected integral relationship with no zero.

b) For $A_1 = A_2 = A/2$

$$\left.\begin{array}{l}\tau_2 = \rho AR/2 \\ \tau_3 = \rho AR/4\end{array}\right\} \tag{19}$$

Thus $\tau_2 = 2\tau_3$

We have two sets of transfer functions:

<u>One-Tank Process</u>

$$\frac{H'(s)}{W_i'(s)} = \frac{K}{s}$$

<u>Two-Tank Process</u>

$$\frac{H_i'(s)}{W_i'(s)} = \frac{K(2\tau_3 s+1)}{s(\tau_3 s+1)}$$

$$\frac{H_2'(s)}{W_i'(s)} = \frac{K}{s(\tau_3 s+1)}$$

Remarks:

- The gain $(K = 1/\rho A)$ is the same for all TF's.
- Also, each TF contains an integrating element.
- However, the two-tank TF's contain a pole $(\tau_3 s+1)$ that will "filter out" changes in level caused by changing $w_i(t)$.
- On the other hand, for this special case we see that the zero in the first tank transfer function $(H_i'(s)/W_i'(s))$ is larger than the pole

  $$2\,\tau_3 > \tau_3$$

  and we should make sure that amplification of changes in $h_1(t)$ caused by the zero do not more than cancel the beneficial filtering of the pole so as to cause the first compartment to overflow easily.
  Now look at more general situations of the two-tank case:

$$\frac{H_1'(s)}{W_i'(s)} = \frac{K(\rho A_2 Rs+1)}{s\left(\dfrac{\rho RA_1A_2}{A}s+1\right)} = \frac{K(\tau_2 s+1)}{s(\tau_3 s+1)} \tag{20}$$

$$\frac{H_2'(s)}{W_i'(s)} = \frac{K}{s(\tau_3 s+1)} \tag{21}$$

For either $A_1 \to 0$ or $A_2 \to 0$,

$$\tau_3 = \frac{\rho RA_1A_2}{A} \to 0$$

Thus the beneficial effect of the pole is lost as the process tends to "look" more like the first-order process.

c) The optimum filtering can be found by maximizing τ_3 with respect to A_1 (or A_2)

$$\tau_3 = \frac{\rho R A_1 A_2}{A} = \frac{\rho R A_1 (A - A_1)}{A}$$

Find max $\tau_3 : \dfrac{\partial \tau_3}{\partial A_1} = \dfrac{\rho R}{A}\left[(A - A_1) + A_1(-1)\right]$

Set to 0: $A - A_1 - A_1 = 0$

$$2A_1 = A$$

$$A_1 = A/2$$

Thus the maximum filtering action is obtained when $A_1 = A_2 = A/2$.

The ratio of τ_2 / τ_3 determines the "amplification effect" of the zero on $h_1(t)$.

$$\frac{\tau_2}{\tau_3} = \frac{\rho A_2 R}{\dfrac{\rho R A_1 A_2}{A}} = \frac{A}{A_1}$$

As A_1 goes to 0, $\dfrac{\tau_2}{\tau_3} \rightarrow \infty$

Therefore the influence of changes in $w_i(t)$ on $h_1(t)$ will be very large, leading to the possibility of overflow in the first tank.

Summing up:

The process designer would like to have $A_1 = A_2 = A/2$ in order to obtain the maximum filtering of $h_1(t)$ and $h_2(t)$. However, the process response should be checked for typical changes in $w_i(t)$ to make sure that h_1 does not overflow. If it does, the area A_1 needs to be increased until that is not
a problem.

Note that $\tau_2 = \tau_3$ when $A_1 = A$, thus someone must make a careful study (simulations) before designing the partitioned tank. Otherwise, leave well enough alone and use the non-partitioned tank.

6.23

The process transfer function is

$$\frac{Y(s)}{U(s)} = G(s) = \frac{K}{(0.1s+1)^2(4s^2+2s+1)}$$

where $K = K_1K_2$

We note that the quadratic term describes an underdamped 2nd-order system since

$$\tau^2 = 4 \qquad \rightarrow \qquad \tau = 2$$

$$2\zeta\tau = 2 \qquad \rightarrow \qquad \zeta = 0.5$$

a) For the second-order process element with τ_2 = 2 and this degree of underdamping $(\zeta = 0.5)$, the small time constant, critically damped 2nd-order process element (τ_1 = 0.1) will have little effect.

In fact, since 0.1 << τ_2 (= 2) we can approximate the critically damped element as $e^{-2\tau_1}$ so that

$$G(s) \approx \frac{Ke^{-0.2s}}{4s^2+2s+1}$$

b) From Fig. 5.11 for $\zeta = 0.5$, $OS \approx 0.15$ or from Eq. 5-53

$$\text{Overshoot} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 0.163$$

Hence y_{max} = 0.163 KM + KM = 0.163 (1) (3) + 3 = 3.5

c) From Fig. 5.4, y_{max} occurs at t/τ = 3K or t_{max} = 6.8 for underdamped 2nd-order process with $\zeta = 0.5$.

Adding in effect of time delay t' = 6.8 + 0.2 = 7.0

d) By using Simulink-MATLAB

$\tau_1 = 0.1$

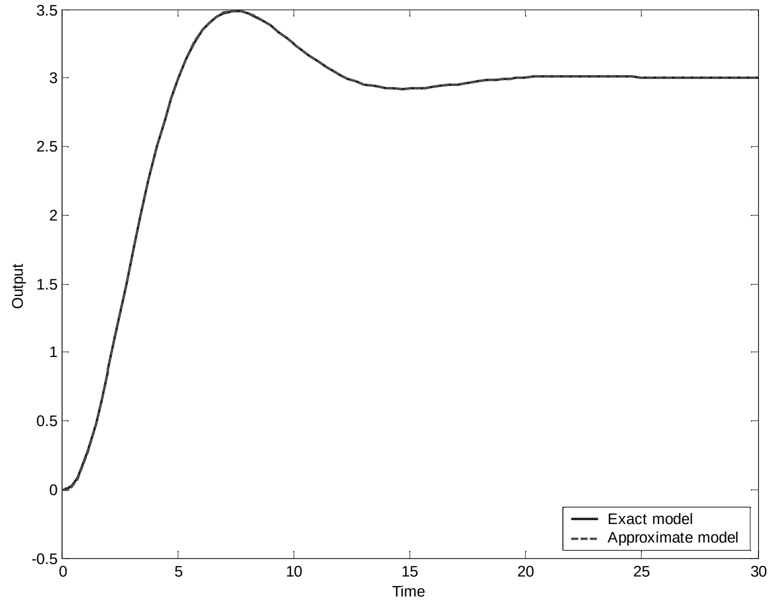


Fig S6.23a. *Step response for exact and approximate model ;* $\tau_1 = 0.1$

$\tau_1 = 1$

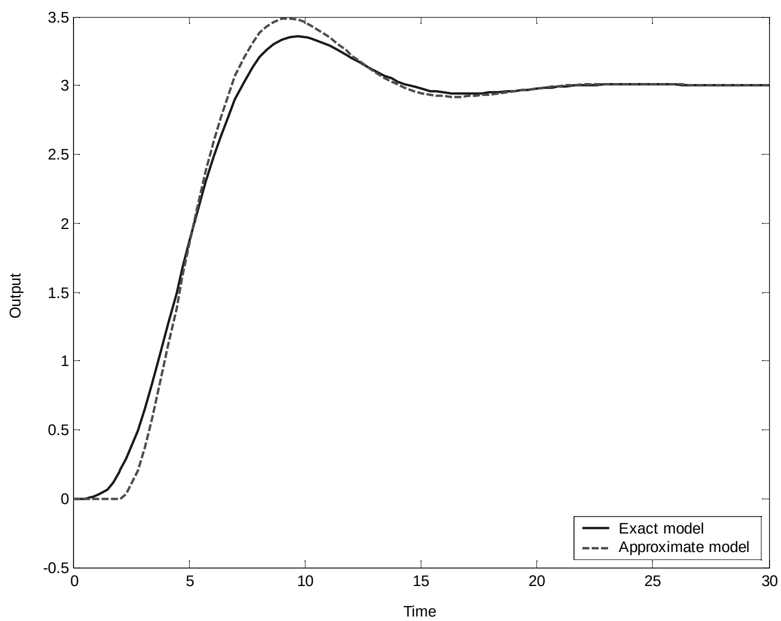


Fig S6.23b. *Step response for exact and approximate model ;* $\tau_1 = 1$

$\tau_1 = 5$

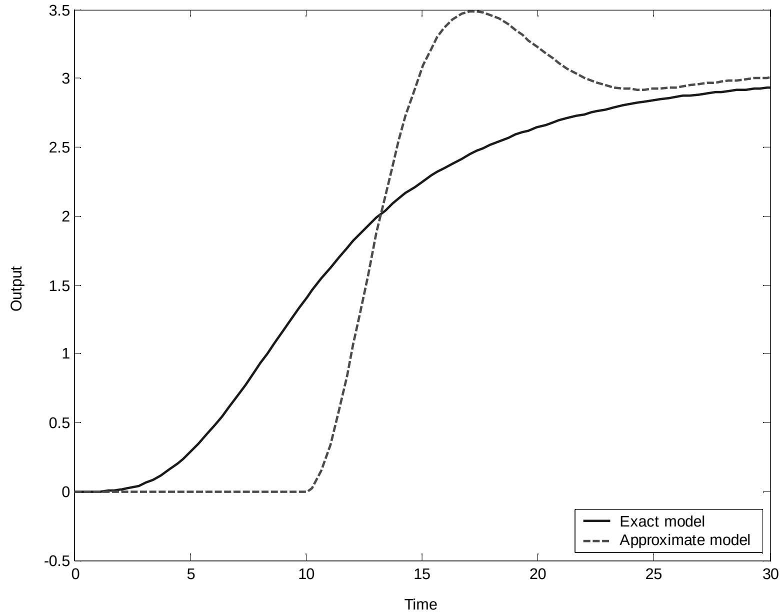


Fig S6.23c. *Step response for exact and approximate model ; $\tau_1 = 5$*

As noted in plots above, the smaller τ_1 is, the better the quality of the approximation. For large values of τ_1 (on the order of the underdamped element's time scale), the approximate model fails.

6.24

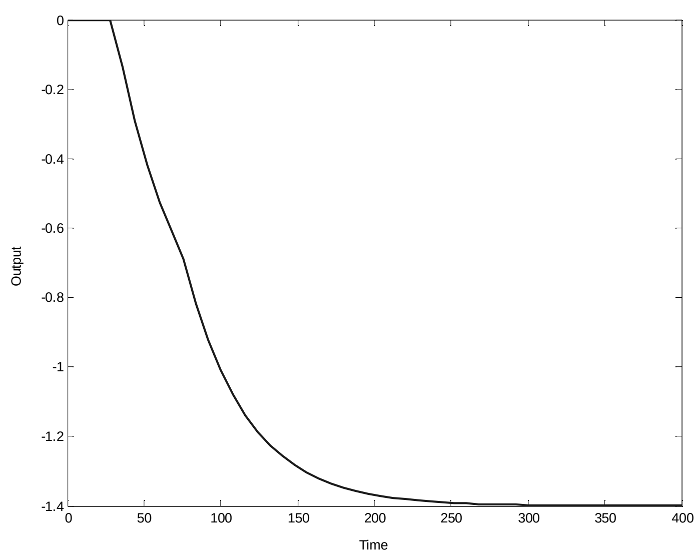
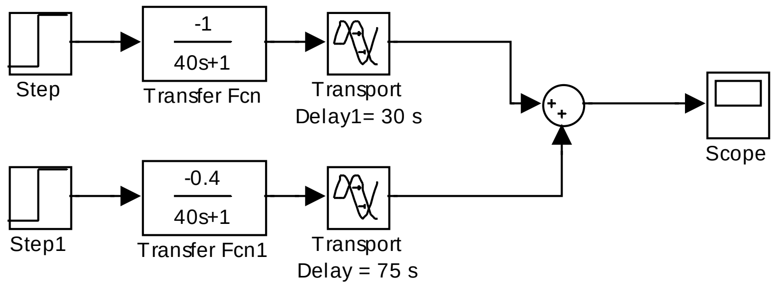


Figure S6.24. *Unit step response in blood pressure.*

The Simulink-MATLAB block diagram is shown below



It appears to respond approx. as a first-order or overdamped second-order process with time delay.