Chapter 7

7.1

In the absence of more accurate data, use a first-order transfer function as

$$\frac{T'(s)}{Q_i'(s)} = \frac{Ke^{-\theta s}}{\tau s+1}$$

$$K = \frac{T(\infty)-T(0)}{\Delta q_i} = \frac{(124.7-120)}{540-500} = 0.118\frac{^\circ\text{F}}{\text{gal/min}}$$

θ = 3:09 am – 3:05 am = 4 min

Assuming that the operator logs a 99% complete system response as "no change after 3:34 am", 5 time constants elapse between 3:09 and 3:34 am.

5τ = 3:34 min − 3:09 min = 25 min
τ = 25/5 min = 5 min

Therefore,

$$\frac{T'(s)}{Q_i'(s)} = \frac{0.188e^{-4s}}{5s+1}$$

To obtain a better estimate of the transfer function, the operator should log more data between the first change in *T* and the new steady state.

7.2

Process gain, $K = \dfrac{h(5.0)-h(0)}{\Delta q_i} = \dfrac{(6.52-5.50)}{30.4\times 0.1} = 0.336\dfrac{\text{min}}{\text{ft}^2}$

a) Output at 63.2% of the total change

= 5.50 + 0.632(6.52-5.50) = 6.145 ft

Interpolating between h = 6.07 ft and h = 6.18 ft

$$\tau = 0.6 + \frac{(0.8-0.6)}{(6.18-6.07)}(6.145-6.07)\,\text{min} = 0.74\,\text{min}$$

b)

$$\left.\frac{dh}{dt}\right|_{t=0} \approx \frac{h(0.2)-h(0)}{0.2-0} = \frac{5.75-5.50}{0.2}\frac{\text{ft}}{\text{min}} = 1.25\frac{\text{ft}}{\text{min}}$$

Using Eq. 7-15,

$$\tau = \frac{KM}{\left(\left.\frac{dh}{dt}\right|_{t=0}\right)} = \frac{0.347\times(30.4\times 0.1)}{1.25} = 0.84\,\text{min}$$

c) The slope of the linear fit between t_i and $z_i \equiv \ln\left[1-\frac{h(t_i)-h(0)}{h(\infty)-h(0)}\right]$ gives an approximation of $(-1/\tau)$ according to Eq. 7-13.

Using $h(\infty) = h(5.0) = 6\ .52$, the values of z_i are

t_i	z_i
0.0	0.00
0.2	-0.28
0.4	-0.55
0.6	-0.82
0.8	-1.10
1.0	-1.37
1.2	-1.63

t_i	z_i
1.4	-1.92
1.6	-2.14
1.8	-2.43
2.0	-2.68
3.0	-3.93
4.0	-4.62
5.0	$-\infty$

Then the slope of the best-fit line, using Eq. 7-6 is

$$slope = \left(-\frac{1}{\tau}\right) = \frac{13S_{tz} - S_t S_z}{13S_{tt} - (S_t)^2} \qquad (1)$$

where the datum at $t_i = 5.0$ has been ignored.

Using definitions,

$$S_t = 18.0 \qquad S_{tt} = 40.4$$
$$S_z = -23.5 \qquad S_{tz} = -51.1$$

Substituting in (1),

$$\left(-\frac{1}{\tau}\right) = -1.213 \qquad \tau = 0.82\,\text{min}$$

d)

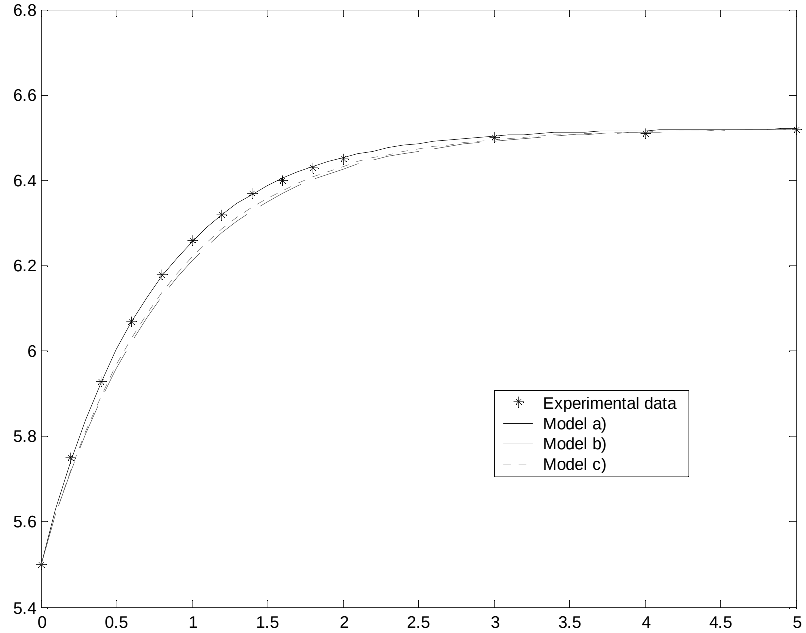

Figure S7.2. *Comparison between models a), b) and c) for step response.*

7.3

a)

$$\frac{T_1'(s)}{Q'(s)} = \frac{K_1}{\tau_1 s+1} \qquad \frac{T_2'(s)}{T_1'(s)} = \frac{K_2}{\tau_2 s+1}$$

$$\frac{T_2'(s)}{Q'(s)} = \frac{K_1 K_2}{(\tau_1 s+1)(\tau_2 s+1)} \approx \frac{K_1 K_2 e^{-\tau_2 s}}{(\tau_1 s+1)} \tag{1}$$

where the approximation follows from Eq. 6-58 and the fact that $\tau_1 > \tau_2$ as revealed by an inspection of the data.

$$K_1 = \frac{T_1(50)-T_1(0)}{\Delta q} = \frac{18.0-10.0}{85-82} = 2.667$$

$$K_2 = \frac{T_2(50)-T_2(0)}{T_1(50)-T_1(0)} = \frac{26.0-20.0}{18.0-10.0} = 0.75$$

Let z_1, z_2 be the natural log of the fraction incomplete response for T_1,T_2, respectively. Then,

$$z_1(t) = \ln\left[\frac{T_1(50) - T_1(t)}{T_1(50) - T_1(0)}\right] = \ln\left[\frac{18 - T_1(t)}{8}\right]$$

$$z_2(t) = \ln\left[\frac{T_2(50) - T_2(t)}{T_2(50) - T_2(0)}\right] = \ln\left[\frac{26 - T_2(t)}{6}\right]$$

A graph of z_1 and z_2 versus t is shown below. The slope of z_1 versus t line is –0.333 ; hence $(1/-\tau_1)=-0.333$ and $\tau_1=3.0$

From the best-fit line for z_2 versus t, the projection intersects $z_2 = 0$ at $t \approx 1.15$. Hence $\tau_2 = 1.15$.

$$\frac{T_1'(s)}{Q'(s)} = \frac{2.667}{3s+1} \qquad (2)$$

$$\frac{T_2'(s)}{T_1'(s)} = \frac{0.75}{1.15s+1} \qquad (3)$$

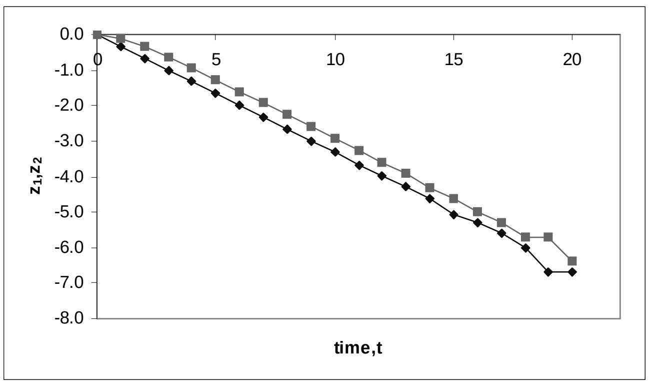

Figure S7.3a. *z_1 and z_2 versus t*

b) By means of Simulink-MATLAB, the following simulations are obtained

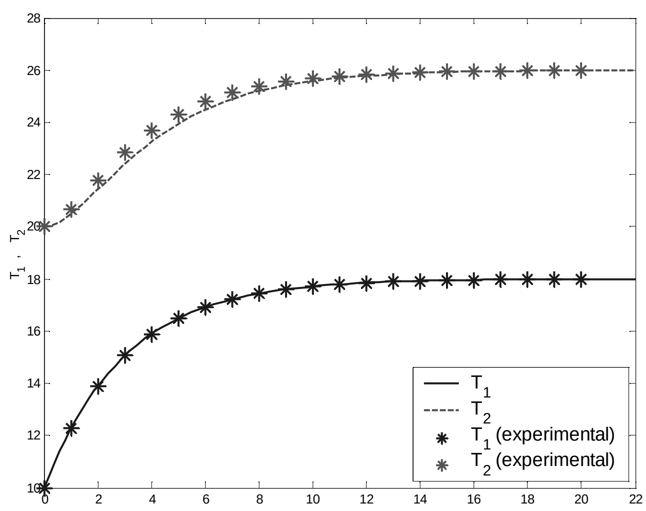

Figure S7.3b. *Comparison of experimental data and models for step change*

7.4

$$Y(s) = G(s)\ X(s) = \frac{2}{(5s+1)(3s+1)(s+1)} \times \frac{1.5}{s}$$

Taking the inverse Laplace transform

$y(t)$ = -75/8*exp(-1/5*t)+27/4*exp(-1/3*t)-3/8*exp(-t)+3 (1)

a) Fraction incomplete response

$$z(t) = \ln\left[1 - \frac{y(t)}{3}\right]$$

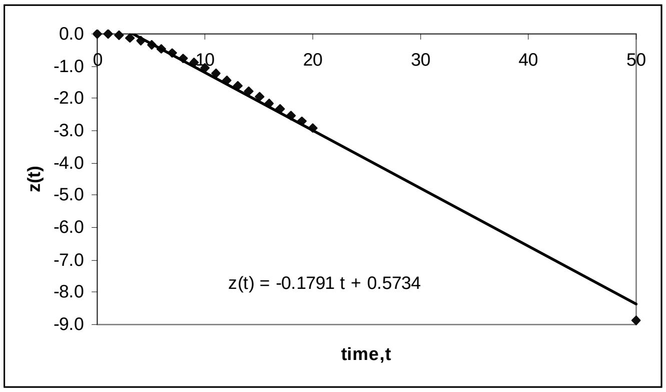

Figure S7.4a. *Fraction incomplete response; linear regression*

From the graph, slope = -0.179 and intercept ≈ 3.2

Hence,

$-1/\tau = -0.179$ and $\tau = 5.6$

$\theta = 3.2$

$$G(s) = \frac{2e^{-3.2s}}{5.6s+1}$$

b) In order to use Smith's method, find t_{20} and t_{60}

$y(t_{20})$= 0.2 × 3 =0.6
$y(t_{60})$= 0.6 × 3 =1.8

Using either Eq. 1 or the plot of this equation, $t_{20} = 4.2$, $t_{60} = 9.0$
Using Fig. 7.7 for $t_{20}/t_{60} = 0.47$
$\zeta = 0.65$, $t_{60}/\tau = 1.75$, and $\tau = 5.14$

$$G(s) \approx \frac{2}{26.4s^2 + 6.68s + 1}$$

The models are compared in the following graph:

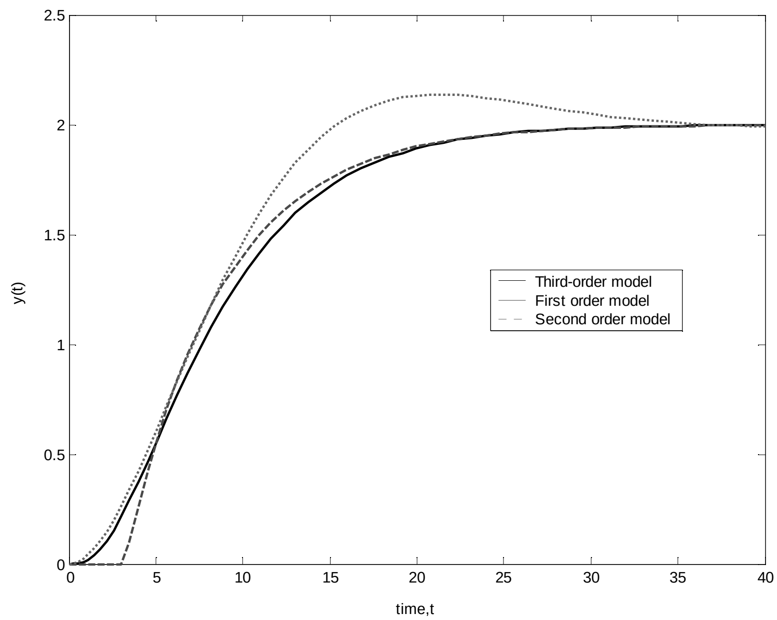

Figure S7.4b. *Comparison of three models for step input*

7.5

The integrator plus time delay model is

$$G(\mathrm{s})\ \frac{K}{s}e^{-\theta s}$$

In the time domain,

$y(t) = 0 \qquad\qquad t < 0$

$y(t)= K\,(t\text{-}\theta) \qquad t \geq 0$

Thus a straight line tangent to the point of inflection will approximate the step response. Two parameters must be found: K and θ (See Fig. S7.5 a)

1.- The process gain K is found by calculating the slope of the straight line.

$$K = \frac{1}{13.5} = 0.074$$

2.- The time delay is evaluated from the intersection of the straight line and the time axis (where $y = 0$).

$$\theta = 1.5$$

Therefore the model is $G(s) = \dfrac{0.074}{s} e^{-1.5s}$

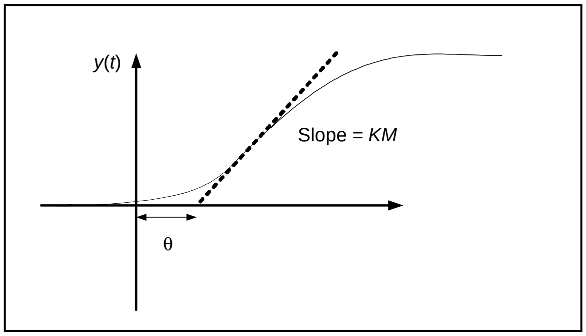

Figure S7.5a. *Integrator plus time delay model*; *parameter evaluation*

From Fig. E7.5, we can read these values (approximate):

Time	Data	Model
0	0	-0.111
2	0.1	0.037
4	0.2	0.185
5	0.3	0.259
7	0.4	0.407
8	0.5	0.481
9	0.6	0.555
11	0.7	0.703
14	0.8	0.925
16.5	0.9	1.184
30	1	2.109

Table.- *Output values from Fig. E7.5 and predicted values by model*

A graphical comparison is shown in Fig. S7.5 b

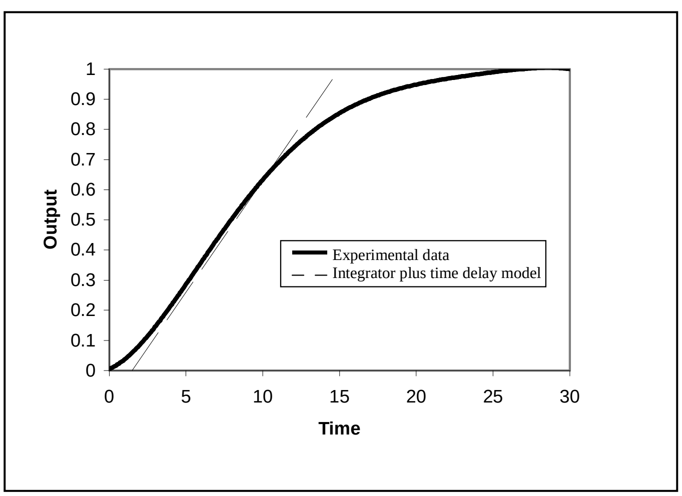

Figure S7.5b. *Comparison between experimental data and integrator plus time delay model.*

7.6

a) Drawing a tangent at the inflection point which is roughly at $t \approx 5$, the intersection with $y(t)=0$ line is at t ≈ 1 and with the $y(t)=1$ line at $t \approx 14$.
Hence $\theta = 1$, $\tau = 14 - 1 = 13$

$$G_1(s) \approx \frac{e^{-s}}{13s+1}$$

b) Smith's method

From the graph, $t_{20} = 3.9$, $t_{60} = 9.6$; using Fig 7.7 for $t_{20}/t_{60} = 0.41$

$\zeta = 1.0$, $t_{60}/\tau = 2.0$, hence $\tau = 4.8$ and $\tau_1 = \tau_2 = \tau = 4.8$

$$G(s) \approx \frac{1}{(4.8s+1)^2}$$

Nonlinear regression

From Figure E7.5, we can read these values (approximated):

Time	Output
0.0	0.0
2.0	0.1
4.0	0.2
5.0	0.3
7.0	0.4
8.0	0.5
9.0	0.6
11.0	0.7
14.0	0.8
17.5	0.9
30.0	1.0

Table.- *Output values from Figure E7.5*

In accounting for Eq. 5-48, the time constants were selected to minimize the sum of the squares of the errors between data and model predictions. Use Excel Solver for this Optimization problem:

$\tau_1 = 6.76$ and $\tau_2 = 6.95$

$$G(s) \approx \frac{1}{(6.95s+1)(6.76s+1))}$$

The models are compared in the following graph:

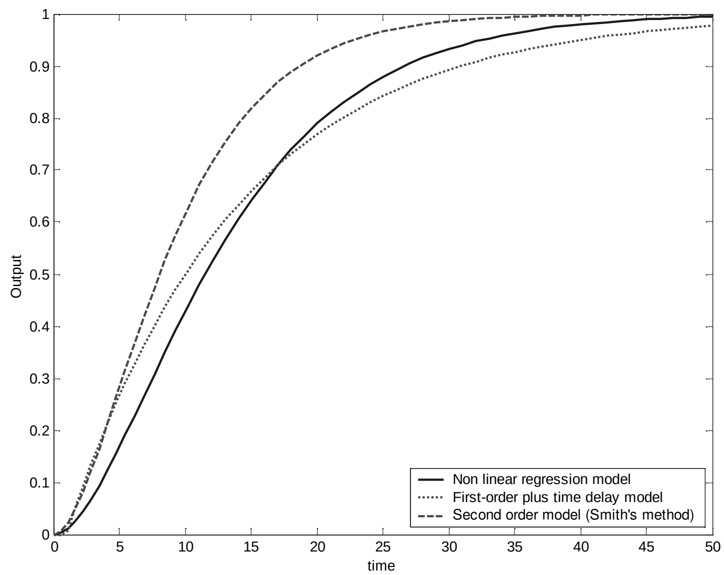

Figure S7.6. *Comparison of three models for unit step input*

7.7

a) From the graph, time delay θ = 4.0 min

Using Smith's method,

from the graph, $t_{20} + \theta \approx 5.6$, $t_{60} + \theta \approx 9.1$

$t_{20} = 1.6$, $t_{60} = 5.1$, $t_{20}/t_{60} = 1.6/5.1 = 0.314$

From Fig.7.7 , $\zeta = 1.63$, $t_{60}/\tau = 3.10$, $\tau = 1.645$

Using Eqs. 5-45, 5-46, $\tau_1 = 4.81$, $\tau_2 = 0.56$

b) Overall transfer function

$$G(s) = \frac{10e^{-4s}}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad , \quad \tau_1 > \tau_2$$

Assuming plug-flow in the pipe with constant-velocity,

$$G_{pipe}(s) = e^{-\theta_p s} \quad , \quad \theta_p = \frac{3}{0.5} \times \frac{1}{60} = 0.1 \min$$

Assuming that the thermocouple has unit gain and no time delay

$$G_{TC}(s) = \frac{1}{(\tau_2 s + 1)} \quad \text{since} \quad \tau_2 << \tau_1$$

Then

$$G_{HE}(s) = \frac{10e^{-3s}}{(\tau_1 s + 1)} \quad \text{, so that}$$

$$G(s) = G_{HE}(s) G_{pipe}(s) G_{TC}(s) = \left(\frac{10e^{-3s}}{\tau_1 s + 1} \right) (e^{-0.1s}) \left(\frac{1}{\tau_2 s + 1} \right)$$

7.8

a) To find the form of the process response, we can see that

$$Y(s) = \frac{K}{s(\tau s + 1)} U(s) = \frac{K}{s(\tau s + 1)} \frac{M}{s} = \frac{K}{(\tau s + 1)} \frac{M}{s^2}$$

Hence the response of this system is similar to a first-order system with a ramp input: the ramp input yields a ramp output that will ultimately cause some process component to saturate.

b) By applying partial fraction expansion technique, the domain response for this system is

$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{\tau s + 1}$ hence $y(t) = -KM\tau + KMt - KM\tau e^{-t/\tau}$

In order to evaluate the parameters K and τ, important properties of the above expression are noted:

1.- For large values of time ($t >> \tau$) , $y(t) \approx y'(t) = KM(t - \tau)$

2.- For $t = 0$, $y'(0) = -KM\tau$

These equations imply that after an initial transient period, the ramp input yields a ramp output with slope equal to KM. That way, the gain K is obtained. Moreover, the time constant τ is obtained from the intercept in Fig. S7.8

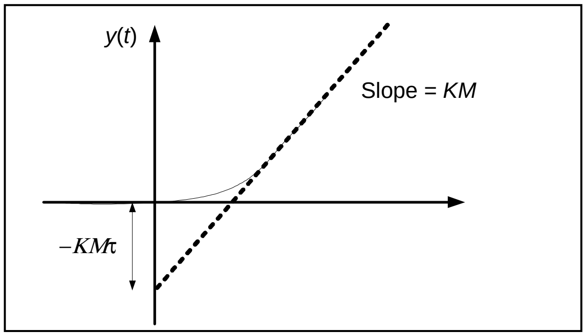

Figure S7.8. *Time domain response and parameter evaluation*

7.9

For underdamped responses,

$$y(t) = KM\left\{1 - e^{-\zeta t/\tau}\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right\} \qquad (5\text{-}51)$$

a) At the response peaks,

$$\frac{dy}{dt} = KM\left\{\frac{\zeta}{\tau}e^{-\zeta t/\tau}\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right.$$

$$\left. - e^{-\zeta t/\tau}\left[-\frac{\sqrt{1-\zeta^2}}{\tau}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \frac{\zeta}{\tau}\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right\} = 0$$

Since $KM \neq 0$ and $e^{-\zeta t/\tau} \neq 0$

$$0 = \left(\frac{\zeta}{\tau} - \frac{\zeta}{\tau}\right)\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \left(\frac{\zeta^2}{\tau\sqrt{1-\zeta^2}} + \frac{\sqrt{1-\zeta^2}}{\tau}\right)\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)$$

$$0 = \sin\left(\frac{\sqrt{1-\zeta^2}}{\tau} t\right) = \sin n\pi \quad , \quad t = n\frac{\pi\tau}{\sqrt{1-\zeta^2}}$$

where n is the number of peak.

Time to the first peak, $\quad t_p = \dfrac{\pi\tau}{\sqrt{1-\zeta^2}}$

b) <u>Graphical approach</u>:

Process gain,

$$K = \frac{w_D(\infty) - w_D(0)}{\Delta Ps} = \frac{9890-9650}{95-92} = 80 \frac{\text{lb}}{\text{hr}/\text{psig}}$$

Overshoot = $\dfrac{a}{b} = \dfrac{9970-9890}{9890-9650} = 0.333$

From Fig. 5.11, $\quad \zeta \approx 0.33$

t_p can be calculated by interpolating Fig. 5.8

For $\zeta \approx 0.33$, $t_p \approx 3.25\ \tau$

Since t_p is known to be 1.75 hr , $\tau = 0.54$

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} = \frac{80}{0.29 s^2 + 0.36 s + 1}$$

<u>Analytical approach</u>

The gain K doesn't change: $K = 80 \dfrac{\text{lb}}{\text{hr}/\text{psig}}$

To obtain the ζ and τ values, Eqs. 5-52 and 5-53 are used:

Overshoot = $\dfrac{a}{b} = \dfrac{9970-9890}{9890-9650} = 0.333 \ = \ \exp(-\zeta\pi/(1-\zeta^2)^{1/2})$

Resolving, $\zeta = 0.33$

$$t_p = \frac{\pi\tau}{\sqrt{1-\zeta^2}} = 1.754 \qquad \text{hence} \qquad \tau = 0.527 \text{ hr}$$

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1} = \frac{80}{0.278s^2 + 0.35s + 1}$$

c) Graphical approach

From Fig. 5.8, $t_s/\tau = 13$ so $t_s = 2$ hr (very crude estimation)

Analytical approach

From settling time definition,

$y = \pm 5\% \, KM$ so $9395.5 < y < 10384.5$

$(KM \pm 5\% \, KM) = KM[\ 1\text{-}e^{(-0.633)}[\cos(1.793t_s)+0.353\sin(1.793t_s)]]$

$1 \pm 0.05 = 1 - e^{(0.633\ t_s)} \cos(1.793\ t_s) + 0.353e^{(-0.633\ t_s)} \sin(1.7973\ t_s)$

Solve by trial and error…………………… $t_s \approx 6.9$ hrs

7.10

a) $$\frac{T'(s)}{W'(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau + 1}$$

$$K = \frac{T(\infty) - T(0)}{\Delta w} = \frac{156 - 140}{80} = 0.2 \frac{^\circ\text{C}}{\text{Kg/min}}$$

From Eqs. 5-53 and 5-55,

Overshoot = $\dfrac{a}{b} = \dfrac{161.5 - 156}{156 - 140} = 0.344$ = $\exp(-\zeta\pi\ (1-\zeta^2)^{1/2}$

By either solving the previous equation or from Figure 5.11, $\zeta = 0.322$ (dimensionless)

There are two alternatives to find the time constant τ :

1.- From the time of the first peak, $t_p \approx 33$ min.

One could find an expression for t_p by differentiating Eq. 5-51 and solving for t at the first zero. However, a method that should work (within required engineering accuracy) is to interpolate a value of ζ=0.35 in Figure 5.8 and note that $t_p/\tau \approx 3$

Hence $\tau \approx \dfrac{33}{3.5} \approx 9.5 - 10\,\text{min}$

2.- From the plot of the output,

$$\text{Period} = P = \frac{2\pi\tau}{\sqrt{1-\zeta^2}} = 67 \text{ min} \quad \text{and hence } \tau = 10 \text{ min}$$

Therefore the transfer function is

$$G(s) = \frac{T'(s)}{W'(s)} = \frac{0.2}{100s^2 + 6.44s + 1}$$

b) After an initial period of oscillation, the ramp input yields a ramp output with slope equal to *KB*. The MATLAB simulation is shown below:

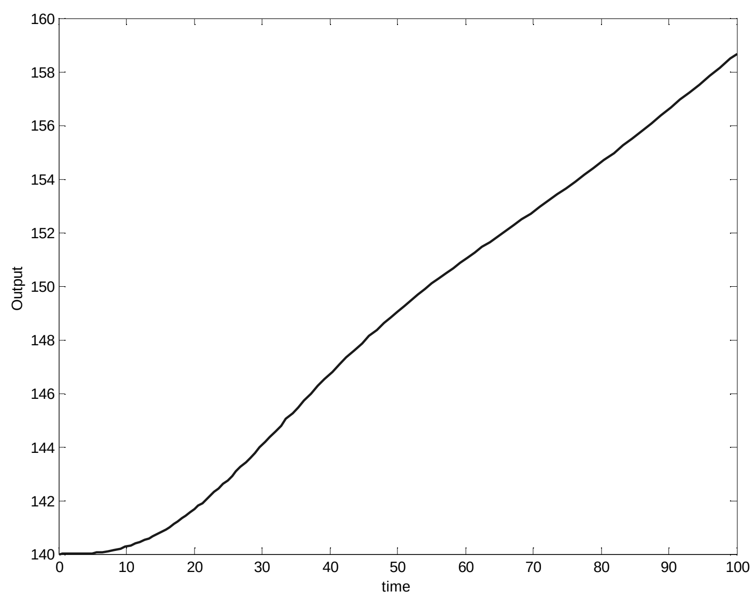

Figure S7.10. *Process output for a ramp input*

We know the response will come from product of G(s) and $X_{ramp} = B/s^2$

Then $Y(s) = \dfrac{KB}{s^2(\tau^2 s^2 + 2\zeta\tau s + 1)}$

From the ramp response of a first-order system we know that the response will asymptotically approach a straight line with slope = KB. Need to find the intercept. By using partial fraction expansion:

$$Y(s) = \frac{KB}{s^2(\tau^2 s^2 + 2\zeta\tau s + 1)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \frac{\alpha_3 s + \alpha_4}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Again by analogy to the first-order system, we need to find only α_1 and α_2. Multiply both sides by s^2 and let $s \rightarrow 0$, $\alpha_2 = KB$ (as expected)

Can't use Heaviside for α_1, so equate coefficients

$$KB = \alpha_1 s(\tau^2 s^2 + 2\zeta\tau s + 1) + \alpha_2(\tau^2 s^2 + 2\zeta\tau s + 1) + \alpha_3 s^3 + \alpha_4 s^2$$

We can get an expression for α_1 in terms of α_2 by looking at terms containing s.

s: $0 = \alpha_1 + \alpha_2 2\zeta\tau \quad \rightarrow \quad \alpha_1 = -KB2\zeta\tau$

and we see that the intercept with the time axis is at $t = 2\zeta\tau$. Finally, presuming that there must be some oscillatory behavior in the response, we sketch the probable response (See Fig. S7.10)

7.11

a) Replacing τ by 5, and K by 6 in Eq. 7-34

$$y(k) = e^{-\Delta t/5} y(k-1) + [1 - e^{-\Delta t/5}]6u(k-1)$$

b) Replacing τ by 5, and K by 6 in Eq. 7-32

$$y(k) = (1 - \frac{\Delta t}{5})y(k-1) + \frac{\Delta t}{5}6u(k-1)$$

In the integrated results tabulated below, the values for $\Delta t = 0.1$ are shown only at integer values of t, for comparison.

t	$y(k)$ (exact)	$y(k)$ (Δt=1)	$y(k)$ (Δt=0.1)
0	3	3	3
1	2.456	2.400	2.451
2	5.274	5.520	5.296
3	6.493	6.816	6.522
4	6.404	6.653	6.427
5	5.243	5.322	5.251
6	4.293	4.258	4.290
7	3.514	3.408	3.505
8	2.877	2.725	2.864
9	2.356	2.180	2.340
10	1.929	1.744	1.912

Table S7.11. *Integrated results for the first order differential equation*

Thus Δt = 0.1 does improve the finite difference model bringing it closer to the exact model.

7.12

To find a_1' and b_1, use the given first order model to minimize

$$J = \sum_{n=1}^{10} (y(k) - a_1' y(k-1) - b_1 x(k-1))^2$$

$$\frac{\partial J}{\partial a_1'} = \sum_{n=1}^{10} 2(y(k) - a_1' y(k-1) - b_1 x(k-1))(-y(k-1) = 0$$

$$\frac{\partial J}{\partial b_1} = \sum_{n=1}^{10} 2(y(k) - a_1' y(k-1) - b_1 x(k-1))(-x(k-1)) = 0$$

Solving simultaneously for a_1' and b_1 gives

$$a_1' = \frac{\sum_{n=1}^{10} y(k)y(k-1) - b_1 \sum_{n=1}^{10} y(k-1)x(k-1)}{\sum_{n=1}^{10} y(k-1)^2}$$

$$b_1 = \frac{\sum_{n=1}^{10} x(k-1)y(k) \sum_{n=1}^{10} y(k-1)^2 - \sum_{n=1}^{10} y(k-1)x(k-1) \sum_{n=1}^{10} y(k-1)y(k)}{\sum_{n=1}^{10} x(k-1)^2 \sum_{n=1}^{10} y(k-1)^2 - \left(\sum_{n=1}^{10} y(k-1)x(k-1) \right)^2}$$

Using the given data,

$$\sum_{n=1}^{10} x(k-1)y(k) = 35.212 \quad , \quad \sum_{n=1}^{10} y(k-1)y(k) = 188.749$$

$$\sum_{n=1}^{10} x(k-1)^2 = 14 \quad , \quad \sum_{n=1}^{10} y(k-1)^2 = 198.112$$

$$\sum_{n=1}^{10} y(k-1)x(k-1) = 24.409$$

Substituting into expressions for a_1' and b_1 gives

$$a_1' = 0.8187 \qquad , \qquad b_1 = 1.0876$$

Fitted model is $\quad y(k+1) = 0.8187y(k) + 1.0876x(k)$

or $\qquad y(k) = 0.8187y(k-1) + 1.0876x(k-1) \qquad (1)$

Let the first-order continuous transfer function be

$$\frac{Y(s)}{X(s)} = \frac{K}{\tau s + 1}$$

From Eq. 7-34, the discrete model should be

$$y(k) = e^{-\Delta t/\tau} y(k-1) + [1 - e^{-\Delta t/\tau}]Kx(k-1) \qquad (2)$$

Comparing Eqs. 1 and 2, for Δt=1, gives

$$\tau = 5 \quad \text{and} \quad K = 6$$

Hence the continuous transfer function is 6/(5s+1)

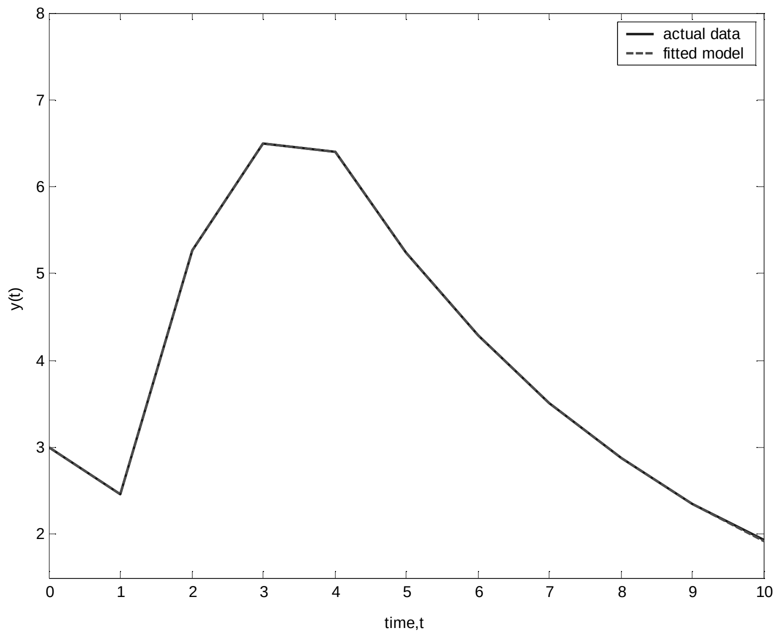

Figure S7.12. *Response of the fitted model and the actual data*

7.13

To fit a first-order discrete model

$$y(k) = a_1' y(k-1) + b_1 x(k-1)$$

Using the expressions for a_1' and b_1 from the solutions to Exercise 7.12, with the data in Table E7.12 gives

$$a_1' = 0.918 \quad , \; b_1 = 0.133$$

Using the graphical (tangent) method of Fig.7.5 .

$$K = 1 \; , \;\; \theta = 0.68, \text{ and } \;\; \tau = 6.8$$

The response to unit step change for the first-order model given by

$$\frac{e^{-0.68s}}{6.8s+1} \quad \text{is} \quad y(t) = 1 - e^{-(t-0.68)/6.8}$$

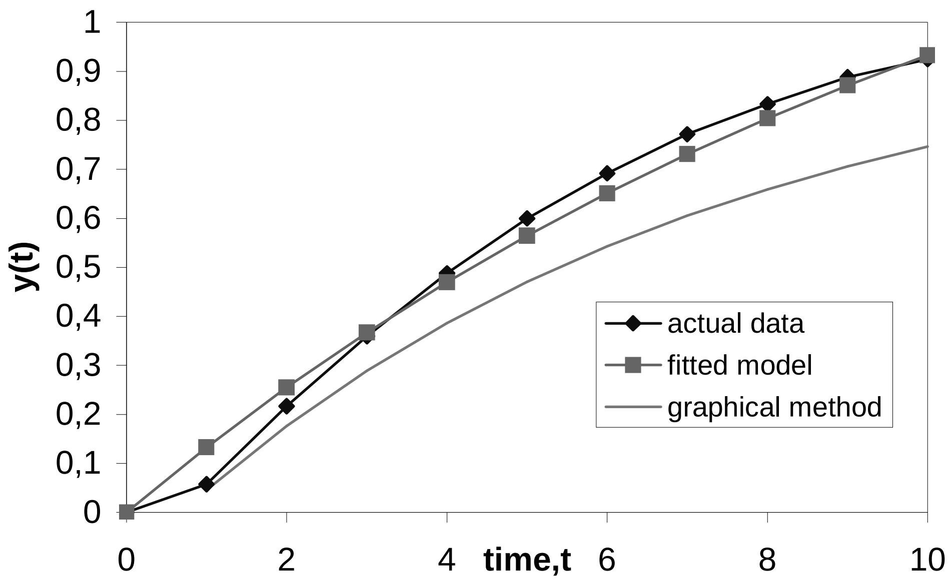

Figure S7.13- *Response of the fitted model, actual data and graphical method*