Chapter 18

18.1

McAvoy has reported the PI controller settings shown in Table S18.1 and the set-point responses of Fig. S18.1a and S18.1b. When both controllers are in automatic with Z-N settings, undesirable damped oscillations result due to the control loop interactions. The multiloop tuning method results in more conservative settings and more sluggish responses.

Controller Pairing	Tuning Method	K_c	τ_I(min)
T_{17} - R	Single loop/Z-N	-2.92	3.18
T_4 - S	Single loop/Z-N	4.31	1.15
T_{17} - R	Multiloop	-2.59	2.58
T_4 - S	Multiloop	4.39	2.58

Table S18.1. *Controller Settings for Exercise 18.1*

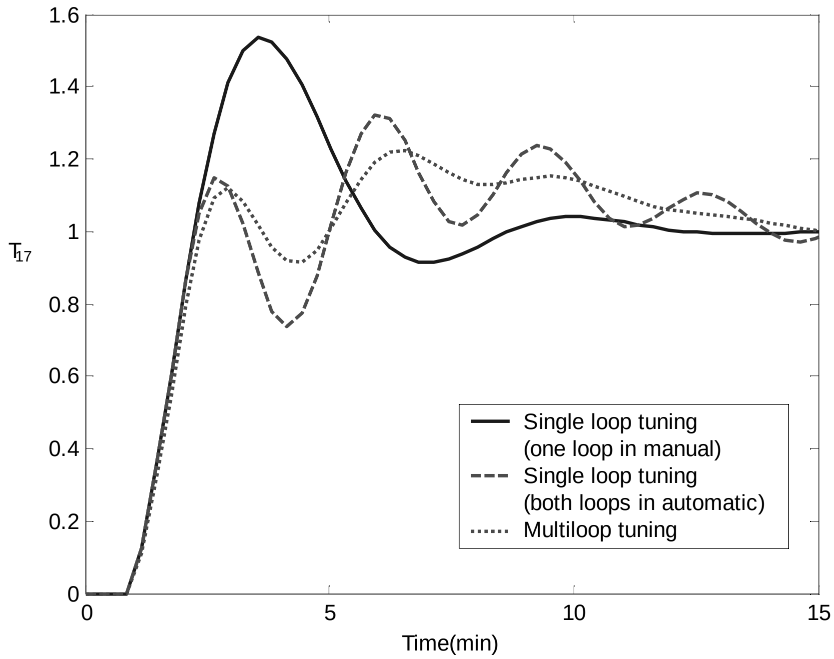


Figure S18.1a. *Set point responses for Exercise 18.1. Analysis for T_{17}*

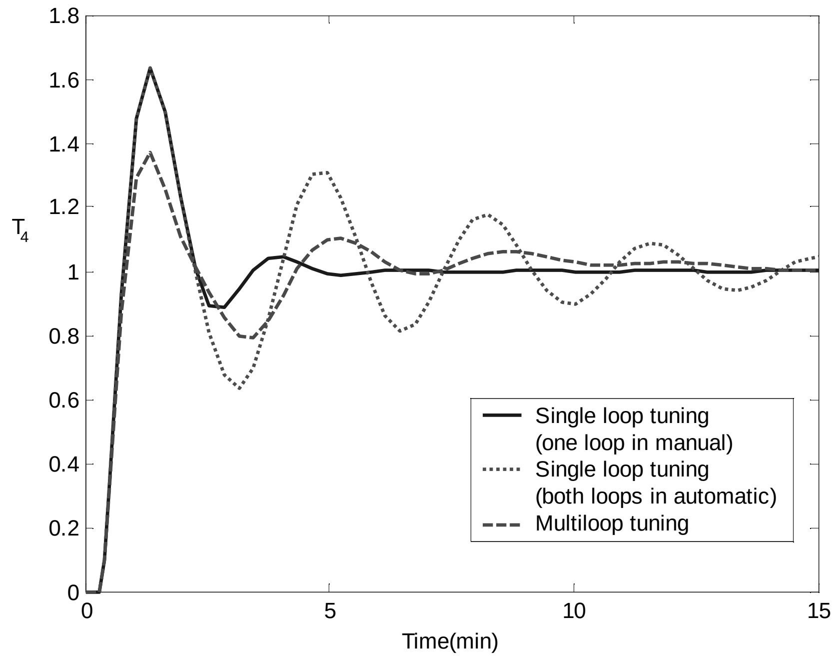


Figure S18.1b. *Set point responses for Exercise 18.1. Analysis for T_4*

18.2

The characteristic equation is found by determining any one of the four transfer functions $Y_1(s)/Y_{sp1}(s)$, $Y_1(s)/Y_{sp2}(s)$, $Y_2(s)/Y_{sp1}(s)$ and $Y_2(s)/Y_{sp2}(s)$, and setting its denominator equal to zero.

In order to determine, say, $Y_1(s)/Y_{sp1}(s)$, set $Y_{sp2} = 0$ in Fig 18.3*b* and use block diagram algebra to obtain

$$C_1(s) = G_{P_{12}} G_{C_1} [R_1(s) - C_1(s)] + G_{P_{11}} M_1(s) \qquad (1)$$

$$M_1(s) = G_{C_2} (-[G_{P_{21}} M_1(s) + G_{P_{22}} G_{C_1} [R_1(s) - C_1(s)]]) \qquad (2)$$

Simplifying (2),

$$M_1(s) = \frac{-G_{C_2} G_{P_{22}} G_{C_1}}{1 + G_{C_2} G_{P_{21}}} [R_1(s) - C_1(s)] \qquad (3)$$

Substituting (3) into (1) and simplifying gives

$$\frac{C_1(s)}{R_1(s)} = \frac{(G_{C_1} G_{P_{12}})(1 + G_{C_2} G_{P_{21}}) - G_{C_1} G_{C_2} G_{P_{11}} G_{P_{22}}}{(1 + G_{C_1} G_{P_{12}})(1 + G_{C_2} G_{P_{21}}) - G_{C_1} G_{C_2} G_{P_{11}} G_{P_{22}}}$$

Therefore characteristic equations is

$(1 + G_{c1} G_{p12})(1 + G_{c2} G_{p21}) - G_{c1} G_{c2} G_{p11} G_{p22} = 0$

If either G_{p11} or G_{p22} is zero, this reduces to

$(1 + G_{c1} G_{p12}) = 0$ or $(1 + G_{c2} G_{p21}) = 0$

So that the stability of the overall system merely depends on the stability of the two individual feedback control loops in Fig. 18.3*b* since the third loop containing G_{p11} and G_{p22} is broken.

18.3

Consider the block diagram for the 1-1/2-2 control scheme in Fig.18.3*a* but including a sensor transfer function (G_{m1},G_{m2}) for each output (y_1,y_2). The following expressions are easily derived,

$\boldsymbol{Y}(s) = \boldsymbol{G}_p(s)\,\boldsymbol{U}(s)$

$$\text{or}\quad \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{p11}(s) & G_{p12}(s) \\ G_{p21}(s) & G_{p22}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \qquad (1)$$

$\boldsymbol{U}(s) = \boldsymbol{G}_c(s)\,\boldsymbol{E}(s)$

$$\text{or}\quad \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} = \begin{bmatrix} G_{c1}(s) & 0 \\ 0 & G_{c2}(s) \end{bmatrix} \begin{bmatrix} E_1(s) \\ E_2(s) \end{bmatrix} \qquad (2)$$

$\boldsymbol{E}(s) = \boldsymbol{Y}_{sp}(s) - \boldsymbol{G}_m(s)\boldsymbol{Y}(s)$

$$\text{or}\quad \begin{bmatrix} E_1(s) \\ E_2(s) \end{bmatrix} = \begin{bmatrix} Y_{sp1}(s) \\ Y_{sp2}(s) \end{bmatrix} - \begin{bmatrix} G_{m1}(s) & 0 \\ 0 & G_{m2}(s) \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} \qquad (3)$$

If Eqs. 1 through 3 are solved for the response of the output to variations of set points, the result is

$\boldsymbol{Y}(s) = \boldsymbol{G}_p(s)\boldsymbol{G}_c(s)\,[\boldsymbol{I} + \boldsymbol{G}_p(s)\boldsymbol{G}_c(s)\boldsymbol{G}_m(s)]^{-1}\,\boldsymbol{Y}_{sp}(s) =$

where $\boldsymbol{I}$ is the identity matrix.

In terms of the component transfer function the matrix

$$\boldsymbol{V} = \boldsymbol{I} + \boldsymbol{G}_p(s)\boldsymbol{G}_c(s)\boldsymbol{G}_m(s) = \begin{bmatrix} 1+h_{11}(s) & h_{12}(s) \\ h_{21}(s) & 1+h_{22}(s) \end{bmatrix}$$

where

$h_{11}(s) = G_{p11}(s)\, G_{c1}(s)\, G_{m1}(s)$
$h_{12}(s) = G_{p12}(s)\, G_{c2}(s)\, G_{m2}(s)$
$h_{21}(s) = G_{p21}(s)\, G_{c1}(s)\, G_{m1}(s)$
$h_{22}(s) = G_{p22}(s)\, G_{c2}(s)\, G_{m2}(s)$

The inverse of $\boldsymbol{V}$, if it exists, is $\boldsymbol{V}^{-1} = \dfrac{1}{\Delta}\begin{bmatrix} 1+h_{22}(s) & -h_{12}(s) \\ -h_{21}(s) & 1+h_{11}(s) \end{bmatrix}$

where $\Delta = (1+h_{11}(s))(1+h_{22}(s)) - h_{12}(s)h_{21}(s)$

By accounting for $\boldsymbol{Y}(s) = [\boldsymbol{G}_p(s)\boldsymbol{G}_c(s)\, \boldsymbol{V}^{-1}(s)]\ \boldsymbol{Y}_{sp}(s)$, the closed-loop transfer functions are (see book notation):

$$T_{11}(s) = \frac{1}{G_{m1}(s)\Delta}\left[h_{11}(s)(1+h_{22}(s)) - h_{12}(s)h_{21}(s)\right]$$

$$T_{12}(s) = \frac{h_{12}(s)}{G_{m2}(s)\Delta}$$

$$T_{21}(s) = \frac{h_{21}(s)}{G_{m1}(s)\Delta}$$

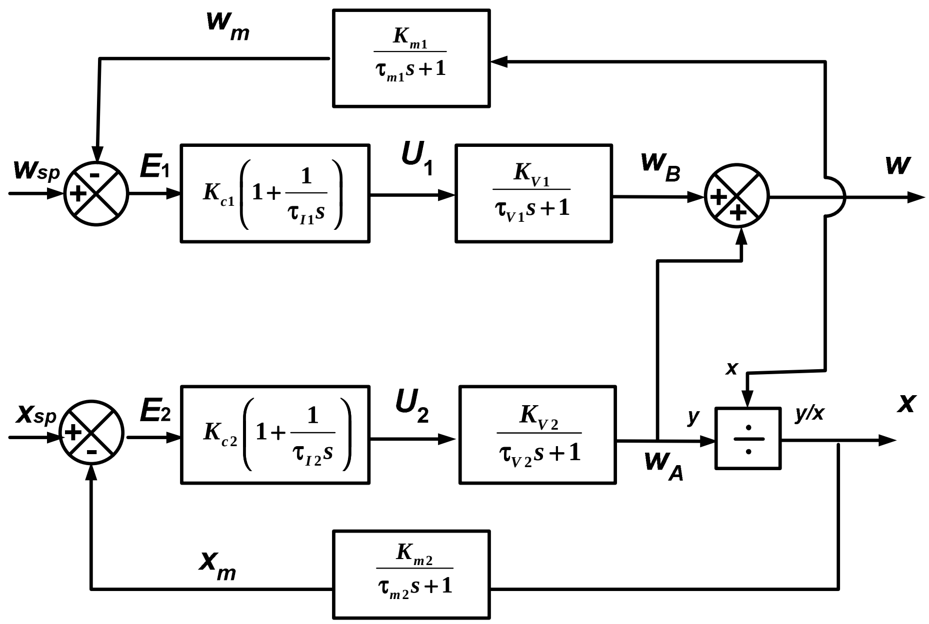
$$T_{22}(s) = \frac{1}{G_{m2}(s)\Delta}\left[h_{22}(s)(1+h_{11}(s)) - h_{21}(s)h_{12}(s)\right]$$

18.4

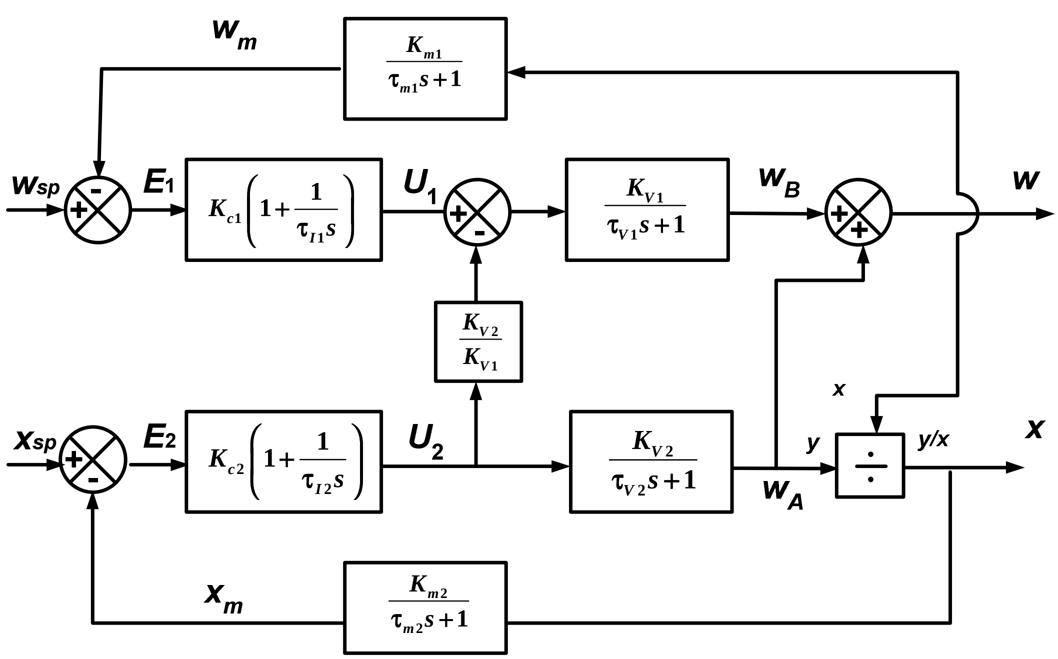
From Eqs. 6-78 and 6-79 a from physical reasoning, its is evident that although h is affected by both the manipulated variables, T is affected only by w_h and is independent of w. Hence, T can be paired only with w_h. Thus, the reasonable pairing for the control scheme is T-w_h, h-w

18.5

a) As shown in Example 18.3, the correct pairing for x = 0.4 case is w-w_B, x-w_A. Therefore, the block diagram is



b) As shown in Example 18.9 , the correct pairing is w-u_1, x-u_2. The block diagram is



18.6

i) Calculate the steady-state gains as

$$K_{11} = \left(\frac{\Delta X_D}{\Delta R}\right)_S = \frac{0.97 - 0.93}{(125 - 175)\ \text{lb/min}} = -8 \times 10^{-4}\ \text{min/lb}$$

$$K_{12} = \left(\frac{\Delta X_B}{\Delta S}\right)_R = \frac{0.96 - 0.94}{(24 - 20)\ \text{lb/min}} = +5 \times 10^{-3}\ \text{min/lb}$$

$$K_{21} = \left(\frac{\Delta X_B}{\Delta R}\right)_S = \frac{0.06 - 0.04}{(175 - 125)\text{lb/min}} = +4 \times 10^{-4}\ \text{min/lb}$$

$$K_{22} = \left(\frac{\Delta X_B}{\Delta S}\right)_R = \frac{0.04 - 0.06}{(24 - 20) lb / \min} = -5 \times 10^{-3}\ \min / lb$$

Substituting into Eq. 18-34,

$$\lambda = \frac{1}{1 - \dfrac{(5*10^{-3})(4*10^{-4})}{(-8*10^{-4})(-5*10^{-3})}} = 2$$

Thus the RGA is

$$\begin{array}{c} \\ x_D \\ x_B \end{array} \begin{array}{c} \quad R \quad\ S \\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \end{array}$$

Pairing for positive relative gains requires X_D-R, X_B-S.

ii) This pairing seems appropriate from dynamic considerations as well; because of the lag in the column, R affects X_D sooner than X_B, and S affects X_B sooner that X_D.

18.7

a) The corresponding steady-state gain matrix is

$$\boldsymbol{K} = \begin{bmatrix} 12.8 & -18.9 \\ 6.6 & -19.4 \end{bmatrix}$$

Using the formula in Eq. 18-34 , we obtain $\lambda_{11} = 2.0$

Thus the RGA is

$$\boldsymbol{\Lambda} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Pairing for positive relative gains requires X_D-R and X_B-S.

b) The same pairing is recommended based on dynamic considerations. The transfer functions between X_D and R contains a smaller dead time and a smaller time constant, so X_D will respond very fast to changes in R. For the pair X_B-S, the time constant is not favorable but the dead time is significantly smaller and the response will be fast as well.

18.8

a) From Eq. 6-89

$$G_{p_{11}}(s) = \frac{(\overline{T}_h - \overline{T})/\overline{w}}{\tau s + 1}, \qquad G_{p_{12}}(s) = \frac{(\overline{T}_c - \overline{T})/\overline{w}}{\tau s + 1}$$

$$G_{p_{21}}(s) = \frac{1/AP}{s}, \qquad G_{p_{22}}(s) = \frac{1/AP}{s}$$

Thus $K_{11} = \dfrac{\overline{T}_h - \overline{T}}{\overline{w}}$, $\quad K_{12} = \dfrac{\overline{T}_c - \overline{T}}{\overline{w}}$

and since G_{p21}, G_{p22} contain integrating elements,

$$K_{21} = \lim_{s \to 0} sG_{P_{21}}(s) = \frac{1}{AP}$$

$$K_{22} = \lim_{s \to 0} sG_{P_{22}}(s) = \frac{1}{AP}$$

Substituting into Eq. 18-34,

$$\lambda = \frac{1}{1 - \dfrac{\overline{T}_c - \overline{T}}{\overline{T}_h - \overline{T}}} = \frac{\overline{T}_h - \overline{T}}{\overline{T}_h - \overline{T}_c}$$

Hence $0 \leq \lambda \leq 1$, and the choice of pairing depends on whether $\lambda > 0.5$ or not. The RGA is

$$\begin{array}{cc} & \begin{array}{cc} w_h & \quad w_c \end{array} \\ \begin{array}{c} T \\ \\ h \end{array} & \begin{bmatrix} \dfrac{\overline{T}_h - \overline{T}}{\overline{T}_h - \overline{T}_c} & \dfrac{\overline{T} - \overline{T}_c}{\overline{T}_h - \overline{T}_c} \\ \dfrac{\overline{T} - \overline{T}_c}{\overline{T}_h - \overline{T}_c} & \dfrac{\overline{T}_h - \overline{T}}{\overline{T}_h - \overline{T}_c} \end{bmatrix} \end{array}$$

b) Assume that $\lambda \geq 0.5$ so that the pairing is T-w_h, h-w_c. Assume valve gains to be unity. Then the ideal decoupling control system will be as in Fig.18.9 where $Y_1 \equiv T$, $Y_2 \equiv h$, $U_1 \equiv w_h$, $U_2 \equiv w_c$, and using Eqs. 18-78 and 18-80,

$$T_{21}(s) = -\frac{(1/AP)s}{(1/AP)s} = -1$$

$$T_{12}(s) = -\frac{[(\overline{T}_c - \overline{T})/\overline{w}]/(\tau s + 1)}{[(\overline{T}_h - \overline{T})/\overline{w}]/(\tau s + 1)} = \frac{\overline{T} - \overline{T}_c}{\overline{T}_h - \overline{T}}$$

c) The above decouplers are physically realizable.

18.9

<u>OPTION A:</u> Controlled variable: T_{17}, T_{24}
Manipulated variables: u_1, u_2

The corresponding steady-state gain matrix is

$$\boldsymbol{K} = \begin{bmatrix} 1.5 & 0.5 \\ 2 & 1.7 \end{bmatrix}$$

Using the formula in Eq.18-34, we obtain $\lambda_{11} = 1.65$
Thus the RGA is

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1.65 & -0.65 \\ -0.65 & 1.65 \end{bmatrix}$$

<u>OPTION B</u>: Controlled variable: T_{17}, T_{30}
Manipulated variables: u_1, u_2

The corresponding steady-state gain matrix is

$$\boldsymbol{K} = \begin{bmatrix} 1.5 & 0.5 \\ 3.4 & 2.9 \end{bmatrix}$$

Using the formula in Eq.18-34, we obtain $\lambda_{11} = 1.64$

Thus the RGA is

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1.64 & -0.64 \\ -0.64 & 1.64 \end{bmatrix}$$

<u>OPTION C</u>: Controlled variable: T_{24}, T_{30}
Manipulated variables: u_1, u_2

The corresponding steady-state gain matrix is

$$\boldsymbol{K} = \begin{bmatrix} 2 & 1.7 \\ 3.4 & 2.9 \end{bmatrix}$$

Using the formula in Eq.18-34, we obtain $\lambda_{11} = 290$

Thus the RGA is

$$\boldsymbol{\Lambda} = \begin{bmatrix} 290 & -289 \\ -289 & 290 \end{bmatrix}$$

Hence options A and B yield approximately the same results. Option C is the least desirable.

By applying Niederlinski's stability theorem for option C:

$$\frac{|K|}{\Pi K_{ii}} = \frac{-0.02}{5.8} < 0$$

Thus the closed-loop system is unstable.

18.10

a) Material balance for each of the two tanks is

$$A_1 \frac{dh_1}{dt} = q_1 + q_6 - \frac{\sqrt{h_1}}{R_1} - K(h_1 - h_2) \quad (1)$$

$$A_2 \frac{dh_2}{dt} = q_2 - \frac{\sqrt{h_2}}{R_2} + K(h_1 + h_2) \quad (2)$$

where A_1, A_2 are cross-sectional areas of tanks 1, 2, respectively. Linearizing, putting in deviation variable form, and taking Laplace transform,

$$A_1 s H_1'(s) = Q_1'(s) + Q_6'(s) - (\frac{1}{2R_1\sqrt{\bar{h}_1}})H_1'(s) - K[H_1'(s) - H_2'(s)]$$

$$A_2 s H_2'(s) = Q_2'(s) - (\frac{1}{2R_2\sqrt{\bar{h}_2}})H_2'(s) + K[H_1'(s) - H_2'(s)]$$

Let $K_1 \equiv \dfrac{1}{2R_1\sqrt{\bar{h}_1}}$ and $K_2 \equiv \dfrac{1}{2R_2\sqrt{\bar{h}_2}}$, and

Solve the above equations simultaneously to get,

$$\begin{aligned}&[(A_1 s + K_1 + K)(A_2 s + K_2 + K) - K^2]H_1'(s) \\ &= (A_2 s + K_2 + K)[Q_1'(s) + Q_6'(s)] + KQ_2'(s)\end{aligned} \quad (3)$$

$$\begin{aligned}&[(A_1 s + K_1 + K)(A_2 s + K_2 + K) - K^2]H_2'(s) \\ &= K[Q_1'(s) - Q_6'(s)] + (A_1 s + K_1 + K)Q_2'(s)\end{aligned} \quad (4)$$

The four steady-state process gains are determined using Eqs. 3 and 4 as

$$K_{11} = \lim_{s\to 0}\left[\frac{H_1'(s)}{Q_1'(s)}\right] = \frac{K_2 + K}{K_1K_2 + K(K_1 + K_2)}$$

$$K_{12} = \lim_{s\to 0}\left[\frac{H_1'(s)}{Q_2'(s)}\right] = \frac{K}{K_1K_2 + K(K_1 + K_2)}$$

$$K_{21} = \lim_{s\to 0}\left[\frac{H_2'(s)}{Q_1'(s)}\right] = \frac{K}{K_1K_2 + K(K_1 + K_2)}$$

$$K_{22} = \lim_{s\to 0}\left[\frac{H_2'(s)}{Q_2'(s)}\right] = \frac{K_1 + K}{K_1K_2 + K(K_1+K_2)}$$

Substituting into Eq. 18-34

$$\lambda = \frac{1}{1 - \dfrac{K^2}{(K_2+K)(K_1+K)}} = \frac{(K_2+K)(K_1+K)}{K_1K_2 + K(K_1+K_2)}$$

Thus RGA is

$$\begin{array}{c} \\ \dfrac{1}{K_1K_2+K(K_1+K_2)} \end{array}\begin{array}{c} \begin{array}{cc} \quad\quad q_1 \quad\quad\quad\quad & \quad\quad\quad\quad q_2 \end{array} \\ \begin{bmatrix} (K_1+K)(K_2+K) & -K^2 \\ -K^2 & (K_1+K)(K_2+K) \end{bmatrix} \end{array}\begin{array}{c} \\ h_1 \\ h_2 \end{array}$$

b) Substituting the given numerical values, the RGA is

$$\begin{array}{cc} & \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{array}{c} h_1 \\ h_2 \end{array} & \begin{bmatrix} 2.50 & -1.50 \\ -1.50 & 2.50 \end{bmatrix} \end{array}$$

For the relative gains to be positive, the preferred pairing is h_1-q_1, h_2-q_2.

18.11

a) Let

$$\underline{Y}(s) = \begin{bmatrix} H_1'(s) \\ H_2'(s) \end{bmatrix}, \underline{U}(s) = \begin{bmatrix} Q_1'(s) \\ Q_2'(s) \end{bmatrix}, D(s) = Q_6'(s)$$

Then by inspection of Eqs. (3) and (4) in the solution to Exercise 18-10,

$$\underline{\underline{G_P}}(s) = \frac{1}{(A_1s+K_1+K)(A_2s+K_2+K)-K^2}\begin{bmatrix} A_2s+K_2+K & K \\ K & A_1s+K_1+K \end{bmatrix}$$

and

$$\underline{G_d}(s) = \frac{1}{(A_1 s + K_1 + K)(A_2 s + K_2 + K) - K^2} \begin{bmatrix} A_2 s + K_2 + K \\ -K \end{bmatrix}$$

where A_1, A_2, K_1, K_2 are as defined in the solution to Exercise 18.10.

b) The block diagram for h_1-q_1 / h_2-q_2 pairing is identical to Fig.18.3*a* with the addition of the load. Thus the signal $D(s)$ passes through a block G_{d1} whose output is added to the summer with output Y_1. Similarly, the summer leading to Y_2 is influenced by the signal D(s) that passes through block G_{d2}.

18.12

$F = 20\, u_1 (P_0 - P_1)$ (1)
$F = 30\, u_2 (P_1 - P_2)$ (2)

Taking P_0 and P_2 to be constant, Eq. 1 gives

$$\left(\frac{\partial F}{\partial u_1}\right)_{u_2} = 20(P_0 - P_1) - 20u_1\left(\frac{\partial P_1}{\partial u_1}\right)_{u_2} \quad (3)$$

and

$$\left(\frac{\partial F}{\partial u_1}\right)_{P_2} = 20(P_0 - P_1) \quad (4)$$

and Eq. 2 gives

$$\left(\frac{\partial F}{\partial u_1}\right)_{u_2} = 30M_2\left(\frac{\partial P_1}{\partial u_1}\right)_{u_2} \quad (5)$$

Substituting for $\left(\frac{\partial P_1}{\partial M_1}\right)_{M_2}$ from (5) into (3) and simplifying

$$\left(\frac{\partial F}{\partial u_1}\right)_{u_2} = \frac{20(P_0 - P_1)}{1 + \frac{20u_1}{30u_2}} \quad (6)$$

Using Eq. 18-24,

$$\lambda_{11} = \frac{(\partial F / \partial u_1)_{u_2}}{(\partial F / \partial u_1)_{P_2}} = \frac{1}{1 + \dfrac{20u_1}{30u_2}} \qquad (7)$$

At nominal conditions

$$u_1 = \frac{F}{20(P_0 - P_1)} = 1/2 \qquad , \qquad u_2 = \frac{F}{30(P_1 - P_2)} = 2/3$$

Substituting into (7), λ_{11} = 2/3 > 0.5. Hence, the best controller pairing is F-u_1, P_1-u_2.

18.13

a) Material balances for the tank,

$$A\frac{dh}{dt} = q_1 + q_2 - q_3 \qquad (1)$$

$$\frac{d(Ahc_3)}{dt} = c_1q_1 + c_2q_2 - c_3q_3 \qquad (2)$$

Substituting for dh/dt from (1) into (2) and simplifying

$$Ah\frac{dC_3}{dt} = (c_1 - c_3)q_1 + (c_2 - c_3)q_2 \qquad (3)$$

Linearizing, using deviation variables, and taking the Laplace transform

$$A\overline{h}sC_3'(s) = (\overline{c_1} - \overline{c_3})Q_1'(s) - \overline{q_1}C_3'(s) + (\overline{c_2} - \overline{c_3})Q_2'(s) - \overline{q_2}C_3'(s)$$

Since $\overline{q_1} + \overline{q_2} = \overline{q_3}$, this becomes

$$\left[\left(\frac{A\overline{h}}{\overline{q_3}}\right)s + 1\right]C_3'(s) = \left(\frac{\overline{c_1} - \overline{c_3}}{\overline{q_3}}\right)Q_1'(s) + \left(\frac{\overline{c_2} - \overline{c_3}}{\overline{q_3}}\right)Q_2'(s) \qquad (4)$$

Similarly from (1),

$$AsH'(s) = Q_1'(s) + Q_2'(s) - Q_3'(s) \qquad (5)$$

Therefore,

$$\underline{\underline{G}}(s) = \begin{bmatrix} \dfrac{H'(s)}{Q_1'(s)} & \dfrac{H'(s)}{Q_3'(s)} \\ \dfrac{C_3'(s)}{Q_1'(s)} & \dfrac{C_3'(s)}{Q_3'(s)} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{As} & -\dfrac{1}{As} \\ \dfrac{(\bar{c}_1 - \bar{c}_3)/\bar{q}_3}{\left(\dfrac{A\bar{h}}{\bar{q}_3}\right)s+1} & 0 \end{bmatrix}$$

Substituting numerical values

$$\underline{\underline{G}}(s) = \begin{bmatrix} \dfrac{0.1415}{s} & \dfrac{-0.1415}{s} \\ \dfrac{0.0075}{1.06s+1} & 0 \end{bmatrix}$$

For the control valves

$$G_v(s) = \frac{0.15}{\left(\dfrac{10}{60}\right)s+1} = \frac{0.15}{0.167s+1} \tag{6}$$

Thus,

$$\underline{\underline{G_P}}(s) = G_v(s)\underline{\underline{G}}(s) = \begin{bmatrix} \dfrac{0.0212}{s(0.167s+1)} & \dfrac{-0.0212}{s(0.167s+1)} \\ \dfrac{0.0011}{(1.06s+1)(0.167s+1)} & 0 \end{bmatrix}$$

b) Since $C_3'(s)/Q_3'(s) = 0$, c_3 is not affected by q_3 and must be paired with q_1. Thus, the pairing that should be used is h-q_3, c_3-q_1.

c) For the pairing determined above, Fig.18.9 can be used with $Y_1 \equiv H'$, $Y_2 \equiv C_3'$, $U_1 \equiv Q_3'$, $U_2 \equiv Q_1'$. Notice that this pairing requires $G_p(s)$ above the switch columns. Then using Eqs. 18-78 and 18-80,

$$T_{21}(s) = -\frac{G_{P_{21}}(s)}{G_{P_{22}}(s)} = -\frac{0}{\left[\dfrac{0.0011}{(1.06s+1)(0.167s+1)}\right]} = 0$$

$$T_{12}(s) = -\frac{G_{P_{12}}(s)}{G_{P_{11}}(s)} = -\frac{0.0212/\left[s(0.167s+1)\right]}{-0.0212/\left[s(0.167s+1)\right]} = 1$$

18.14

In this case, an RGA analysis is not needed. The manipulated and controlled variables are:

Controlled variables: F_1, P_1 and I
Manipulated variables: m_1, m_2, m_3

Basically, the pairing could be done based on dynamic considerations, so that the time constants and dead times in the response must be as low as possible.

The level of the interface "I" may be easily controlled with m_3 so that any change in the set-point is controlled by opening or closing the valve in the bottom of the decanter.

The manipulated variable m_1 could be used to control the inflow rate F_1. If F_1 is moved away from its set-point, the valve will respond quickly to control this change.

The decanter overhead pressure P_1 is controlled by manipulating m_2. That way, pressure changes will be quickly treated. This control configuration is also used in distillation columns.

18.15

OPTION A: Controlled variable: Y_1, Y_2
Manipulated variables: U_1, U_2

The corresponding steady-state gain matrix is

$$\boldsymbol{K} = \begin{bmatrix} 3 & -0.5 \\ -10 & 2 \end{bmatrix}$$

Using the formula in Eq.18-34, we obtain $\lambda_{11} = 6$

Thus the RGA is

$$\boldsymbol{\Lambda} = \begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix}$$

OPTION B: Controlled variable: Y_1, Y_2
Manipulated variables: U_1, U_3

The corresponding steady-state gain matrix is

$$\boldsymbol{K} = \begin{bmatrix} 3 & 1/2 \\ -10 & 4 \end{bmatrix}$$

Using the formula in Eq.18-34, we obtain $\lambda_{11} = 0.71$

Thus the RGA is

$$\boldsymbol{\Lambda} = \begin{bmatrix} 0.71 & 0.29 \\ 0.29 & 0.71 \end{bmatrix}$$

OPTION C: Controlled variable: Y_1, Y_2
Manipulated variables: U_2, U_3

The corresponding steady-state gain matrix is

$$\boldsymbol{K} = \begin{bmatrix} -0.5 & 1/2 \\ 2 & 4 \end{bmatrix}$$

Using the formula in Eq.18-34, we obtain $\lambda_{11} = 0.67$

Thus the RGA is

$$\boldsymbol{\Lambda} = \begin{bmatrix} 0.67 & 0.33 \\ 0.33 & 0.67 \end{bmatrix}$$

By accounting for Bristol's original recommendation, the controlled and manipulated variables are paired so that the corresponding relative gains are positive and as close to one as possible. Thus, OPTION B leads to the best control configuration.

18.16

The process scheme is shown below

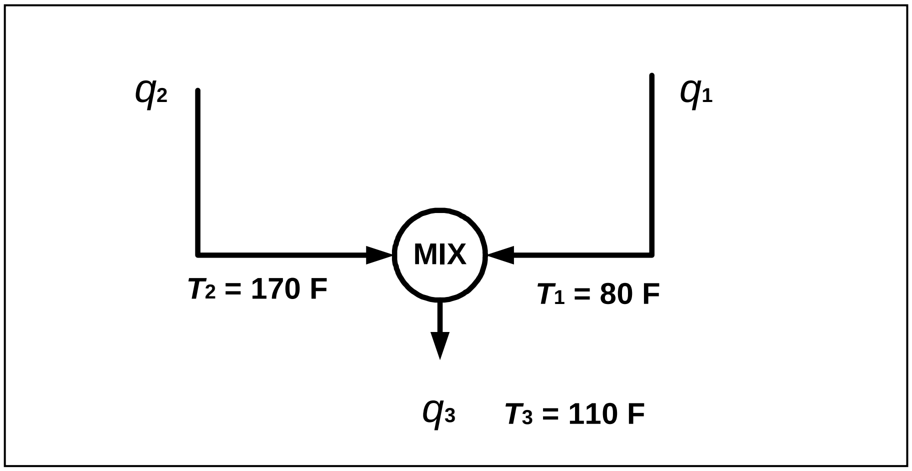


Figure S18.16. *Process scheme*

a) Steady state material balance:

$$q_1 + q_2 = q_3 \tag{1}$$

Steady state energy balance:

$$q_1C(T_1\text{-}T_{ref})+q_2C(T_2\text{-}T_{ref}) = q_3C(T_3\text{-}T_{ref}) \tag{2}$$

By substituting (1) in (2) and solving:

$q_1 = 1$ gpm
$q_2 = 2$ gpm

b) The steady-state gain matrix ***K*** must be calculated :

$$\begin{bmatrix} T_3' \\ q_3' \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} q_1' \\ q_2' \end{bmatrix} \tag{3}$$

From (1), it follows that $K_{21}=K_{22}=1$. From (2),

$$q_3T_3 = q_1T_1 + q_2T_2 \tag{4}$$

Substitute (1) and rearrange,

$$T_3 = \frac{q_1}{q_1 + q_2}(T_1 + T_2) \tag{5}$$

$$K_{11} = \left(\frac{\partial T_3}{\partial q_1}\right)_{q_2} = (T_1 + T_2)\left[\frac{(q_1 + q_2) - q_1}{(q_1 + q_2)^2}\right] = \frac{(T_1 + T_2)q_2}{(q_1 + q_2)^2}$$

$$K_{12} = \left(\frac{\partial T_3}{\partial q_2}\right)_{q_1} = (T_1 + T_2)\left[-\frac{q_1}{(q_1 + q_2)^2}\right]$$

RGA analysis:

$$\lambda_{11} = \frac{1}{1 - \frac{K_{12}K_{21}}{K_{11}K_{22}}} = \frac{1}{1 - \left(-\frac{q_1}{q_2}\right)} = \frac{q_2}{q_2 + q_1} \qquad \rightarrow \qquad \lambda_{12} = 1 - \lambda_{11} = \frac{q_1}{q_2 + q_1}$$

Thus the RGA is,

$$\mathbf{\Lambda} = \begin{array}{c} \\ T_3 \\ q_3 \end{array} \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} \frac{q_2}{q_2 + q_1} & \frac{q_1}{q_2 + q_1} \\ \frac{q_1}{q_2 + q_1} & \frac{q_2}{q_2 + q_1} \end{pmatrix} \end{array}$$

Substitute numerical values for numerical conditions,

$$\mathbf{\Lambda} = \begin{array}{c} \\ T_3 \\ q_3 \end{array} \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{array}$$

Pair : T_3 - q_2 / q_3 - q_1

18.17

a) Dynamic Model:

Mass Balance:

$$\rho A\frac{dh}{dt} = (1-f)w_1 + w_2 - w_3 \quad (1)$$

Energy Balance: ($T_{ref} = 0$)

$$\rho C_p A\frac{d(hT_3)}{dt} = C_p(1-f)w_1T_1 + C_p w_2T_2 - C_p w_3T_3 - UA_c(T_3 - T_c) \quad (2)$$

Mixing Point:

$$w_4 = w_3 + fw_1 \quad (3)$$

Energy Balance on Mixing Point:

$$C_p w_4T_4 = C_p w_3T_3 + C_p fw_1T_1 \quad (4)$$

Control valves:

$$U = C_3 X_c \quad (5)$$

$$w_3 = x_3(C_1h - C_2 fw_1) \quad (6)$$

b) Degrees of freedom:

Variables: 14

h, w_1, w_2, w_3, w_4 ,T_1, T_2, T_3, T_4, T_c, x_c, x_3, f, U

Equations: 6

Degrees of freedom = $N_V – N_E$ = 8

Specified by the environment: 4 (T_c, w_1, T_1, T_2)

Manipulated variables: 4 (f, w_2, x_c, x_3)

c) Controlled variables:

h Guidelines #2 and 5 (i.e., G2 and G5)

T_4 G3 and G5

w_4 G3 and G5

w_2 (or T_3) G4 and G5 (or G2 and G5)

d) RGA

At steady state, (1) and (2) become:

$$0 = (1-f)w_1 + w_2 - w_3 \tag{7}$$

$$0 = C_p(1-f)w_1T_1 + C_p w_2 T_2 - C_3 w_3 T_3 - UA_c(T_3 - T_c) \tag{8}$$

Rearrange (8) and substitute (5),

$$T_3 = \frac{C_p(1-f)w_1 + C_p w_2 T_2 - C_3 x_c A_c T_c}{C_3 w_3 + C_3 x_c A_c} \tag{9}$$

Rearrange (7)

$$w_3 = (1-f)w_1 + w_2 \tag{10}$$

Substitute (10) into (9),

$$T_3 = \frac{C_p(1-f)w_1 + C_p w_2 T_2 + C_3 x_c A_c T_c}{C_3(1-f)w_1 + C_3 w_2 + C_3 x_c A_c} \tag{11}$$

Substitute (10), (3) and (11) into (4),

$$(w_3 + fw_1)T_4 = w_3T_3 + fw_1T_1 \tag{12}$$

or

$$\left[(1-f)w_1 + w_2 + fw_1\right]T_4 = fw_1T_1 +$$

$$+ \left[(1-f)w_1 + w_2\right]\left[\frac{C_p(1-f)w_1 + C_p w_2 T_2 - C_3 x_c A_c T_c}{C_3(1-f)w_1 + C_3 w_2 + C_3 x_c A_c}\right] \tag{13}$$

Rearrange,

$$T_4 = \frac{fw_1T_1}{w_1+w_2} + \left[\frac{(1-f)w_1+w_2}{w_1+w_2}\right]\left[\frac{C_p(1-f)w_1+C_pw_2T_2-C_3x_cA_cT_c}{C_3(1-f)w_1+C_3w_2+C_3x_cA_c}\right]$$

(14)

Rearrange (6),

$$h = \frac{w_3 + x_3C_2fw_1}{x_3C_1} \tag{15}$$

Substitute (10) into (15),

$$h = \frac{(1-f)w_1 + w_2 + x_3C_2fw_1}{x_3C_1} \tag{16}$$

Rewrite (14) as,

$$T_4 = \frac{fw_1T_1}{w_1+w_2} + \left[\frac{E_1+E_8f+w_2}{w_1+w_2}\right]\left[\frac{E_2f+E_3w_2+E_4}{E_5f+E_6w_2+E_7}\right] \tag{17}$$

where:

$$\left.\begin{array}{lll} E_1 = w_1 & E_2 = -C_pw_1 & E_3 = C_pT_2 \\ E_4 = C_3X_cAT_c + C_pw_1 & & E_5 = -C_3w_1 \\ E_6 = C_3 & E_7 = C_3X_cA + C_3w_1 & E_8 = -w_1 \end{array}\right\} \tag{18}$$

Can write (17) as,

$$T_4 = \frac{fw_1T_1}{w_1+w_2} +$$

$$+\frac{\overbrace{E_8E_2f^2 + (E_3E_8+E_2)fw_2 + (E_1E_3+E_4)w_2 + (E_1E_2+E_8E_4)f + E_1E_4}^{F_1}}{\underbrace{E_6w_2{}^2 + (w_1E_6+E_7)w_2 + w_1E_5f + E_5w_2f + E_7w_1}_{F_2}}$$

(19)

Thus

$$\frac{\partial T_4}{\partial f} = K_{11} = \frac{w_1T_1}{w_1+w_2} + \frac{2E_BE_2f + (E_3E_8+E_2)w_2 + E_1E_2 + E_8E_4}{[F_2]}$$

$$-\frac{(F_1)[w_1 E_5 + E_5 w_2]}{F_2^{\ 2}} \tag{20}$$

Similarly

$$\frac{\partial T_4}{\partial f} = K_{12}$$

From (16)

$$\frac{\partial h}{\partial f} = K_{21} = \frac{x_3 C_2 w_1 - w_1}{x_3 C_1}$$

$$\frac{\partial h}{\partial w_2} = K_{22} = \frac{1}{x_3 C_1}$$

Then

$$\Lambda = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix}$$

where

$$\lambda = \frac{1}{1 - \dfrac{K_{12}K_{21}}{K_{11}K_{22}}}$$

e) It will be difficult to control T_4 because neither x_3 nor f has a large steady-state effect on T_4.

18.18

Multiloop PI control system reported by Lee et al:

	K_c	τ_I
X_D-R control loop	0.850	7.21 min
X_B-S control loop	-0.089	8.86 min

a) PERFECT PROCESS MODEL

Dynamic decouplers:

$$T_{21}(s) = -\frac{G_{p21}(s)}{G_{p22}(s)} = -\frac{6.6(14.4s+1)}{(10.9s+1)(-19.4)}e^{-4s} = \frac{-95.04s-6.6}{-211s-19.4}e^{-4s}$$

$$T_{12}(s) = -\frac{G_{p12}(s)}{G_{p11}(s)} = -\frac{-18.9(16.7s+1)}{(21s+1)(12.8)}e^{-2s} = \frac{315.63s+18.9}{268.8s+12.8}e^{-2s}$$

Static decouplers:

$$T_{21}(s) = -\frac{K_{p21}(s)}{K_{p22}(s)} = -\frac{6.6}{(-19.4)}$$

$$T_{12}(s) = -\frac{K_{p12}(s)}{K_{p11}(s)} = -\frac{-18.9}{(12.8)}$$

By using Simulink-MATLAB, unit set-point responses are shown below. Both X_D setpoint change and X_B setpoint change are considered separately.

a.1) <u>X_D set-point change</u>:

a.1.1.- Conventional multiloop PI control:

***X_D* response**:

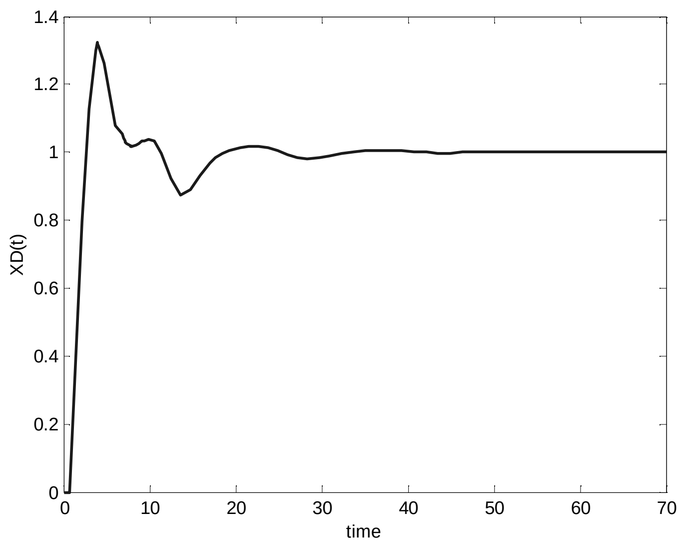


Figure S18.18a. *X_D response to X_D set-point change; perfect model*

***X_B* response:**

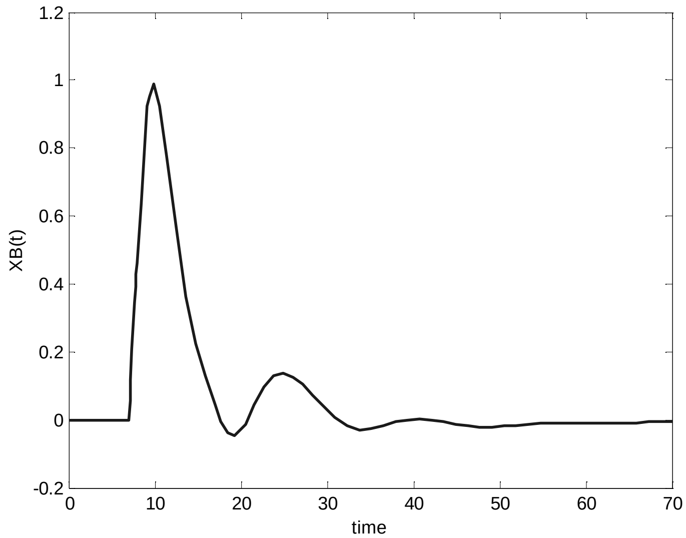


Figure S18.18b. *X_B response to X_D set-point change; perfect model*

a.1.2.- Static decoupler:

***X_D* response**:

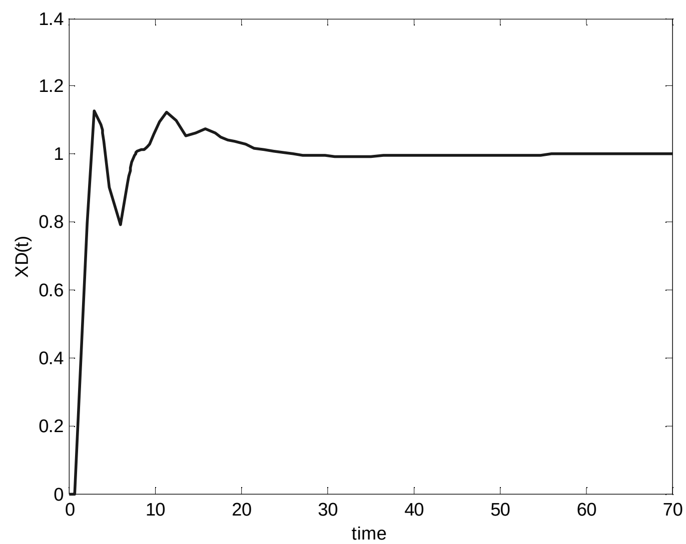


Figure S18.18c. *X_D response to X_D set-point change; perfect model.*

***X_B* response**:

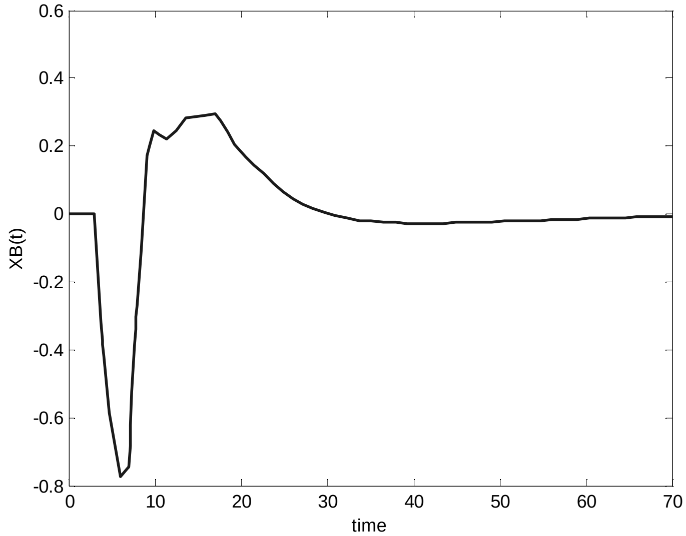


Figure S18.18d. *X_B response to X_D set-point change; perfect model*

a.1.3.- Dynamic decoupler:

***X_D* response**:

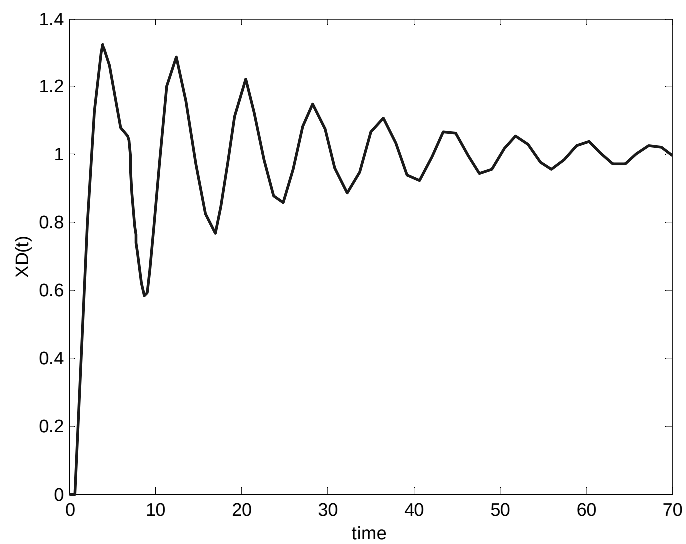


Figure S18.18e. *X_D response to X_D set-point change; perfect model*

***X_B* response**:

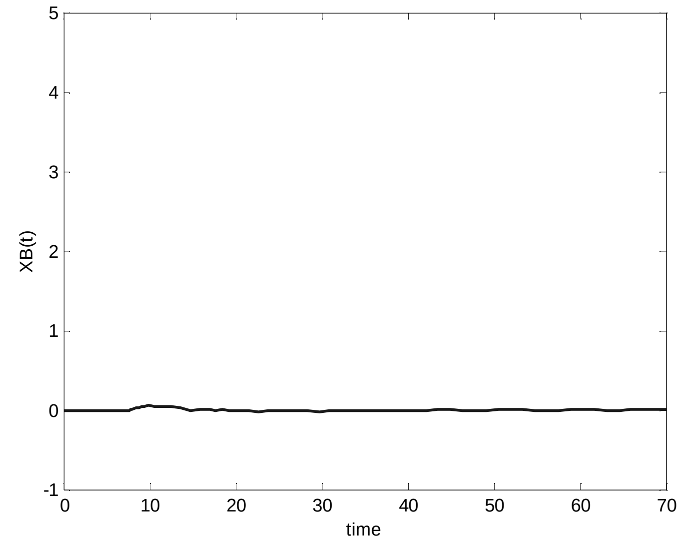


Figure S18.18f. *X_B response to X_D set-point change; perfect model*

As noted in simulations above, the static decoupler provides better X_D response (less oscillatory) than the dynamic decoupler. However, the dynamic decoupler provides perfect control of X_B during the setpoint change in X_D.

a.2) <u>X_B set-point change</u>

a.1.1.- Conventional multiloop PI control:

***X_D* response**:

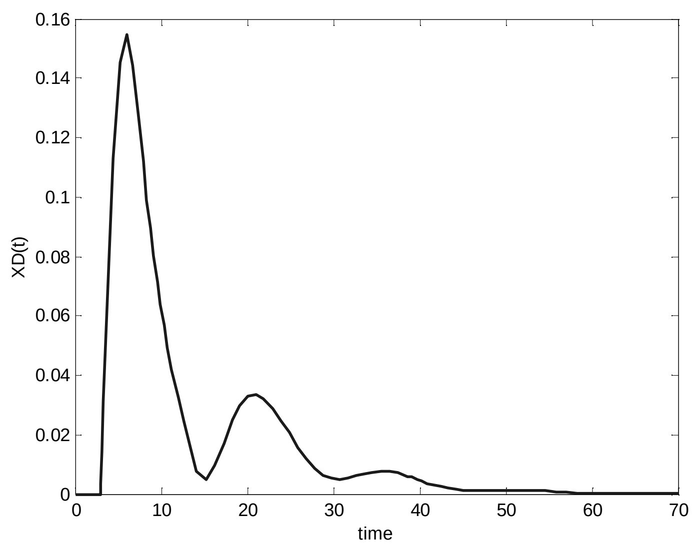


Figure S18.18g. *X_D response to X_B set-point change; perfect model*

***X_B* response**

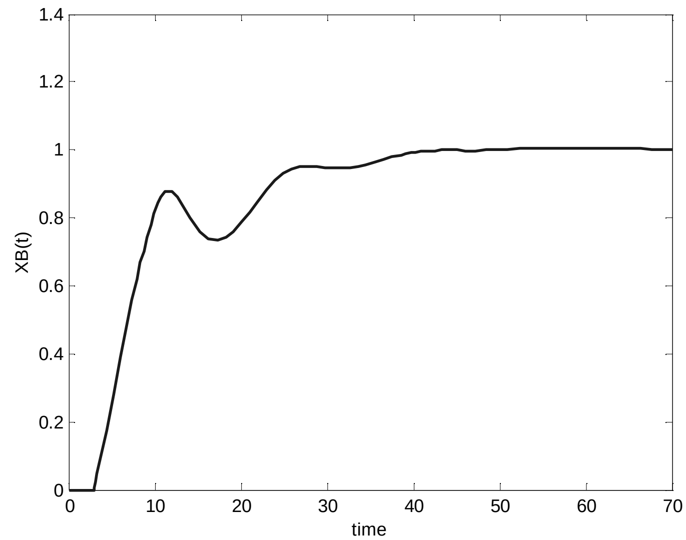


Figure S18.18h. *X_B response to X_B set-point change; perfect model*

a.1.2.- Static decoupler:

***X_D* response**:

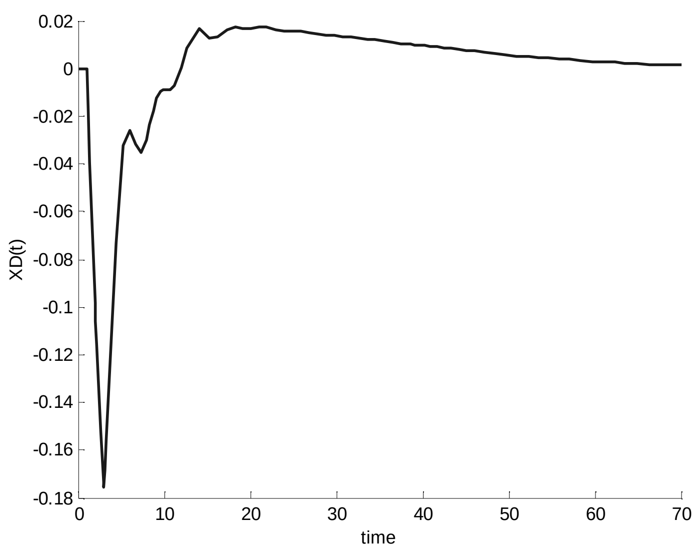


Figure S18.18i. *X_D response to X_B set-point change; perfect model*

***X_B* response**:

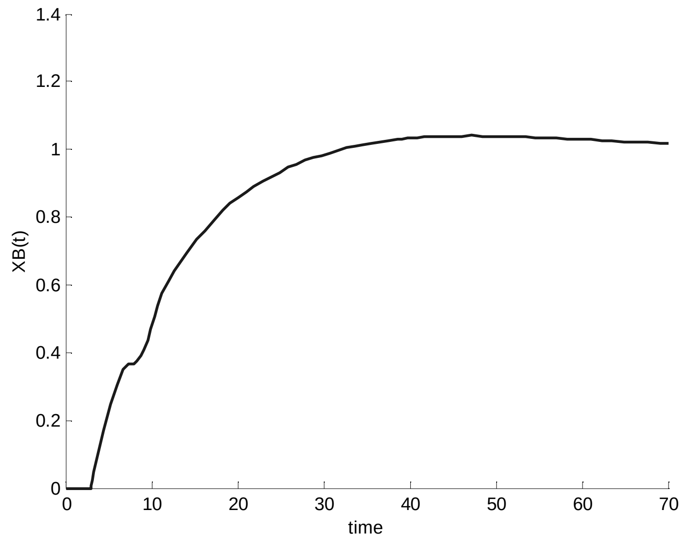


Figure S18.18j. *X_B response to X_B set-point change; perfect model*

a.1.3.- Dynamic decoupler:

***X_D* response**:

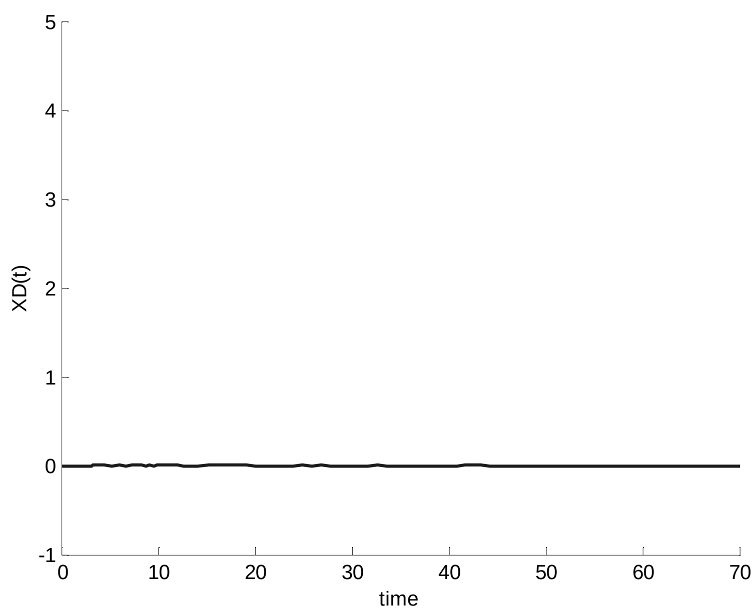


Figure S18.18k. *X_D response to X_B set-point change; perfect model*

***X_B* response**:

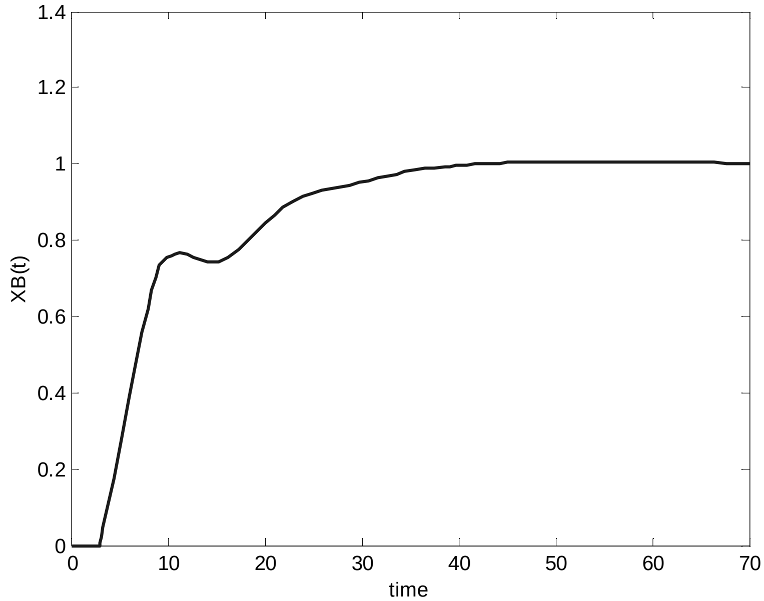


Figure S18.18l. *X_B response to X_B set-point change; perfect model*

In simulations above, in this case the dynamic decoupler provides better performance. In addition, this dynamic decoupler provides perfect control of X_B during the setpoint change in X_D .

b) MODEL ERROR

Degree of robustness to model errors is evaluated for the dynamic decoupler. Only X_D set-point change is considered. Furthermore, different model errors are analyzed:

b.1) +20 % K_{11} $\rightarrow$ $K_{11} = 15.36$

***X_D* response**:

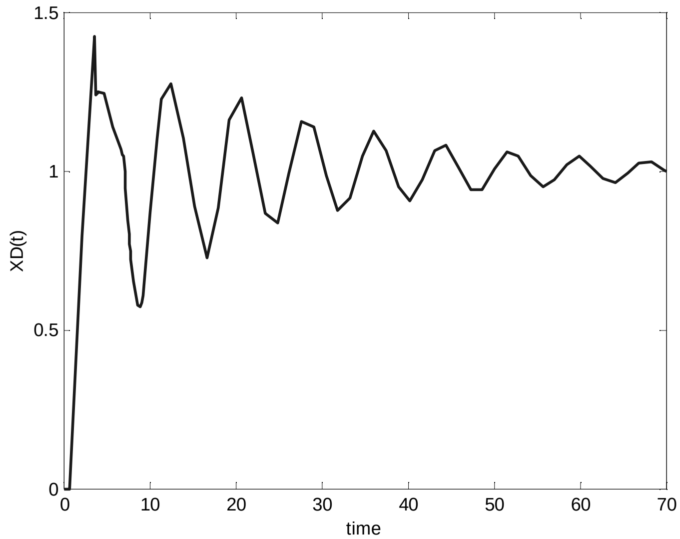


Figure S18.18m. *X_D response to X_D set-point change; +20% K_{11}*

***X_B* response**:

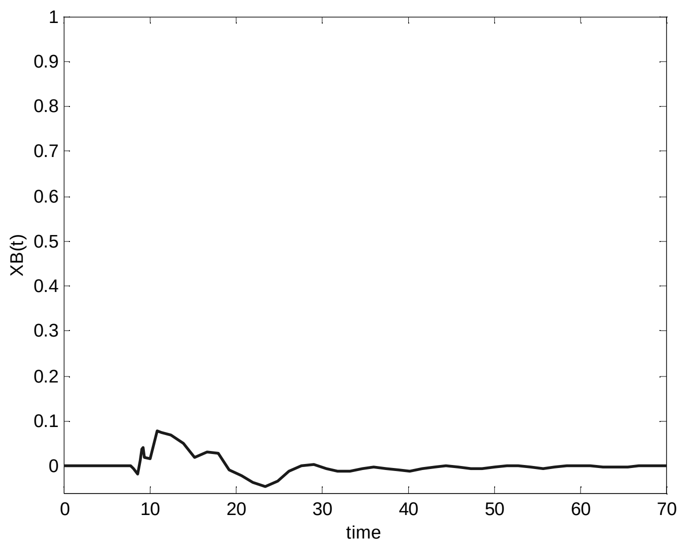


Figure S18.18n. *X_B response to X_D set-point change; +20% K_{11}*

b.2) +20 % K_{11} +20 % K_{22} $\rightarrow$ K_{11} = 15.36 and K_{22} = -23.28

***X_D* response**:

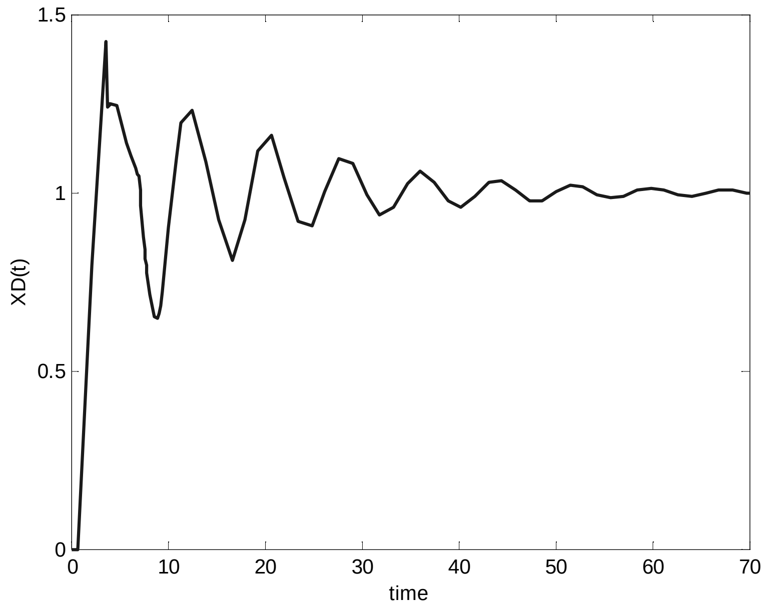


Figure S18.18o. *X_D response to X_D set-point change; +20% K_{11} and K_{22}*

***X_B* response**:

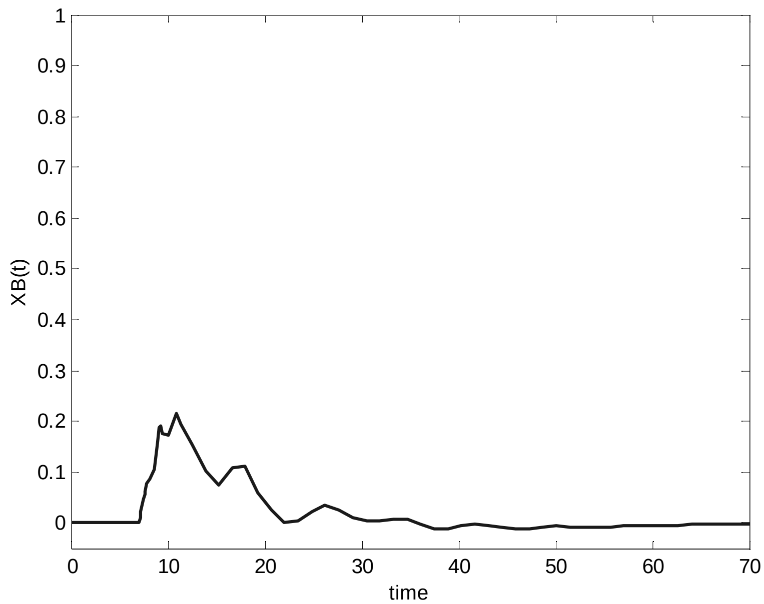


Figure S18.18p. *X_B response to X_D set-point change; +20% K_{11} and K_{22}*

b.3) +20 % τ_{11} → τ_{11} = 20.04

***X_D* response**:

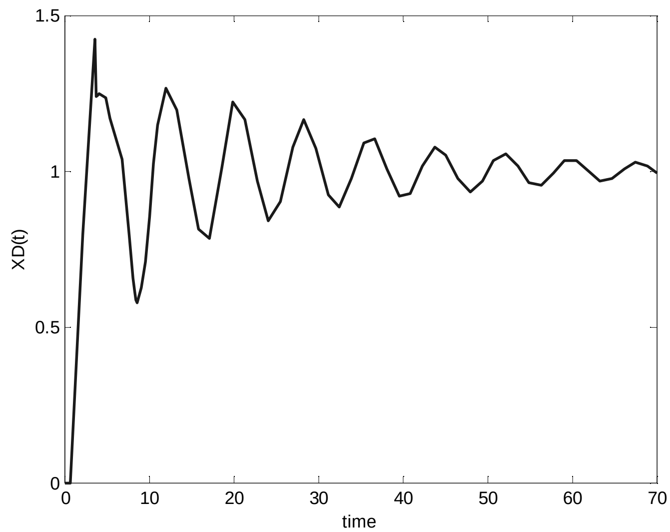


Figure S18.18q. *X_D response to X_D set-point change; +20% τ_{11}*

***X_B* response**:

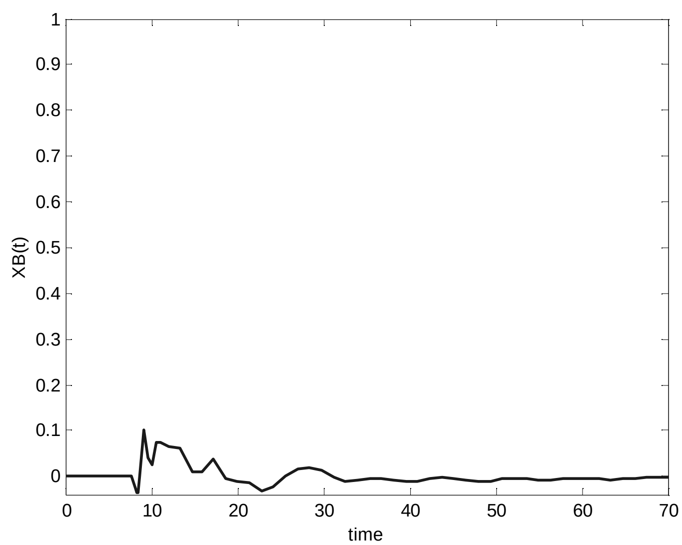


Figure S18.18r. *X_B response to X_D set-point change; +20% τ_{11}*

b.4) +20 % τ_{11} +20 % τ_{22} → τ_{11} = 20.04 and τ_{22} = 17.28

***X_D* response**:

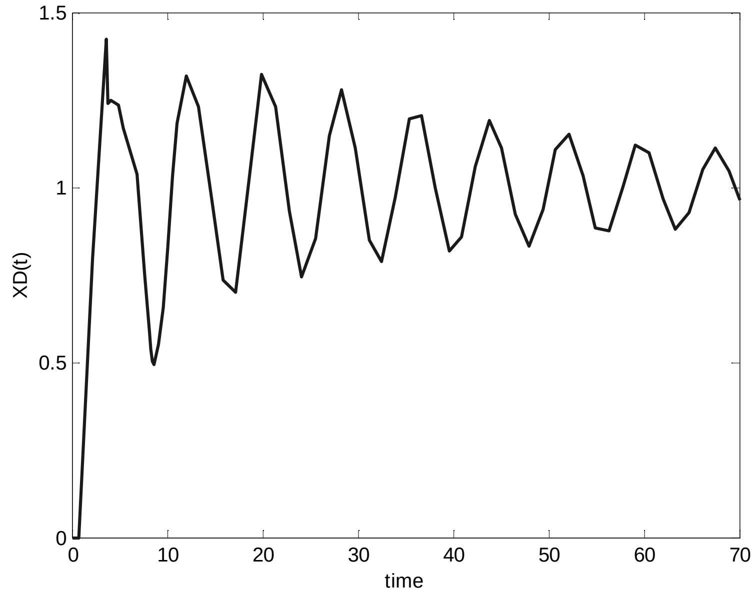


Figure S18.18s. *X_D response to X_D set-point change; +20% τ_{11} and τ_{22}*

***X_B* response**:

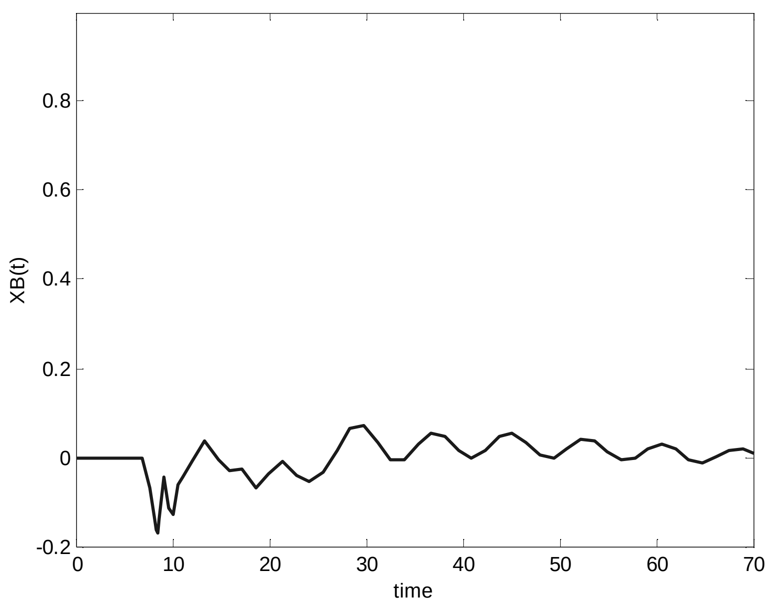


Figure S18.18t. *X_B response to X_D set-point change; +20% τ_{11} and τ_{22}*

As noted in simulations above, model errors in K_{11} , K_{22} and τ_{11} are not so important, although the dynamic decoupler does not provide perfect control for X_B in this case. System looks to be more sensitive to errors in τ_{22}.

18.19

a)

i) Static considerations:

Pairing according to RGA elements closest to +1:

$H_1 - Q_3$, pH_1-Q_1, H_2-Q_4, $pH_2 - Q_6$

ii) Dynamic considerations:

The some pairing results in the smallest time constants for tank 1. It is also dynamically best for tank 2 because it avoids the large θ/τ ratio of 0.8.

iii) Physical considerations

The proposed pairing makes sense because the controlled variables for each tank are paired with the inlet flows for that some tank.

Because pH is more important than level, we might use the pairing, $H_1 - Q_1$ / pH_1-Q_3 , for the first tank to provide better pH control due to the smaller time delay (0.5 vs. 1.0 min).

b) The new gain matrix for the 2×2 problem is

$$\boldsymbol{K} = \begin{bmatrix} 0.42 & 0.41 \\ -0.32 & 0.32 \end{bmatrix}$$

From Eq. 18-34,

$$\lambda_{11} = \frac{1}{1 - \dfrac{(0.41)(-0.32)}{(0.42)(0.32)}} = 0.506$$

Thus

$$\mathbf{\Lambda} = \begin{bmatrix} 0.506 & 0.494 \\ 0.494 & 0.506 \end{bmatrix}$$

RGA pairing: $H_2 - Q_4$ / pH$_2$-Q_6. The pairing also avoids the large delay of 0.8 min.

18.20

Since level is tightly controlled, there is a no accumulation, and a material balance yields:

Overall: $w_F - E\, w_S - w_P \approx 0$ (1)

Solute: $w_F x_F - w_P x_P \approx 0$ (2)

Controlled variable: x'_P, w'_F

Manipulated variables: w'_P, w'_s

From (1):

$$w_F = w_S\, E + w_P$$

From (2):

$$x_P = \frac{x_F}{w_P} w_F = \frac{x_F}{w_P}(w_s E + w_P) \tag{3}$$

Using deviation variables:

$$w'_F = w'_s E + w'_P$$

Linearizing (3):

$$x_P = \bar{x}_P + \left.\frac{\partial x_P}{\partial w_P}\right|_{\bar{w}_P, \bar{w}_s} (w'_P) + \left.\frac{\partial x_P}{\partial w_s}\right|_{\bar{w}_P, \bar{w}_s} (w'_s)$$

$$x'_P = \left(\frac{-x_F E \bar{w}_s}{\bar{w}_P{}^2}\right) w'_P + \left(\frac{x_F E}{\bar{w}_P}\right) w'_s \tag{5}$$

Then the steady-state gain matrix is

$$\begin{array}{c|cc} & w_P' & w_s' \\ \hline x_P' & \left(\dfrac{-x_F E\overline{w}_s}{\overline{w}_P^{\;2}}\right) & \left(\dfrac{x_F E}{\overline{w}_P}\right) \\ w_F' & 1 & \text{E} \end{array}$$

By using the formula in Eq.18-34, we obtain

$$\lambda_{11} = \frac{1}{1+\dfrac{\overline{w}_P}{E\overline{w}_s}} = \frac{E\overline{w}_s}{E\overline{w}_s + \overline{w}_p} = \lambda_{22}$$

$$\lambda_{12} = \lambda_{21} = 1 - \lambda_{11} = \frac{\overline{w}_p}{E\overline{w}_s + \overline{w}_p}$$

So the RGA is

$$\mathbf{\Lambda} = \begin{bmatrix} \dfrac{E\overline{w}_s}{E\overline{w}_s + \overline{w}_p} & \dfrac{\overline{w}_P}{E\overline{w}_s + \overline{w}_p} \\ \\ \dfrac{\overline{w}_P}{E\overline{w}_s + \overline{w}_p} & \dfrac{E\overline{w}_s}{E\overline{w}_s + \overline{w}_p} \end{bmatrix}$$

So, if $E\overline{w}_s > \overline{w}_P$, the pairing should be x_P' - w_P' / w_F' - w_s'
So, if $E\overline{w}_s < \overline{w}_P$, the pairing should be x_P' - w_s' / w_F' - w_P'

18.21

a) The corresponding steady-state gain matrix is

$$\boldsymbol{K} = \begin{bmatrix} -0.04 & -0.0005 \\ 0.22 & -0.02 \end{bmatrix}$$

Using the formula in Eq. 18-34, we obtain $\lambda_{11} = 1.16$

Thus the RGA is

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1.16 & -0.16 \\ -0.16 & 1.16 \end{bmatrix}$$

b) Pairing for positive relative gains requires y_1-u_1 and y_2-u_2.

18.22

For higher-dimension process (n>2) the RGA can be calculated from the expression

$$\lambda_{ij} = K_{ij} H_{ij}$$

where H_{ij} is the (i,j) element of $H = (\boldsymbol{K}^{-1})^T$

By using MATLAB,

$$\boldsymbol{K}^{-1} = \begin{bmatrix} 62.23 & -122.17 & 58.02 \\ -84.47 & 170.83 & -83.43 \\ 1.95 & -14.85 & 13.09 \end{bmatrix}$$

$$\boldsymbol{H} = \begin{bmatrix} 62.23 & -84.47 & 1.95 \\ -122.17 & 170.83 & -14.85 \\ 58.02 & -83.43 & 13.09 \end{bmatrix}$$

Thus the RGA is

$$\boldsymbol{\Lambda} = \begin{bmatrix} 210.34 & -211.18 & 1.89 \\ -390.95 & 406.58 & -14.642 \\ 181.60 & -194.39 & 13.80 \end{bmatrix}$$

This RGA analysis shows the control difficulties for this process because of the control loop interactions. For instance, if the pairings are 1-3, 2-2, 3-1, the third loop will experience difficulties in closed-loop operation. But other options not be better.

SVA analysis:

Determinant of $\boldsymbol{K} = |\boldsymbol{K}| = 0.0034$
The condition number = CN = 1845

Since the determinant is small, the required adjustments in *U* will be very large, resulting in excessive control actions. In addition, this example shows the ***K*** matrix is poorly conditioned and very sensitive to small variations in its elements.

18.23

Applying SVA analysis:

Determinant of $\mathbf{K} = |\boldsymbol{K}| = -6.76$
The condition number = CN = 542.93

The large condition number indicates poor conditioning. Therefore this process will require large changes in the manipulated variables in order to influence the controlled variables. Some outputs or inputs should be eliminated to achieve better control, and singular value decomposition (SVD) can be used to select the variables to be eliminated.

By using the MATLAB command SVD, singular values of matrix ***K*** are:

$$\Sigma = \begin{bmatrix} 21.3682 & & & \\ & 6.9480 & & \\ & & 1.1576 & \\ & & & 0.0394 \end{bmatrix}$$

Note that $\sigma_3/\sigma_4 > 10$, then the last singular value can be neglected. If we eliminate one input and one output variable, there are sixteen possible pairing shown in Table S18.23, along with the condition number CN.

Pairing number	Controlled variables	Manipulated variables	CN
1	y_1,y_2,y_3	u_1,u_2,u_3	114.29
2	y_1,y_2,y_3	u_1,u_2,u_4	51.31
3	y_1,y_2,y_3	u_1,u_3,u_4	398.79
4	y_1,y_2,y_3	u_2,u_3,u_4	315.29
5	y_1,y_2,y_4	u_1,u_2,u_3	42.46
6	y_1,y_2,y_4	u_1,u_2,u_4	30.27
7	y_1,y_2,y_4	u_1,u_3,u_4	393.20
8	y_1,y_2,y_4	u_2,u_3,u_4	317.15
9	y_1,y_3,y_4	u_1,u_2,u_3	21.21
10	y_1,y_3,y_4	u_1,u_2,u_4	16.14
11	y_1,y_3,y_4	u_1,u_3,u_4	3897.2
12	y_1,y_3,y_4	u_2,u_3,u_4	693.25
13	y_2,y_3,y_4	u_1,u_2,u_3	24.28
14	y_2,y_3,y_4	u_1,u_2,u_4	20.62
15	y_2,y_3,y_4	u_1,u_3,u_4	1332.7
16	y_2,y_3,y_4	u_2,u_3,u_4	868.34

Table S18.23. *CN for different 3x3 pairings.*

Based on having minimal condition number, pairing 10 (y_1-u_1,y_3-u_2,y_4-u_4) is recommended. The RGA for the reduced variable set is

$$\mathbf{\Lambda} = \begin{bmatrix} 1.654 & -0.880 & 0.226 \\ -0.785 & 3.742 & -1.957 \\ 0.1312 & -1.8615 & 2.7304 \end{bmatrix}$$