

Mathmmatical Method



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Introduction

not important

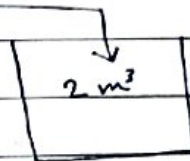
Lee 1

Mathematical Methods in Chemical Engineering

Flow = $0.01 \text{ m}^3/\text{min}$

Ex 1:

We need to fill a 2 m^3 tank by water.
The inlet water flowrate $0.01 \text{ m}^3/\text{min}$. How
long it takes to fill the tank? $t = ?$



Sol:

$$IN - out + G - C = acc$$

$$F_{in} - 0 + 0 - 0 = acc$$

$$\downarrow \frac{dV}{dt} \left[\frac{\text{m}^3}{t} \right]$$

$$\frac{dV}{dt} = F_0$$

$$\int_0^2 dV = \int_0^t F_0 dt$$

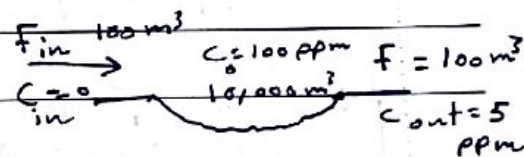
$$V = F_0 t$$

$$\frac{2}{0.01} = \frac{0.01 t}{0.01}$$

$$t = 200 \text{ min} \\ \text{or } 3.3 \text{ hr}$$

Ex 2:

Suppose you are swimming in a lake. $V = 10,000 \text{ m}^3$. polluted by
dye, $C_0 = 100 \text{ ppm}$. Fresh water is introduced at a flow rate
of $100 \text{ m}^3/\text{day}$. How long the initial conc. reducing to $C_p = 5 \text{ ppm}$?



Sol:

$$IN - out + G - C = acc$$

$$F_{in} C_{in} - F_{out} C_{out} + 0 - 0 = \frac{dVC}{dt}$$

$$- F_{out} C_{out} = V \frac{dC}{dt}$$

$$C_p = 5 \int \frac{dC}{C} = \int_0^t \frac{-F}{V} dt$$

$$C_0 = 100$$

$$\ln C \Big|_{100}^5 = \frac{-F}{V} t \Big|_0^t \rightarrow \ln \frac{5}{100} = \frac{-100}{10,000} t \rightarrow 2.995 = 0.01 t$$

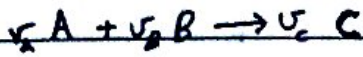
$$t = 299.5 \text{ day}$$

$$K_1 = \frac{1}{t} \rightarrow \text{time}$$

Introduction not important

lec 2

Fluid flow with chemical Rxn.



$$\text{Rate} = \frac{-1}{\nu_A} \frac{dC_A}{dt} = \frac{-1}{\nu_B} \frac{dC_B}{dt} = \frac{1}{\nu_C} \frac{dC_C}{dt}$$

$$\rightarrow = K C_A^{\nu_A} C_B^{\nu_B}$$

المعادلة:

$$F_0 C_{A_0} - F C_A - K C_A V = V \frac{dC_A}{dt}$$

$[m^3] [mol] \rightarrow [mol]$

$$[\frac{mol}{t}] - [\frac{mol}{t}] - [\frac{1}{t}] [\frac{mol}{m^3}] = \frac{mol}{t}$$

$$F_0 C_{A_0} - F C_A - K C_A V = V \frac{dC_A}{dt}$$

المعادلة:

$$\rightarrow \frac{dC_A}{dt} = \frac{F_0}{V} C_{A_0} - \frac{F}{V} C_A - K C_A$$

R: Rxn constant

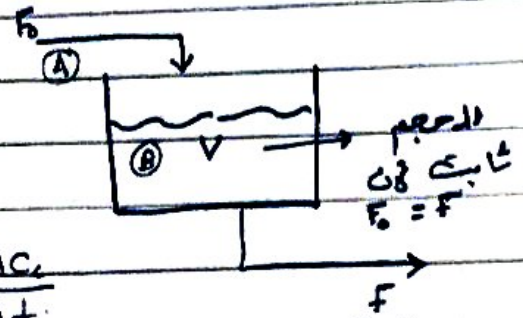
$$\rightarrow \frac{dC_A}{dt} = \frac{F_0}{V} [C_{A_0} - C_A] - R C_A$$

$$\rightarrow \frac{dC_A}{dt} = \frac{F_0}{V} C_{A_0} - [\frac{F_0}{V} + R] C_A$$

$$\rightarrow \frac{dC_A}{dt} + \underbrace{[\frac{F_0}{V} + R]}_{\alpha} C_A = \underbrace{\frac{F_0}{V} C_{A_0}}_{B}$$

$$C_A + \alpha C_A = B C_{A_0}$$

$$[y + \alpha y = B C_{A_0}] \rightarrow \text{first order ordinary}$$



تكون المتغيرات A, B, C
تغير بمرور الوقت
المعادلة

المركب هو المتغير

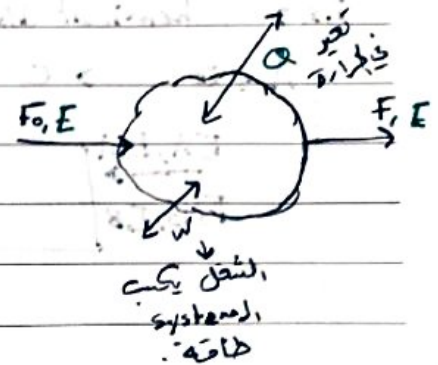


Introduction not important

lec 3

~~Energy balance on systems~~

E.B $\begin{cases} \rightarrow \text{within the system} \\ \rightarrow \text{across the borders} \end{cases}$



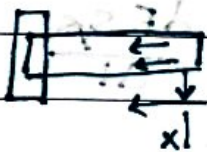
① within the system

$$F_o E_o - F_i E_i = \frac{dE_{\text{sys}}}{dt}$$

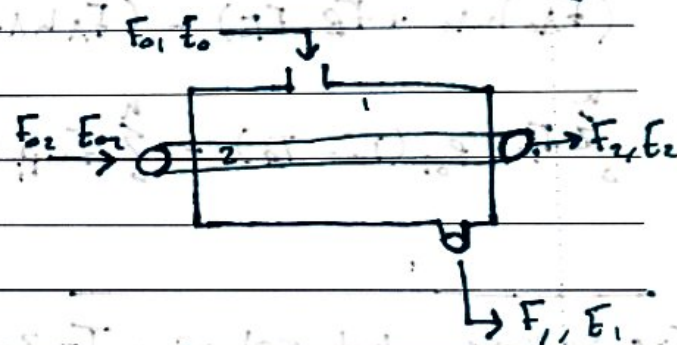
$$E = KE, PE, U$$

②

* conduction flux, $q = -k \frac{dT}{dx}$

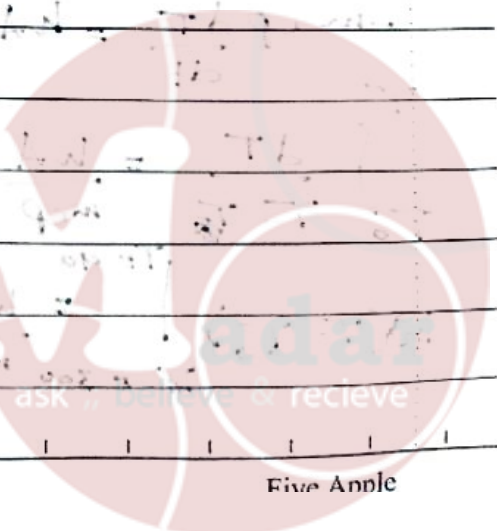
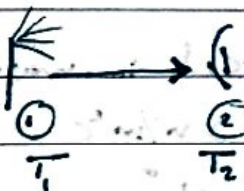


↓
تدفق الحرارة
في اتجاه x



* convective flux, $q = h(T_1 - T_2)$

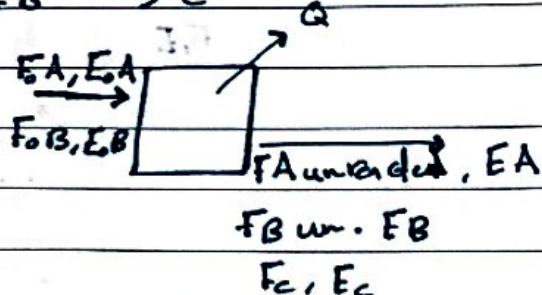
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تدفق الحرارة
في اتجاه x



Introduction not important

lec 3

Ex: Chemical Rxn. with heat transfer



$$(F_A E_A + F_B E_B) - (F_C E_C + F_{B \text{ unreacted}} E_A) + Q = \frac{dE_{\text{system}}}{dt}$$

$$E = U, p \Delta V = \Delta h = h = c_p \Delta T$$

↓
Enthalpy

$$(F_A h_A + F_B h_B) - (F_C h_C + F_{B \text{ unreacted}} h_A) + Q = \frac{dE_{\text{sys}}}{dt}$$

$$\sum F_i c_{p,i} (T_i - T_R) - \sum F_j c_{p,j} (T_j - T_R) + Q = c_p \frac{dT}{dt}$$

Ex: Suppose a hot sphere, $T = 300^\circ\text{C}$ is suddenly immersed in water

@ 20°C . The convective heat transfer coeff, $h = 58 \text{ W/m}^2\text{C}$.

what will the time needed to cool this sphere to 90°C ,

$m_{\text{sphere}} = 6 \text{ kg}$, $c_{p, \text{sphere}} = 896 \text{ J/Kg.C}$, $A_{\text{sphere}} = 0.08 \text{ m}^2$.

$$-m c_p \frac{dT}{dt} = h A (T - T_\infty)$$

$$\int_{T_0}^T \frac{dT}{T - T_\infty} = \frac{h A}{m c_p} \int_0^t dt$$

$$\ln(T - T_\infty) \Big|_{T_0=300}^{T=90} = \frac{h A}{m c_p} t$$

$$\frac{\ln(90 - 20)}{(300 - 20)} = \frac{h A}{m c_p} t$$

$$t = 1600 \text{ s}$$

let's start

lec 4

- ordinary differential equations, ODE

$y = f(x)$, y : dependent variable

x : independent

ODE: $\frac{dy}{dx^n} = f(x, y)$

ODE (order)
 $\swarrow$ 1st
 $\searrow$ higher

Ex: 1st $\rightarrow \frac{dy}{dx} = 3xy$

PDE (Linearity)
 $\swarrow$ linear
 $\searrow$ non linear

Ex: linear $\rightarrow \dot{y} + y = x$
 non linear $\rightarrow (y\dot{y})^2 + y = xy$

ODE (degree)
 $\swarrow$ 1st
 $\searrow$ higher

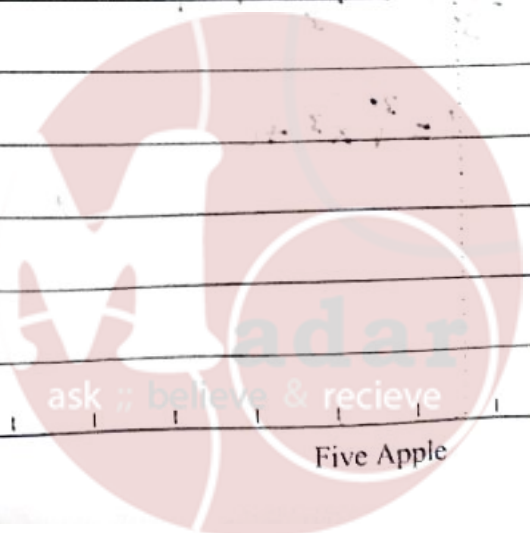
Ex: 1st $\frac{dy}{dx} = y$
 higher $(\frac{dy}{dx})^2 + y = 0$

higher $\rightarrow \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x$

ODE
 $\swarrow$ constant
 $\searrow$ non constant (coeff)

Ex: const $\rightarrow \frac{dy}{dx} + y = 3$

non constant $\rightarrow \frac{dy}{dx} + y = xy$



Soln of 1st order ODE

① Direct method

$$\dot{y} = f(x)$$

Ex $\frac{dy}{dx} = 3x^2$

$$dy = 3x^2 dx$$

$$\int dy = \int 3x^2 dx$$

$$y = x^3 + C$$

② Separation of variable

$$\dot{y} = f(x, y)$$

$$\frac{dy}{dx} = F(x) g(y)$$

$$\int \frac{dy}{g(y)} = \int F(x) dx$$

Ex:

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

$$\int y^2 dy = \int x^2 dx$$

$$\frac{y^3}{3} = \frac{x^3}{3} + C$$

$$y = \sqrt[3]{x^3 + C}$$

③ Integrating factor

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$M \frac{dy}{dx} + M P(x)y = M Q(x)$$

$$\frac{d(My)}{dx} = \frac{dM}{dx} y + M \frac{dy}{dx}$$

$$My = \int \frac{dM}{dx} y + M P(x)y$$

Ex: $\frac{dy}{dx} + 2y = e^{-2x}$

$$M = e^{2x}$$

$$M \frac{dy}{dx} + M P(x)y = \frac{d(My)}{dx}$$

$$M \frac{dy}{dx} + y \frac{dM}{dx}$$

$$M P(x)y = y \frac{dM}{dx}$$

$$M P(x) = \frac{dM}{dx}$$

$$\int \frac{P(x)}{\frac{dM}{dx}} dx = \int \frac{dM}{M}$$

$$\ln M = \int P(x) dx$$

$$M = \int P(x) dx$$

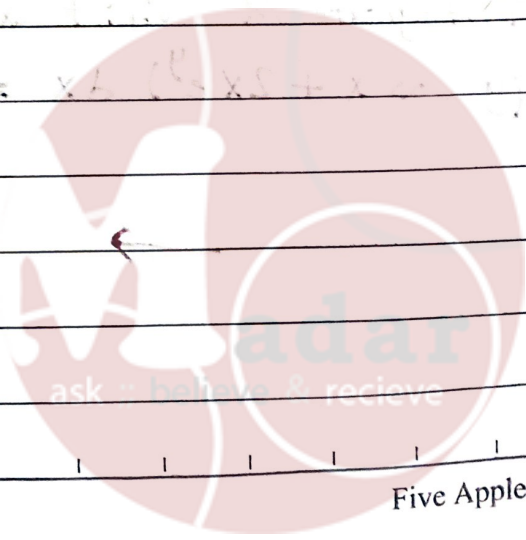
$$y = \frac{1}{M} \int M Q(x) dx$$

Ex: $\frac{dy}{dx} + \boxed{} + 2y = e^{-2x}$

$$\mu = e^{\int P(x) dx} = e^{\int 2 dx} = e^{2x}$$

$$y = \frac{1}{\mu} \int \mu Q(x) dx \rightarrow \frac{1}{e^{2x}} \int e^x e^{-2x} dx = \frac{1}{e^{2x}} x + C$$

$$= \boxed{\frac{x}{e^{2x}} + C}$$



(H) Exact Solution:

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

$$M(x,y) + N(x,y) \frac{dy}{dx} = 0$$

$$M_y \stackrel{?}{=} N_x \rightarrow (M_y = N_x) \rightarrow \text{Exact}$$

- Steps to solve:

① Integrate M w.r.t (x)

② Differential ~~of~~ w.r.t (y) $\rightarrow M$

③ Equate w.r.t N

$$\text{Ex: } \underbrace{(y \cos x + 2x e^y)}_{M(x,y)} + \underbrace{(\sin x + x^2 e^y + 2)}_{N(x,y)} \frac{dy}{dx} = 0$$

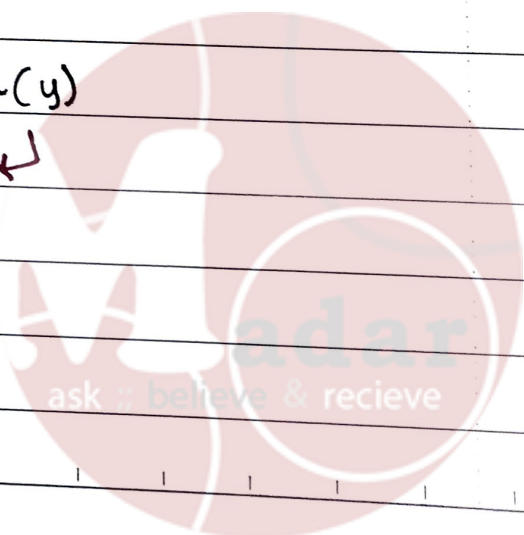
$$M_y = \cos x + 2x e^y \rightarrow M_y = N_x \therefore \text{Exact soln.}$$
$$N_x = \cos x + 2x e^y$$

by steps:

① Integrate M w.r.t (x)

$$\int (y \cos x + 2x e^y) dx = y \sin x + x^2 e^y + h(y)$$

const y $\rightarrow$



② Differential w.r.t y

$$= \sin x + x^2 e^y + h'(y)$$

③ Equate w.r.t N

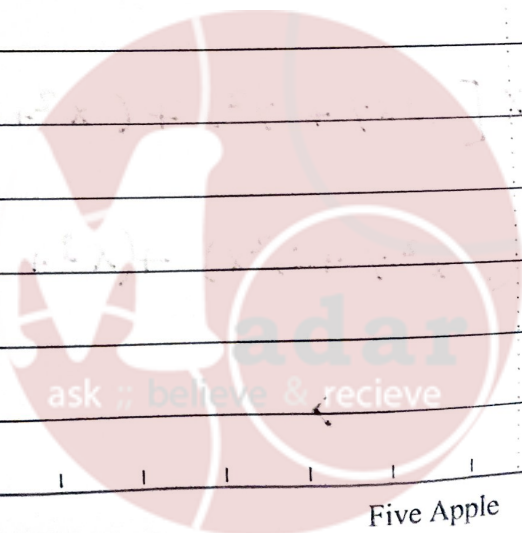
$$\sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y + 2$$

$$h'(y) = 2$$

- Integrate The remaining w.r.t y

$$h(y) = 2y$$

$$\rightarrow f(x, y) = y \sin x + x^2 e^y + 2y$$



⑤ Non-Exact Equations:

$$M(x,y) + N(x,y) \frac{dy}{dx} = 0$$

$M(x) \leftarrow M_y \neq N_x \rightarrow \text{Non-Exact}$

$$\boxed{M = c e^{\int \frac{M_y - N_x}{N} dx}} \rightarrow \text{انت من هنا}$$

or

$M(y) \leftarrow \boxed{M = c e^{\int \frac{M_y - N_x}{N} dy}}$

انت من هنا

$$\text{Ex: } (3xy + y^2) + (x^2 + xy) \frac{dy}{dx} = 0$$

$$\begin{aligned} M_y &= 3x + 2y \\ N_x &= 2x + y \end{aligned} \quad \left. \vphantom{\begin{aligned} M_y &= 3x + 2y \\ N_x &= 2x + y \end{aligned}} \right\} \neq \text{Non-Exact}$$

- Assume Integrating factor $M = f(x)$

$$M = c e^{\int \frac{M_y - N_x}{N} dx} \rightarrow M = c e^{\int \frac{3x + 2y - 2x - y}{x^2 + xy} dx}$$

$$\rightarrow M = c e^{\int \frac{x+y}{x(x+y)} dx} = c e^{\int \frac{1}{x} dx} = c e^{\ln x} = c x \rightarrow \boxed{M = x}$$

$$x \left[(3xy + y^2) + (x^2 + xy) \frac{dy}{dx} = 0 \right]$$

$$= (3x^2y + y^2x) + (x^3 + x^2y) \frac{dy}{dx} = 0$$

→



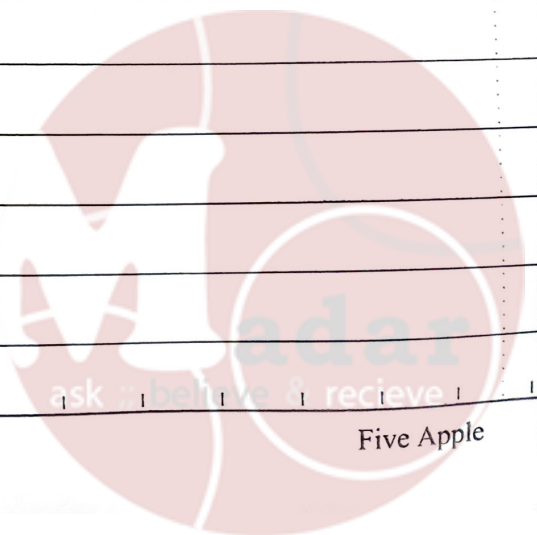
$$M_y = 3x^2 + 2xy \quad N_x = 3x^2 + 2xy \rightarrow M_y = N_x \rightarrow \text{Exact}$$

∴ Now we can solve it with Exact method.

في هذا امثال جوبينا $M(x)$ ونزبطه في
 زميلة اخرى لو ما زبطت بنجرب $M(y)$

عند التعويض بـ x $\int \frac{M_y - N_x}{N} dx$ وبعد ناتج التكاملي

لو خلعت فيها x فقط نكمل الحل ورج نزبطه لو صيرنا
 x, y ما في داعي للضرب ما رج نزبطه.
 وهذا ينطبق على $M(y)$.



⑥ Homogeneous Equations:

$$\frac{dy}{dx} = f(x, y) \rightarrow \frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

$\Rightarrow$ Homogeneous Equation

Ex: $\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}$

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{2xy}{x^2}$$

$$= \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)$$

$$= g\left(\frac{y}{x}\right)$$

$$\rightarrow \frac{dy}{dx} = f(x, y) = g\left(\frac{y}{x}\right)$$

$$= g(v) \quad v = \frac{y}{x}$$

$$y = vx$$

$$\frac{dy}{dx} = v \frac{dx}{dx} + x \frac{dv}{dx}$$

$$= v + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = g(v)$$

$$x \frac{dv}{dx} = g(v) - v$$

$$\boxed{\frac{dv}{g(v) - v} = \frac{dx}{x}}$$

بالضرب في x

$$\frac{dy}{dx} = f(x, y) = g\left(\frac{y}{x}\right) = g(v)$$

$$g(v) = v + x \frac{dv}{dx}$$

$$\rightarrow \boxed{\frac{dv}{g(v) - v} = \frac{dx}{x}}$$

$$\left(\frac{y}{x}\right)^2 \frac{dy}{dx} = \frac{y^2}{x^2} + 2\left(\frac{y}{x}\right)$$

$$g(v) = v^2 + 2v$$

بالعوض في المعادلة

$$\frac{dv}{[v^2 + 2v] - v} = \frac{dx}{x}$$

$$\rightarrow \int \frac{dv}{v^2 + v} = \int \frac{dx}{x}$$

نكامل بالجزء

$$\ln v - \ln(v+1) = \ln cx$$

$$\ln \frac{v}{v+1} = \ln cx$$

$$\frac{v}{v+1} = cx$$

$$\rightarrow g(v) = v + x \frac{dv}{dx} = \frac{y}{x} = cx$$

$$\frac{y}{x} = \frac{y}{x} + 1$$

$$\frac{y}{x} = \frac{y}{x} cx + cx$$

$$\frac{y}{x} + cy + cx$$

$$y = cy + cx^2$$

$$y(1 - cx) = cx^2$$

$$y = \frac{cx^2}{1 - cx}$$

ask

Five Apple

Ex:

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)$$

$$= v^2 + v = g(v)$$

تعويض:

$$\frac{dv}{v^2 + v} = \frac{dx}{x} \rightarrow \int \frac{dv}{v^2 + v} = \int \frac{dx}{x}$$

$$\frac{-1}{v} = \ln cx$$

$$\rightarrow \frac{-1}{\frac{y}{x}} = \ln cx$$

$$\rightarrow \frac{-x}{y} = \ln cx$$

$$y = \frac{-x}{\ln cx}$$



⑦ Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Examples:

$$* \frac{dy}{dx} + 5y = x^2 y^2$$

$$* x^2 \frac{dy}{dx} + 3xy = y$$

$$* \frac{dy}{dx} + P(x)y = Q(x)y^n$$

$$\rightarrow \frac{dy}{dx} + P(x)y = Q(x)$$

$\rightarrow$ use integrating factor

by derivation

$$\boxed{\frac{1}{1-n} \frac{dv}{dx} + P(x)v = Q(x)} = \boxed{\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)}$$

$$M = \exp \int (1-n)P(x) dx$$

$$v = \frac{1}{M} \int M(1-n)Q(x) dx = y^{1-n}$$

Ex:

$$\frac{\dot{y} + \frac{4}{x}y}{y^2} = x^3 y^2$$

یعنی $Q(x)$ به y وابسته است
Bernoulli
Equation

$$y^{-2} \dot{y} + \frac{4}{x} y^{-1} = x^3$$

$$n=2, v = y^{1-n} = y^{-1}$$

$$\frac{1}{1-n} \frac{dv}{dx} + \bar{p}(x)v = \bar{Q}(x)$$

$$\frac{1}{1-2} \frac{dv}{dx} + \frac{4}{x} v = x^3$$

$$-\frac{dv}{dx} + \frac{4}{x} v = x^3$$

$$\frac{dv}{dx} - \frac{4}{x} v = -x^3$$

$$\mu = e^{\int -\frac{4}{x} dx}$$

$$\mu = e^{-4 \ln x}$$

$$v = \frac{1}{\mu} \int \mu (C_1 - u) Q(x) dx = y^{-1}$$

$$\frac{1}{e^{-4 \ln x}} \int e^{-4 \ln x} (-1) x^{-3} dx = \frac{-1}{x^{-4}} \int x^{-4} x^{-3} dx$$

$$= \frac{-1}{x^{-4}} (\ln x + C) \Rightarrow -x^4 (\ln x + C)$$

$$y^{-1} = -x^4 (\ln x + C) \Rightarrow y = \frac{1}{-x^4 (\ln x + C)}$$

ask :: believe & recieve

$$\text{Ex: } y \frac{dy}{dx} + x^2 y^2 = 3xy^5$$

divide by y

$$\frac{dy}{dx} + x^2 y = 3xy^4$$

$$v = y^{1-n} = y^{-3}$$

$$n = 4$$

$$\frac{1}{1-n} \frac{dv}{dx} + p(x)v = Q(x)$$

$$\rightarrow \frac{-1}{3} \frac{dv}{dx} + x^2 v = 3x \quad * -3$$

$$= \frac{dv}{dx} + x^2 v (-3) = -9x$$

$P(x)$

$Q(x)$

$$= \frac{dv}{dx} + 3x^2 v = -9x \quad (\text{by integrating factor})$$

$$\mu = \exp \int P(x) dx = \exp \int -3x^2 dx = \cancel{1/x} \cancel{1/x} \cancel{1/x}$$

$$= e^{-x}$$

$$v = \frac{1}{\mu} \int \mu Q(x) dx \rightarrow \int e^{-x} - 9x dx$$

$$y^3 = \frac{1}{\mu} \int e^{-x} - 9x dx$$

$$y^3 = \frac{1}{\text{soln}}$$

$$y = \sqrt[3]{\frac{1}{\text{soln}}}$$



⑧ Riccati Equation :

$$\frac{dy}{dx} = p(x)y^2 + Q(x)y + R(x)$$

if $p(x) = -1$

$$\frac{dy}{dx} + y^2 = Q(x)y + R(x)$$

Assume $y = \frac{1}{u} \frac{du}{dx}$

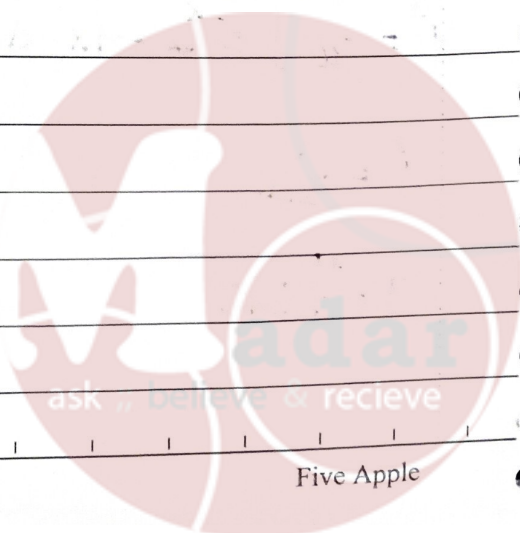
$$\frac{dy}{dx} = \frac{1}{u} \frac{d^2u}{dx^2} - \frac{1}{u^2} \left(\frac{du}{dx} \right)^2$$

$$\left(\frac{1}{u} \frac{d^2u}{dx^2} - \frac{1}{u^2} \left(\frac{du}{dx} \right)^2 \right) + \left(\frac{1}{u} \frac{du}{dx} \right)^2 = Q(x) \frac{1}{u} \frac{du}{dx} + R(x)$$

$$\frac{1}{u} \frac{d^2u}{dx^2} - Q(x) \frac{1}{u} \frac{du}{dx} - R(x) = 0$$

$$\frac{d^2u}{dx^2} - Q(x) \frac{du}{dx} - R(x)u = 0$$

$$u'' - Q(x)u' - R(x)u = 0 \rightarrow 2^{\text{nd}} \text{ order}$$



First order of 2nd degree

$$\left(\frac{dy}{dx}\right)^2 - \alpha \frac{dy}{dx} + qy = g(x)$$

Assume $p = \frac{dy}{dx}$

$$p^2 - \alpha p + qy = g(x)$$

$$p^2 - \alpha p + qy - g(x) = 0$$

$$p = \frac{\alpha \pm \sqrt{\alpha^2 - 4(qy - g(x))}}{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $ax^2 + bx + c$

Ex:

$$\left(\frac{dy}{dx}\right)^2 - 2\left(\frac{dy}{dx}\right) + y = x - 1$$

$$p^2 - 2p + y + 1 - x = 0$$

$$p = \frac{2 \pm \sqrt{4 - 4(y + 1 - x)}}{2}$$

$$p = \frac{2 \pm \sqrt{4 - 4y - 4 + 4x}}{2}$$

$$p = \frac{2 \pm \sqrt{4(x - y)}}{2}$$

$$p = 1 \pm \sqrt{x - y} = \frac{dy}{dx}$$

Let $u = x - y$

$$\frac{du}{dx} = \frac{dx}{dx} - \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = 1 - \frac{du}{dx}}$$

$$\frac{dy}{dx} = 1 - \frac{du}{dx}$$

$$\frac{dy}{dx} = 1 \pm \sqrt{x - y} = 1 \pm \sqrt{u}$$

$$x - \frac{du}{dx} = 1 \pm \sqrt{u}$$

$$-\frac{du}{dx} = \pm \sqrt{u} \rightarrow \frac{du}{dx} = \mp \sqrt{u}$$

$$\frac{du}{\sqrt{u}} = \mp dx$$

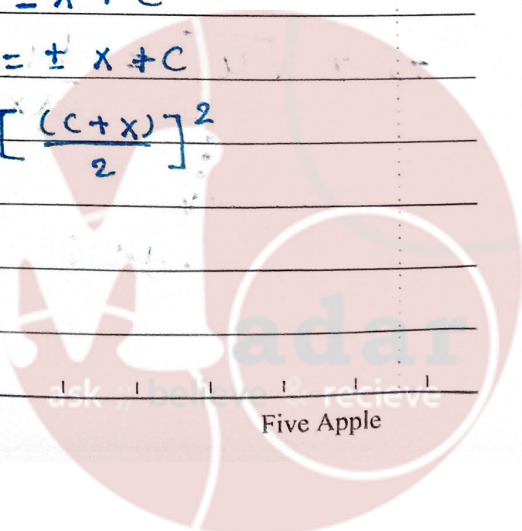
$$\int \frac{du}{\sqrt{u}} = \mp \int dx$$

$$\frac{u^{\frac{1}{2}}}{\frac{1}{2}} = \mp x + C$$

$$2\sqrt{u} = \mp x + C$$

$$2\sqrt{x - y} = \mp x + C$$

$$y = x - \left[\frac{(C \mp x)}{2}\right]^2$$



Example:

$$2\left(\frac{dy}{dx}\right)^2 - 4\frac{dy}{dx} + y = 0$$

~~$$2\left(\frac{dy}{dx}\right)^2 - 4\frac{dy}{dx} + y = 0$$~~

$$2p^2 - 4p + y = 0$$

$$p = \frac{4 \pm \sqrt{16 - 4 \times 2 \times y}}{2 \times 2}$$

$$= \frac{4 \pm \sqrt{16 - 4 \times 2 \times y}}{4}$$

$$= \frac{4 \pm \sqrt{4(4 - 2y)}}{4}$$

$$= \frac{4 \pm 2\sqrt{4 - 2y}}{4}$$

$$p = 1 \pm \frac{1}{2}\sqrt{4 - 2y} = \frac{dy}{dx}$$

$$u = 4 - 2y \quad (\text{sub. } 2y \text{ to } 4)$$

~~$$\frac{du}{dx} = -2\frac{dy}{dx}$$~~

$$\frac{dy}{dx} = -\frac{1}{2} \frac{du}{dx}$$

$$\frac{dy}{dx} = 1 \pm \frac{1}{2}\sqrt{4 - 2y}$$

$$\left(1 \pm \frac{1}{2}\sqrt{4 - 2y} = -\frac{1}{2} \frac{du}{dx}\right) \times -2$$

$$-2 \pm \sqrt{4 - 2y} = \frac{du}{dx}$$



...
JLS

Example:

$$2\left(\frac{dy}{dx}\right)^2 - 4\left(\frac{dy}{dx}\right) = 0$$

$$p = \frac{dy}{dx}$$

$$2p^2 - 4p = 0$$

$$p = \frac{4 \pm \sqrt{16 - 0}}{2 \times 2}$$

$$p = \frac{4 \pm 4}{4}$$

$$p = 0 \text{ or } p = 2$$

$$* ax^2 + bx + c = 0$$

$$b^2 - 4ac > 0 \rightarrow \text{Real Root}$$

$$b^2 - 4ac < 0 \rightarrow \text{imaginary Root}$$



$$\text{Ex: } 2x^2 - x + 2 =$$

$$x = \frac{1 \pm \sqrt{1 - 4 \times 2 \times 2}}{4}$$

$$x = \frac{1 \pm \sqrt{-15}}{4} = \frac{1 \pm \sqrt{-1}\sqrt{15}}{4}$$



Initial value problems

$$\frac{dy}{dx} = f(x)$$

$$y(0) = \text{value}$$

Ex₁: (Separation of variables)

$$\dot{y} + \frac{3y}{t} = 0 \text{ variable}$$

$$y(1) = 2$$

$$\Rightarrow \frac{dy}{dt} + \frac{3y}{t} = 0$$

$$\rightarrow \frac{dy}{dt} = -\frac{3y}{t}$$

$$\int \frac{dy}{y} = \int -\frac{3dt}{t}$$

$$\ln y = -3 \ln t + C$$

$$\ln y = \ln t^{-3} + C$$

$$y = C_2 t^{-3}$$

$$\text{at } t=1 \rightarrow y=2$$

$$2 = C_2 \cdot 1^{-3}$$

$$\boxed{C_2 = 2}$$

* general Soln :

$$y = 2t^{-3}$$

Ex₂: $\frac{dy}{dx} + 3x^2 y = 6x^2$
 $y(0) = 3$

(Integrating factor)

$$\rightarrow P(x) = 3x^2$$

$$Q(x) = 6x^2$$

$$\mu = \exp \int P(x) dx$$

$$= \exp \int 3x^2 dx$$

$$= e^{x^3}$$

$$y_1 = \frac{1}{\mu} \int \mu Q(x) dx$$

$$= e^{-x^3} \int e^{x^3} \cdot 6x^2 dx$$

$$\rightarrow y = x^3$$

$$\frac{dy}{dx} = 3x^2 \cdot dx$$

$$dx = \frac{dy}{3x^2}$$

$$\int e^{x^3} \cdot 6x^2 \frac{dy}{3x^2} = e^y = 2e^{x^3} + C$$

$$\frac{dy}{dx} = 3x^2 \cdot dx$$

$$y = e^{-x^3} (2e^{x^3} + C)$$

$$y = e^{-x^3} \cdot 2e^{x^3} + C \cdot e^{-x^3}$$

$$y = 2 + C \cdot (1) \quad (x=0 \rightarrow y=3)$$

$$3 = 2 + C$$

$$\boxed{C=1}$$

$$* \boxed{y = 2 + e^{-x^3}}$$

Example 3:

$$(3x^2y - 1)dx + (x^3 + 6y - y^2)dy = 0$$

$$M \quad y(0) = 3 \quad N$$

$$M_y = 3x^2 \quad (\text{Exact equation})$$

$$N_x = 3x^2$$

$$\textcircled{1} \int M dx = \int (3x^2y - 1) dx = yx^3 - x + g(y)$$

$$\textcircled{2} S = x^3y - x + g(y) \rightarrow \frac{dS}{dy} = x^3 + g'(y)$$

$$\textcircled{3} x^3 + g'(y) = x^3 + 6y - y^2$$

$$\int g'(y) = \int (6y - y^2) dy$$

$$g(y) = 3y^2 - \frac{y^3}{3} + C$$

$$\Rightarrow yx^3 - x + 3y^2 - \frac{y^3}{3} + C = f(x, y) = 0$$

$$27 - 9 + C = 0$$

$$C = -18$$

$$\Rightarrow \boxed{yx^3 - x + 3y^2 - \frac{y^3}{3} - 18}$$



Example 4:

$$\frac{dy}{dx} + y = y^2$$

(Bernoulli Equation)

$$y(0) = \frac{1}{2}$$

$$p(x) = 1$$

$$Q(x) = 1$$

$$n = 2$$

$$v = y^{1-n} = y^{-1}$$

$$\frac{1}{1-n} \frac{dv}{dx} + p(x)v = Q(x)$$

$$\frac{dv}{dx} - p(x)v = -Q(x)$$

$$\frac{dv}{dx} - v = -1$$

$$\frac{dv}{dx} = v - 1$$

$$\int \frac{dv}{v-1} = \int dx$$

$$\ln(v-1) = (x+C)$$

$$v-1 = ce^x$$

$$v = 1 + ce^x$$

$$1 + ce^x = y^{-1}$$

$$* \boxed{y = \frac{1}{1+ce^x}}$$

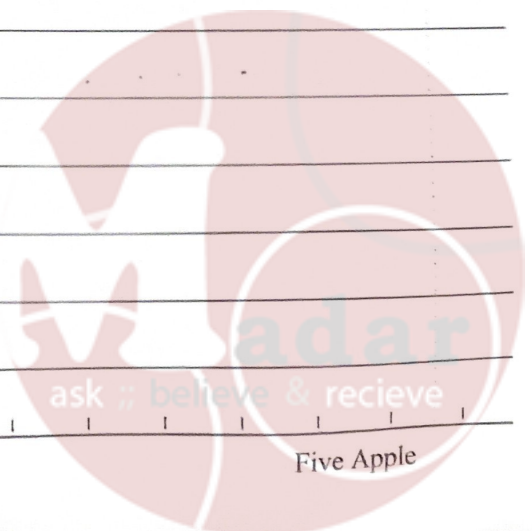
$$\text{at } x=0, y = \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{1+c}$$

$$1+c = 2$$

$$\boxed{c=1}$$

$$\Rightarrow \boxed{y = \frac{1}{1+e^x}}$$



① $y'' = f(y, y')$

Ex: $-1 + y\ddot{y} + \dot{y}^2 = 0$

$-\ddot{y} + 2\dot{y} - y = 0$

→ soln: $\dot{y} = p$

$\ddot{y} = \dot{p} = \frac{dp}{dx} = \frac{dp}{dy} \frac{dy}{dx} = \frac{dp}{dy} \dot{y} = \frac{dp}{dy} p$

تعويض

$1 + y\ddot{y} + \dot{y}^2 = 0$

$1 + y p \frac{dp}{dy} + p^2 = 0$

$1 + y p \frac{dp}{dy} = -p^2$

$\cancel{dy} \times y p \frac{dp}{\cancel{dy}} = \frac{-(1+p^2)}{-(1+p^2)} \times dy$

$\frac{y p dp}{y(1+p^2)} = \frac{dy}{y}$

$\frac{p dp}{(1+p^2)} = \frac{dy}{y}$

$\int \frac{dy}{y} + \int \frac{p dp}{(1+p^2)} = 0$

$2 \left(\ln y + \frac{1}{2} \ln(1+p^2) \right) = C$

$\ln y^2 + \ln(1+p^2) = C_1$
→

$\ln y^2 + \ln(1+p^2) = C_1$

$\ln[y^2(1+p^2)] = C_1$

$y^2(1+p^2) = \exp C_1 = \alpha^2$

$y^2 \left(1 + \left(\frac{dy}{dx} \right)^2 \right) = \alpha^2$

$1 + \left(\frac{dy}{dx} \right)^2 = \frac{\alpha^2}{y^2}$

$\frac{dy}{dx} = \sqrt{\frac{\alpha^2}{y^2} - 1}$

$\frac{dy}{dx} = \sqrt{\frac{\alpha^2 - y^2}{y^2}}$

$\frac{dy}{dx} = \frac{1}{y} \sqrt{\alpha^2 - y^2}$

$\int \frac{y dy}{\sqrt{\alpha^2 - y^2}} = \int \frac{\sqrt{\alpha^2 - y^2}}{\sqrt{\alpha^2 - y^2}} dx$

$$\text{Ex: } \ddot{y} + y = 0$$

$$\dot{y} = p, \quad \ddot{y} = p \frac{dp}{dy}$$

$$p \frac{dp}{dy} + y = 0$$

$$p \frac{dp}{dy} = -y$$

$$\int p \, dp = \int -y \, dy$$

$$\frac{p^2}{2} = -\frac{y^2}{2} + C$$

$$p^2 = -y^2 + C_1$$

$$\left(\frac{dy}{dx}\right)^2 = C_1 - y^2$$

$$\frac{dy}{dx} = \sqrt{C_1 - y^2}$$

$$\frac{dy}{dx} = \sqrt{a^2 - y^2}$$

$$\int \frac{dy}{\sqrt{a^2 - y^2}} = \int dx$$

$$\text{Ex:}$$

$$\ddot{y} + y(\dot{y})^2 = 0$$

$$p \frac{dp}{dy} + yp^2 = 0$$

$$\frac{dp}{dy} + yp = 0$$

$$\frac{dp}{dy} = -yp$$

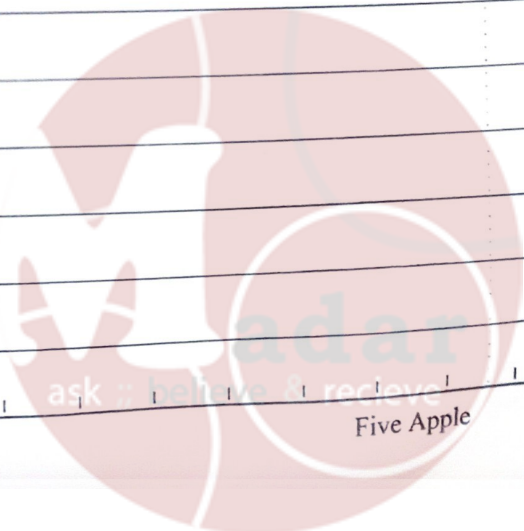
$$\int \frac{dp}{p} = \int -y \, dy$$

$$\ln p = -\frac{y^2}{2} + C$$

$$p = \exp\left(C - \frac{y^2}{2}\right)$$

$$\frac{dy}{dx} = \exp\left(C - \frac{y^2}{2}\right)$$

$$\int \frac{dy}{\exp\left(C - \frac{y^2}{2}\right)} = \int dx$$



② $\ddot{y} = f(x, y)$

Ex:

$$\ddot{y} + 2x\dot{y} = x$$

Let $p = \bar{y}$
 $\bar{y} = \frac{dp}{dx}$

$$\frac{dp}{dx} + 2xp = x \text{ (Integrating factor)}$$

$$H = \exp \int 2x \, dx = e^{x^2}$$

$$P = \frac{1}{\sigma^2} \int_0^{\infty} x e^{-x^2/\sigma^2} dx$$

$$\rho = \frac{1}{2} + A e^{-x^2} = y$$

۱) کمال لخص علی

$$y = \frac{1}{2}x + A \int e^{-x^2} dx + B$$

$$E_{x_2} : \ddot{y} + x = 0$$

$$\frac{dp}{dx} = y''$$

$$\frac{dp}{dx} + x = 0$$

$$\frac{dp}{dx} = -x$$

$$dP = -x dx$$

$$p = \frac{-x^2}{2} + C$$

$$\frac{dy}{dx} = -\frac{x^2}{2} + C$$

$$y = -\frac{x^3}{6} + cx + c_1$$

$$y = -\frac{x^3}{6} + cx + c_1$$

$$Ex_2: \ddot{y} + \dot{y} = x$$

$$\text{Let } p = \hat{y}$$

$$\frac{dp}{dx} = \hat{y}$$

$$\frac{dp}{dx} + p = x \text{ (Integrating factors)}$$

$$M = \exp \int_1 dx = e^x$$

$$p = \frac{1}{e^x} \int e^x x \, dx$$

$$p = e^{-x} [e^x (x-1)] + A e^{-x}$$

$$\frac{dy}{dx} = (x-1) + Ae^{-x}$$

$$dy = ((x-1) + 4e^{-x}) dx$$

$$y = \left(\frac{x^2}{2} - x \right) - A e^{-x} + B$$

③ المذكور انظروا قبل

④ $\ddot{y} = f(\dot{y}, y, x)$

with constant coefficient

$\rightarrow \ddot{y} + a_1 \dot{y} + a_0 y = g(x)$

$\begin{aligned} &\ddot{y} + a_1 \dot{y} + a_0 y = 0 \Rightarrow y_c(x) \quad \text{or constant} \\ &\quad \downarrow \text{complementary} \\ &\ddot{y} + a_1 \dot{y} + a_0 y = g(x) \Rightarrow y_p(x) \\ &\quad \downarrow \text{particular} \\ &\boxed{y = y_c + y_p} \end{aligned}$

we assume $y = e^{rx}$
we get the value of r

(imaginary root)

$y = e^{rx}, \dot{y} = r e^{rx}, \ddot{y} = r^2 e^{rx}$

complementary $\downarrow$ في

$r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} = 0$
 $e^{rx} [r^2 + a_1 r + a_0] = 0$
 $e^{rx} \neq 0$

So $r^2 + a_1 r + a_0 = 0$

$r = -a_1 \pm \frac{\sqrt{a_1^2 - 4a_0}}{2}$

قوة داخل الجذر

Case 3 $\rightarrow r = \alpha \pm \beta i$

$y_c = [A \cos \beta x + B \sin \beta x] e^{\alpha x}$

* How to get Particular soln?

$\begin{array}{cc} g(x) & y_p(x) \\ \textcircled{1} & g(x) = c \rightarrow c/a_0 \\ \textcircled{2} & g(x) = cx \rightarrow \frac{c}{a_0} \left[x - \frac{a_1}{a_0} \right] \end{array}$

$\textcircled{3} \quad g(x) = cx^2 \rightarrow \frac{c}{a_0} \left[x^2 - \frac{2a_1}{a_0} x + \frac{(a_1^2 - a_0)}{a_0^2} \right]$

- Case 1 $\Rightarrow r_1$ and r_2 are

real and distinct $\rightarrow$ مختلف

$y_c = A e^{r_1 x} + B e^{r_2 x}$

$Cx^2 + x$ $\rightarrow$ $3x^2 + 2x$

مع بعض بعض $\rightarrow$ $3x^2 + 2x$

- Case 2 $\Rightarrow r_1, r_2 \Rightarrow$ real, $r_1 = r_2 = r$

$y_c = (A + Bx) e^{rx}$

ask : believe & recieve

Five Apple

Ex₁: $\ddot{y} - 3\dot{y} + 2y = 10$ (complementary)
 $y = e^x, \quad y = r e^x, \quad y = r^2 e^x$

$$r^2 e^{rx} - 3r e^{rx} + 2e^{rx} = 0$$

$$e^{rx}(r^2 - 3r + 2) = 0$$

$$(r-1)(r-2) = 0 \rightarrow r = 1, 2$$

$$y_c = A e^x + B e^{2x}$$

$$y_p = \frac{C}{a_0} = \frac{10}{2} = 5$$

$$y = y_c + y_p = A e^x + B e^{2x} + 5$$

Ex₂: $\ddot{y} - 3\dot{y} + 2y = 10x$ (particular)

$y_c = A e^x + B e^{2x} \rightarrow y_c$ نجد

$g(x) = c$ تعبر

$y_p = ?$

وهي نفس المثال الأول

$$y_p = \frac{C}{a_0} \left(x - \frac{a_1}{a_0} \right)$$

$$y_p = \frac{10}{2} \left(x - \frac{-3}{2} \right)$$

$$y_p = 5(x + 1.5)$$

$$y = A e^x + B e^{2x} + 5(x + 1.5)$$

Ex₃: $\ddot{y} - 3\dot{y} + 2y = 10x^2$

$$y_c = A e^x + B e^{2x}$$

$$y_p = \frac{C}{a_0} \left[x^2 - \frac{2a_1}{a_0} x + \frac{2(a_1^2 - a_0)}{a_0^2} \right]$$

$$= \frac{10}{2} \left[x^2 - 2 \cdot \frac{-3}{2} x + \frac{2(9-2)}{4} \right]$$

$$y_p = 5(x^2 + 3 + \frac{14}{4})$$

$$y = A e^x + B e^{2x} + 5x^2 + 5(3 + \frac{14}{4})$$

$$\text{Ex 4: } \ddot{y} + 2\dot{y} + y = g(x)$$

$$y_c = (r^2 + 2r + 1)e^{rx} = 0$$

$$(r+1)(r+1) = 0$$

$$\boxed{r = -1}$$

$$y_c = (A + Bx)e^{-x}$$

$$\text{Ex 5: } \ddot{y} + 2\dot{y} + 2y = g(x)$$

$$y_c = e^{rx}(r^2 + 2r + 2) = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = -1 \pm i$$

$$= \alpha + \beta i$$

$$y_c = (A \cos \beta x + B \sin \beta x)e^{\alpha x}$$

$$= (A \cos x + B \sin x)e^{-x}$$



③ Homogeneous eq.

$$x\ddot{y} = f(\dot{y}, \frac{y}{x})$$

Ex:

$$① x\ddot{y} + \dot{y} + \frac{y}{x} = 0$$

$$② x^2\ddot{y} + x\dot{y} + y = 0 \quad \leftarrow * x$$

$$③ x\ddot{y} + \dot{y} + 5 = (\frac{y}{x})^2$$

$$\frac{y}{x} = v(x) \quad \text{نقوض}$$

$$\rightarrow y = v(x) \cdot x$$

$$\dot{y} = x\dot{v}(x) + v(x)$$

$$\ddot{y} = x\ddot{v}(x) + 2\dot{v}(x)$$

← (الفرض الأول)

Example:

$$(x^3\ddot{y} + x^2\dot{y}^2 - y^2 = 0) \div x^2$$

$$x\ddot{y} + \dot{y}^2 - (\frac{y}{x})^2 = 0 \quad \text{فلما نتربح}$$

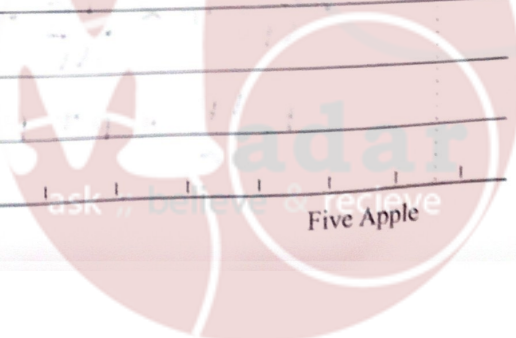
نقوضه:

$$x[x\ddot{v} + 2\dot{v}] + [x^2\dot{v}^2 + 2xv\dot{v} + v^2] - v^2 = 0$$

$$x^2\ddot{v} + 2x\dot{v} + x^2\dot{v}^2 + 2xv\dot{v} = 0 \rightarrow \text{محلنا إلى صيغة فقط في } (x, v)$$

$$x = e^t \Rightarrow t = \ln x \Rightarrow \frac{1}{x} = \frac{1}{e^t} = e^{-t}$$

← (الفرض الثاني)



$$\frac{dr}{dx} = \dot{r} = \frac{dr}{dt} \cdot \frac{dt}{dx} = \frac{dr}{dt} \left(\frac{1}{x} \right)$$

$$\frac{d^2r}{dx^2} = \ddot{r} = \frac{d}{dx} \left[\frac{dr}{dx} \right] = \frac{d}{dt} \left[\frac{dr}{dt} \left(\frac{1}{x} \right) \right] \frac{dt}{dx}$$

$$= \left[\frac{d^2r}{dt^2} \left(\frac{1}{x} \right) + \frac{d}{dx} \left(\frac{1}{x} \right) \frac{dr}{dt} \right] \frac{dt}{dx}$$

$$= \left[\frac{d^2r}{dt^2} (e^{-t}) - e^{-t} \frac{dr}{dt} \right] \left(\frac{1}{x} \right)$$

$$= \left[\frac{d^2r}{dt^2} \left(\frac{1}{x} \right) - \left(\frac{1}{x} \right) \frac{dr}{dt} \right] \left(\frac{1}{x} \right)$$

$$= \left(\frac{1}{x} \right)^2 \left[\frac{d^2r}{dt^2} - \frac{dr}{dt} \right] = \frac{d^2r}{dx^2}$$

$$x p = \frac{dr}{dt} \rightarrow \text{(العرض بـ x)}$$

$$\frac{d^2r}{dx^2} = \frac{d}{dt} \left(\frac{dr}{dt} \right) = \frac{dp}{dt} = \frac{dp}{dr} \frac{dr}{dt} = p \frac{dp}{dr}$$

$$\text{So } \left[\frac{d^2r}{dx^2} = p \frac{dp}{dr} \right] \rightarrow \text{مفاد}$$

أضرب

$$\rightarrow x^2 \left[\frac{1}{x^2} \cdot \left(\frac{d^2r}{dt^2} - \frac{dr}{dt} \right) \right] + 2x \frac{dr}{dx} \cdot \frac{1}{x} + x^2 \left(\frac{dr}{dt} \right) \left(\frac{1}{x^3} \right)$$

$$x^2 \left(\frac{dr}{dt} \right)^2 \cdot \frac{1}{x^2} + 2x r \frac{dr}{dt} \cdot \frac{1}{x} = 0$$

$$\frac{d^2r}{dt^2} + 2 \frac{dr}{dt} + \left(\frac{dr}{dt} \right)^2 + 2r \frac{dr}{dt} = 0$$

$$p \frac{dp}{dr} + p + p^2 + 2rp = 0 \quad (\text{Integrating factor})$$

⑤ Euler - Cauchy Eq.

$$ax^2\ddot{y} + bx\dot{y} + cy = 0$$

(4) خاصية من

مضروب y و x^2 مضروب $\dot{y}$ و x

$$y = x^r, \dot{y} = r x^{r-1}, \ddot{y} = r(r-1) x^{r-2}$$

نفس الشيء

$$\begin{aligned} a x^2 [r(r-1) x^{r-2}] + b x r x^{r-1} + c x^r &= 0 \\ = [ar(r-1)] x^r + br [x^r] + c x^r &= 0 \\ = [ar(r-1) + br + c] x^r &= 0 \end{aligned}$$

↳ always not zero

$$\begin{aligned} \text{So } ar(r-1) + br + c &= 0 \\ = ar^2 - ar + br + c &= 0 \\ = ar^2 + (b-a)r + c &= 0 \end{aligned}$$

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$

based on r we will have

* two distinct values of r ; r_1, r_2

$$y_1 = x^{r_1}, y_2 = x^{r_2}$$

* two roots (similar); $r_2 = r_1 = r$

$$y_1 = x^r$$

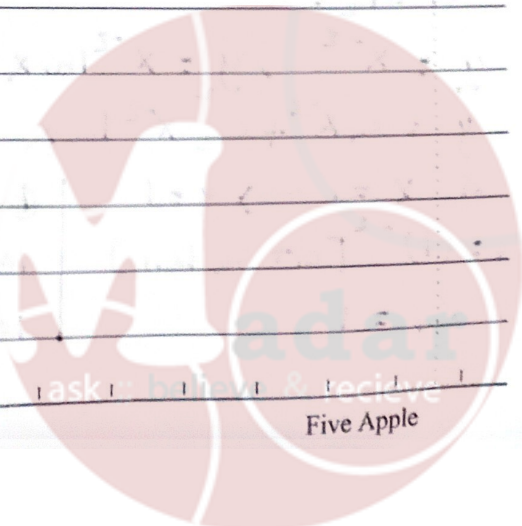
$$y_2 = x^r \ln x$$

* complex roots, $r_1 = \alpha + \beta i$

$$r_2 = \alpha - \beta i$$

$$y_1 = x^\alpha \cos(\beta \ln x)$$

$$y_2 = x^\alpha \sin(\beta \ln x)$$



Example:

$$x^2 \ddot{y} + x \dot{y} - y = 0, \quad y(10) = 3, \quad y(0) = 1 \rightarrow \text{جوابی$$

$a=1, b=1, c=-1$ C_1/C_2

$$r = \frac{-(1-1) \pm \sqrt{0^2 - 4(1)(-1)}}{2}$$

$$= \frac{0 \pm \sqrt{4}}{2} = \pm \frac{2}{2} = \pm 1$$

$$r_1 = 1, r_2 = -1$$

$$y_1 = x, y_2 = x^{-1}$$

$$y = C_1 x + C_2 x^{-1}$$

① $x=0 \rightarrow y=1$

$$1 = C_1(0) + C_2 \frac{1}{0} \rightarrow C_2 = 0$$

② $x=10, y=3$

$$C_1 = \frac{3}{10}$$

Ex: $x^2 \ddot{y} + 5x \dot{y} + 4y = 0, y(1)=1, y(2)=10$

$$r = \frac{-4 \pm \sqrt{4^2 - 4(1)(4)}}{2(1)}$$

$$r_1 = r_2 = -2$$

$$y_1 = x^{-2}, y_2 = x^{-2} \ln x$$

$$y = C_1 x^{-2} + C_2 x^{-2} \ln x = x^{-2} [C_1 + C_2 \ln x]$$

① $x=1 \rightarrow y=1$

$$1 = (1)^{-2} [C_1 + C_2 \ln 1]$$

$$C_1 = 1$$

② $x=2 \rightarrow y=10$

$$10 = (1)[2]^{-2} + C_2 [2]^{-2} \ln 2$$

$$10 = \frac{1}{4} + C_2 \frac{1}{4} \ln 2$$

$$C_2 = 56.3$$

$$y = x^{-2} [1 + 56.3 \ln x]$$

Ex₃:

$$x^2 \ddot{y} - x \dot{y} + 5y = 0$$

$$y(10) = 50$$

$$y(30) = 600$$

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= 1 \pm 2i$$

$$= \alpha \pm \beta i$$

$$\text{So } y_1 = x^1 \cos(2 \ln x)$$

$$y_2 = x^1 \sin(2 \ln x)$$

$$y = x^1 [C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x)]$$

$$\text{a) } x = 10 \rightarrow y = 50$$

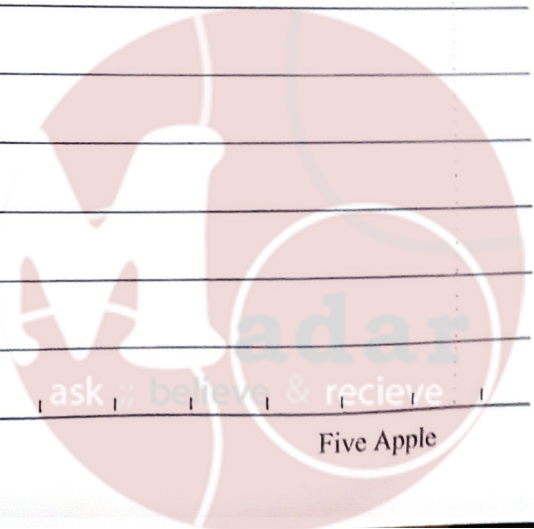
$$\text{b) } x = 30 \rightarrow y = 600$$

$$50 = 10 [C_1 \cos(2 \ln 10) + C_2 \sin(2 \ln 10)]$$

$$600 = 30 [C_1 \cos(2 \ln 30) + C_2 \sin(2 \ln 30)]$$

↓
لازم بالادسية

→ معادله
برجسته
صفه آخره



الدكتور سمح ندخل معنا ورقة
لامتحان الميد 📇



Integrating factor

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$M = \exp \int P(x) dx$$

$$y = \frac{1}{M} \int M Q(x) dx$$

Exact Solution

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

$$My = Nx \rightarrow \text{Exact}$$

- steps:
- ① Integrate M w.r.t (x)
 - ② Differentiate w.r.t (y)
 - ③ Equate w.r.t N

نكتب $N(y)$ يكون

if $My \neq Nx \rightarrow \text{Non-Exact}$

$$* M = Ce^{\int \frac{My - Nx}{N} dx}$$

(لو طرح فطير x بيحرب شرط)

$$* M = Ce^{\int \frac{My - Nx}{-M} dy}$$

(لو طرح فطير y بيحرب شرط)

* نضرب M بالمعادلة ونضرب

فرد اخرب $My = Nx$

Homogeneous Eq.

$$\frac{dy}{dx} = f(x, y) \rightarrow \frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

$$\left[v = \frac{y}{x} \right] \rightarrow y = vx$$

نقوض

$$\frac{dr}{[g(r)] - r} = \frac{dx}{x}$$

نكامل او بنزع قيمة

عشان نخلص بدلالة

Initial Value

فقط بوظنا قيمة مثل $u(0) = 2$

لنجد قيمة C في النهاية

زما بداية لكل منر باحد

الطرف السابقة

Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Integrating factor

$$\text{Ex: } y + \frac{4}{x}y = x^3 y^2$$

نكتب y بكون y^n

$$n=2, v=y^{1-n}$$

$$= y^{-1}$$

نقوض في

$$\frac{1}{1-n} \frac{dv}{dx} + P(x)v = Q(x)$$

$$-\frac{dv}{dx} + \frac{4}{x}v = x^3$$

نغير المتغير

$$\frac{dv}{dx} - \frac{4}{x}v = -x^3$$

$$\text{Integrating factor } M = \exp \int -\frac{4}{x} dx = e^{-4 \ln x}$$

$$v = \frac{1}{e^{-4 \ln x}} \int e^{-4 \ln x} (-x^3) dx$$

$$v = \frac{-1}{x^{-4}} \int x^{-4} \cdot x^3 dx$$

$$v = -x^4 (\ln x + C)$$

$$y = \frac{1}{-x^4 (\ln x + C)}$$

$$y^{-1} = -x^4 (\ln x + C)$$

$$y = \frac{1}{-x^4 (\ln x + C)}$$

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Riccati Equation

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x)$$

$$\text{Ex: } \frac{dy}{dx} = 2y^2 + y + 1$$

$$\text{Assume } y = \frac{1}{u} \frac{du}{dx}$$

$$P(x)=2, Q(x)=1, R(x)=1$$

$$u'' - P(x)u' - R(x)u = 0$$

$$\rightarrow u'' - u' - u = 0$$

$$u = e^{rx}, u' = r e^{rx}, u'' = r^2 e^{rx}$$

$$\rightarrow r^2 e^{rx} - r e^{rx} - e^{rx} = 0$$

$$= e^{rx} (r^2 - r - 1) = 0$$

$$\therefore r^2 - r - 1 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r_1 = 1.6, r_2 = -0.6$$

$$u = C_1 e^{1.6x} - C_2 e^{-0.6x}$$

$$y = \frac{1}{u} \frac{du}{dx}$$

$$\frac{du}{dx} = 1.6 C_1 e^{1.6x} - 0.6 C_2 e^{-0.6x}$$

$$y = \frac{1.6 C_1 e^{1.6x} - 0.6 C_2 e^{-0.6x}}{C_1 e^{1.6x} - C_2 e^{-0.6x}}$$

$$y = \frac{1.6 C_1 e^{1.6x} - 0.6 C_2 e^{-0.6x}}{C_1 e^{1.6x} - C_2 e^{-0.6x}}$$

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$$y = \frac{1.6 C_1 e^{1.6x} - 0.6 C_2 e^{-0.6x}}{C_1 e^{1.6x} - C_2 e^{-0.6x}}$$

$$y = \frac{1.6 C_1 e^{1.6x} - 0.6 C_2 e^{-0.6x}}{C_1 e^{1.6x} - C_2 e^{-0.6x}}$$

$$y = \frac{1.6 C_1 e^{1.6x} - 0.6 C_2 e^{-0.6x}}{C_1 e^{1.6x} - C_2 e^{-0.6x}}$$

$$y = \frac{1.6 C_1 e^{1.6x} - 0.6 C_2 e^{-0.6x}}{C_1 e^{1.6x} - C_2 e^{-0.6x}}$$

First order of 2nd degree

$$\left(\frac{dy}{dx}\right)^2 - x \frac{dy}{dx} + y = g(x)$$

$$\text{Ex: } \left(\frac{dy}{dx}\right)^2 - 2 \frac{dy}{dx} + y = x-1$$

$$\text{Assume } p = \frac{dy}{dx}$$

$$\rightarrow p^2 - 2p + y - x + 1 = 0$$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = 1 \pm \sqrt{x-y}$$

$$\text{Let } u = x-y$$

$$\frac{du}{dx} = \left(\frac{dx}{dx} - \frac{dy}{dx}\right) = 1 - p$$

$$\frac{dy}{dx} = 1 - \frac{du}{dx} \rightarrow \text{rem } \frac{dy}{dx} = 1 \pm \sqrt{x-y} = p$$

$$1 \pm \sqrt{u} = 1 - \frac{du}{dx} \rightarrow \frac{du}{dx} = \pm \sqrt{u}$$

$$\rightarrow \frac{du}{\sqrt{u}} = \pm dx = 2\sqrt{u} = \pm x + C$$

$$\rightarrow \int \frac{du}{\sqrt{u}} = \pm \int dx = 2\sqrt{u} = \pm x + C$$

$$2\sqrt{x-y} = \pm x + C$$

$$y = x - \left[\frac{(C \pm x)}{2}\right]^2$$

$$y = x - \left[\frac{(C \pm x)}{2}\right]^2$$

$$y = x - \left[\frac{(C \pm x)}{2}\right]^2$$

$$y = x - \left[\frac{(C \pm x)}{2}\right]^2$$

① $\ddot{y} = f(y, \dot{y})$

Assume $\dot{y} = p$

$\dot{y} = \frac{dp}{dy} p$

Ex: $\ddot{y} + y(y)^2 = 0$

نضرب *

$p \frac{dp}{dy} + y p^2 = 0$

p تفصل، ونقسم على p

$\frac{dp}{dy} = -y p$

$\Rightarrow \int \frac{dp}{p} = -y dy$

$= \ln p = -\frac{y^2}{2} + C$

$\Rightarrow p = \exp(-\frac{y^2}{2} + C)$

$\Rightarrow \dot{y} = \exp(-\frac{y^2}{2} + C)$

$\Rightarrow \int \frac{dy}{\exp(-\frac{y^2}{2} + C)} = \int dx$

② $\ddot{y} = f(x, \dot{y})$

Let $p = \dot{y}$

$\dot{y} = \frac{dp}{dx}$

Ex: $\ddot{y} + \dot{y} = x$

$\Rightarrow \frac{dp}{dx} + p = x$

(integrating factor)

$M = \exp \int dx = e^x$

$P = \frac{1}{e^x} \int e^x x dx$

$P = e^{-x} [e^x (x-1)] + A e^{-x}$

$\frac{dy}{dx} = (x-1) + A e^{-x}$

$dy = ((x-1) + A e^{-x}) dx$

$y = \frac{x^2}{2} - x + A e^{-x} + B$

④ $\ddot{y} = f(y, \dot{y}, x)$

$\Rightarrow \ddot{y} + a_1 \dot{y} + a_0 y = g(x)$



$y = y_c + y_p$

Let $y = e^{rx}$, $\dot{y} = r e^{rx}$, $\ddot{y} = r^2 e^{rx}$

نضرب في e^{-rx} ونقسم على e^{-rx}

① r_1 and r_2 are distinct

$y_c = A e^{r_1 x} + B e^{r_2 x}$ ③ $r = \alpha \pm \beta i$

② $r_1 = r_2$ $y_c = [(A \cos \beta x + B \sin \beta x)] e^{\alpha x}$

$y_c = (A+Bx) e^{rx}$

$\frac{y_p}{g(x)}$ y_p C/a_0 $C/a_0 [x - \frac{a_1}{a_0}]$ $C/a_0 [x^2 - \frac{2a_1}{a_0}x + \frac{2(a_1^2 - a_0)}{a_0^2}]$

لدينا $g(x) = 10x$ y_c نضرب y_c في e^{-rx}

⑤ Euler-Cauchy Eqn.

$ax^2 \ddot{y} + bx \dot{y} + cy = 0$

x مضروب بـ y و x^2 مضروب بـ $\ddot{y}$

$y = x^r$, $\dot{y} = r x^{r-1}$, $\ddot{y} = r(r-1) x^{r-2}$

نضرب في x^2 ونقسم على x^{r-2}

if $r_1 \neq r_2$

$y_1 = c_1 x^{r_1}$ $y_2 = c_2 x^{r_2}$

if $r_1 = r_2$

$y_1 = c_1 x^r$

$y_2 = c_2 x^r \ln x$

if we have complex root $\alpha \pm \beta i$

$y_1 = c_1 x^\alpha \cos(\beta \ln x)$

$y_2 = c_2 x^\alpha \sin(\beta \ln x)$

$y = y_1 + y_2$

⑥ momentum in =

$L \times W \times T \times x_2 \mid_x = 0$

② momentum out =

$L \times W \times T \times x_2 \mid_x$

③ gravitational force

$= L \times W \times p \times \Delta x \times g \times \cos \theta$

Balance

in-out + forces = 0

$L W T x_2 \mid_x - L W T x_2 \mid_x + L W p \Delta x g \cos \theta = 0$

$\lim_{\Delta x \rightarrow 0} \left(\frac{L W T x_2 \mid_x - L W T x_2 \mid_x}{\Delta x} \right) + L W p g \cos \theta = 0$

$-\frac{d(L W T x_2)}{dx} + L W p g \cos \theta = 0$

$-L W \frac{dT x_2}{dx} + L W p g \cos \theta = 0$

$\frac{dT x_2}{dx} + L W p g \cos \theta = 0$ ①

$\frac{d^2 V}{dx^2} + \frac{p g}{m} \cos \theta$ ②

conditions V_x زخم أفقي

$V_x = -\frac{p g \cos \theta}{2m} x^2 + \frac{p g \cos \theta \Delta x}{m}$

$V_x = -\frac{p g \cos \theta}{m} \frac{x^2}{2} + c_1 x + c_2$

نضرب

believe & recieve

Series Solution :

$$P_2(x) \ddot{y} + P_1(x) \dot{y} + P_0(x) y = 0$$

Assume : $y = x^c$
 $\dot{y} = c x^{c-1}$
 $\ddot{y} = c(c-1) x^{c-2}$

نقوم بتعويض هذه المعادلات

$$P_2(x) c(c-1) x^{c-2} + P_1(x) c x^{c-1} + P_0(x) x^c = 0$$

Example:

$$2x \ddot{y} + \dot{y} + y = 0 \quad / \text{let } y = x^c, \dot{y} = c x^{c-1}, \ddot{y} = c(c-1) x^{c-2}$$

$$\rightarrow 2x c(c-1) x^{c-2} + c x^{c-1} + x^c = 0$$

$$= 2c(c-1) x^{c-1} + c x^{c-1} + x^c = 0 \quad (* \text{ جمعنا الأسس للحد الأول})$$

$$= [2c(c-1) + c] x^{c-1} + 1 x^c = 0 \quad (\text{عامل مشترك}) \leftarrow$$

$$f(c) x^r + g(c) x^{r+p} = 0 \quad (g(c) = 1) *$$

$$\left. \begin{array}{l} r = c-1 \\ r+p = c \end{array} \right\} \rightarrow p = c - r \rightarrow p = c - (c-1) \rightarrow \boxed{p=1}$$

I have to assume $f(c) = 0$ قيمة p دائماً، الفرق بين قوة x بعد أخذ العامل المشترك، بالتالي عنّا x^{c-1} و x^c
 $c - (c-1) = 1$

$$A_{k+1} = (-1)^{k+1} A_0 \frac{g(c_1) g(c_1+p) g(c_1+2p) \dots}{f(c_1+p) f(c_1+2p) f(c_1+3p) \dots}$$

Case 1: roots are distant and don't differ by integer.

$$y = A y_1 + B y_2 \quad (\text{fractions})$$

$$y_1 = A_0 x^{c_1} + a_1 x^{c_1+p} + a_2 x^{c_1+2p} + \dots$$

القوة لـ x

$$y_2 = b_0 x^{c_2} + b_1 x^{c_2+p} + b_2 x^{c_2+2p} + \dots$$

Example:

$$2x \ddot{y} + \dot{y} + y = 0$$

$$\text{Assume } y = x^c, \dot{y} = c x^{c-1}, \ddot{y} = c(c-1) x^{c-2}$$

$$2x c(c-1) x^{c-2} + c x^{c-1} + x^c = 0$$

$$2c(c-1) x^{c-1} + c x^{c-1} + x^c = 0$$

$$x^{c-1} [2c(c-1) + c] + x^c = 0$$

$P(c)$

$Q(c)$

$$P=1$$

زى
ما طرقتها
بالحل الأول

$$f(c) = 0$$

$$2c^2 - 2c + c = 0$$

$$c(2c - 2 + 1) = 0 \rightarrow c = 0, \frac{1}{2}$$

$$A_{k+1} = (-1)^{k+1} A_0 \frac{g(c_1) g(c_1+p) g(c_1+2p) \dots g(c_1+kp)}{f(c_1+p) f(c_1+2p) f(c_1+3p) \dots f(c_1+(k+1)p)}$$

$$A_{k+1} = (-1)^{k+1} A_0 \frac{1}{f(c_1+p) f(c_1+2p) f(c_1+3p) \dots f(c_1+(k+1)p)}$$

① $c = 0$

$$k=0 \rightarrow A_1 = (-1)^1 A_0 \frac{1}{f(c_1+1)} = -A_0 \frac{1}{1} \rightarrow A_1 = -A_0$$

بل اول وحدة $c=0$ لو كانت
منوخذ اول وده كذا.

$$k=1 \rightarrow A_2 = (-1)^2 A_0 \frac{1}{f(c_1+p) f(c_1+2p)} \rightarrow A_2 = \frac{1}{6} A_0$$

$$k=2 \rightarrow A_3 = (-1)^3 A_0 \frac{1}{f(c_1+p) f(c_1+2p) f(c_1+3p)} \rightarrow A_3 = -\frac{1}{90} A_0$$

$$y_1 = A_0 x^{c_1} + A_1 x^{c_1+p} + A_2 x^{c_1+2p} + \dots$$

$$= A_0 x^0 + A_1 x^1 + A_2 x^2 + \dots$$

ask | believe | recieve

Five Apple

$$\textcircled{1} c = \frac{1}{2}$$

$$K=0 \rightarrow b_1 = (-1)^1 b_0 \frac{1}{\Gamma(c_2+1)} = -1 b_0 \times \frac{1}{3} \rightarrow b_1 = -\frac{1}{3} b_0$$

$$K=1 \rightarrow b_2 = (-1)^2 b_0 \times \frac{1}{\Gamma(c_2+1) \Gamma(c_2+2)} = b_0 \times \frac{1}{3 \times 10} \rightarrow b_2 = \frac{1}{30} b_0$$

$$K=2 \rightarrow b_3 = (-1)^3 b_0 \times \frac{1}{\Gamma(c_2+1) \Gamma(c_2+2) \Gamma(c_2+3)} = -1 b_0 \times \frac{1}{3 \times 10 \times 21} = -b_0 \times \frac{1}{630} \rightarrow b_3 = -\frac{1}{630} b_0$$

$$\rightarrow y_2 = b_0 x^{\frac{1}{2}} + \frac{-1}{3} b_0 x^{1.5} + \frac{1}{30} b_0 x^{2.5} - \frac{1}{630} b_0 x^{3.5}$$

$$\rightarrow y_1 = A_0 x^0 + -A_0 x^1 + \frac{1}{6} A_0 x^2 - \frac{1}{90} A_0 x^3$$

$$y = A y_1 + B y_2$$

$$y = A \left(A_0 x^0 + -A_0 x^1 + \frac{1}{6} A_0 x^2 - \frac{1}{90} A_0 x^3 \right) + B \left(b_0 x^{\frac{1}{2}} - \frac{1}{3} b_0 x^{1.5} + \frac{1}{30} b_0 x^{2.5} - \frac{1}{630} b_0 x^{3.5} \right)$$



Taylor Series:

$$f(x) = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3$$

Ex: $f(x) = 3 - e^{-x}$ at $a = 0 \rightarrow$ (هو محدد بالسؤال)

$$f'(x) = e^{-x}$$

$$f''(x) = -e^{-x}$$

$$f'''(x) = e^{-x}$$

$$f(x) \Big|_{a=0} = 2 + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 = y$$

$$= 2 + x - \frac{1}{2} x^2 + \frac{1}{6} x^3 = y$$

$$P_2(x) \ddot{y} + P_1(x) \dot{y} + P_0(x) y = 0$$

Example: $2x\ddot{y} + \dot{y} + y = 0 \rightarrow$

نفس المثل
السابقة

$$y = x^c / \dot{y} = c x^{c-1} / \ddot{y} = c(c-1) x^{c-2}$$

$$2x [c(c-1)] x^{c-2} + c x^{c-1} + x^c = 0$$

$$2c(c-1) x^{c-1} + c x^{c-1} + x^c = 0$$

(Case 1)

بعد ذلك بتقدير

$$y_1 = A_0 [1 - x + \frac{1}{6} x^2 + \dots]$$

* to prove: $2x\ddot{y} + \dot{y} + y = 0$

$$\dot{y}_1 = -A_0 (1 + \frac{1}{3} x)$$

$$\ddot{y}_1 = A_0 \frac{1}{3}$$

$$2x (A_0 \frac{1}{3}) + -A_0 + \frac{1}{3} x + A_0 - A_0 x + \frac{A_0}{6} x^2$$

$$\frac{2}{3} A_0 x - A_0 + \frac{1}{3} A_0 x + A_0 - A_0 x + \frac{A_0}{6} x^2 = 0$$

من هنا يتبين ان المعادلة

Laplace transformation:

A method that converts ODE into a simple algebraic Equation and PDE to ODE.

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

Ex₁: $f(t) = 1$

$$L[1] = \int_0^{\infty} e^{-st} (1) dt = \left[-\frac{1}{s} e^{-st} \right]_0^{\infty} = -\frac{1}{s} \left[\cancel{e^{-\infty}} - \cancel{e^0} \right] = \left[\frac{1}{s} \right]$$

Ex₂: $f(t) = K$

$$L(K) = \int_0^{\infty} e^{-st} K dt = K \cdot \int_0^{\infty} e^{-st} dt = \left[\frac{K}{s} \right]$$

Ex₃: $f(t) = e^{\alpha t}$

$$L(e^{\alpha t}) = \int_0^{\infty} e^{-st} e^{\alpha t} dt = \int_0^{\infty} e^{(\alpha-s)t} dt = \left[\frac{1}{\alpha-s} e^{(\alpha-s)t} \right]_0^{\infty} = \left[\frac{+1}{s-\alpha} \right]$$



Laplace

$$L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

Ex 4:

$$f(t) = kt$$

$$L[kt] = \int_0^{\infty} kt e^{-st} dt = k \int_0^{\infty} t e^{-st} dt \text{ (by parts)} = \frac{k}{s^2}$$

Ex 5:

$$f(t) = \sin(\omega t)$$

note: $i = \sqrt{-1}$ (constant)

$$\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

$$L[\sin \omega t] = \int_0^{\infty} e^{-st} \left[\frac{e^{i\omega t} - e^{-i\omega t}}{2i} \right] dt = \frac{1}{2i} \int_0^{\infty} e^{-st} [e^{i\omega t} - e^{-i\omega t}] dt$$

$$= \frac{1}{2i} \left[\int_0^{\infty} e^{-st} e^{i\omega t} dt - \int_0^{\infty} e^{-st} e^{-i\omega t} dt \right]$$

$$= \frac{1}{2i} \left[\int_0^{\infty} e^{(i\omega - s)t} dt - \int_0^{\infty} e^{-(i\omega + s)t} dt \right]$$

$$= \frac{1}{2i} \left\{ \left[\frac{1}{i\omega - s} e^{(i\omega - s)t} \right]_0^{\infty} - \left[\frac{1}{-(i\omega + s)} e^{-(i\omega + s)t} \right]_0^{\infty} \right\}$$

$$= \frac{1}{2i} \left\{ \left[\frac{1}{i\omega - s} e^{-(s - i\omega)t} \right]_0^{\infty} - \left[\frac{1}{-(i\omega + s)} e^{-(i\omega + s)t} \right]_0^{\infty} \right\}$$

$$= \frac{1}{2i} \left[\frac{-1}{i\omega - s} - \frac{1}{i\omega + s} \right] = \frac{1}{2i} \left[\frac{-(i\omega + s) - (i\omega - s)}{(i\omega - s)(i\omega + s)} \right]$$

$$* i = \sqrt{-1} \rightarrow i^2 = -1$$

$$L[\sin \omega t] = \frac{1}{2i} \left[\frac{-2i\omega}{-(\omega^2 + s^2)} \right] = \frac{\omega}{\omega^2 + s^2}$$

Five Apple

ask, believe & receive

Ex 6:

$$f(t) = t e^{-\alpha t}$$

$$L[f(t)] = \int_0^{\infty} f(t) e^{-st} dt = \int_0^{\infty} t e^{-\alpha t} e^{-st} dt = \int_0^{\infty} t e^{-(s+\alpha)t} dt \quad (\text{by parts})$$

$$u = t$$
$$du = dt$$

$$dv = e^{-(s+\alpha)t}$$

$$v = \frac{-1}{(s+\alpha)} e^{-(s+\alpha)t}$$

$$= \left[\frac{-t}{s+\alpha} e^{-(s+\alpha)t} \right]_0^{\infty} - \int_0^{\infty} \frac{-1}{(s+\alpha)} e^{-(s+\alpha)t} dt$$

$$\rightarrow = \frac{1}{(s+\alpha)^2}$$

Ex 7

General rule:

$$L[t^n e^{-\alpha t}] = \frac{n!}{(s+\alpha)^{n+1}}$$



$$F(t) \rightarrow F(s)$$

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

Example:

$$\frac{dx}{dt} ?$$

$$\int_0^{\infty} \frac{dx}{dt} e^{-st} dt$$

$$u = e^{-st} \quad dv = \frac{dx}{dt} dt$$

$$du = -s e^{-st} \quad v = x$$

$$= x e^{-st} \Big|_0^{\infty} - \int_0^{\infty} x(-s) e^{-st} dt$$

$$= -x(t=0) + s \int_0^{\infty} x e^{-st} dt$$

$$\int_0^{\infty} \frac{dx}{dt} e^{-st} dt = s X(s) - x(t=0)$$

Example:

$$\frac{d^2 x}{dt^2} ?$$

$$\rightarrow \int_0^{\infty} \frac{d^2 x}{dt^2} e^{-st} dt$$

$$u = e^{-st} \quad dv = \frac{d^2 x}{dt^2} dt$$

$$du = -s e^{-st}$$

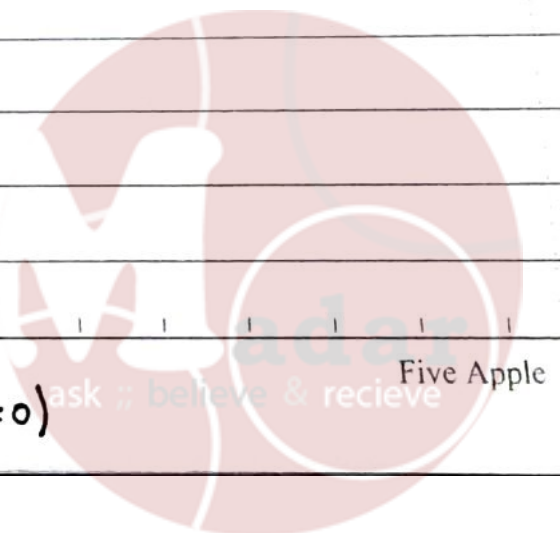
$$\frac{dx}{dt}$$

$$= e^{-st} \frac{dx}{dt} \Big|_0^{\infty} + \int_0^{\infty} -s e^{-st} \frac{dx}{dt} dt$$

رجوع إلى أول مثال
بـ $-s$

$$= -\dot{x}(t=0) + s [s X(s) - x(t=0)]$$

$$\int_0^{\infty} \frac{d^2 x}{dt^2} e^{-st} dt = s^2 X(s) - s x(t=0) - \dot{x}(t=0)$$



Example:

$$\frac{d^2 y}{dt^2} + 4y = 3, \quad y(0) = \dot{y}(0) = 1$$

$$\rightarrow [s^2 X(s) - s y(0) - \dot{y}(0)] + 4 y(s) = \frac{3}{s}$$

$$(s^2 + 4) y(s) = \frac{3}{s} + s + 1$$

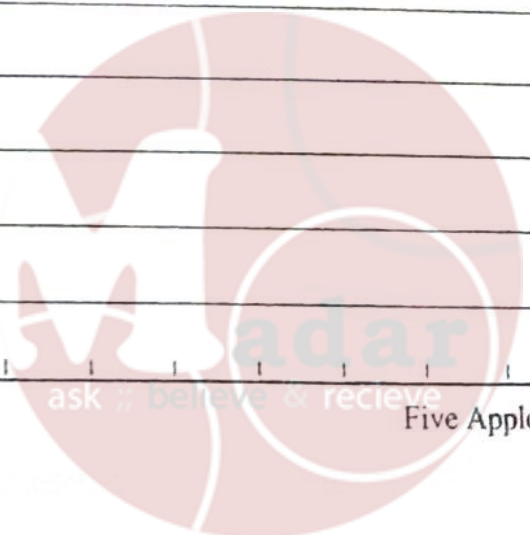
$$y(s) = \frac{3}{s} + s + 1$$

$$\frac{3}{s^2 + 4} = \frac{3}{s(s^2 + 4)} + \frac{s}{s^2 + 4} + \frac{1}{s^2 + 4}$$

From tables:

LaPlace

بجول ل



Example :

$$\ddot{y} + \dot{y} + y = 1, \quad y(0) = \dot{y}(0) = 0$$

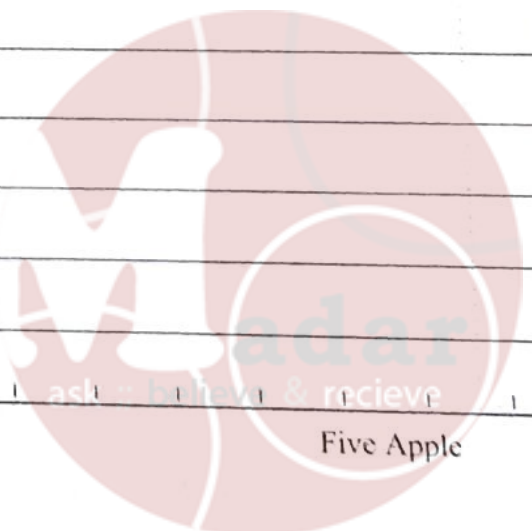
$$[s^2 y(s) - s y(0) - \dot{y}(0)] + [s y(s) - y(0)] + y(s) = \frac{1}{s}$$

$$s^2 y(s) + s y(s) + y(s) = \frac{1}{s}$$

$$y(s) [s^2 + s + 1] = \frac{1}{s}$$

$$y(s) = \frac{1}{s(s^2 + s + 1)} \rightarrow \text{Partial fractions}$$

table 1, no $\dot{y}$ + 1 ~~clerk~~



Laplace transformation Integration

$$\mathcal{L} \left[\int x(t) dt \right] = \int_0^{\infty} \left(\int x(t) dt \right) e^{-st} dt$$

$$u = \int x dt \quad dv = e^{-st} dt$$

$$du = x dt \quad v = \frac{1}{s} e^{-st}$$

$$\begin{aligned} \int_0^{\infty} \left(\int x(t) dt \right) e^{-st} dt &= \left[\frac{-e^{-st}}{s} \int x dt \right]_0^{\infty} - \int_0^{\infty} \frac{-1}{s} e^{-st} x dt \\ &= \frac{1}{s} \int x dt \Big|_{t=0} + \frac{1}{s} x(s) \end{aligned}$$

Partial fraction

$$F(s) = \frac{g(s)}{(s-a)^{m_1} (s-b)^{m_2} \dots}$$

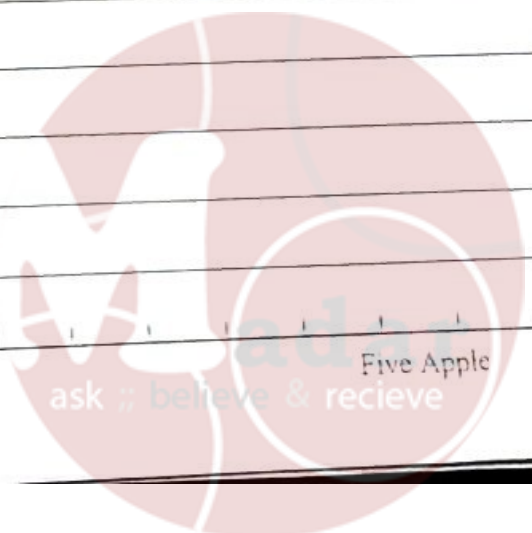
Ex₁:

$$f(s) = \frac{1}{(s-1)^2 (s-2)}$$

$$\text{Ex}_2: f(s) = \frac{3}{(s-1)(s-2)}$$

$$\text{Ex}_3: f(s) = \frac{s^2+3}{(s+9)^2}$$

$$\text{Ex: } \frac{1}{(s-1)^3} = \frac{A_1}{(s-1)^3} + \frac{A_2}{(s-1)^2} + \frac{A_3}{(s-1)}$$



$$F(s) = \frac{Z(s)}{(s-p_1)^n (s-p_2)(s-p_3)}$$

$$= \frac{A_1}{(s-p_1)^n} + \frac{A_2}{(s-p_1)^{n-1}} + \frac{A_3}{(s-p_1)^{n-2}} + \frac{B_1}{(s-p_2)} + \frac{C_1}{(s-p_3)}$$

المشتق m مرة
 $A_m = \lim_{s \rightarrow p_1} \left[\frac{d^{m-1}}{ds^{m-1}} \left[(s-p_1)^n F(s) \right] \right] \frac{1}{(m-1)!}$
 $m = 1, 2, 3, \dots, n$

$$A_1 = \lim_{s \rightarrow p_1} \left[\frac{d^{1-1}}{ds^{1-1}} \left[(s-p_1)^n F(s) \right] \right] \frac{1}{1-1!}$$

$$= \lim_{s \rightarrow p_1} \left[(s-p_1)^n F(s) \right]$$

المشتق 2 مرة
 $A_2 = \lim_{s \rightarrow p_1} \left[\frac{d^{2-1}}{ds^{2-1}} \left[(s-p_1)^n F(s) \right] \right] \frac{1}{2-1!}$

$$A_3 = \lim_{s \rightarrow p_1} \left[\frac{d^2}{ds^2} \left[(s-p_1)^n F(s) \right] \right] \frac{1}{(3-1)!}$$

المشتق 0 مرة
 $B_1 = \lim_{s \rightarrow p_2} \left[(s-p_2) F(s) \right]$

$$C_1 = \lim_{s \rightarrow p_3} \left[(s-p_3) F(s) \right]$$



Ex:

$$F(s) = \frac{1}{(s-a)(s-b)} = \frac{A_1}{s-a} + \frac{B_1}{s-b}$$

$$A_1 = \lim_{s \rightarrow a} [(s-a) F(s)] = \lim_{s \rightarrow a} \left[\cancel{s-a} \frac{1}{(\cancel{s-a})(s-b)} \right]$$
$$= \lim_{s \rightarrow a} \left(\frac{1}{s-b} \right)$$

$$A_1 = \left[\frac{1}{a-b} \right]$$

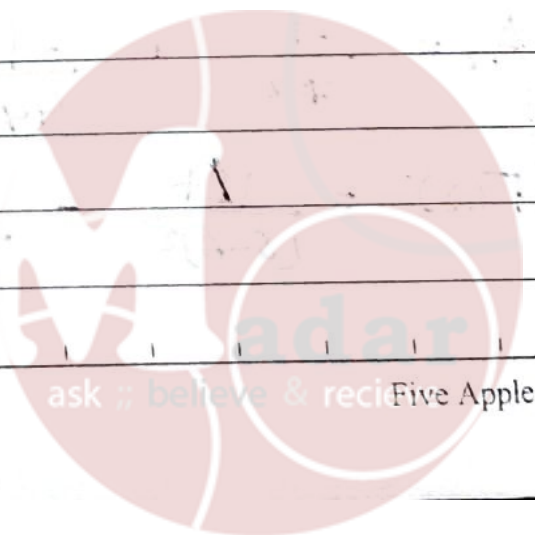
$$B_1 = \lim_{s \rightarrow b} \left[\cancel{(s-b)} \frac{1}{(s-\cancel{b})(s-\cancel{b})} \right] = \lim_{s \rightarrow b} \frac{1}{s-a} = \left[\frac{1}{b-a} \right]$$

$$\text{Ex: } F(s) = \frac{s^2+6}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$A = \lim_{s \rightarrow 1} \left[\cancel{(s-1)} \frac{s^2+6}{(\cancel{s-1})(s-2)} \right] = \lim_{s \rightarrow 1} \frac{s^2+6}{s-2} = -7$$

$$A_2 = \lim_{s \rightarrow 2} \left[\cancel{(s-2)} \frac{s^2+6}{(s-1)(\cancel{s-2})} \right] = 10$$

$$F(s) = \frac{-7}{s-1} + \frac{10}{s-2}$$



Example:

$$F(s) = \frac{1}{(s-a)^2(s-b)} = \frac{A_1}{(s-a)^2} + \frac{A_2}{(s-a)} + \frac{B_1}{(s-b)}$$

$$A_1 = \lim_{s \rightarrow a} \left[\frac{d^{+1}}{ds^{1-1}} (s-a)^2 F(s) \right] \frac{1}{1!}$$

$$A_1 = \lim_{s \rightarrow a} \left[(s-a)^2 \frac{1}{(s-a)^2(s-b)} \right] = \lim_{s \rightarrow a} \frac{1}{s-b} = \frac{1}{a-b}$$

$$A_2 = \lim_{s \rightarrow a} \left[\frac{d^{2-1}}{ds^{2-1}} (s-a)^2 \frac{1}{(s-a)^2(s-b)} \right]$$

$$\lim_{s \rightarrow a} \left[\frac{d}{ds} \left(\frac{1}{s-b} \right) \right] = \lim_{s \rightarrow a} \left[\frac{(-1)(1)}{(s-b)^2} \right] = \frac{-1}{(a-b)^2}$$

$$B_1 = \lim_{s \rightarrow b} \left[(s-b) \frac{1}{(s-b)^2(s-b)} \right] = \frac{1}{(b-a)^2}$$

Ex: $F(s) = \frac{1}{(s-6)^2(s+1)}$

$$F(s) = \frac{A_1}{(s-6)^2} + \frac{A_2}{(s-6)} + \frac{B_1}{s-(-1)}$$

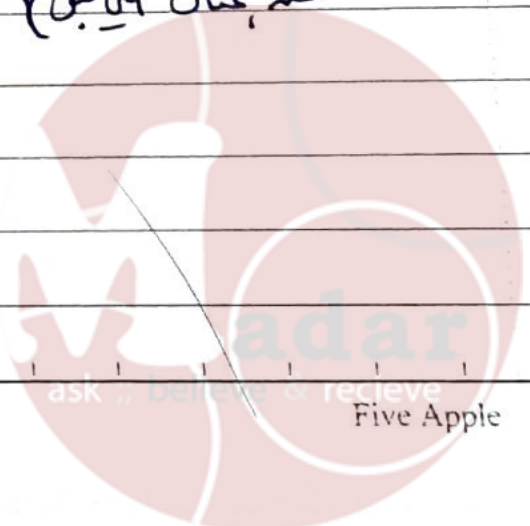
$$A_1 = \frac{1}{6-(-1)} = \frac{1}{7}$$

$$B_1 = \frac{1}{(-1-6)^2} = \frac{1}{49}$$

$$A_2 = \frac{-1}{7^2} = \frac{-1}{49}$$

$$F(s) = \frac{1/7}{(s-6)^2} - \frac{1/49}{s-6} + \frac{1/49}{s+1}$$

طبیقتاً مباشره
منه لیکال له قیمن



واجب شرحه الدكتور في المحاضرة
وطلب نكّله 📇





$$*1, F(CA_0 - CA_1) - KVCA_1 = V \frac{dCA_1}{dt} \rightarrow V \frac{dCA_1}{dt} = \frac{FCA_0}{V} - \left[\frac{F+KV}{V} \right] CA_1$$

$$*2, F(CA_1 - CA_2) - KVCA_2 = V \frac{dCA_2}{dt}$$

$$*3, F(CA_2 - CA_3) - KVCA_3 = V \frac{dCA_3}{dt}$$

→ (for V)

$$*1, \frac{dCA_1}{dt} = D_1 CA_0 - [R_1] CA_1$$

$$*2, \frac{dCA_2}{dt} = D_1 CA_1 - [R_1] CA_2$$

$$*3, \frac{dCA_3}{dt} = D_1 CA_2 - [R_1] CA_3$$

time ↗

$$\rightarrow CA_1(0) = CA_2(0) = CA_3(0) = 0$$

* Laplace transformation

$$① \cancel{SCA_1(s)} - \cancel{CA_1(0)} = \frac{D_1 CA_0}{s} - R_1 CA_1(s)$$

$$② SC A_2(s) - \cancel{CA_2(0)} = D_1 CA_1(s) - R_1 CA_2(s)$$

$$③ SC A_3(s) - \cancel{CA_3(0)} = D_1 CA_2(s) - R_1 CA_3(s)$$

$$SC A_1(s) = \frac{D_1 CA_0}{s} - R_1 CA_1(s) \Rightarrow (s + R_1) CA_1(s) = \frac{D_1 CA_0}{s}$$

$$SC A_2(s) = D_1 CA_1(s) - R_1 CA_2(s) \Rightarrow (s + R_1) CA_2(s) = D_1 CA_1(s)$$

$$SC A_3(s) = D_1 CA_2(s) - R_1 CA_3(s) \Rightarrow (s + R_1) CA_3(s) = D_1 CA_2(s)$$

→



$$CA_1(s) = \frac{D_1 CA_0}{(s)(s+R_1)}$$

$$CA_1(s) = \frac{A}{s-0} + \frac{B}{s+R_1}$$

$$A = \lim_{s \rightarrow 0} \frac{s D_1 CA_0}{(s)(s+R_1)} = \frac{D_1 CA_0}{R_1} = K_1$$

$$B = \lim_{s \rightarrow -R_1} (s+R_1) \frac{D_1 CA_0}{(s)(s+R_1)} = -\frac{D_1 CA_0}{R_1} = -K_1$$

$$CA_1(s) = \frac{K_1}{s} - \frac{K_1}{s+R_1}$$

$$\rightarrow CA_1(t) = K_1 - K_1 e^{-R_1 t}$$

$$CA_1(t) = K_1 [1 - e^{-R_1 t}]$$

$$CA_2(s) = \frac{D_1}{(s+R_1)} CA_1(s) = \frac{D_1}{s+R_1} \frac{D_1 CA_0}{s} = \frac{D_1^2 CA_0}{(s+R_1)s}$$

$$= \frac{D_1^2 CA_0}{(s+R_1)^2 s} = \frac{A_1}{(s+R_1)^2} + \frac{A_2}{(s+R_1)} + \frac{B_1}{s}$$

$$A_1 = \lim_{s \rightarrow -R_1} [(s+R_1)^2 \frac{D_1^2 CA_0}{(s+R_1)^2 s}] = \frac{D_1^2 CA_0}{-R_1} = -K_2$$

$$A_2 = \lim_{s \rightarrow -R_1} \frac{d}{ds} [(s+R_1)^2 \frac{D_1^2 CA_0}{(s+R_1)^2 s}]$$

$$= \lim_{s \rightarrow -R_1} d \left(\frac{D_1^2 CA_0}{s} \right)$$

$$= -\frac{D_1^2 CA_0}{s^2} \bigg|_{-R_1} \rightarrow -\frac{D_1^2 CA_0}{R_1^2} = -K_3$$



$$B_1 = \lim_{s \rightarrow 0} \frac{s P_1^2 C A_0}{(s+R_1)^2 s} = \frac{D_1^2 C A_0}{R_1^2} = K_3$$

$$C A_2(s) = \frac{-K_2}{(s+R_1)^2} + \frac{K_1}{s+R_1} + \frac{K_3}{s}$$

$$C A_2(t) = -K_2 [t e^{-R_1 t}] + -K_2 [e^{-R_1 t}] + K_3$$

$$C A_3(s) = \frac{D_1 C A_2(s)}{(s+R_1)}$$

$$\text{note: } C A_2(s) = \frac{D_1^2 C A_0}{s(s+R_1)^2}$$

$$\rightarrow C A_3 = \frac{D_1^3 C A_0}{s(s+R_1)^2} = \frac{A_1}{(s+R_1)^3} + \frac{A_2}{(s+R_1)^2} + \frac{A_3}{s+R_1} + \frac{B_1}{s}$$

$$A_1 = \lim_{s \rightarrow -R_1} \left((s+R_1)^3 \frac{D_1^3 C A_0}{s(s+R_1)^2} \right) = \frac{-D_1^3 C A_0}{R_1} = -K_4$$

$$\begin{aligned} A_2 &= \lim_{s \rightarrow -R_1} \frac{d}{ds} \left((s+R_1)^3 \frac{D_1^3 C A_0}{s(s+R_1)^2} \right) = \lim_{s \rightarrow -R_1} \frac{d}{ds} \left(\frac{D_1^3 C A_0}{s} \right) \\ &= -\frac{D_1^3 C A_0}{s^2} \Big|_{-R_1} = -\frac{D_1^3 C A_0}{R_1^2} \\ &= -K_5 \end{aligned}$$

→



$$A_3 = \lim_{s \rightarrow -R} \frac{d^2}{ds^2} \left(\cancel{(s+R_1)^3} \frac{D_1^3 c A_0}{s(s+R_1)^3} \right) =$$

$$\lim_{s \rightarrow -R} \frac{d^2}{ds^2} \left(\frac{D_1^3 c A_0}{s} \right) = -\frac{D_1^3 c A_0}{R^3} = -K_6$$

$$B_1 = \lim_{s \rightarrow 0} \left(\cancel{s} \frac{D_1^3 c A_0}{s(s+R_1)^3} \right) = \frac{D_1^3 c A_0}{R_1^3} = K_6$$

$$c A_3(s) = \frac{-K_4}{(s+R_1)^3} - \frac{K_5}{(s+R_1)^2} - \frac{K_6}{s+R_1} + \frac{K_6}{s}$$

$$c A_3(t) = -K_4 [t^2 e^{-R_1 t}] - K_5 [t e^{-R_1 t}] - K_6 [e^{-R_1 t}] + K_6$$



الدكتور سمح ندخل ورقة للفاينل 📇

*الدقتر ناقص تشابتر المصفوفات



Series Solution:

$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0$
 $Ex: 2x^2y'' + y' + y = 0$ / let $y = x^c$
 $2xc(c-1)x^{c-2} + c x^{c-1} + x^c = 0$ $y = c(x)$
 $2c(c-1)x^c + c x^c + x^c = 0$ $y = c(c-1)x^{c-2}$
 $[2c(c-1) + c]x^c + x^c = 0$
 $f(c)x^c + g(c)x^c = 0$
 $f(c) = 2c(c-1) + c = 2c^2 - c$
 $g(c) = 1$
 $A_{k+1} = (-1)^{k+1} A_0 \frac{g(c_1) g(c_1+p) g(c_1+2p)}{f(c_1+p) f(c_1+2p) f(c_1+3p)}$

Laplace integration:

$L\{f(x+t)dt\} = \int_0^\infty f(x+t)dt \cdot e^{-st}$
 $u = x+t \quad dv = e^{-st}$
 $du = dt \quad v = \frac{1}{-s} e^{-st}$
 $= \left[\frac{1}{-s} f(x+t) e^{-st} \right]_0^\infty - \int_0^\infty \frac{1}{-s} e^{-st} f'(x+t) dt$
 $= \frac{1}{s} f(x) + \frac{1}{s} L\{f'(x)\}$

Partial Fractions:

$F(s) = \frac{Z(s)}{(s-p_1)^2(s-p_2)} = \frac{A_1}{(s-p_1)^2} + \frac{A_2}{s-p_1} + \frac{B_1}{s-p_2}$
 $Ex: y' + y = 0$
 $y = P, \quad y' = \frac{dP}{dy} P$
 $P \frac{dP}{dy} + P^2 = 0$
 $\frac{dP}{dy} + P = 0$
 $\int \frac{dP}{P} = - \int \frac{dy}{y}$
 $\ln P = - \ln y + C$
 $P = \frac{1}{y} e^C$
 $P = \frac{1}{y}$

$P = y, \quad y' = \frac{dy}{dx}$
 $Ex: y' + y = x$
 $\frac{dy}{dx} + y = x$
 $\int \frac{dy}{y} + \int \frac{1}{y} dy = \int \frac{x}{y} dx$
 $\ln y + y = \frac{1}{2} x^2 + C$
 $y = \frac{1}{2} x^2 + C$

$A_{k+1} = (-1)^{k+1} A_0 \frac{g(c_1) g(c_1+p) g(c_1+2p)}{f(c_1+p) f(c_1+2p) f(c_1+3p)}$

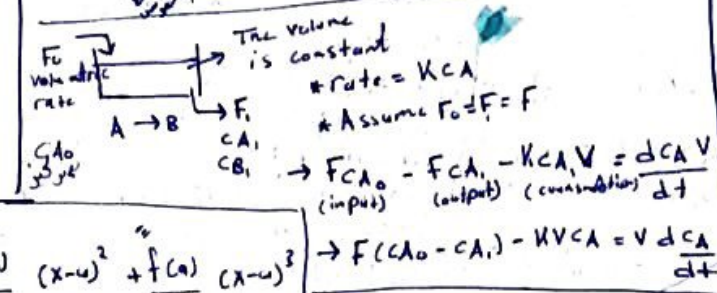
Case 1

$f(w) = 0 \rightarrow c_1 \neq c_2$
 $y_1 = A_0 x^{c_1} + A_1 x^{c_1+p} + A_2 x^{c_1+2p}$
 $y_2 = B_0 x^{c_2} + B_1 x^{c_2+p} + B_2 x^{c_2+2p}$
 $y = A y_1 + B y_2$

$A_m = \lim_{s \rightarrow p} \left[\frac{d^{m-1}}{ds^{m-1}} ((s-p)^m F(s)) \right] \frac{1}{(m-1)!}$
 A is constant

Euler-Cauchy Eq.

$ax^2y'' + bxy' + cy = 0$
 $y = x^r, \quad y' = r x^{r-1}, \quad y'' = r(r-1)x^{r-2}$
 $ax^2(r(r-1)x^{r-2}) + b x(r x^{r-1}) + c x^r = 0$
 $a r(r-1) + b r + c = 0$



if $n \neq r_2$
 $y_1 = c_1 x^{r_1}, \quad y_2 = c_2 x^{r_2}$
 if $r_1 = r_2$
 $y_1 = c_1 x^{r_1}$
 $y_2 = c_2 x^{r_1} \ln x$
 $y_1 = c_1 x^{\alpha} \cos(\beta \ln x)$
 $y_2 = c_2 x^{\alpha} \sin(\beta \ln x)$
 $y = y_1 + y_2$

Taylor Series

$f(x) = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots$

Laplace transformation

$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s)$
 $f(t) = \sin(wt) \quad * \sin(wt) = \frac{e^{iwt} - e^{-iwt}}{2i}$
 $L\{\sin(wt)\} = \int_0^\infty e^{-st} \frac{e^{iwt} - e^{-iwt}}{2i} dt$
 $= \frac{1}{2i} \left[\int_0^\infty e^{-(s-iw)t} dt - \int_0^\infty e^{-(s+iw)t} dt \right]$
 $= \frac{1}{2i} \left[\frac{1}{-(s-iw)} - \frac{1}{-(s+iw)} \right]$
 $= \frac{1}{2i} \left[\frac{1}{-s+iw} - \frac{1}{-s-iw} \right] = \frac{1}{2i} \left[\frac{-2iw}{(s-iw)(s+iw)} \right] = \frac{w}{s^2 + w^2}$

① $F(sA_0 - CA_1) - KVC A_1 = V \frac{dCA_1}{dt}$
 ② $F(sCA_1 - CA_2) - KVC A_2 = V \frac{dCA_2}{dt}$
 ③ $F(sCA_2 - CA_3) - KVC A_3 = V \frac{dCA_3}{dt}$
 ④ $\frac{dCA_1}{dt} = D_1 CA_0 - [R_1] CA_1$
 ⑤ $\frac{dCA_2}{dt} = D_2 CA_1 - [R_2] CA_2$
 ⑥ $\frac{dCA_3}{dt} = D_3 CA_2 - [R_3] CA_3$

Integration by Parts

$u \frac{dv}{dx} \rightarrow u v - \int v \frac{du}{dx}$
 $x \ln x \rightarrow \frac{dx}{dt}$

$Ex: y'' + y' + y = 1, \quad y(0) = y'(0) = 0$
 $\rightarrow [s^2 y(s) - s y(0) - y'(0)] + [s y(s) - y(0)] + y(s) = \frac{1}{s}$
 $s^2 y(s) + s y(s) + y(s) = \frac{1}{s}$
 $y(s) (s^2 + s + 1) = \frac{1}{s}$
 $y(s) = \frac{1}{s(s^2 + s + 1)} \rightarrow \text{partial fraction}$

Laplace

① $sCA_1(s) - CA_1(0) = \frac{D_1 CA_0}{s} - R_1 CA_1(s)$
 ② $sCA_2(s) - CA_2(0) = \frac{D_2 CA_1}{s} - R_2 CA_2(s)$
 ③ $sCA_3(s) - CA_3(0) = \frac{D_3 CA_2}{s} - R_3 CA_3(s)$
 ④ $(s+R_1) CA_1(s) = \frac{D_1 CA_0}{s}$
 ⑤ $(s+R_2) CA_2(s) = \frac{D_2 CA_1}{s}$
 ⑥ $(s+R_3) CA_3(s) = \frac{D_3 CA_2}{s}$
 $CA_1(s) = \frac{D_1 CA_0}{s} \cdot \frac{1}{s+R_1} = \frac{A}{s} + \frac{B}{s+R_1}$
 ⑦ $B = A$

matrix
المصفوفة
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1) Integrating Factor:
 $\frac{dy}{dx} + P(x)y = Q(x)$
 $M = \exp(\int P(x) dx)$
 $y = \frac{1}{M} \int M Q(x) dx$

3) Bernoulli Eq:
 $M = \frac{M y^N - N y}{N}$
 $M = \frac{M y^N - N y}{-N}$
 $\frac{dM}{dx} = -\frac{M}{N}$

6) Riccati Eq:
 $\frac{dy}{dx} = p(x)y^2 + q(x)y + R(x)$
 $Ex: \frac{dy}{dx} = 2y^2 + y + 1$
 $y = \frac{1}{u} \frac{du}{dx}$
 $P(x)=2, Q(x)=1, R(x)=1$
 $u' - \phi(x)u - R(x)u = 0$
 $\rightarrow u' - u - u = 0$
 $u = e^x, u = e^{-x}, u = e^{2x}$
 $\rightarrow e^x (r^2 - r - 1) = 0$
 $r = \frac{-1 \pm \sqrt{1+4}}{2}$

7) First order
 $(\frac{dy}{dx})^2 = x \frac{dy}{dx} + y = g(x)$
 $Ex: (\frac{dy}{dx})^2 - 2 \frac{dy}{dx} + y = x - 1$
 $Assume p = \frac{dy}{dx}$
 $\rightarrow p^2 - 2p + y - x + 1 = 0$
 $p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
 $p = 1 \pm \sqrt{x-y} = \frac{dy}{dx}$
 $Let u = x-y$
 $\frac{du}{dx} = \frac{dx}{dx} - \frac{dy}{dx}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} F \\ R \end{bmatrix} = \begin{bmatrix} aF + bR \\ cF + dR \end{bmatrix}$
 2×2 2×1 2×1
 2×2 2×1 2×1

4) Homogeneous
 $\frac{dy}{dx} = f(\frac{y}{x})$
 $* v = \frac{y}{x} \rightarrow y = vx$
 $\frac{dv}{dx} = \frac{dy}{dx} - \frac{y}{x}$

5) Bernoulli Eq:
 $\frac{dy}{dx} + p(x)y = q(x)y^a$
 $u = y^{1-a}$
 $* y = \frac{1}{u} \frac{du}{dx}$
 $\frac{du}{dx} = \frac{1}{1-a} \frac{dy}{dx}$
 $y = \frac{1}{u} \frac{du}{dx}$

$\frac{dy}{dx} = 1 - \frac{y}{x}$
 $\frac{1}{x} \frac{du}{dx} = 1 - \frac{u}{x}$
 $\pm \frac{du}{dx} = -\frac{u}{x}$
 $\pm \frac{du}{u} = -\frac{dx}{x}$
 $\pm \ln u = -\ln x$
 $u = \frac{1}{x}$
 $y = \frac{1}{x}$

Eigen Value
 $Ax = \lambda x$
 A matrix constant
 λ variable
 x constant
 $Ex: \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

1) Write Equations
 $a_{11}x_1 + a_{12}x_2 = \lambda x_1$
 $a_{21}x_1 + a_{22}x_2 = \lambda x_2$

2) $(A - \lambda I) = 0$
 $\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
 $\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3) Determinant
 $(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$
 $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$

4) Find λ
 $\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
 $\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

6) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

7) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

8) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

9) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

10) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

11) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

12) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

13) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

14) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

15) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

16) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

17) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

18) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$

19) Find x
 $y = \frac{a_{12}}{a_{11} - \lambda} x$
 $y = \frac{a_{21}}{a_{22} - \lambda} x$