Mathmatical Notes

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Concept :-

Solu Equations

Anatytically ← → Numerically

Equations → Variables

Dependent ← → independent

في كل المعادلات ← only one

at Least one

Dep = f (ind)

Differential equation → involves derivatives اي معادلة تحتوي اشتقاقات

لا يشترط عدد الاشتقاقات ← { variation of the dependent with

اقل شي مرة وحدة. (at Lest one). respect to the independent

مثلاً ↓

$y = f(x)$

dep ↵ ↳ in

$\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{\partial^2 v}{\partial x \partial y}$

$\boxed{I}$

Differential equation

Ordinary → only one depe ← Partial

only one ind Variable

at least two ind variable

Example:-

1) $\frac{dy}{dx} = 2x + y$

y dep → 1

x id → 1 (ordinary Deff. Ɛ)

لانها تحتوي ع اشتقاق

2) $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} = 15x$

x dep → 1

t ind → 1

ordinary . D. Ɛ

3) $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = v$

v dep

x, y ind → 2

Partial . D . Ɛ .

No.

$\boxed{2}$ order Diff.eq → the highest derivative

اعلى درجة اشتقاق

Ex:- $\frac{d^2y}{dx^2}$: Second order

$\left(\frac{dy}{dx}\right)^2$: first order

1) $\frac{dy}{dx} = 2x + y$ → 1st order ordinary D.E

2) $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} = 15$: 2nd order ordinary D.E

3) $\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2} = v$: 2nd order partial D.E

$\boxed{3}$ Degree

The power of the highest derivative.

← تخلص من ال Fraction
الكسور

1) $\frac{dy}{dx} = 2x + y$: 1st order ordinary D.E of first degree.

لانها اعلى اشت ←

2) $\left(\frac{d^2x}{dt^2}\right)^{\boxed{1}} - \left(2\frac{dx}{dt}\right)^1 = (15x)^1$: 2nd order ordinary D.E of first degree.

3) $\left(\frac{\partial^2 v}{\partial x^2}\right)^1 + \left(\frac{\partial^2 v}{\partial y^2}\right)^1 = (v)^1$: 2nd order partial D.E of first degree.

4) $\frac{d^2x}{dt^2} = \left(1 + \left(\frac{dx}{dy}\right)^2\right)^{3/2}$ clear the Fraction

$$\left(\frac{d^2x}{dt^2}\right)^2 = \left(1+\left(\frac{dx}{dt}\right)^2\right)^3$$: 2^{nd} o.O.D.E of 2^{nd} degree

↳ the highest ✓

$\boxed{4}$ Solution : any function which satisfies the equation reduces it to an identity.

← اي معادلة لم يحط عليها قيد ، دقيقها Balance.

→ $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15y = 0$: 2^{nd} order. O.D.E of 1^{st} degree.

1) $y_1 = 5e^{5x}$

2) $y_2 = e^{-3x}$

3) $y_3 = e^{x}$

which of these is the soluation of the equation?

→ نعوض substitute ب y.

Left = right ⇒ y is soluation

Solu:-

1) $y_1' = 5e^{5x}$, $y_1'' = 25e^{5x}$

$$25e^{5x} - 2\times5e^{5x} - 15e^{5x} \overset{?}{=} 0$$

$$0 = 0$$

∴ y_1 is a solution

2) $y_2' = -3e^{-3x}$, $y_2'' = 9e^{-3x}$

$+9e^{-3x} - 2 * -3e^{-3} - 15e^{-3x} \overset{?}{=} 0$ → Balance

$0 = 0$

∴ y_2 is asolution

3) $y_3' = e^{x}$, $y_3'' = e^{x}$

$e^{x} - 2e^{x} - 15e^{x} \overset{?}{=} 0$ → Balance

$-16e^{x} = 0$

∴ y_3 is not a solution.

⇒ $\underbrace{y_1 , y_2}$ ⇒ not the final solution,

parts of the solution   لازم معها condition

Condition ⇒ $y(0) = 2$, $y'(0) = 5$

هل ال solution تحقق ال cond ؟

$y_1 = e^{5x}$ → $y(0) = e^{0} = 1$ → لم تحقق ال cond

→ $y'(0) = 5e^{0} = 5$

⇒ لحتى تكون final الحل لازم تكون general

نقطة 5 .

No.

$\boxed{5}$ General Solution : Solution which contains arbitrary constants

* يحتوي على ارقام تتغير حسب ال con

number of the arbitray → depend on order

1st order ⇒ one arbitrary constant

2nd order ⇒ two arbitrary constant

Ex :- $y'' - 2y' - 15y = 0$

Part Solution:-

two arbitrary ⇒ 2nd order

$$\left.\begin{array}{l} y_1 = e^{5x} \\ y_2 = e^{-3x} \end{array}\right\} \rightarrow y = c_1 y_1 + c_2 y_2$$

$$y_g = c_1 e^{5x} + c_2 e^{-3x}$$ ⇒ general Solution

→ (linear compination)

تحقق المعادلة وال 2 condition

→ general Solution

$$y' = 5 c_1 e^{5x} - 3 c_2 e^{-3x}$$

$$y'' = 25 c_1 e^{5x} + 9 c_2 e^{-3x}$$

$$\rightarrow 25 \cancel{c_1} e^{5x} + 9 c_2 e^{-3x} - 10 \cancel{c_1} e^{5x} + 6 c_2 e^{-3} - 15 \cancel{c_1} e^{5x} - 15 c_2 e^{-3x} \overset{?}{=} 0$$

$$0 = 0 \ \checkmark$$

$\therefore y_g$ is a Solution

$\rightarrow y(0) = 2 \rightarrow y_g(0) = c_1 e^{0} + c_2 e^{0}$

$2 = c_1 + c_2$ ---- ①

$\rightarrow y'(0) = 5 \rightarrow y'_g = 5c_1 e^{5x} - 3c_2 e^{-3x}$

$y'(0) = 5c_1 - 3c_2$

$5 = 5c_1 - 3c_2$ ---- ②

$5 = 5(2 - c_2) - 3c_2$

$5 = 10 - 5c_2 - 3c_2$

$8c_2 = 5 \Rightarrow c_2 = \frac{5}{8}$

$c_1 = 2 - \frac{5}{8} \Rightarrow c_1 = \frac{11}{8}$

$y_g = \frac{11}{8} e^{5x} + \frac{5}{8} e^{-3x}$ → particular solution (unique)

$\Downarrow$

$y(0) = 2$ ✓ تحقق :

$y'(0) = 5$ ✓

$y'' - 2y' - 15y = 0$ ✓

$y' = \frac{55}{8} e^{5x} - \frac{15}{8} e^{-3x}$, $y'' = \frac{275}{8} e^{5x} + \frac{45}{8} e^{-3x}$

$\Rightarrow \frac{275}{8} e^{5x} + \frac{45}{8} e^{-3x} - \frac{110}{8} e^{5x} + \frac{30}{8} e^{-3x} - \frac{165}{8} e^{5x} - \frac{75}{8} e^{-3x} \overset{?}{=} 0$

$0 = 0$ ✓

$\Rightarrow y(0) = \frac{11}{8} + \frac{5}{8} = 2$ ✓

$\Rightarrow y'(0) = \frac{55}{8} - \frac{15}{8} = \frac{40}{8} = 5$ ✓

No.

ملاحظات (من محاضرة الـ ٨ !!

arbitrary an number

بتحقق المعادلة

ولا لاتحقق ال con

$y = \boxed{2}e^{5x} - \boxed{3}e^{-3x}$

$y' = 10e^{5x} + 9e^{-3x}$, $y'' = 50e^{5x} - 27e^{-3}$

$50e^{5x} - 27e^{-3} - 20e^{5x} + 18e^{-3x} - 30e^{5x} + 45e^{-3x}$

$0 = 0$ ✓

$y(0) = 2e^{0} - 3e^{0} = -1$ ✗

حققت المعادلة ولكن لم يحقق

conditions

Find the general solution ?

بتخلي ال (c_2 , c_1) ← او بتغطي اي رقم لأن تحقق المعادلة.

Find the particular solution?

بنعطي قيمة ال c_1 , c_2

بصير اسمه

constants

Eqution without condition not find the particular solution just the general solution

لأن ال parti فريدة ما في غيره

⇒ any other solution achieves 3quation?

No ⇒ لأن تغير ال condit بتغير

ال arbitrary وبالتالي الحل دائماً واحد فقط.

٣ انواع so :-

بتحول إلى

1) general ⇒ $y_g = \overset{\text{اي رقم}}{c_1 y_1 + c_2} y_2$

2) Particualr ⇒ $y_p = \frac{4}{8} y_1 + \frac{5}{6} y_2$

3) Part of solution y_1 , y_2 ⇒ لا تحقق ال conditions

No.

⊛ Particular solution

كل حل" لا يحتوى arbitrary

"No arbitrary constants exist"

if I have general solution + conditions ⇒ particular

6 types of solution:-

البلاست

لسن

Implicit and Explicit solution

لما لا نستطيع الفصل.

فصل ال independ

عن depend

كل وحدة في side

Example:-

- lets $x = e^{5t}$ → عبارة عن y p

ماخذ arbitarary

x dep
t ind } separated each other ? Yes

not mixed with each other ? No

∴ explicit solution in x

- $y^3 - 3x + 3y = 5$

if y depn → لا نستطيع ان نجعل ال

x indep   depn في طرف واحد

$$y = \frac{5 - y^3 + 3x}{3}$$

∴ implicit solution

convert

if y indep

x dep

$$x = \frac{+5 + y^3 + 3y}{3}$$ → ∴ explicit solution in x

No.

* ليس جميع المعادلة يمكن تحويلها إلى explicit
ولكن ال explicit هي أكمل من الحل فقط

ordinary

$\boxed{7}$ linear O.D.E and non-linear

المعادلة الخطية .

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x)$$

↳ general form of alinear O.D.E of order $\underline{\underline{n}}$.

$\sim\sim \Rightarrow$ They are all function of $\underline{\underline{x}}$ only $<$ constant / function

Look at the power of the depnednt only ⌉

1- all terms containing (y) must be raised to power 1

2- No multiplication of dependent terms

$x \quad y\frac{dy}{dx} \quad , \quad e^x\frac{dy}{dx}$ ✓ $, \quad \frac{dy}{dt}\cdot\frac{dy}{dt}$ (x) $\left(\frac{dy}{dx}\right)^2$ (x)

3- only explicit appearance of the dependent variable

$y^2, \sqrt{2}, \sin(y), \text{con}(y), \ln(y)$

$y'' + 5y' + 6y = 0 \rightarrow$ linear

$''y + 5y + 6y^2 = 0 \rightarrow$ non-linear power $\underline{\underline{2}}$

$y'' + 6y' + 6y = \ln(y) \rightarrow$ non-linear

لأن ال

$yy'' + 6y' + 5y = e^x \rightarrow$ non-linear

$\boxed{8}$ Initial value problem

Boundary value problem

* $y'' - 2y' - 15y = 0$ $y(0) = 2$, $y'(0) = 5$

$x = x$

$y_g = c_1 e^{5x} + c_2 e^{-3x}$ → the similar

→ Initial value

$c_1 = \frac{11}{8}$, $c_2 = \frac{5}{8}$

$y_p = \frac{11}{8} e^{5x} + \frac{5}{8} e^{-3x}$

* $y'' - 2y' - 15y = 0$ $y(0) = 2$ $y(2) = -4$

$y_g = c_1 e^{5x} + c_2 e^{-3x}$ $x \neq x$

$2 = c_1 + c_2$ --- ① not the similar

$-4 = c_1 e^{10} + c_2 e^{-6}$ ... ② → Boundary value

y_p ~~=x~~ = -----

⊛ مانفرق : حيث الحل

⊛ هذا التصنيف ندييه أ الا بعد 2^{nd} order لان 1^{st} ماعنا غير 1 condition

↳ initial value problem.

$\boxed{9}$ Singular solution

Non-singular solution

$y'' - 2y' - 15y = 0$

$y_g = c_1 e^{5x} + c_2 e^{-3x}$

$y(0) = 2$, $y'(0) = 5$

$y_p = \frac{11}{8} e^{5x} + \frac{5}{8} e^{-3x}$ ← Non-singular (unique)

نفرض انو Solution

$y = \sin(x) + 5$ → singular Solution.

غير مشتق من ال general Solution

هاد طالع من التجربة وليس من الحل الرياضي.

Example

$(y')^2 - xy' + y = 0$

$y_{\boxed{g}} = cx - c^2$ c is the arbitrary constant.

نتأكد انو Solution

$y' = c$

$\Rightarrow c^2 - xc + cx - c^2 \overset{?}{=} 0$

$0 = 0$ ✓

افرض انو ① $y_p = 2x - 4$ → non-singular / obtaind form y_g

$y' = 2$, $(2)^2 - x(2) + 2(x) - 4 \overset{?}{=} 0$

$0 = 0$ ✓

خرجت عن y_g

② افرض انو $y_2 = \frac{x^{2}}{4}$ → singular not obtain

$y'_2 = \frac{x^2}{2}$, $(\frac{x}{2})^2 - x(\frac{x}{2}) + \frac{x^2}{4} \overset{?}{\neq} 0$ ✓

$0 = 0$

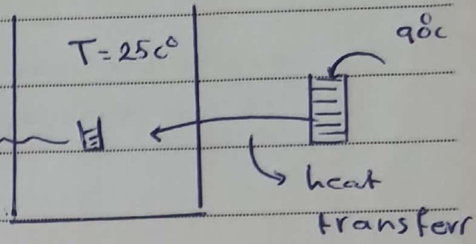
$\boxed{10}$ Mathmatical Modelis

⇐ نموذج رياضي يمثل تجربة فيزيائية

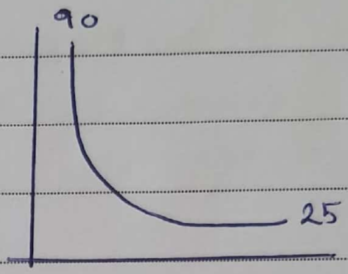
⇐ اهم شيء تحديد ال dep و ind (المتغيرات)

singular ⇒ because we can't ~~be~~ obtain it from the general solution.

No.



دع نقل درجة الحرارة
مع الزمن من ٩٠......25
بعد فترة



احسب عند

$T(t=20)$

دع نكتب النموذج رياضي

* اهم شيء مين يتغير مع مين (الحرارة تتغير مع الزمن)

↳ فيه اشتقاق

(فيه تتغير مع الزمن) not stady state

$T \rightarrow$ dep

$t \rightarrow$ inde

not arbitrary

$$\frac{dT}{dt} = k\, f(t) \xrightarrow{\text{بعد الحل}} T_g = C\, e^{-kt} \rightarrow T_g = C_1\, e^{-4t}$$

general solution

لازم يعطينا conditions

وهنا فيه هذه الاسئلة احنا لازم نطلع ال conditions

*ملاحظة: مانقدر نحل معادلة model إلا بوجود conditions.

$T(0) = 90$, $T(\infty) = 25$ → ما بستخدمها

بعد فترة / ولكن ال D.E فقط 1order لذلك نستخدم 1 conditions

No. ....

حلو ⇒ First order ordinary D.E

any function ↖

$$\frac{dy}{dx} = f(\underset{\text{ind}}{x}, \underset{\hookrightarrow \text{dep}}{y})$$

← دح نتعامل مع linear

ول non رح نحاول نحولها
الى linear

1. seperable — القدرة ع فصل dep عن ال in
2. Reduction to sperable. (by assumptions)
   ↳ لتحويل non-seperable
   الى sperable
3. Exact D.E
4. Integrating factor ⇒ function
5. linear D.E
6. Applications (mixing, reaction, heat transferr)

1. Seperable 1^{st} o.D.E

$$\frac{dy}{dx} = f(x,y) = F(\underset{\substack{\downarrow \\ \text{only } x}}{x}) \cdot G(\underset{\substack{\downarrow \\ y}}{y})$$

فصل كل اقتران لحاله. — بعدين لما افصلها بشاوله
اجهز التكامل

$$dy = F(x) \cdot G(y)\, dx$$

$$\frac{dy}{G(y)} = F(x)\, dx$$

$$\int \frac{dy}{G(y)} = \int F(x) \cdot dx \quad + \underset{\searrow \text{ arbitrary constant.}}{c}$$

Example:-

$$\frac{dy}{dx} = \frac{x^2+1}{2-y}$$

كلمة لتقسيم split ⟶ Sepeable D.E

$$f(x,y) = \underset{\sim\sim\sim}{\overset{F(x)}{(x^2+1)}} \cdot \overset{G(y)}{\left(\frac{1}{2-y}\right)}$$

$$\int (2-y)\,dy = \int (x^2+1)\,dx$$

$$2y - \frac{y^2}{2} = \frac{x^3}{3} + x + c$$ → ~~ex~~Implicit solution.

ما بنرق ex او ln

$$y\left(2-\frac{1}{2}y\right) = \frac{1}{3}x^3 + x + c$$

$$y_y = \frac{\left(\frac{1}{3}x^3 + x + c\right)}{\left(2-\frac{1}{2}y\right)}$$

give conditions $y = 4$ when $x = -3$ $\quad y(-3) = 4$

$$2(4) - \frac{16}{2} = \frac{(-3)^3}{3} - 3 + c$$

$$\cancel{8 - 8} = \frac{-27}{3} - 3 + c$$

$$c = 12$$

Example:-

$$x\frac{dy}{dx} - y = 2x^2y$$

$$\Rightarrow x\frac{dy}{dx} = 2x^2y + y \Rightarrow \frac{dy}{dx} = \frac{2x^2y+y}{x}$$

$$\frac{dy}{dx} = \cancel{y}\,\overset{F(x)}{\overset{\downarrow}{\left(\frac{2x^2+1}{x}\right)}}\ \overset{G(y)}{\overset{\downarrow}{y}}$$

$$\int \rightarrow \quad \frac{2x^2+1}{x}\cdot dx = \frac{dy}{y}$$ ⇝ seperated

$$\int 2x + \frac{1}{x}\cdot dx = \int \frac{1}{y}\,dy$$

$c + x^2 + \ln(x) = \ln(y)$ general solution

$\ln(x) - \ln(y) = c_1 - x^2$

$e \rightarrow \ln\left(\frac{x}{y}\right) = c_1 - x^2$

$\frac{x}{y} = e^{(c_1 - x^2)} \rightarrow \frac{x}{y} = \underbrace{e^{c_1}}_{\text{constant}} \cdot e^{-x^2}$

$\frac{x}{y} = A e^{-x^2}$

$\Rightarrow y = \frac{x}{A e^{-x^2}} \Rightarrow B x e^{x^2}$ explicit solution.

2. Reduction to seperable. [non-seperable ⇒ seperable]

لازم يكون هاد المتغير ضمن الاقتران

Ⓐ An equation of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ can be reduced to sperable using this assumption:-

$\frac{y}{x} = v$ or $\underset{\hookrightarrow \text{ind}_1}{y = v x}\} \rightarrow$ Assuption

يعني $y = vx \quad \frac{dy}{dx} = f\left(\frac{y}{x}\right)$

نعوضها هون يجب ان نشتق. Substitue

$$\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx} \Rightarrow \frac{dy}{dx} = v + \frac{dv}{dx}x$$

$$v + x\frac{dv}{dx} = f(v) \Rightarrow x\frac{dv}{dx} = f(v) + v$$ اضرب x

$$\frac{dv}{dx} = \frac{f(v) - v}{x}$$

$$\frac{dv}{dx} = \underbrace{\frac{1}{x}}_{F(x)} \cdot \underbrace{(f(v) - v)}_{G(y)}$$

$\int \Rightarrow \frac{dv}{f(v)-v} = \frac{dx}{x}$ seperated

نكمل الحل حسب ال f

Example:-

$$\frac{dy}{dx} = \frac{x-y}{x+y}$$

(اقسم على x للحصول $\frac{y}{x}$

$$\frac{dy}{dx} = \frac{1 - y/x}{1 + y/x} \Rightarrow f(y/x)$$

let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

نعوض

$$v + x\frac{dv}{dx} = \frac{1-v}{1+v} \Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{x} \cdot \left(\frac{1-v-v(1+v)}{1+v} \right)$$

$\int \Rightarrow \frac{1+v}{1-2v-v^2} \cdot dv = \frac{dx}{x}$ seperated

$\Rightarrow \ln(x) + C =$ ودا التكامل .

$\int \frac{1+v}{1-2v-v^2} \cdot dv$ $\qquad$ $y = 1-2v-v^2$

$\frac{dy}{dv} = -2-2v$

$\rightarrow \int \frac{\cancel{1+v}}{y} \cdot \frac{dy}{-2(\cancel{1+v})}$ $\qquad$ $dv = \frac{dy}{-2(1+v)}$

$= -\frac{\ln(y)}{2} = -\frac{1}{2}\ln(1-2v-v^2)$

$*2 \Rightarrow -\frac{1}{2}\ln(1-2v-v^2) = \ln(x) + c$

$e \Rightarrow -\ln(1-2v-v^2) = 2\ln(x)^{2} + \boxed{2c} \to c_1$

$\Rightarrow e^{-\ln(1-2v-v^2)} = e^{(\ln x^2 + c_1)}$

$\Rightarrow \frac{1}{1-2v-v^2} = e^{\ln x^2} \cdot \boxed{e^{c}} \to c_2$

$\Rightarrow \frac{1}{1-2v-v^2} = c_2 x^2$

$x^2(1-2v-v^2) = \boxed{\frac{1}{c_2}} \to c_3$

$x^2(1-2(y/x)-(y/x)^2) = c_3$ $\qquad$ final Solution.

وحد مقامات $\qquad$ explicit solution.

$\cancel{x^2}\left(\frac{x^2-2yx-y^2}{\cancel{x^2}}\right) = c_3$

$x^2 - 2yx - y^2 = c_3$

No.

عادي تكون ضمن ارقام،

Ⓑ if $\frac{dy}{dx} = \sqrt[n]{f(x+y)}$, this form can be reduced by assumption of $x + y = v^{\textcircled{2} \rightarrow n}$

* حسب الجذر.

$$\xRightarrow{\text{derive}} \quad 1 + \frac{dy}{dx} = \frac{dv2v}{dx} \rightarrow \frac{dy}{dx} = 2v\frac{dv}{dx} - 1$$

عوضها $\Rightarrow 2v.\frac{dv}{dx} - 1 = \sqrt{f(v)}$

$$\Rightarrow 2v.\frac{dv}{dt} = \underbrace{\sqrt{f(v)} + 1}_{F(v)}$$

$$\frac{2v.dv}{\sqrt{f(v)} + 1} \quad dv = dt \Rightarrow \text{seperated.}$$

Example:-

$$\frac{dy}{dx} = \sqrt{x+y}$$

$$\text{let} = x + y = v^2 \Rightarrow \frac{2v.d\cancel{y}v}{dt} = 1 + \frac{d\overset{y}{\cancel{x}}}{dx}$$

$$\rightarrow \frac{dy}{dt} = \frac{2v.dv}{dx} - 1$$

$$2v.\frac{dv}{dt} - 1 = \sqrt{v^2} \Rightarrow 2v.\frac{dv}{dt} = \cancel{\sqrt{v^2}} + 1$$

$$\frac{2v.dv}{\cancel{\sqrt{v^2}} + 1} = dx$$

$$\Rightarrow \frac{2v.dv}{v+1} = dx$$

$$\int \frac{2v}{v+1}\,dv = \int \left(2 - \frac{2}{v+1}\right) dv$$

من نفس الدرجة ← قسمة.

$$v+1 \,\overline{\big)\, 2v}^{\;2}$$
$$\underline{-\cancel{2v}-2}$$
$$-2$$

$$\Rightarrow 2v - 2\ln(v+1)\,.$$

$$\Rightarrow 2v - 2\ln(v+1) = x + c$$

$$2[v - \ln(v+1)] = x + c$$

$$2[\sqrt{x+y} - \ln(\sqrt{x+y}+1) = x + c$$ explicit solution

Example:-

- $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$

find a solution

- $(y + \sqrt{x^2+y^2})\,dx - x\,dy = 0 \;,\; y(1) = 0$

1) $2xy\,dy = -(x^2 - 3y^2)\,dx$

$$\frac{dy}{dx} = \frac{-(x^2-3y^2)}{2xy}$$

$\div x^2$

$$\frac{dy}{dx} = \frac{-(1 - 3(y/x)^2)}{2\frac{y}{x}}$$

$$y = vx \rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\cancel{v} + x\frac{dv}{dx} = \frac{-1+3v^2}{2v} - v$$
$$\cancel{-v}$$

$$x\frac{dv}{dx} = \frac{-1+3v^2-2v^2}{2v} \rightarrow \int \frac{2v}{v^2-1}\,dv = \int \frac{dx}{x}$$

$$\ln(v^2-1) = \ln(x) + c$$

$$v^2 - 1 = c_1 x \rightarrow \sqrt{v^2} = \sqrt{c_1x - 1} \rightarrow \frac{y}{x} = \sqrt{c_1x-1}$$

$$(y + \sqrt{x^2+y^2})\,dx = x\,dy$$

$\div x$

$$\left(\frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}\right)dx = dy$$

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}$$

$$v = \frac{y}{x}$$

$$y = vx \rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2}$$

$$\cancel{v} + x\frac{dv}{dx} = \cancel{v} + \sqrt{1+v^2}$$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{1}{x}\,dx$$

sin $\rightarrow$ $\sin^{-1} v = \ln(x) + c$

$$v = \sin(\ln(x) + c)$$

$$\frac{y}{x} = \sin(\ln(x)+c)$$

$y(1) = 0$

zero = $\sin(c)$

c = zero, 180°

No.

عادي تكون فيه ارقام

لست جذور ، رقم صحيح .

ⓒ if $\frac{dy}{dX} = [f(x+y)]^n$, where n is apossitve integer.

→ let $x+y=V$

$1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

$\frac{dv}{dx} - 1 = [f(v)]^n$

$\frac{dv}{dX} = [f(v)]^n + 1$

$\frac{dv}{(f(v))^2+1} = dx$ - - - seperable.

Ex:-

$\frac{dy}{dX} = (x+y)^2$

let $v = x+y$

$\frac{dv}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dv}{dx} - 1$

$\frac{dv}{dX} - 1 = v^2$

$\rightarrow \int dX = \int \frac{dv}{v^2+1}$

$x + c = \arctan(v)$

$\arctan(x+y) = X + c$

$X+y = \tan(x+c)$

$y = \tan(x+c) - X$ explicit solution.

$\frac{dy}{dx} = (2x+2y)^2$

let $v = 2x+2y$

$\frac{dv}{dx} = 2 + 2\frac{dy}{dx}$

$\frac{dy}{dx} = (\frac{dv}{dx} - 2)\frac{1}{2}$

$\frac{1}{2}(\frac{dv}{dx} - 2) = v^2$

$2v^2 = \frac{dv}{dx} - 2$

$\frac{dv}{dx} = 2v^2 + 2$

$\frac{dv}{\cancel{2(v^2)}}$ $\frac{dv}{(2v^2+2)} = dX$

$\int \frac{dv}{v^2+1} = \int 2\,dX$

$\arctan(v) = 2x + c$

$y = \frac{1}{2}(\tan($ [illegible] $2x)$

① if $\frac{dy}{dx} = \sin(f(x+y))$ or cos, tan, sec, cosec

كلهم نفس الطريقة

let $f(\underbrace{x+y}_{\theta}) = v$

Example:-

$$\frac{dy}{dx} = \tan(x+y) - 1$$

let $x+y = v \Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

$$\frac{dv}{dx} \cancel{-1} = \tan(v) \cancel{-1}$$

$$\Rightarrow \frac{dv}{dx} = \tan(v) \Rightarrow \frac{dv}{\tan(v)} = dx$$

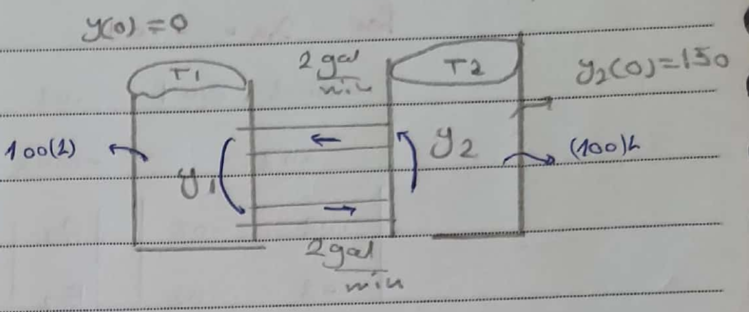
$\ln(\sin(v)) = x + c$

$\ln(\sin(x+y)) = x + c$

$\sin(x+y) = e^{x+c} = e^{x} \cdot \boxed{e^{c}} \rightarrow c_1$

$\sin(x+y) = c_1 e^{x}$

Example.



time ⇒ equalprian state.?

قبل ما نفتح ال valve :-

Consumtion ما ض وحتي خارجه

بس input و output

↓ الزائد ↓ الزائد

tank 1 قد الداخل ← الخارج

$$\frac{dy_1}{dt} = \frac{2y_2}{100} - \frac{2}{100}y_1$$

$$\frac{dy_2}{dt} = \frac{2}{100}y_1 - \frac{2}{100}y_2$$

$y_1(0) = 0$, $y_2(0) = 150$

$\frac{\cancel{gal}/min}{\cancel{gal}}$ * concetrati

✓ = concetvarin / min

$$y_1' = -.02\,y_1 + 0.02\,y_2 .$$

$$y_2' = 0.02\,y_1 - 0.02\,y_2$$

Det = 0

هنه يعني ان دسه صد eigven value

صفر = ← نتوحد الثا

ما شغل حلقه system

صفر شي

رقم ✓

$$\begin{vmatrix} -.02 & .02 \\ .02 & -.02 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} \to \begin{vmatrix} y_1' \\ y_2' \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -.02-\lambda & .02 \\ .02 & -.02-\lambda \end{vmatrix} \Rightarrow (-.02-\lambda)(-.02-\lambda) - (.02 * .02) = 0$$

$$\lambda^2 + 0.04\lambda = 0$$

$$\lambda(\lambda + .04) = 0 \to \lambda_1 = 0$$

$$\lambda_2 = -.04$$

for $\lambda_1 = 0$

$$\begin{vmatrix} -.02-0 & .02 \\ .02 & -.02-0 \end{vmatrix} \begin{vmatrix} A_1 \\ B_1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$\Rightarrow -0.02A_1 + .02B_1 = 0 \to A_1 = B_1$ if $A_1 = 1, B_1 = 1$

$$y_1 = \begin{vmatrix} 1 \\ 1 \end{vmatrix} e^{0x}$$

for $\lambda_2 = -.04$

$$\begin{vmatrix} -0.02+0.04 & .02 \\ .02 & -0.02+0.04 \end{vmatrix} \begin{vmatrix} A_2 \\ B_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} .02 & .02 \\ .02 & .02 \end{vmatrix} \begin{vmatrix} A_2 \\ B_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \Rightarrow \begin{matrix} 0.02A_2 + .02B_2 = 0 \\ A_2 = -B_2 \end{matrix}$$

if $A_2 = -1 \therefore B_2 = 1$

$$y_2 = \begin{vmatrix} -1 \\ 1 \end{vmatrix} e^{-0.04x}$$

$$y_g = c_1 \begin{vmatrix} 1 \\ 1 \end{vmatrix} e^{0x} + c_2 \begin{vmatrix} -1 \\ 1 \end{vmatrix} e^{-.04t}$$

$y_{1g} = c_1 - c_2 e^{-.04t}$, $y_{2g} = c_1 e^{0x} + c_2 e^{-.04x}$.

$\rightarrow$ $0 = c_1 - c_2$ ... ① $\qquad 1500 = c_1 + c_2$ ... ②

$150 = 2c_1 \rightarrow c_1 = 75$

$c_2 = 75$

$y_1 g = 75 - 75 e^{-.04t}$, $y_{2g} = 75 + 75 e^{-.04t}$. final Solution.

No.

Ⓔ if $\frac{dy}{dx} = f(ax+by+k)$ $\longrightarrow$ Polynomial

العربية كلاي ؟

let $ax+by+k = v$

$$a + b\frac{dy}{dx} = \frac{dv}{dx} \rightarrow \frac{dy}{dx} = \frac{1}{b}\left[\frac{dv}{dx} - a\right]$$

$$\frac{1}{b}\left[\frac{dv}{dt} - a\right] = f(v) \rightarrow \frac{dv}{dx} = bf(v) + a$$

$$\int \frac{dv}{bf(v)+a} = \int dx$$ seperated

Example:-

$\frac{dy}{dx} = (x+y-7)^{②}$ $\longrightarrow$ اي رقم ولكنه ليس كسر

$a = 1$, $b = 1$, $k = -7$

$$x+y-7 = v \Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dt} - 1$$

$$\frac{dv}{dx} - 1 = v^2$$

$$\frac{dv}{dx} = v^2 + 1 \Rightarrow \int \frac{dv}{v^2+1} = \int dx$$ seperated.

$$\arctan(v) = x + c$$

tan → $\arctan(x+y-7) = x + c$ لهون صحّ

$$x+y-7 = \tan(x+c)$$

Ⓕ if $(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0$

a_1, a_2, b_1, b_2

***Case 1** if $\frac{a_2}{a_1} = \frac{b_2}{b_1}$ $\Rightarrow$ let $z = a_1x + b_1y$

$$\frac{dz}{dx} = a_1 + b_1\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{b_1}\left[\frac{dz}{dx} - a\right]$$

المعادلة الأصلية ⇒ $\frac{dy}{dx} = \frac{-(a_1x + b_1y + c_1)}{(a_2x + b_2y + c_2)}$

$$\frac{1}{b_1}\left[\frac{dz}{dt} - a\right] = -\frac{(a_1x + b_1y + c_1)}{(a_2x + b_2y + c_2)}.$$

$$\frac{1}{b_1}\left[\frac{dz}{dx} - a\right] = -\frac{(z + c_1)}{\text{constant}\,(z + c_2)}$$ seperated.

Example:-

$(x + 2y + 3)\,dx + (2x + 4y - 1)\,dy = 0$

$a_1 = 1$, $a_2 = 2$, $b_1 = 2$, $b_2 = 4$

$\frac{a_2}{a_1} = \frac{b_2}{b_1}$ → let $z = x + 2y$

$\frac{dz}{dx} = 1 + 2\frac{dy}{dx}$ ⇒ $\frac{dy}{dx} = \frac{1}{2}\left[\frac{dz}{dx} - 1\right]$

$$\frac{dy}{dx} = \frac{x + 2y + 3}{-(2x + 4y - 1)}$$

$$\frac{1}{2}\left[\frac{dz}{dx}-1\right] = \frac{-(z+3)}{2z-1}$$

$$\frac{dz}{dx} = \frac{-2z-6}{2z-1}+1$$

$$\frac{dz}{dx} = \frac{-\cancel{2z}-6+\cancel{2z}-1}{2z-1} = \frac{-7}{2z-1}$$

$$\int \rightarrow \quad (2z-1)\,dz = -7\,dx$$

$$\frac{\cancel{2}z^2}{\cancel{2}} - z = -7X + c$$

$$(x+2y)^2 - (x+2y) = -7X + c$$

$$x^2 + 4xy + 4y^2 - x - 2y + 7x = c$$

$$x^2 + 4y^2 + 4xy + 6X - 2y = c$$ final Solution implicit.

* Case 2:-

if $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$

let $x = X + h$, $y = Y + k$

new variable ↲ h constant, k constant

To obtain the values of h and k solve:-

$a_1h + b_1k + c_1 = 0$ ....(1

$a_2h + b_2k + c_2 = 0$ ....(2

لايجاد h, k ⇐ نعوض في المعادلتين الخطيتين

بدل x←h , y←k ونحل المعادلتين

رقم + X = x , رقم + Y = y نعوضهم في المعادلة → Seperable

→ احد ال case السابقات.

No.

Ec:-

$$(5x + 2y - 1)\,dx + (2x + y + 1)\,dy = 0$$

$$\frac{a_2}{a_1} \overset{?}{=} \frac{b_2}{b_1} \Rightarrow \frac{2}{5} \neq \frac{1}{2}$$

let $x = X + h \;,\; y = Y + k$

$5h + 2k - 1 = 0$ ----(1

$-2 \times 2h + k + 1 = 0$ ----(2

$\Downarrow$

$5h + \cancel{2k} - 1 = 0$

$-4h - \cancel{2k} - 2 = 0$

$h - 3 = 0 \Rightarrow h = 3 \Rightarrow 15 + 2k - 1 = 0 \Rightarrow k = -7$

$x = X + 3 \quad , \quad y = Y + k$

$1 = \frac{dX}{dx} \Rightarrow dx = dX \quad , \quad dy = dY$

$$\Rightarrow (5(X+3) + 2(Y-7) - 1)\,dX + (2(X+3) + (Y-7) + 1)\,dY = 0$$

← لازم كل Constant يروحوا

$$\Rightarrow (5X + \cancel{15} + 2Y - \cancel{14} \cancel{-1})\,dx + (2X + \cancel{6} + Y - \cancel{7} + \cancel{1})\,dY = 0$$

$$(5X + 2Y)\,dX + (2X + Y)\,dY = 0$$

$$\frac{dY}{dX} = -\frac{[5X + 2Y]}{[2X + Y]} \Rightarrow \text{case } Ⓑ \rightarrow 2$$

X ÷

$$\frac{dY}{dx} = \frac{-[5x+2Y]}{[2X+Y]} = \frac{-[5+2\frac{Y}{x}]}{[2+\frac{Y}{x}]}$$

let $\frac{Y}{x} = v \Rightarrow \frac{dY}{dx} = v + \frac{x\,dv}{dx}$

$$\Rightarrow v + x\frac{dY}{dx} = \frac{-5-2v}{2+v}$$

$$x\frac{dY}{dx} = \frac{-5-2v-2v-v^2}{2+v}$$

$\int \rightarrow \left[\frac{2+v}{v^2+4v+5}\right] dv = -\frac{1}{x}\,dx$ Seperated

$$\frac{1}{2}\ln(v^2+4v+5) = -\ln(x)+c$$

$$\frac{1}{2}\ln\left(\left(\frac{Y}{x}\right)^2+4\left(\frac{Y}{x}\right)+5\right) = -\ln(x)+c$$

$Y = y+7$, $X = x-3$

$$\frac{1}{2}\ln\left[\left(\frac{y+7}{x-3}\right)^2+4\left(\frac{y+7}{x-3}\right)+5\right] = -\ln(x-3)+c \quad \#$$

No.

Exact Differential Equation

$$\frac{dy}{dx} = f(x,y) \rightarrow \underbrace{M(x,y)}_{\substack{M \\ A}}\,dx + \underbrace{N(x,y)}_{\substack{N \\ B}}\,dy = 0$$

مربوطة

Find $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$

$\frac{\partial A}{\partial y}$, $\frac{\partial B}{\partial x}$ → الحل التوقع ← (عكس المربوط)

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ 1^{st} O.D.E is exact.

if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ 1^{st} O.D.E is not exact.

How do we obtain solution for exact D.E ?

1) A) find $u(x,y) = \int M\,dx + k(y)$

constant ← لكن في y لما اشتق لل x بروح

B) to find $k(y) \rightarrow \frac{\partial u}{\partial y} = N(x,y) \Rightarrow \frac{\partial}{\partial y}\left[\int M\,dx\right] + \frac{dk}{dy} = n$

c) $u(x,y) = \int M\,dx + k(y) = c$

2) Another appoach

في دراسة نبدأ في الاسهل
M او N للتكامل

A) $u(x,y) = \int N\,dy + l(x)$

B) $\frac{du}{dx} = M \Rightarrow \frac{\partial}{\partial x}\left[\int N\,dy\right] + \frac{dl}{dx} = M$

c) $u(x,y) = \int N\,dy + d(x) = c$ → constant .

Ex:-

$$\frac{dy}{dx} = \frac{-(x^3 + 3xy^2)}{3x^2y + y^3}$$

* ممكن نقسم x^3 بيصير :-

$$\frac{-(1 - 3(\frac{y}{x})^2)}{3\frac{x}{x} + (\frac{y}{x})^3}$$

$$\underbrace{(x^3 + 3xy)}_{M}dx + \underbrace{(3x^2y + y^3)}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} , \frac{\partial N}{\partial x}$$

constant constant,

$$\boxed{x^3} - \boxed{3x}y^2$$

$$\Rightarrow \frac{\partial M}{\partial y} = +6xy \qquad , \qquad \frac{\partial N}{\partial x} = 6xy$$

$\therefore$ Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6xy$ $\therefore$ exact D.E

A) $u(x,y) = \int M dx + k(y)$

$$= \int (x^3 + 3xy)\, dx + k(y)$$

$$u = \frac{x^4}{4} + \frac{3}{2}x^2y^2 + k(y)$$

know the cosnt

B) $\frac{\partial u}{\partial y} = N$

$\underline{k(y)}$ لانها ←

بالغلط :(

$$0 + \frac{3}{2}x^2(2y) + \frac{dk}{dy} = 3x^3y + y^3 \Rightarrow 3x^2\cancel{y} + y^3 = 3\cancel{x^2}y + \frac{\cancel{d}k}{dy}$$

$$\Rightarrow y^3 = \frac{dk}{dy} \dashrightarrow \text{sepe..}$$

$$\int dk = \int y^3 dy \Rightarrow K_{(y)} = \frac{y^4}{4} + c.$$

نعوضها ضد $\underline{u}$

c) $u(x,y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + C_0 = C$

$u(x,y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 = C_1$

$x^4 + 6x^2y^2 + y^4 = 4C_1 \hookrightarrow C_2$

the other approach:-

A) $u(x,y) = \int N\,dy + \ln(x)$

$= \int 3x^2y + y^3\,dy + l(x)$

$u = \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + l(x)$

B) $\frac{\partial u}{\partial x} = M$

$3\cancel{x}y^2 + 0 + \frac{dL}{dx} = 3\cancel{x}y^2 + x^3$

$\int dL = \int x^3\,dx \Rightarrow L = \frac{x^4}{4} + C_0$

C) $u(x,y) = \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + \frac{x^4}{4} + C_0 = C$

$u(x,y) = 6x^2y^2 + y^4 + x^4 = 4C_1$

Example :-

$$\underbrace{(2xy)}_{M}\,dx + \underbrace{(x^2+\cos(y))}_{N}\,dy = 0$$

$$\frac{6M}{\partial y} = 2x \quad , \quad \frac{\partial N}{\partial x} = 2x$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x \quad \text{is exact}$$

A) $u(x,y) = \int M\,dx + k(y)$

$$= \int (2xy)\,dx + ky$$

$$= x^2y + k(y)$$

B) $\frac{\partial u}{\partial y} = N$

$$\cancel{x^2} + \frac{dk}{dy} = \cancel{x^2} + \cos(y)$$

$$\int dk = \int \cos(y)\,dy$$

$$k = +\sin(y) + C_0$$

C) $u(x,y) = x^2y + \sin(y) + C_0 = C$

$x^2y + \sin(y) = C_1$ → general Solution.

Ho:- $\frac{dy}{dx} = \frac{2x\cos(y) + 3x^2y}{y + x^2\sin(y) - x^3}$ $\quad y(0) = 2$, find the particular solu?

Example:-

$$\underbrace{y}_{M}\,dx - \underbrace{x}_{N}\,dy = 0$$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -1$ $\therefore \neq$ is not exact.

multiply the D.E by $(1/x^2)$ ⇒ factor (function)

$$\underbrace{\frac{y}{x^2}}_{M}\,dx - \underbrace{\frac{1}{x}}_{N}\,dy = 0$$

$\frac{\partial M'}{\partial y} = \frac{1}{x^2}$, $\frac{\partial N'}{\partial x} = --\frac{1}{x^2}$ $\therefore =$ is exact.

⇒ The function $(1/x^2)$ that converts the non-exact to exact is called the integrating factor (If).

- How do we obtain the If ? * مش كل المعادلات لها هاد العامل.

- Try first ① $R_1 = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$, if R_1 is function of x only the int.f =

$$e^{\int R_1 \, \underline{dx}}$$

- Try second ② $R_2 = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$, if R_2 is function of y only the If =

$$e^{\int R_2 \, \underline{dy}}$$

* اذا طلع في x او y خلص ماشي If

⇒ $R_1 = \frac{1}{-x}(1 - -1)$ ⇒ $\frac{-2}{x}$ function $f(x)$ only

$\therefore e^{\int \frac{-2}{x} dx}$ ⇒ $e^{-2\ln x}$ ⇒ $\cancel{e^{\ln}} x^{-2}$ ⇒ $\frac{1}{x^2}$

Example:-

$$\underbrace{2xy}_{M}\,dx + (\underbrace{4y + 3\overset{x}{\cancel{y}}^2}_{N})\,dy = 0 \quad , y(0,2) = -1.5$$

find Solution?

$$\frac{\partial M}{\partial y} = 2x \;,\; \frac{\partial N}{\partial x} = 6x \qquad \therefore \text{ is not exact.}$$

$$\Rightarrow R_1 = \frac{1}{4y+3x^2}\left(2x - 6x\right)$$

$$= \frac{-4x}{4y+3x^2} \leftarrow f(x,y)$$

$$\Rightarrow R_2 = \frac{1}{2xy}\,(6x - 2x)$$

$$= \frac{\overset{2}{\cancel{4}}\cancel{x}}{\cancel{2}\cancel{x}y} = \frac{2}{y} \leftarrow f(y)$$

$$\therefore e^{\int \frac{2}{y}\,dy} \Rightarrow e^{2\ln(y)} \Rightarrow \cancel{e}^{\ln(y)^2} \Rightarrow y^2$$

$$\Rightarrow (\underbrace{2xy^3}_{M'})\,dx + (\underbrace{4y^3 + 3x^2y^2}_{N'})\,dy = 0$$

$$\frac{\partial M'}{\partial y} = 6xy^2 \;,\; \frac{\partial N'}{\partial x} = 6xy^2 \qquad \therefore \text{exact}$$

$$A)\int N'\,dy + \overset{L(x)}{\cancel{K(y)}}$$

$$u = \int \overset{4y^3+3x^2y^2}{\cancel{2xy^3}}\,dy + \overset{L(x)}{\cancel{K(y)}}$$

$$= y^3 + x^2y^3 + L(x)$$

B) $\frac{\partial u}{\partial x} = M$

$0 + \cancel{2y^3x} + \frac{dL}{dx} = \cancel{2xy^3} \Rightarrow \frac{dL}{dx} = 0$

اي رقم مشتقة L ⇐ 0

$\int dL = \int 0 \, dx$

$L = \text{constant} + c_0$

$u(x,y) = y^4 + y^3x^2 + c + c_0 = c_1$

$\Rightarrow y^4 + y^3x^2 = c_2$ general solution.

$(-1.5)^4 + (-1.5)^3(0.2)^2 = c_2$

$c = 4.9275$

particular solution $y^4 + y^3x^2 = 4.9275$

H.W:-

$(\underbrace{2y + xy}_{M})dx + (\underbrace{2x}_{N})dy = 0$ find the general solution?

$\frac{\partial M}{\partial y} = 2 + x \quad , \quad \frac{\partial N}{\partial x} = 2 \quad \Rightarrow$ not exact.

$R = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2x}(\cancel{2} + x - \cancel{2}) = \frac{x}{2x} = \frac{1}{2}$

$R_2 = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{2y+2\cancel{x}}(2-2-x) = \frac{1}{2y+2x} ($

⇒ linear D.E (first)

$$\frac{dy}{dx} + P(x)\,y = r(x)$$

$\frac{dy}{dx}$ ⇐ لازم واحد.

Case 1 if $r(x) = 0$, the equation will homogeneous

$$\frac{dy}{dx} + P(x)y = 0 \Rightarrow \frac{dy}{dx} = -P(x)y \Rightarrow \frac{dy}{y} = -P(x)\,dx$$

spereated

$$\ln y = \int -P(x)\,dx + c$$

$$y = e^{\int -P(x)dx}\,\overset{\rightarrow c_1}{\boxed{e^{c}}}$$

$$\boxed{y = c_1\,e^{\int -P(x)dx}}$$ general solution.

Case 2 if $r(x) \neq 0$, the equation will non homogeneous

$$\frac{dy}{dx} + P(x)y = r(x) \Rightarrow \frac{dy}{dx} = r(x) - P(x)y$$

$$\Rightarrow dy = [r(x) - P(x)y]\,dx$$

$$⊛ \Rightarrow \underbrace{[P(x)y - r(x)]}_{M}\,dx + \underbrace{(1)}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = P(x), \quad \frac{\partial N}{\partial x} = 0 \quad \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

∴ non - Exact.

⊛ Factor:-

$$R_1 = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{1}\,(P(x) - 0) = P(x)$$ function in (x) only.

$$IF \Rightarrow e^{\int R_1 dx} = e^{\int P(x)dx}$$ ⇒ maltiply ⊛

No.

$$e^{\int p(x)dx}\left[p(x)y - r(t)\right]dx + e^{\int p(x)dx}(1)\,dy = 0$$

$$\left[\underbrace{p(x)y\,e^{\int p(x)dx} - r(t)\,e^{\int p(x)dx}}_{M}\right]dx + \underbrace{e^{\int p(x)dx}}_{N}\,dy = 0$$

Check exact or not:-

$$\frac{\partial M}{\partial y} = p(x)\,e^{\int p(x)dx} \quad , \quad \frac{\partial N}{\partial x} = p(x)\,e^{\int p(x)dy}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{Exact}$$

To find the solution:-

$$\int N\,dy \Rightarrow \int \underbrace{e^{\int p(x)dx}}_{\text{constant}}\,dy \Rightarrow u = y\,e^{\int p(x)dx} + h(x) \quad *$$

$$\frac{\partial u}{\partial x} = M$$

$$p(x)\cancel{y}\,e^{\int p(x)dx} + \frac{dh}{dx} = p(x)\cancel{y}\,e^{\int p(x)dx} - r(t)\,e^{\int p(x)d(x)}$$

$$dh = \left(-r(t)\,e^{\int p(x)dx}\right)dx$$

$$\therefore h = \int -r(t)\,e^{\int p(x)dx}\,dx + c \qquad * \text{ نعوض في}$$

$$u = y\,e^{\int p(x)dx} + \int -r(t)\,e^{\int p(x)dx} + c_0 = c$$

$$y\,e^{\int p(x)dx} = c + \int +r(x)\,e^{\int p(x)dx}$$

$$\div\, e^{\int p(x)dx}$$

$$y = \frac{1}{e^{\int p(x)dx}} \left[\int r(x)\, e^{\int p(x)dx} + c_1 \right]$$

$$\Rightarrow y = e^{-\int p(x)dx} \left[\int r(x)\, e^{\int p(x)dx} + c_1 \right]$$ The general solution for the first O.D.E linear and non-Homogene

if $r(x) = 0$:-

$$y = c_1 e^{-\int p(x)dx}$$

Example:-

$$(x^2+1)\frac{dy}{dx} + 4xy = x \qquad y(2)=1$$

linear ⇒ not multiply (y), Power → (1), Sin (x)

non-homogeneous ⇒ $r(x) \neq 0$

form:- لازم المعامل واحد.

$$\frac{dy}{dx} + \frac{4x}{x^2+1}\, y = \frac{x}{x^2+1}$$

$\nearrow$ $p(x)$ $\qquad$ $\nearrow$ $r(x)$

$$p(x) = \frac{4x}{x^2+1} \quad , \quad r(x) = \frac{x}{x^2+1}$$

$$y = e^{-\int p(x)dx} \left[\int r(x)\, e^{\int p(x)dx} + c_1 \right]$$

$$\int p(x)dx \Rightarrow \int \frac{4x}{x^2+1} dx \Rightarrow 2\ln(x^2+1) \Rightarrow \ln(x^2+1)^2$$

$$\Rightarrow y = \cancel{e}^{-\cancel{\ln}(x^2+1)^2} \left[\int \frac{x}{x^2+1}\, \cancel{e}^{\cancel{\ln}(x^2+1)^2} + c_1 \right]$$

No.

$$y = \frac{1}{(x^2+1)^2}\left[\int \frac{x}{\cancel{x^2+1}} \cdot (x^2+1)^{\overset{①}{\cancel{2}}}\, dx + c_1\right]$$

$\hookrightarrow (x^3+x)$

$$y \Rightarrow \frac{1}{(x^2+1)^2}\left[\frac{x^4}{4} + \frac{x^2}{2} + c_1\right]$$

$$y(x^2+1)^2 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + c_1$$ general solution.

$y(2)=1 \Rightarrow x=2,\ y=1$

$$1(5)^2 = \frac{1}{\cancel{4}} \times \overset{\cancel{8}4}{\cancel{16}} + \frac{1}{\cancel{2}} \times \overset{2}{\cancel{4}} + c_1$$

$c_1 = 19$

$$y_p(x^2+1)^2 = \frac{x^4}{4} + \frac{x^2}{2} + 19$$ ⇒ final Solution.

Example:-

$$\frac{dy}{dx} = e^{2x} + y$$ linear.

$$\frac{dy}{dt} - (1)y = e^{2x}$$

$P(x) = -1$, and $r(x) = e^{2x}$ 1^{st}.o.DE linear non-Hom

$$y = e^{-\int p(x)dt}\left[\int e^{\int p(x)dx} \cdot r(x) \cdot dx + c_1\right]$$

No.

$\int p(x)\,dx = \int -1\,dt = -x.$

$\therefore\ y = e^{-(-x)}\left[\int \overbrace{e^{-x}\cdot e^{2x}}^{e^{x}}\,dx + c\right]$

$y = e^{x}\left[e^{x} + c_1\right]$

$y_g = e^{2x} + c_1 e^{x}$

Reduction to linear D.E (Bernoli Equation)

⊛ $\frac{dy}{dt} + P(t)y = g(t)y^{a} \Rightarrow$ non-linear.

if $a=0$ or $\Rightarrow$ $\frac{dy}{dt} + P(t)y = g(t)$ linear non hom.

if $a=1$ $\Rightarrow$ $\frac{dy}{dx} + P(x)\,y = g(t)\,y \Rightarrow \frac{dy}{dx} + (P(t) - g(t))\,y = 0$

linear and Hom.

if $a \neq 0$ or 1 $\rightarrow$ we have non linear equation

But This non-linear equation can be reduced to linear equation if we assume $u(t) = y^{1-a}$.

$\frac{du}{dt} = (1-a)\,y^{(\cancel{1}-a)\cancel{-1}}\,\frac{dy}{dx}$

$* \quad \frac{dy}{dt} = g(t)y^{a} - P(t)y$

$\frac{du}{dt} = (1-a)\,y^{-a}\left(g(t)y^{a} - P(t)y\right)$

$$\frac{dy}{dt} = (1-a)\left[g(x) - P(x)\, \underset{\hookrightarrow u(x)}{y^{1-a}} \right]$$

$$\frac{dy}{dt} = (1-a)\left[g(x) - P(x)\, u \right]$$

$$\frac{dy}{dt} = (1-a)\, g(x) - (1-a)\, P(x)\, u$$

$$\rightarrow \frac{dy}{dt} + \underbrace{(1-a) P(x)}_{\mu(x)} u = \underbrace{(1-a) g(x)}_{G(x)}$$

$\frac{du}{dt} + \mu(x)\, u = G(x)$ linear in u and non-Hom.

Solution:-

$$u \leftarrow y = e^{-\int(\mu(x) dx} \left[e^{\int \mu(x) dx} \cdot G(t)\, dx + c_1 \right]$$

Eample:-

$$\frac{dy}{dx} + \underset{p(x)}{y} = \underset{g(x)}{y^{②}} \longrightarrow \text{non-linear}$$

$a = 2$ let $u = y^{1-a} = y^{1-2} = \frac{1}{y}$

$G(x) = 1$, $p(x) = 1$

$$\frac{du}{dx} + (1-a)\, P(x)\, u = (1-a)\, g(x)$$

$$\frac{du}{dx} - 1 \cdot (1) \cdot u = (1-a)(1) \Rightarrow \frac{du}{dt} - \underset{\mu(x)}{(1)\, u} = \cancel{(1-a)} \underset{G(x)}{-1}$$

$$u = e^{-\int \mu\, d(x)} \left[\int e^{\int \mu(x) dx} \cdot G(x) \cdot dx + c \right]$$

$\int \mu(t)\,dt = \int -1\,dt = -x$

$u = e^{x}\left[\int e^{-x}\cdot(-1)\,dx + c\right]$

$u = e^{x}\left[(1)\,e^{-x} + c\right]$

$u = 1 + c\,e^{x}$

$\downarrow$

$\frac{1}{y} = 1 + c\,e^{x}$ final solution.

$y = \frac{1}{1+ce^{x}}$ explicit solution.

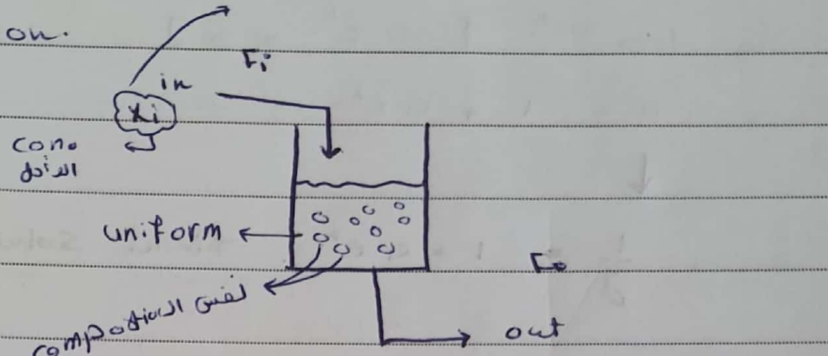
Applications :-

ترجمة السؤال إلى معادلات

- mixing in tanks.
- Cooling
- Sample reaction.

هاد لا يتغير

- mixing ⇒



لم لا يحدث تفاعل

x_i ⇒ concentration الطارة

$F_i = F_o$ → state.

$F_o < F_i$ → varible ← $F_1 \neq F_o$

Volume in tank constant. لم (يعني بيبي دخل طلع ما أثر بالكمية)

رح يزيد (ضل منه المادة الداخلة)

رح يقل (طلعت المادة بيبي جوا)

- حسب ما فهمت هو بكونه لدخل بس ف نحط عليه اثر

تركيز الـ salt. zero

Q :- A tank initally contains 50 gallans of pure water. Starting at time $t=0$, a brine contanining 2lb of dissolved salt per gallon flows into the tank at the rate of 3 gallon per minute. The mixture is kept uniform by stirring and the well-stirred mixture simulanceously flows out of the tank at the same rate.

دخلنا مادة تركيز

متجانسا

خلط

تقليب

[illegible]

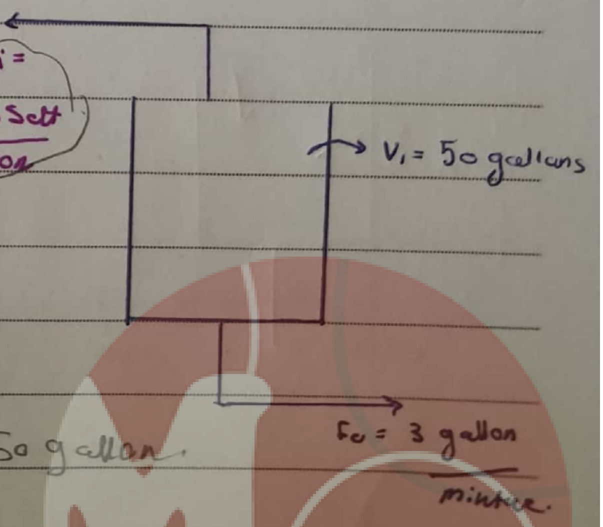
الخليط

لا يتغير مع الوقت،

2lb كل gallon فيه

$F_i \Leftarrow \frac{3 \text{ gallon}}{\text{minute}}$ ما يتغيروا.

$x_i = \frac{2 \text{ lb salt}}{\text{gallon}}$ لا يتغير مع الوقت،



let x is the amount of salts in the tanke at any atime.

$\frac{dx}{dt}$ ← تتغير مع الزمن.

كم الـ volume مع تغير الزمن؟ صفر

$$\frac{dv}{dt} = 0 \quad , \quad V_i = V_f = 50 \text{ gallon.}$$

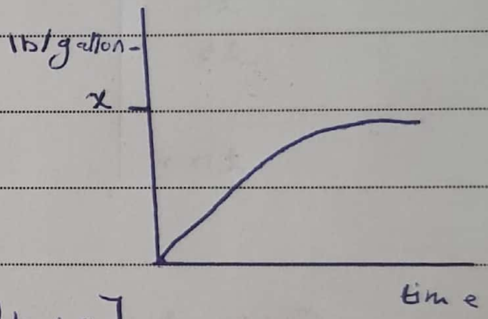
$x(t) = ?$

$\frac{dx}{dt} = X_{in} - X_{out}$.

$X_{in} = \frac{2 \text{ lb salt}}{\text{gallon}} * 3 \frac{\text{gallons}}{\text{minute}} = \frac{6 \text{ lb}}{\text{minute}}$

$X_{out} = \frac{x \text{ lb}}{50 \text{ gallon}} * 3 \frac{\text{gallon}}{\text{minute}} = \frac{3x}{50}$

$\frac{dx}{dt} = 6 - \frac{3x}{50}$, condition $x|_{t_0} = 0$ (pure water)



$\frac{dx}{dt} + \frac{3}{50}x = 6$ → linear-non hom

$P(t)$ ↩ $\quad$ ↳ $r(t)$

$x(t) = e^{-\int P(t)dt} \left[\int e^{\int p(t)dt} \cdot r(t) dt + c \right]$

$\int P(t)dt = \int \frac{3}{50} dt = \frac{3}{50}t$

$x(t) = e^{-\frac{3}{50}t} \left[\int e^{\frac{3}{50}t} \cdot 6 \, dt + c \right]$

$e^{-\frac{3}{5}t} \left[\cancel{\frac{3}{50}} \frac{50}{3} \cdot 6 \, e^{\frac{3}{50}t} + c \right]$

$x(t) = 100 + c e^{-\frac{3}{50}t}$ general (a)

$x|_{t=0} = 0 \Rightarrow 0 = 100 + ce^{0} \quad c = -1000$

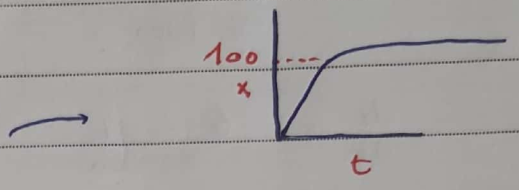
$x(t) = 100 - 100 e^{-\frac{3}{50}t}$ ⟸ particular

(b) $(x = t = 25)$

$$x(25) = 100 - e^{-\frac{3}{50} \cdot 25} = 77.69 \text{ lb solut}$$

(c) $$x = 100 - 100\, e^{-(\infty)} = 100 \qquad \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

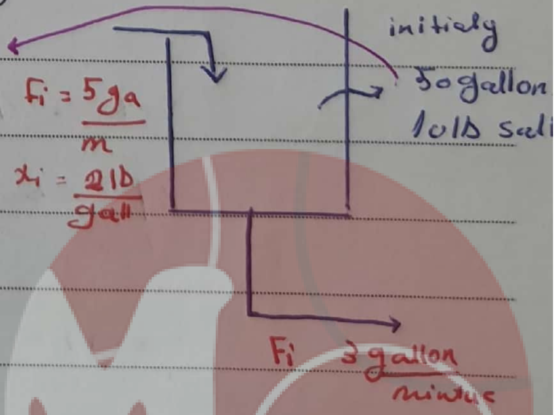
t	x(t)
0	0
50	
20	
25	
⋮	
200	



Q2:- A large tank initialy contains 50 gallons of brine in which there is dissolved 10 lb of salt. A brine con~~ta~~taning 2 lb of dissolved salt per gallon flows into the tank at the rate of 5 gallon per minute. the mixture is kept uniform by stirring and the stirred mixture simultaneously flows out at the slower rate of 3 gallon per minute

⇒ How much salt is in the tank at any time?

$x(t=0) = 10$



$\text{Volume} = 50 + 2t$

$x_i = 5 * 2 = 10$

$x_{out} = 3 * (50 + 2t)$

$$\frac{dx}{dt} = 10 - \frac{3x}{50+2t} \quad , \quad x(t=0) = 10$$

$$\frac{dx}{dt} + \frac{3x}{50+2t} = 10 \quad \rightarrow$$

Q3:- Newton's Law of cooling

$$\frac{dT}{dt} = -k\,(T - T_0)$$

↳ surrounding Tamp.

if the object T is grater than T_0 ⟹ cooling Law

$$\frac{dT}{dt} = k\,(T - T_0)$$

الحرارة رح تنزل

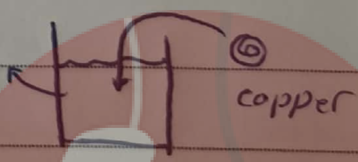
if the object T is Lower then T_0 ⇒ Heating Low

$$\frac{dT}{dt} = k\,(T_0 - T)$$

water $T(0) = 100°c$

find the time → $T = 31c°$

$T(30)c$



$T(t=0) =$

$T(t = 3\text{ minute})$

$= 70\ c$

Coolings:-

$$\frac{dT}{dt} = k\,(T-30)$$

$$\underset{p(t)}{\frac{dT}{dt} - kT} = \underset{r(t)}{-30k}$$

oR $\frac{dT}{dt} = k\,(T-30) \Rightarrow \int \frac{dT}{T-30} = \int k\,dt$

$\ln(T-30) = kt + c \Rightarrow$ 2 unknown $\Rightarrow$ 2 conditions.

$$T-30 = e^{kt}\cdot \boxed{e^{c}} \rightarrow a$$

$$T = ae^{kt} + 30$$

$T(t=0) = 100 \Rightarrow 100 = 30 + ce^{0} \Rightarrow a = 70$

$T = 30 + 70e^{kt}$

$T(t=3) = 70 \Rightarrow 70 = 30 + 7e^{3k} \Rightarrow \frac{40}{70} = e^{3k}$

$k = -.0.186$

$\therefore T(t) = 30 + 70\,e^{-.186}$

$T(t=?) = 31$

$31 = 30 + 70\,e^{-.186} \Rightarrow t = 22.78$ minute

Secand O.O.D.E of first degree.

$$y'' + P(x)y' + q(x)y = r(x)$$

Coefficient ←

P(x) and q(x) and r(x) They are all function of x only.

2nd order, linear

Case 1: if $r(x) = \text{zero}$, ∴ 2nd O.O.D.E homogenouse

Case 2: if $r(x) \neq \text{zero}$, ∴ 2nd O.O.DE non-homogenouse.

3 ⇒ if P(x) and q(x) are not constant

∴ 2nd O.O.D.E linear with variable coefficients

4 ⇒ if P(x) and q(x) are constant

∴ 2nd O.O.D.E linear with constant ″.

1) $y'' + 3xy' + x^3y = e^x$

2nd O.O.D.E linear, variable cofficients, non-hom. ✓

2) $y'' + 4y = e^{-x}.\sin(x)$

2nd O.O.D.E linear, constant cofficient, non-hom ✓

$p(x)=4$ $q(x)=0$

3) $x y'' y + (y'^2) + 2yy' = 0$

2nd O.O.D.E, non linear, Variable ″, homo ✗

3) $(1-x^2)y'' - 2xy' + 6y = 0$ لانو رح تقسم

2nd O.O.D.E, linear, variable, homo

Solution:-

$y'' - y = 0$ homogeneous ① and linear ②

$y_1 = e^{x}$ and $y_2 = e^{-x}$

$y_1' = e^{x} , y_1'' = e^{x} \Rightarrow e^{x} - e^{x} \overset{?}{=} 0$ ✓ Solution

$y_2' = -e^{-x} , y_2'' = e^{-x} \Rightarrow +e^{-x} - e^{-x} \overset{?}{=} 0$ ✓ solution

$y_g = c_1 y_1 + c_2 y_2$ $c_1, c_2 \Rightarrow$ any number. obtain the conditions.

$= c_1 e^{x} + c_2 e^{-x}$

فرضاً $\Rightarrow 4e^{x} + 5e^{-x} \Rightarrow y_p$

نتأكد :-

$y_p' = 4e^{x} - 5e^{-x} , y_p'' = 4e^{x} + 5e^{-x}$

$4e^{x} + 5e^{-x} - 4e^{x} - 5e^{-x} \overset{?}{=} 0$ ✓ $\Rightarrow$ solution.

Theorem 1:-

For a homogeneous (1) linear (2) equation, any linear combination of two solutions on an open interval is again a solution.

let $y'' + p(x)y' + q(x)y = 0$ --- ①

and y_1 and y_2 be solution of ②

$y_g = c_1 y_1 + c_2 y_2$ (linear combination)

انا اشتقتها وعوضتها فيه ① المفروض تحققها

No.

$y'_g = c_1 y' + c_2 y''$, $y''_g = c_1 y''_1 + c_2 y''_2$

$c_1 y''_1 + c_2 y''_2 + P(x) [c_1 y'_1 + c_2 y'_2] + q(x) [c_1 y_1 + c_2 y_2] = 0$

$c_1 [y''_1 + P(x) y'_1 + q(x) y_1] + c_2 [y''_2 + P(x) y'_2 +$
$q(x) y_2] \overset{?}{=} 0$ ✓ الموضوع صفر

Since y_1 and y_2 are solutions then y_g is also a solution.

هل يشترط linea, hom ؟

ما بزبط ع هدول this is not true for a non-homogeneuos or non-linear.

Eample:-

ليس ضمن القاعدة

$y'' + y = 1$ ⇒ 2^{nd}. o.o.D.E linear ✓ but non-homo x

$y_1 = 1 + \cos(x)$, $y_2 = 1 + \sin(x)$

$y'_1 = -\sin(x)$, $y'_2 = \cos(x)$

$y''_1 = -+ \cos(x)$, $y''_2 = -\sin(x)$

$-\cancel{\cos(x)} + 1 + \cancel{\cos(x)} = 1$, $-\cancel{\sin(x)} + 1 + \cancel{\sin(x)} = 1$

$\therefore y_2$ is Solution , y_2 is a Solution

let $y_g =$ is the linear combinations of y_1 and y_2 :-

$y_g = c_1 y_1 + c_2 y_2$

$y_g = c_1 (1 + \cos(x)) + c_2 (1 + \sin(x))$

$y'_g = c_1 (-\sin(x)) + c_2 (\cos(x))$

$y''_g = c_1 (-\cos(x)) + c_2 (-\sin(x))$

$$y'' + y = 1 \Rightarrow -c_2 \cancel{\sin}(x) - c_1 \cancel{\cos}(x) + c_1 + c_1 \cancel{\cos}(x)$$

$$+ c_2 + c_2 \cancel{\sin}(x) \overset{?}{=} 1$$

c_1, c_2 ممكن يكونوا اي ارقام لذلك العبارة ليست دائماً صحيحة.

$$c_1 + c_2 = 1$$

↳ this is not always true becouse c_1 and c_2 are arbitrary constants and can be any number $[-\infty, +\infty]$

In this case the linear combination is not a solution.

non-hom ↵

Example:-

$y''y - xy' = 0 \Rightarrow$ 2^{nd} o.o.D.E non linear x

homogeneous ✓

$y_1 = x^2$, $y_2 = 1$

$y_1' = 2x$ $\quad y_2' = 0$

$y_1'' = 2$ $\quad y_1'' = 0$

$2 * x^2 - x * 2x \overset{?}{=} 0$ $\quad 0 \cdot (1) - x(0) \overset{?}{=} 0$

$2x^2 - 2x^2 = 0$ ✓

$\therefore y_1$ is a solution $\quad y_2$ is a solution

linear combintions $y_g = c_1 y_1 + c_2 y_2$

$$y_g = c_1 x^2 + c_2$$

$$y_g' = 2c_1 x + 0$$

$$y_g'' = 2c_1$$

$$2c_1 \cdot (c_1 x^2 + c_2) - x \cdot (2c_1 x) \overset{?}{=} 0$$

$$2c_1^2 x^2 + 2c_1 c_2 - 2c_1 x^2 \overset{?}{=} 0$$

لو نحط اي ارقام ل c_1 ، c_2

No.

فرضاً $c_1 = 1 \quad , c_2 = 3$

$$2c_1^2 x^2 + 2c_1c_2 - 2c_1x^2 \overset{?}{=} 0$$

$$2y^2 + 6 - \cancel{2}x^2 \overset{?}{=} 0 \quad \Rightarrow \text{not solution.}$$

General solution:- linear combination.

$y_g = c_1y_1 + c_2y_2$ and y_1 and y_2 are called parts of the solutions

c_1 and c_2 are arbitrary constants and can be obtained if we have conditions:-

1) initial value problem
2) Boundary value problem.

نفس الحال ولكن مضروب بعدد. او العكس.

(1) if $\dfrac{y_1}{y_2}$ = constant then y_1 and y_2 are proportional.

(2) if $\dfrac{y_1}{y_2} \neq$ constant then y_1 and y_2 are not proportional.

(3) y_1 and y_2 are called a basis of solution of a differ equation if they are not proprtional (y_1/y_2 or $y_2/y_1 \neq$ constast)

(4) y_1 and y_2 are called linearly ~~in~~dependent if y_1 and y_2 are proportional

$\dfrac{y_1}{y_2}$ or $\dfrac{y_2}{y_1}$ = constant linearly dependent

(5) y_1 and y_2 are called linearly independent if y_1 and y_2 are not proportional $\dfrac{y_1}{y_2}$ or $\dfrac{y_2}{y_1} \neq$ constant liearly indep.

(6) a general solution can be constructed from linearly independent solution only ← 2 لازمن من solution

Example:-

$y'' - y = 0 \qquad y(0) = 5 \quad , \quad y'(0) = 3$

$y_1 = e^x \quad , \quad y_2 = e^{-x}$

check independence :-

$\frac{y_1}{y_2} = \frac{e^x}{e^{-x}} = e^{2x} \neq \text{constant} \Rightarrow y_1$ and y_2 are linearly independent

$y_g = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 e^{-x}$

$5 = c_1 + c_2 \quad --- \boxed{1}$

$y_g' = c_1 y_1' + c_2 y_1'$

$3 = c_1 - c_2 \quad --- \boxed{2}$

$c_1 = 4 \quad , \quad c_2 = 1$

$y_p = 4e^x + e^{-x}$

Example

$y'' - y = 0 \qquad y(0) \quad , \quad y'(0) = 3$

$y_1 = e^x \quad , \quad y_2 = L e^x$

↳ constant.

check

$\frac{y_1}{y_2} = \frac{e^x}{Le^{+x}} \rightarrow \frac{1}{L}$ or $\frac{y_2}{y_1} = L$ = constant

y_1 and y_2 are (linearly dependent)

هاد الحكي غلط $\quad y_g = c_1 y_1 + c_2 y_2 \Rightarrow (c_1 + c_2 L) e^x$

$y_g' = (c_1 + c_2 L) e^x$

$y_g'' = (c_1 + c_2 L) e^x$

No.

$$\Rightarrow (c_1 + c_2 L) e^x - (c_1 + c_2 L) e^x \overset{?}{=} 0 \quad ✓$$

ما بيصير مش ممكن .
$5 = c_1 + c_2 L \quad --- ①$
$3 = c_1 + c_2 L \quad --- ②$

ما نقدر نطلع ال general solution .

و نعيد النظر linear او not

" Reduction of $\underline{\underline{2}}^{nd}$. O.O.D.E to a $\underline{\underline{1}}^{st}$ O.O.D.E "

Case 1:- if in a 2^{nd}. O.O.D.E the dependent variable (y) does not appear expilicitly , then the equation will be of the form $f(x, y', y'') = 0$, then we can redouce this to a 1^{st}.O.O by assuming $y' = \frac{dy}{dx} = z$

Example:-

$$x \frac{d^2y}{dx} - 2 \frac{dy}{dx} = 0$$

ظهور ال x

$$\hookrightarrow f(x, y', y'') = 0$$

$$\text{let} \quad \frac{dy}{dx} = z \quad , \quad \frac{d^2y}{dx^2} = \frac{dz}{dx}$$

$$x . \frac{dz}{dx} - 2z = 0$$

$$x \frac{dz}{dx} = 2z \qquad 1^{st} . O.O.D.E$$

$$\frac{dz}{\cancel{dx}\, z} = \frac{2}{x} dx \quad \Rightarrow \quad \ln z = 2 \ln x + c$$

$$\rightarrow z = c_1 x^2$$

$$\frac{dy}{dt} = c_1 x^2 \Rightarrow \int dy = \int c_1 x^2 \, dx$$

$$y = \frac{c_1}{3} x^3 + c_2 \rightarrow y = c_3 x^3 + c_2$$

Case 2:- if in a second o.o.D.E, the independ variable (x) does not appear explicitly in the D.E, $f(y, y', y'') = 0$ then ~~you have~~ we can assume $y'' = \frac{dz}{dy} \cdot z$ where $z = \frac{dy}{dx}$ to reduce the 2nd o.o.D.E to the 1st o.o.D.E

Example

$$y'' + e^{2y} (y')^3 = 0 \qquad 2^{nd} \text{ o.o.D.E}$$

↳ $f(y'', y, y') = 0$

let $y'' = \frac{dz}{dy} \cdot z \Rightarrow z = \frac{dy}{dx}$

$$\frac{dz}{dy} \frac{z}{z} + e^{2y} \frac{(z)^3}{z} = 0$$

$$\frac{dz}{dy} \cdot \cancel{z} + e^{2y} z^2 = 0$$

$$\frac{dz}{dy} = -e^{2y} z^2 \Rightarrow \int \frac{dz}{z^2} = \int -e^{2y} \, dy$$

$$-\frac{1}{z} = -\frac{1}{2} e^{2y} + c \Rightarrow z = \frac{1}{2e^{2y}} - c$$

$$z = \frac{1}{2e^{2y}} + c_1$$

$$\frac{dy}{dx} = \frac{1}{\frac{1}{2}e^{2y} + c_1} \Rightarrow \int (\tfrac{1}{2} e^{2y} + c_1)\, dy = \int dx + c$$

$$\frac{1}{4} e^{2y} + ay = x + c_2$$

(1) Homogeneous 2^{nd} o.o.D.E with $\overset{(1)}{}$ costna $\overset{(2)}{}$ coefficients:-

$$y'' + a y' + by = 0$$

a,b ⇒ constant

r(x) = 0 Homo.

$$a y'' + b y' + cy = 0$$ العامل لازم دايما.

normally ↓

$$y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$$

let $y = e^{\lambda x}$ where λ is a constant.

$$y' = \lambda e^{\lambda x}$$

$$y'' = \lambda^2 e^{\lambda x} \Rightarrow \lambda^2 e^{\lambda x} + a \lambda e^{\lambda x} + b e^{\lambda x} = 0$$

$$e^{\lambda x} [\lambda^2 + a\lambda + b] = 0$$

≠zero ↲ ↳ must equal zero

now $\lambda^2 + a\lambda + b = 0$ المعادلة المميزة

is called the characteristic equation or

auxiliary equation from which the values of مساعدة.

λ will be obtained

✓

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4c}}{2}$$

$$a\lambda^2 + b\lambda + c = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

non-normallized

$\lambda_1 = -\frac{a}{2} \mp \sqrt{\frac{a^2-4c}{4}}$ positive ✓

$\lambda_2 = -\frac{a}{2} \mp \sqrt{\frac{a^2-4c}{4}}$ $= 0$ ✓

$= - \Rightarrow$ ما بنحلها

depends on a^2-4c there will be 3 different cases:-

Case 1:-

if $(a^2-4c) > 0$, the we have two real distinct root ← حقيقتين مختلفتين

(λ_1 and λ_2 different from each othere then

$y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$ → اذا بعوضهم لازم تحقق المعادلة

$\frac{y_1}{y_2} = \frac{e^{\lambda_1 x}}{e^{\lambda_2 x}} = e^{(\lambda_1 - \lambda_2)x} \Rightarrow \lambda_1 \neq \lambda_2$

$e^{\lambda_3 x}$ which not constant

not proportinal $\Rightarrow$ they are indep.

يعني بنقدر نحل عل هديك الطريقة ←

y_1 and y_2 are independent, the linear combination can be used to write the general solution

$y_g = c_1 y_1 + c_2 y_2 = c_1 e^{\lambda x} + c_2 e^{\lambda_2 x}$

Example:-

$\ddot{y} - y = 0$ 2nd. o.o. D.E with constant cofficiant and homogen

let $y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \Rightarrow (\lambda^2 - 1) e^{\lambda x} = 0$

zero = ↲ ↳ $\neq 0$

$\lambda^2 + 0\lambda + 1 = 0$

$a = 0$, $b = -1$

$\lambda = \frac{-\overset{0}{a} \mp \sqrt{(0)^2 - 4(-1)}}{2} = \pm 1$

$\lambda_1 = +1$, $\lambda_2 = -1$

real distinct roots

$\left.\begin{array}{l} y_1 = e^{\lambda_1 x} = e^{+x} \\ y_1 = e^{\lambda_2 x} = e^{-x} \end{array}\right\}$ $\frac{y_1}{y_2} = \frac{e^{x}}{e^{-x}} = e^{2x} \neq$ constant

linearly ind.

$\therefore y_g = c_1 e^{x} + c_2 e^{-x}$

↳ general solution

Example:-

$y'' + 6y' + 9y = 0$ 2nd homo - constant coff.

let $y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

$\Rightarrow \lambda^2 e^{\lambda x} + 6\lambda e^{\lambda x} + 9 e^{\lambda x} = 0$

$e^{\lambda x} (\lambda^2 + 6\lambda + 9) = 0$

↳ zero

$\lambda = \frac{-6 \mp \sqrt{36 - 4\times 9}}{2} = \frac{-6 \mp 0}{2} = -3$

$\lambda_1 = \lambda_2 = -3$ repeated roots.

Case 2:- ↲

$\left.\begin{array}{l} y_1 = e^{\lambda x} = e^{-3x} \\ y_2 = e^{\lambda x} = e^{-3x} \end{array}\right\}$ $\frac{y_1}{y_2} = \frac{e^{-3x}}{e^{-3x}} = 1$

depend ↲

↳ no linear combaination,

Example:-

$\ddot{y} + 2.2\dot{y} + 0.4y = 0.$ Homo of constant coff.

let $y = e^{\lambda x}$

$'y = \lambda^2 e^{\lambda x}$

$''y = \lambda^2 e^{\lambda x}$

$$\lambda^2 e^{\lambda x} + 2.2\,\lambda e^{\lambda x} + 0.4\,e^{\lambda x} = 0$$

$$e^{\lambda x}\,(\lambda^2 + 2.2\,\lambda + 0.4) = 0$$

$\hookrightarrow$ = zero

$$\lambda = \frac{-2.2 \pm \sqrt{(2.2)^2 - 4 * .4}}{2}$$

$\left.\begin{aligned} \lambda_1 &= -.2 \\ \lambda_2 &= -2 \end{aligned}\right\}$ real distinat Values

$\therefore y_g = a\,e^{-.2x} + c_2\,e^{-2x}$ general Solution.

شكله ما في depends وماله اشتقاق

~~linear com~~ ~~بينهم~~

Case 2:- if $(\cancel{b}\ a^2 - 4b) = 0$

$\lambda_1 = \lambda_2 = \frac{-a}{2}$, then $y_1 = e^{-\frac{a}{2}x}$ but $y_2 \neq e^{-\frac{a}{2}x}$

How do we obtain another part of solution (independnt)

let $y_2 = u y_1$ where $y_1 = e^{-\frac{a}{2}x}$, u = another indep. variable.

Solution

function only(x)

$f(x)$

Derivation with x

بدنا نطلع قيمة الـ u

$'y_2 = u\,'y_1 + 'u\,y_1$

$''y_2 = u\,''y_1 + 'u\,'y_1 + 'u\,'y_1 + ''u\,y_1$

$''u\,'y_1 + 2\,'u\,'y_1 + ''u\,y_1$

$\Rightarrow$ Substitute these derivatives in $y'' + ay' + by = 0$

$$u y_1'' + 2 u' y_1' + u'' y_1 + a (u y_1' + u' y_1) + b (u y_1) = 0$$

$$y_1 u'' + \{ 2 y_1' + a y_1 \} u' + \{ y_1'' + y_1' a + b y_1 \} u = 0$$

$\neq$ Zero ↲ ↳ $\neq$ Zero

موجود بـ ال orginall

لأن y_1 solution.

↳ مش بالأصل صفر.

$\boxed{*}$ $y_1 = e^{-\frac{a}{2}x}$

$y_1' = -\frac{a}{2} e^{-a/2 x}$

$y_2'' = \frac{a^2}{2} e^{-\frac{a}{2}x}$

$y_1'' + a y_1' + by = 0$ (المعادلة الأصلية)

$$\frac{a^2}{4} e^{-\frac{a}{2}x} + a \left(-\frac{a}{2} \right) e^{-\frac{a}{2}x} + b\, e^{-\frac{a}{2}x} \overset{?}{=} 0$$

$$e^{-\frac{a}{2}x} \left(\frac{a^2}{4} - \frac{a^2}{2} + b \right\} \overset{?}{=} 0$$

$a^2 - 4b = 0$

$b = \frac{a^2}{4}$

<<هاد الشرط>>

$$\left\{ \frac{a^2}{4} - \frac{a^2}{2} + \frac{a^2}{4} \right\} = ✓ = 0$$

↳ لذلك لا نفرضها فوق انها صفر.

$\boxed{*}$ $2 y_1' + a y_1 \overset{?}{=} 0?$ $\quad y_1 = e^{-\frac{a}{2}x}$, $y_1' = -\frac{a}{2} e^{-\frac{a}{2}x}$

$$2 \left(-\frac{a}{2} e^{-a/2 x} \right) + a\, e^{-\frac{a}{2}x} \overset{?}{=} \text{Zero} \Rightarrow (-a + a)\, e^{-\frac{a}{2}x} \overset{?}{=} 0$$

* $y_1 u'' = 0 \Rightarrow u'' = 0 \Rightarrow \frac{d^2u}{dx^2} = 0 \rightarrow \int d^2u = \int 0\, dt$

$y_1 \neq 0$ ↲

$\Rightarrow \int du = \int c_1 dt + c_2 \Rightarrow u = c_1 x + c_2 + c_3$

$\therefore u = c_1 x + c_4$

Substitue back in $y_2 = u y_1 \rightarrow y_2 = (c_1 x + c_4) y_1$

No.

$$y_g = Ay_1 + By_2$$

لازم بس $= Ay_1 + B(c_1 x + c_2)y_1$

$$\underset{\sim}{A}y_1 + Bc_1 x + \underset{\sim}{Bc_4}\, y_1$$

~~[illegible]~~ لنفترد التحكم

$$A + Bc_4 = L$$

$$Bc_1 = M$$

$$y_g = \overset{c_1}{L}y_1 + \overset{c_2}{M}\,x y_1$$

$$\boxed{(L + Mx)\, e^{-\frac{9}{2}x}} \quad \text{general Solution}$$

هاد الحل دائماً متبع.

$$y_1 = L e^{-\frac{9}{2}x}$$

$$y_2 = M\, x\, e^{-\frac{9}{2}x}$$

$$\frac{y_1}{y_2} = \frac{L \cancel{e^{-\frac{9}{2}x}}}{M \cancel{e^{-\frac{9}{2}x}}} = \frac{L}{M \cancel{x}} \neq \text{constant} \quad \therefore y_2 \text{ is indepdent on } y_1.$$

$\frac{L}{M}$

Example:-

$$\ddot{y} + 6\dot{y} + 9y = 0 \quad , \text{ let } y = e^{\lambda x}$$

$\vdots$

$$(\lambda^2 + 6\lambda + 9)\, e^{\lambda x} = 0 \Rightarrow \lambda^2 + 6\lambda + 9 = 0$$

$$\lambda = \frac{-6 \mp \sqrt{0}}{2} = -3 \quad \therefore \lambda_1 = \lambda_2 = -3$$

repeted root.

$$y_1 = e^{-3x} \quad , \quad y_2 = x e^{-3x}$$

$$y_g = (c_1 + c_2 x)\, e^{-3x} \quad \text{general solution.}$$

Example:-

$\ddot{y} + 4\dot{y} + 4y = 0$ Homo. of constant coffi

let $y = e^{\lambda x}$

$\dot{y} = \lambda e^{\lambda x}$

$\ddot{y} = \lambda^2 e^{\lambda x}$

$(\lambda^2 + 4\lambda + 4) e^{\lambda x} = 0$

zero ↲ ↳ ≠ zero

$\lambda = \frac{\mp 4 \mp \sqrt{16 - 4.4}}{2} = 2$ $\lambda_1 = \lambda_2 = 2$ repeted roots

$y_1 = e^{2x}$, $y_2 = x e^{2x}$

$y_g = (c_1 + x c_2) e^{2x}$

Case 3 : if $a^2 - 4b < 0$ (-ve)

$$\lambda = \frac{-a \mp \sqrt{a^2 - 4b}}{2} = \frac{-a}{2} \mp \sqrt{\frac{a^2 - 4b}{4}} \rightarrow -ve$$

$$\lambda = \frac{-a}{2} \mp \sqrt{\frac{(-1)(4b - a^2)}{4}} = \frac{-a}{2} \mp \underbrace{\sqrt{\frac{4b - a^2}{4}}}_{\omega} \cdot \underbrace{\sqrt{-1}}_{i}$$

+ve

$i^2 = -1$

$\lambda = \frac{-a}{2} \mp \omega i$

$\lambda_1 = \frac{-a}{2} + \omega i$ $\Rightarrow$ for $y_1 = e^{\lambda_1 x} = e^{(\frac{-a}{2} + \omega i)x} = e^{\frac{-a}{2}x + \omega i x} \cdot e$

$\lambda_2 \; \frac{-a}{2} - \omega i$ $f y_2 = e^{\lambda_2 x} = e^{(\frac{-a}{2} - \omega i)x} = e^{\frac{-a}{2}x} \cdot e^{-\omega i x}$

$$y_g = c_1 [e^{\frac{-a}{2}x} \cdot e^{\omega i x}] + c_2 [e^{\frac{-a}{2}x} \cdot e^{-\omega i x}]$$

No.

$$y_g = e^{-\frac{9}{2}x}\left[c_1 e^{wix} + c_2 e^{-wix}\right]$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{if} \quad \theta = wx$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

* اذا طلب drivation

بدنا نعمل كل هاد

ما طلب ← اذا فقط الجواب

$$e^{wxi} = \cos(wx) + i\sin(wx)$$
$$e^{-wix} = \cos(wx) - i\sin(wx)$$

$$y_g = e^{-\frac{9}{2}x}\left[c_1\{\cos(wx) + i\sin(wx)\} + c_2\{\cos(wx) - i\sin(wx)\}\right]$$
$$= e^{-\frac{9}{2}x}\left[\underline{c_1\cos(wx)} + c_1 i\sin(wx) + \underline{c_2\cos(wx)} - c_2 i\sin(wx)\right]$$

$$= e^{-\frac{9}{2}x}\left[\underbrace{(c_1+c_2)}_{A}\cos(wx) + \underbrace{(c_1-c_2)i}_{B}\sin(wx)\right]$$

$$y_g = e^{-\frac{9}{2}x}\left[A\cos(wx) + B\sin(wx)\right]$$

Example:-

$$\ddot{y} + 2\dot{y} + 10y = 0 \quad , \text{let} \quad y = e^{\lambda x}$$

$$\left[\lambda^2 + 2\lambda + 10\right]e^{\lambda x} = 0$$

$$\lambda = \frac{2 \mp \sqrt{4-40}}{2} = \frac{--2 \mp \sqrt{-36}}{2}$$

$$\lambda = \frac{2}{2} \mp \sqrt{\frac{36}{4}}\,.\,i = 1 \mp 3i$$

real part

$$\lambda_1 = \textcircled{1} + 3i \qquad \text{Real} = -\frac{9}{2} = 1 \; , \; w = 3$$
$$\lambda_2 = 1 - 3i$$

$y_g = e^{-\frac{q}{2}x} \left[c_1 \cos(wx) + c_2 \sin(wx) \right]$

$y_g = e^{\omega x} \left[c_1 \cos(3x) + c_2 \sin(3x) \right]$

Example :-

$y'' + 2y' + 5y = 0 \quad , \; y(0) = 1 \; , \; y'(0) = 5$

$\Rightarrow$ Homo & constant coff

let $y = e^{\lambda x}$

$'y = \lambda e^{\lambda x}$

$''y = \lambda^2 e^{\lambda x}$

$(\lambda^2 + 2\lambda + 5) e^{\lambda x} = 0$

$$\lambda = \frac{-2 \mp \sqrt{4-20}}{2} = -1 \mp 2i$$

$\lambda_1 = -1 + 2i \; , \; \lambda_2 = -1 - 2i \; , \;$ real $-\frac{q}{2} = -1 , \; w = 2$

$y_g = e^{-x} \left[c_1 \cos(2x) + c_2 \overset{\sin}{\cancel{\cos}}(2x) \right]$

$1 = \overset{0}{e} \left[c_1 \overset{1}{\cos(0)} + c_2 \cancel{\sin(0)} \right]$

$c_1 = 1$

$y_g' = -e^{-x} \left[\cos(2x) + c_2 \sin(2x) \right] + e^{-x} \left[-2\sin(2x) + 2c_2 \cos(2x) \right]$

$5 = -1 \left[\overset{1}{\cos(0)} + c_2 \cancel{\sin(0)} \right] + 1 \left[-2 \cancel{\sin(0)} + 2c_2 \overset{1}{\cos(0)} \right]$

$5 = -1 + 2c_2 \Rightarrow c_2 = 3$

$y_{gp} = e^{-x} \left[\cos(2x) + 3 \sin(2x) \right]$

No.

(2) 2^{nd}. o.o.D.E Homogeneous with variable coefficients.

لـ مش كل شي رح نحلو

$$\underline{x^2}y'' + \underline{ax}y' + \underline{b}y = 0 \quad \Rightarrow \quad \text{Euler - Cauchy equation.}$$

coeff ↲

let $y = x^m$ where m is a ~~fixed~~ constant

$$y' = m\,x^{(m-1)} \quad , \quad y'' = m(m-1)\,e^{m-2} = (m^2-m)\,x^{m-2}$$

Substitute

y, y', and y'' into Euler equation.

$$x^2\left[(m^2-m)\,x^{m-2}\right] + a\cancel{x}\,m\,x^{m-1} + b\,x^m = 0$$

$$\underbrace{x^m \cdot \cancel{x^{-2}}} \qquad \underbrace{x^m \cdot \cancel{x^{-1}}}$$

$$\Rightarrow (m^2-m)\,x^m + a\,m\,x^m + b\,x^m = 0$$

$$[m^2 - m + am + b]\,x^m = 0$$

$$[m^2 + (a-1)m + b]\,x^m = 0$$

$= \text{Zero}$ ↲ ↳ will never be Zero.

$\therefore$ $m^2 + (a-1)m + b = 0$ is the characteristic equation

$$m = \frac{-(a-1) \mp \sqrt{(a-1)^2 - 4b}}{2}$$

Case 1:- if $(a-1)^2 - 4b >$ Zero , $\overset{m_1}{\cancel{\lambda_1}} = \frac{-(a-1)}{2} + \sqrt{\frac{(a-1)^2-4b}{4}}$

⇒ two real distinct root. $\overset{m_2}{\cancel{\lambda_2}} = \frac{-(a-1)}{2} - \sqrt{\frac{(a-1)^2-4b}{4}}$

then $y_1 = x^{m_1} = x^{[\;\;]}$

$y_2 = x^{m_2} = x^{[\;\;]}$

$$\frac{y_1}{y_2} = \frac{x^{m_1}}{x^{m_2}} = x^{m_1-m_2} \qquad (m_1 - m_2) \neq 0$$

$\therefore$ independent Solution.

$$y_g = c_1 y_1 + c_2 y_2 .$$

Example:-

$x^2 y'' - 2.5 y' - 2y = 0$ Eulear Equation.

let $y = \cancel{m^x} x^m$, $y' = m x^{m-1}$, $y'' = (m^2 - m) x^{m-2}$

لـ عوض رح يطلع :-

$$\Rightarrow [m^2 - (2.5 - 1) m + (-2)] x^m = 0$$

$$[m^2 - 3.5m - 2] x^m = 0$$

$$m = \frac{3.5 \mp \sqrt{(3.5)^2 - 4(-2)}}{2} = \frac{3.5}{2} \mp \sqrt{\frac{(3.5)^2 + 8}{4}}$$

$m_1 = 4$, $m_2 = -0.5$

$$\left.\begin{array}{l} y_1 = x^m = x^4 \\ y_2 = x^{m_2} = x^{-0.5} \end{array}\right\} \frac{y_1}{y_2} = \frac{x^4}{x^{-.05}} = x^{4-0.5} \neq \text{constant}$$

$\therefore$ independent.

$$y_g = c_1 y_1 + c_2 y_2 = c_1 x^4 + c_2 x^{-0.5} = c_1 x^4 + \frac{c_2}{\sqrt{x}}$$

Case 2:- if $(a-1)^2 - 4b = \text{zero}$, repeated roots $\overset{m_1}{\cancel{x}} = \overset{m_2}{\cancel{x}} = \frac{1-a}{2}$

$y_1 = x^{m_1} = x^{m_2} = x^{\frac{1-a}{2}}$, we can't obtain an independent y_2 from the same root.

let $y_2 = u y_1$, $y_2' = u y_1' + u' y_1$

$y_2'' = u y_1'' + u' y_1' + u' y_1' + u'' y_1$

$y_2'' = u y_1'' + 2u' y_1' + u'' y_1$

Substitute in Euler equation :-

$x^2 y'' + a x y' + b y = 0$

$x^2\,[u''y_1 + \underline{2u'y_1'} + \underline{u y_1''}] + ax\,[\underline{u'y_1} + \underline{u y_1'}] + \underline{b y_1 u} = 0$

$x^2 u'' y_1 + [2 x^2 y_1' + axy_1]\,u' + [x^2 y_1'' + a\cancel{x y_1'} + b y_1]\,u = 0$

لـ نتأكد اذا = صفر.

↳ = zero.

because y_1 is a solution.

* $y = x^{\frac{1-a}{2}}$, $y' = \frac{1-a}{2}\; x^{-\frac{1-a}{2}}$

$x^{\frac{1-a}{2}} \cdot x^{\cancel{-1}} \sim x^{\frac{1-a}{2}} \cdot x^{-\frac{2}{2}}$

$\cancel{2}\, x^{\cancel{2}\,1} \left(\frac{1-a}{\cancel{2}}\right) . x^{-\frac{1-a}{2}} + a x\; x^{\frac{1-a}{2}}$

$= (1-a)\, x .\, x^{\frac{1-a}{2}} + a x . x^{\frac{1-a}{2}}$

$= (1)\, x\; x^{\frac{1-a}{2}} - a\,\cancel{x}.\, x^{\frac{1-a}{2}} + a\cancel{x}\; x^{\frac{1-a}{2}}$

$\Rightarrow\; [2 x^2 y_1' + a x y_1] = x y_1$

$\underset{\sim\sim\sim\sim\sim\sim\sim\sim}{x^2 u'' y_1 + x y_1 u' = 0} \;\rightarrow\; y_1 (x^2 u'' + x u') = 0 \;\rightarrow\; y_1 \neq 0$

$\Rightarrow\; u'' x^2 + x u' = 0 \;\Rightarrow\; f(x, u'', u') = 0$ u is missing

to solve this equation

let $z = u' \;\Rightarrow\; z' = u''$

$x^2 z' + xz = 0$ (reduced to a 1st o.o.D.E)

$\cancel{x}^2 z' = -\cancel{x} z \;\Rightarrow\; x\frac{dz}{dx} = -z \;\Rightarrow\; \int \frac{dz}{z} = \int \frac{-dx}{x}$

$\Rightarrow\; \ln(z) = -\ln(x) + c$

$z = c\,\frac{1}{x} \;\Rightarrow\; \frac{c}{x} \;\Rightarrow\; \frac{du}{dx} = \frac{c}{x}$ seperable

$\int du = \int \frac{c}{x}\,dx + c_2 \;\Rightarrow\; u = c \ln x + c_2$

NO.

$$\therefore y_2 = 4 y_1$$

$$= (c_1 \ln x + c_2)\, y_1$$

$$y_g = A y_1 + B y_2$$

$$= A \cancel{y} x^{\frac{1-a}{2}} + B\,[4 \ln x + c_2]\, x^{\frac{1-a}{2}}$$

$$L \quad = \underset{\sim}{A}\, x^{\frac{1-a}{2}} + \underbrace{B c_1}_{N} \ln x + \underset{\sim}{B c_2}\, x^{\frac{1-a}{2}}$$

$$= L\, x^{\frac{1-a}{2}} + N \ln x \cdot x^{\frac{1-a}{2}}$$

$$= [L + N \ln x]\, x^{\frac{1-a}{2}}$$

⇒ for a repeated case $m_1 = m_2$

$$y_g = [[L + M \ln(x)]\, x^{m_1}]$$

Example:-

$x^2 y'' - 3 x y' + 4 y = 0$ Eulear + Homo

let $y = x^m$

$$[m^2 + (a-1)m + b]\, x^m = 0$$

$$[m^2 + (-3-1)m + 4]\, x^m = 0$$

$$[m^2 - 4m + 4]\, x^m = 0$$

$$m = \frac{4 \pm \sqrt{(4)^2 - 4(4)}}{2} = 2 \qquad \therefore m_1 = m_2 = 2$$

$y_g = [L + M \ln(x)]\, x^2$ → general solution.

No.

Case 3 : if $(a-1)^2 - 4b < \text{zero}$, imaginary roots

$$m = \frac{1-a}{2} \mp \sqrt{\frac{(a-1)^2 - 4b}{4}} = \frac{1-a}{2} \mp \sqrt{\frac{(-1)(4b-(a-1)^2)}{4}}$$

$$m = \frac{1-a}{2} \mp \sqrt{-1} \cdot \sqrt{\frac{4b-(a-1)^2}{4}} \xrightarrow{} +ve$$

$$\mu = \frac{1-a}{2} \quad \text{and} \quad v = \sqrt{\frac{4b-(a-1)^2}{4}}$$

$$\therefore m_1 = \mu + iv \;, \quad m_2 = \mu - iv$$

$$y_1 = x^{m_1} = x^{m+iv} = x^{m} \cdot x^{iv}$$

$$y_2 = x^{m_2} = x^{\mu - iv} = x^{\mu} \cdot x^{-iv}$$

$$x^{iv} = \cos(v \ln x) + i \sin(v \ln x)$$

$$x^{-iv} = \cos(v \ln x) - i \sin(v \ln x)$$

$$y_1 = x^{m} \cdot x^{iv} = \cancel{m} x^{\mu} [\cos(v \ln x) + i \sin(v \ln x)]$$

$$y_2 = x^{\mu} \cdot x^{-iv} = x^{\mu} [\cos(v \ln x) - i \sin(v \ln x)]$$

$$y_g = c_1 y_1 + c_2 y_2$$

$$= c_1 [x^{\mu} (\cos(v \ln x) + i \sin(v \ln x)] + c_2 [x^{\mu} (\cos(v \ln x) - i \sin(v \ln x)]$$

$$y_g = x^{\mu} [c_1 \cos(v \ln x) + c_1 i \sin(v \ln x) + c_2 \cos(v \ln x) - c_2 i \sin(v \ln x)]$$

$$y_g = x^{m} [\underbrace{(c_1 + c_2)}_{A} \cos(v \ln x) + \underbrace{(c_1 - c_2) i}_{B} \sin(v \ln x)]$$

$$y_g = x^{\mu} [A \cos(v \ln x) + B \sin(v \ln x)]$$

No.

Example:-

$x^2 y'' + 7xy' + 13y = 0 \Rightarrow$ Huler

let $y = x^m$

$y_1 = m x^{(m-1)}$

$''y_2 = (m^2-m) x^{m-2}$

$\cancel{x^2} (m^2-m) x^m \cdot \cancel{x^{-2}} + 7\cancel{x} \cdot m x^m \cdot \cancel{x^{-1}} + 13 x^m = 0$

$x^m [m^2 - m + 7m + 13]$

$[m^2 + 6m + 13] x^n = 0$

$m = \frac{-6 \mp \sqrt{36-52}}{2}$ $u = \frac{-6}{2} = -3, v = \sqrt{\frac{52-36}{4}} = 2$

$\therefore m = u \mp vi$ $m_1 = -3+2i$

$= -3 \mp 2i$ $m_2 = -3-2i$

General Solution $= x^m [A \cos(v \ln x) + B \sin(v \ln x)]$

$= x^{-3} [A \cos(2 \ln x) + B \sin(2 \ln x)]$

Example

$4x^2 y'' + 8xy' - 15y = 0 \Rightarrow$ not Normallize

$x^2 y'' + 2xy' - \frac{15}{4} y = 0$ $a = 2$

$b = -\frac{15}{4}$

$\Rightarrow [m^2 + (a-1)m + b] x^u = 0$

$[m^2 + (2-1)m - \frac{15}{4}] x^m = 0$

$[m^2 + m - \frac{15}{4}] m x^m = 0$

$m = \frac{-1}{2} \mp \sqrt{\frac{(1)^2 + 4(\frac{15}{4})}{4}} \Rightarrow \frac{-1 \mp 4}{2}$ $m_1 = -2.5$

$m_2 = 1.5$

$y = c_1 x^{1.5} + c_2^{-2.5}$

Example:-

$x * \quad x y'' + 4 y' = 0 \quad \Rightarrow$ Huler

$$x^2 y'' + 4 x y' + 4 \cdot (0) y = 0$$

$$[m^2 + (4-1) m + 0] \, x^m = 0$$

$$[m^2 + 3m + 0] \, x^m = 0$$

$$m = \frac{-3 \mp \sqrt{9}}{2} = -1.5 \mp 1.5$$

$$m_1 = 0 \quad , \quad m_2 = -3$$

$y_1 = x^0 = 1 \quad , \quad y_2 = x^{-3}$

$y_g = c_1 + c_2 x^{-3}$

رونس كان

Uniqueness Theory (Wronskian)

$y_g = c_1 y_1 + c_2 y_2$

في بعض الحالات يوجد اكثر من basis solution

نستخدم هذه الطريقة لمعرفة اذا بعطيت او لا

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

2x2 square matrix

if $w = \text{zero}$, then the y_1 and y_2 are dependent ⇒ (x) general

if $w \neq \text{zero}$, then y_1 and y_2 are linearly indepedent ⇒(✓) general

Reduction of order :-

$$\dot{y} + \underset{\sim}{P(x)}\,\acute{y} + \underset{\sim}{q(x)}\,y = 0 \Rightarrow \text{not Huler}$$

not constant and not Hular ⇒ any fuction.

و باي طريقة هيتحلها

if y_1 is Known , Then one can get a 2nd basis of solution by leting. $y_2 = u y_1$

$$\acute{y}_2 = u\acute{y}_1 + \acute{u}y_1 \quad , \quad \ddot{y}_2 = u\ddot{y}_1 + \underbrace{\acute{u}\acute{y}_1 + \acute{u}\acute{y}_1}_{2\acute{u}\acute{y}_1} + \ddot{u}y_1$$

$$\searrow \quad \underline{u\ddot{y}_1} + 2\acute{u}\acute{y}_1 + \ddot{u}y_1 + P(x)[\underline{u\acute{y}_1} + \acute{u}y_1] + \underline{q(x)uy_1} = 0$$

نقسم على ,

$$y_1\ddot{u} + [2\acute{y}_1 + P(x)y_1]\acute{u} + [\ddot{y}_1 + \cancel{P(x)}\acute{y}_1 + q(x)y_1]\overset{u}{\cancel{y}} = 0$$

↳ becaus y_1 is solution

$$\ddot{u} + \left[\frac{2\acute{y}_1}{y_1} + P(x)\right]\acute{u} = 0$$

$f(\ddot{u}, \acute{u}, x) = 0$ $\quad u$ is missing

let $Z = \frac{du}{dx}$, $\frac{dZ}{dx} = \frac{d^2u}{dx^2}$

$$\searrow \frac{dz}{dx} + \left[\frac{2\acute{y}_1}{y_1} + P(x)\right]z = 0 \qquad 1^{st}.\text{O.O.D.E in } z.$$

$$\frac{dz}{dx} = -\left[\frac{2\acute{y}_1}{y_1} + P(x)\right]z$$

$$\Rightarrow \int \frac{dz}{z} = \int \left[-\frac{2y_1'}{y_1} - P(x) \right] dx + c$$

$$\ln z = \int -\frac{2y_1'}{y_1}\, dx - \int P(x)\, dx = -2 \ln y_1 - \int P(x) dx$$

$$\Rightarrow \boxed{z} = \frac{1}{y_1^2} \cdot e^{-\int P(x) dx}$$

$$\frac{du}{dx} = \frac{1}{y_1^2} \cdot e^{-\int P(x) dx}$$ seperable.

$$du = \left[\frac{1}{y_1^2} \cdot e^{-\int P(x) dx} \right] dx$$

$$u = \int \frac{e^{-\int P(x) dx}}{y_1^2}\, dx$$

$$\therefore\ y_2 = u y_1 = \left(\int \frac{e^{-\int P(x) dx}}{y_1^2}\, dx \right) \cdot y_1$$

Example:-

$$(x^2-1) y'' - 2x y' + 2y = 0$$ not Hular, not constant eff.

$y_1 = x$ is a solution.

$$y'' - \frac{2x}{x^2-1} y' + \frac{2}{x^2-1} y = 0$$

✓ $P(x) = \frac{-2x}{x^2-1}$, $q(x) = \frac{2}{x^2-1}$

$y_1 = x$, $y_1' = 1$, $y_2'' = 0$

$(x^2-1)(0) - 2x(1) + 2(x) = 0 \Rightarrow$ ✓ y_1 is a slowtion.

To find the second part (y_2) is independente

$y_2 = u\, y_1$

$$u = \int \frac{e^{-\int p(x)dx}}{y_1^2} \, dx$$

$$\int P(x)\,dx = \int -\frac{2x}{x^2-1} = -\ln(x^2-1)$$

$$u = \int \frac{e^{\ln(x^2-1)}}{x^2} \cdot dx = \int 1 - \frac{1}{x^2} \cdot dx$$

$$u = x + \frac{1}{x}$$

$$y_2 = u y_1 = [x + \frac{1}{x}] \cdot x = x^2 + 1$$

2nd part of solution

$y_1 = x$, $y_2 = x^2 + 1$

$$w = \begin{vmatrix} y_1 & y_2 \\ \acute{y} & \acute{y}_2 \end{vmatrix} = \begin{vmatrix} x & x^2+1 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 - 1 = x^2 - 1 \neq \text{Zero}$$

$\therefore y_1$ and y_2 are independent.

$$y_g = c_1 x + c_2 (x^2 + 1)$$

2^{nd} O.O.D.E Non-homogeneous

$$y'' + P(x)y' + q(x)y = r(x) \quad \Rightarrow \quad r(x) \neq 0$$

1- $P(x)$ and $q(x)$ → constant

احنا بس اخدنا ال Huler

2- $P(x)$ and $q(x)$ → arbitrary function of x [Huler]

A general solution of the non-homogeneous equation is a solution of the form $y_g = y_h + y_p$

y_h → is the solution of the homo-D.E $y'' + P(x)y' + q(x) = 0$

$y_h = c_1 y_1 + c_2 y_2$

« 6 cases »

y_p → any solution of the D.E (Non-hom) containing no arbitrary constant, and this part of the general solution is called a particular solution (because no arbitrary)

y_p can be obtain by:-

تعتمد على حدود معينة اذا مش موجودة نحل على التانية ،

1) undetemined coefficients (special cases)

2) Variation of parameters (General)

→ Solution by undetermined coefficient:-

we used applies to equation of the form

$$y'' + ay' + by = r(x)$$

1) constant coefficients

2) special forms of $r(x)$ مش أي $r(x)$

No.

$r(x)$ ↓

⇒ (exponential , polynomiale , cosine, sine ... or sum or product of such function)
↳ يا جمع يا ضرب

Rules of the method:-

A) Term in $r(x)$	choice of y_p
Constant in the function ⤳ $k\,e^{\nu x}$	Ⓒ $e^{\nu x}$, c is the undetermined Coeffi
$k\,x^{n}$ ($n = 0, 1, 2, 3$)	بنضل ننزل لحد الصفر Ⓐ x^{n} + ③ x^{n-1} + ... + Ⓜ x^{0} ↳ undetermined coffc
$k\cos(\omega x)$ $k\sin(\omega x)$	Ⓐ $\cos(\omega x)$ + Ⓑ $\sin(\omega x)$
$k\,\underbrace{e^{\nu x}}_{exp}\,\underbrace{\cos(\omega x)}_{cos}$	$1\,e^{\nu x}\,[A\cos(\omega x) + B\sin(\omega x)]$ $e^{\nu x}\,[M\cos(\omega x) + N\sin(\omega x)]$
$k\,\underbrace{e^{\nu x}}_{exp}\,\underbrace{[x^{2} + 3x]}_{poly}$	$L\,e^{\nu x}\,[A x^{2} + B x + C]$ $[M x^{2} + N x + Z C]\,e^{\nu x}$

Example:-

(1) $y'' + 4y = 8x^{2}$ Constant coe of non-homo
↳ $r(x)$ poly

$y_g = y_h + y_p$

$y_h \Rightarrow y'' + 4y = 0$ (Homo. parts) of constant.

let $y = e^{\lambda x}$

No.

$[\lambda^2+4]\,e^{\lambda x}=0$

$\lambda^2+4=0 \rightarrow \lambda=-4$ $\qquad -\frac{a}{2}=0$, $w=2$

$\lambda=0\mp 2i$ $\qquad \sim$

$y_h = e^{-\frac{a}{2}x}\,[A\cos(wx)+B\sin(wx)]$

$= e^{\cdot}\,[A\cos(2x)+B\sin(2x)]$

To find y_p:-

Let $y_p = c_1x^2+c_2x^{1}+c_3x^{0}$ $\qquad c_1, c_2, c_3$ are

$r(x)=8x^2$ $\qquad$ undeterumined coeffici

we find the values of c_1, c_2, c_3

by subsitituting y_p into the differentianed

Equation

$y_p = c_1x^2+c_2x+c_3$

$y'_p = 2c_1x+c_2$

$y''_p = 2c_1$

Substitute into the $y''+4y=8x^2$

$2c_1+4[c_1x^2+c_2x+c_3]=8x^2$

$2c_1+\underline{4c_1x^2}+\underline{4c_2x}+4c_3=8x^2 \curvearrowright \underline{8x^2}+\underline{0x}+0 \Rightarrow$

$\rightarrow 4c_1x^2=8x^2 \Rightarrow 4c_1=8 \Rightarrow c_1=2$

$\rightarrow 4c_2x=0x \Rightarrow 4c_2=0 \Rightarrow c_2=0$

$2c_1+4c_3=0 \Rightarrow 4c_3=-2c_1=-4 \Rightarrow c_3=-1$

$y_p = 2x^2-1$ (it has no arbitrary constant)

↲ عشان ما يصير في الحل أكثر من constant وحيد فقط

2nd equation.

No.

$\therefore y_g = y_h + y_p$

$= A\cos(2x) + B\sin(2x) + (2x^2-1)$

• لما تكون فيه conditions بعوضه فيه y_g
وليس فيه ال y_h.

في الحالة مع constant coefficient

B) if aterm in your choice (y_p) happens to be asolutions of the homogeneous part of the solution (y_h), then multiply your choice of y_p by (x) or x^2 or x^n, it his solution corresponds to double roots.

$y_h = (c_1 + c_2 x) e^{\lambda x}$

ما يكون فيه تكرار ضربنا بـ x لما تكرارين بـ x^2 وهكذا

* بعد اما اطلع y_h و y_p
اذا فيه اي تكرار فيه y_p
كان صحح

$y'' - 3y' + 2y = e^{x}$ ↝ $r(x)$ non-homo ⇒ undetermined & constant coefficient.

لان من الحالات المذكورة.

$y_g = y_h + y_p$

$y'' - 3y' + 2y = 0$ homo parts & constant

let $y = e^{\lambda x}$

$[\lambda^2 - 3\lambda + 3] e^{\lambda x} = 0$

$\lambda = \dfrac{3 \mp \sqrt{9-8}}{2} = \dfrac{3 \mp 1}{2}$ $\lambda_1 = 2$, $\lambda_2 = 1$

$y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$= c_1 e^{2x} + c_2 e^{x}$

No.

for $y_p \Rightarrow r(x) = e^x$ exp

let $y_p = Ae^x$ (modify) ← لا نقبل الحل لأن هذا الجزء مكرر في

underter ↲ ال homo parts.

$y_p = Axe^x$ → ليس مكرر ؟ نعم

(accepted) اذاً لا ترفض كمان مرة.

$y_p' = Ae^x + Axe^x$

$y_p'' = Ae^x + Ae^x + Axe^x$

$= 2Ae^x + Axe^x$ Substitue in $y'' - 3y' + 2y = e^x$

$2Ae^x + Axe^x - 3[Ae^x + Axe^x] + 2Axe^x = e^x$

$2Ae^x + \cancel{A}xe^x - 3Ae^x - 3\cancel{A}xe^x + 2\cancel{A}xe^x = e^x$

$2Ae^x - 3Ae^x = e^x$

$(2-3)Ae^x = e^x$

$-Ae^x = 1e^x \Rightarrow -A = 1 \Rightarrow A = -1$

$y_p = -xe^x$

$y_g = c_1e^{2x} + c_2e^x - xe^x$ → لو طلعنا ال wronskin رح يكون

الجواب $\neq 0$ اذا هاذ هو ال general

solution.

1) undetiermined coefficients:-

$\dot{y} + a y' + by = \underset{\frown}{r(x)}$ special function.

$y_g = y_h + y_p$

↳ no arbitrary.

Example:-

$\dot{y} - 2y' + y = e^x + x$

$r(x) \Rightarrow Ae^x + c_1 x + c_2$

For y_h = $\ddot{y} - 2\dot{y} + y = \text{zero}$ homo + constant.

Let $y = e^{\lambda x}$

$(\lambda^2 - 2\lambda + 1) e^{\lambda x} = 0$

$$\lambda = \frac{2 \pm \sqrt{4-4}}{2} = 1$$

$y_h = (c_1 + c_2 x) e^x = c_1 e^x + c_2 x e^x$

for $y_p \Rightarrow r(x) = \underset{\sim}{Ae^x} \cancel{+} c_1 x + c_2$

↳ same the $c_1 e^x$ → $A x e^x$

$y_p = \underset{\sim}{A x e^x} + c_1 x + c_2$

↳ same the $c_2 x e^x$ ⇒ $A x^2 e^x$

$y_p = A x^2 e^x + c_1 x + c_2$

$y_p = Ax^2e^x + 2Axe^x + B$

$\acute{y}_p = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$

$= Ax^2e^x + 4Axe^x + 2Ae^x$

Substituted:-

$$\ddot{y} - 2\acute{y} + y = e^x + x$$

$\cancel{Ax^2e^x} + 4Axe^x + 2A\cancel{e^x}$ ~~e^x~~ $- \cancel{2Ax^2e^x} - 4Axe^x - 2B + \cancel{Ax^2e^x} + Bx + D$

$= \cancel{e^x} + x + 0$

$2Ae^x = e^x$ / $Bx = x$ / $-2B + D = 0$

$\Rightarrow A = \frac{1}{2}$ $B = 1$ $-2(1) + D = 0$

$D = 2$

$y_p = \frac{1}{2}x^2e^x + x + 2$

$y_g = c_1e^x + c_2xe^x + \frac{1}{2}x^2e^x + x + 2$

Example:-

$$\ddot{y} + 2\acute{y} + 5y = 16e^x + \sin(2x)$$

$y_g = y_h + y_p$

$\ddot{y} + 2\acute{y} + 5y = 0$ homo of constant parts ←

let $y = e^{\lambda x}$

$(\lambda^2 + 2\lambda + 5)e^{\lambda x} = 0$

$\lambda = \frac{2 \mp \sqrt{4-20}}{2}$ $\frac{-a}{2} = -1$, $w = \frac{\sqrt{16}}{2} = 2$

$\lambda = -1 \mp 2i$

$y_h = e^{-x}\left[A\cos(2x) + B\sin(2x)\right]$

for $y_p \Rightarrow r(x) = 16e^{x} + \sin(2x)$

$y_p = Ae^{x} + k_1\cos(2x) + k_2\sin(2x)$

No repetion

$y_p' = Ae^{x} - 2k_1\sin(2x) + 2k_2\cos(2x)$

$y_p'' = Ae^{x} - 4k_1\cos(2x) - 4k_2\sin(2x)$

Substituted:-

$y'' + 2y' - 5y = 16e^{x} + \sin(2x)$

$$\underline{Ae^{x}} - 4k_1\underline{\cos(2x)} - \underline{4k_2\sin(2x)} + \underline{2Ae^{x}} - \underline{4k_1\sin(2x)} + \underline{4k_2\cos(2x)} + \underline{5Ae^{x}}$$
$$+ 5k_1\underline{\cos(2x)} + \underline{5k_2\sin(2x)} = \underline{16e^{x}} + \sin(2x) + 0\cos(2x)$$

$\rightarrow (-4k_1 + 5k_1 + 4k_2)\cos(2x) = 0\cos(2x)$

$(k_1 + 4k_2)\cos(2x) = 0\cos(2x)$

$k_1 + 4k_2 = 0$ ----- ①

$\rightarrow (-4k_1 - 4k_2 + 5k_2)\sin(2x) = \sin(2x)$

$-4k_1 + k_2 = 1$ ---- ②

$4 \times (\cancel{k_1} + 4k_2 = 0)$

$-4\cancel{k_1} + k_2 = 1$

$\rightarrow 17k_2 = 1 \rightarrow \boxed{k_2 = \frac{1}{17}}$, $k_1 + \frac{4}{17} = 0$

$\rightarrow k_1 = \frac{-4}{17}$

$\rightarrow 8Ae^{x} = 16e^{x}$

$8A = 16 \rightarrow A = 2$

$\rightarrow y_p = 2e^{x} + \frac{-4}{17}\cos(2x) + \frac{1}{17}\sin(2x)$

$$y_g = y_h + y_p$$

$$= e^{-x}\left[A\cos(2x) + B\sin(2x)\right] + 2e^{x} - \frac{4}{17}\cos(2x) + \frac{1}{17}\sin(2x)$$

2) Variation of Parameters:-

$$y'' + P(x)\,y' + q(x)\,y = r(x)$$

constant ليس ضروري يكون

$P(x)$, $q(x)$ are arbitrary variable function.

$r(x)$ any function of the independent variable ($x^{\frac{3}{2}}$, $x^{\frac{1}{2}}$ ...)

↳ ليست من undete

We use this method when the function have roots or negativ sin or cos ....

$$y_g = y_h + y_p$$

نحصل عليها ال y_p

For $y_h = c_1 \boxed{y_1} + c_2 \boxed{y_2}$

$$y_p = y_1 \int \frac{W_1}{W}\, r(x)\, dx + y_2 \int \frac{W_2}{W}\, r(x)\, dx$$

$$W = \begin{vmatrix} y_1 ① & ⊖\; y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$$

Example:-

$$y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}$$

$$\to e^{-2x} \cdot x^{-2}$$

exp . poly

* اذا كان ضيـالنه نحل
بطريقة ال Variation

$y_h \Rightarrow y'' + 4y' + 4y = 0$ homo of constant.

$$\text{let } y = e^{\lambda x} \Rightarrow \underset{\text{zero}}{\underbrace{(\lambda^2 + 4\lambda + 4)}} e^{\lambda x} = 0$$

$$\lambda = \frac{-4 \mp \sqrt{16-16}}{2} \quad , \quad \lambda_1 = \lambda_2 = -2$$

$$y_h = (c_1 + c_2 x) e^{\lambda x}$$

$$= c_1 \underset{y_1}{\boxed{e^{-2x}}} + c_2 \underset{y_2}{\boxed{x e^{-2x}}}$$

y_1 ← فقط بدون ال constant

لان y_p ماجها

constant.

$$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix}$$

$$= e^{-4x} - 2e^{-4x}x + 2xe^{-4x} = e^{-4x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{-2x} \\ 1 & e^{-2x} - 2xe^{-2x} \end{vmatrix} = -xe^{-2x}$$

$$W_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & 1 \end{vmatrix} = e^{-2x}$$

No.

$$y_p = e^{-2x}\int \frac{-x\cancel{e^{-2x}}}{\cancel{e^{-4x}}}\cdot\left(\cancel{e^{-2x}}\cdot x^{-2}\right)dx$$

$$+\ x e^{-2x}\int \frac{e^{-2x}}{e^{-4x}}\cdot\left(e^{-2x}\cdot x^{-2}\right)dx$$

$$= e^{-2x}\int -\frac{1}{x}\,dx \ +\ x e^{-2x}\int \frac{1}{x^2}\,dx$$

$$= -e^{-2x}\ln(x) + \cancel{x}e^{-2x}\left[-\frac{1}{\cancel{x}}\right]$$

$$y_p = -e^{-2x}\left(\ln(x)+1\right)$$

$$y_g = y_h + y_p$$

$$= c_1 e^{-2x} + c_2 x e^{-2x} - e^{-2x}\left(\ln(x)+1\right)$$

$$= c_1 \underbrace{e^{-2x}} + c_2 x e^{-2x} - e^{-2x}\ln(x) - \underbrace{e^{-2x}}$$

تكرار (x)، لايجوز لان بصير dependent.

$$= \underbrace{(c_1-1)}_{c_3}e^{-2x} + c_2 x e^{-2x} - e^{-2x}\ln(x)$$

Example:-

$$x^2 y'' - 2x y' + 2y = 5x^3\cos(x)$$

لازم (1)

$$y'' - \frac{2}{x}y' + \frac{2}{x^2} = 5x\cos(x)$$

$$\leftarrow y'' - \frac{2}{x}y' + \frac{2}{x^2} = 0$$

$$x^2 y'' - 2x y' + 2y = 0$$

let $y = x^m$

$$(m^2 + (a-1)m + b)\,x^m = 0$$

$$(m^2 + (-2-1)m + 2)\,x^m = 0$$

$$(m^2 - 3m + 2)\,x^m = 0$$

$$m = \frac{3 \mp \sqrt{9-8}}{2} = \frac{3 \mp 1}{2} \Rightarrow m_1 = 2, \; m_2 = 1$$

$$y_h = c_1 x^2 + c_2 x$$

$$W = \begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix} = x^2 - 2x^2 = -x^2$$

$$W_1 = \begin{vmatrix} 0 & x \\ 1 & 1 \end{vmatrix} = -x$$

$$W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & 1 \end{vmatrix} = x^2$$

$$y_p = x^2 \int \frac{-x}{-x^2} \cdot 5x\cos(x)\,dx + x \int \frac{x^2}{-x^2} \cdot 5x\cos(x)\,dx$$

$$y_p = x^2 \int 5 \cancel{x} \cos(x)\,dx - x \int 5x\cos(x)\,dx$$ تكامل بالأجزاء .

$$y_p = 5x^2 \sin(x) - 5x\left[x\sin(x) + \cos(x)\right]$$

$$y_p = -5x\cos(x)$$

$$\therefore y_g = c_1 x^2 + c_2 x - 5x\cos(x)$$

No.

زي الـ 2nd

homo ← ٣ او ٤ او ٥
higher
Non

Higher O.D.E (linear)

٣ او ٤ او ٥ ----

$$y^{(n)} + P_{n-1}(x)\, y^{n-1} \cdots\cdots + P(x)\, y' + P_0(x)\, y = r(x)$$ general form

n^{th} order, linear

if $r(x) = 0$ (homogeneous)

if $r(x) \neq 0$ Non-homogeneous.

4th order. D.E

$$y'''' + P_3(x)\, y''' + P_2(x)\, y'' + P_1(x)\, y = r(x)$$

عدد الحلول = عدد الـ n

if for a differential equation of n^{th} order, we have (n) solution $y_1, y_2, y_3 \cdots y_n$ and if they are basis then by the saperposition principle or linearly principle, the general solution will be :-

$$y_g = c_1 y_1 + c_2 y_2 + c_3 y_3 \cdots\cdots + c_n y_n$$

$c_1, c_2, c_3, - c_n$ are the arbitrary constant

→ a particular solution is obtained if :-

اذا عوضنا بال conditions ← we assign spcific value to the n^{th} constant.

→ Linear indepedence of solutions :- (wronskian)

2nd → y_1, y_2 $\frac{y_1}{y_2}$ or $\frac{y_2}{y_1}$ ≠ constant

n^{th} →

n × n
لازم [illegible] ←

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_3 \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & & & & \\ y_1^{n-1} & y^{n-1} & y^{n-1} & & y_n^{n-1} \end{vmatrix} \quad \boxed{n \times n}$$

square

Example:-

4^{th}

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & \overset{\boxed{'''}}{y_4} \end{vmatrix}$$

n - 1

$$\Rightarrow y_1 \begin{vmatrix} y_2' & y_3' & y_4' \\ y_2'' & y_3'' & y_4'' \\ y_2''' & y_3''' & y_4''' \end{vmatrix} - y_2 \begin{vmatrix} y_1' & y_3' & y_4' \\ y_1'' & 2_3'' & y_4'' \\ y_1''' & y_3''' & y_4''' \end{vmatrix} + y_3 \begin{vmatrix} y_1' & y_2' & y_4' \\ y_1'' & y_2'' & y_4'' \\ y_1''' & y_2''' & y_4''' \end{vmatrix}$$

$$- y_4 \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} \rightarrow$$

بتصغير

3rd.

$3^{rd} \rightarrow$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$\Rightarrow y_1 [y_2' y_3'' - y_2'' y_3'] - y_2 [y_1' y_3'' - y_1'' y_3'] + y_3 [y_1' y_2'' - y_1'' y_2']$$

1) Gauss Elimination

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \Rightarrow \begin{vmatrix} y_1 & y_2 & y_3 \\ 0 & y_{21}' & y_{31}' \\ 0 & 0 & y_{32}' \end{vmatrix} = y_1 \, y_{21}' \, y_{32}''$$

(اشرط تحول) Lower triangle

(ليا قواعد معينه) = zero

No. ....

لا ترتبط لكل الحالات فقط حالات معينة زي exp

2) Vandermode determiant

Ex:-

$$\omega = \begin{vmatrix} e^{-2x} & e^{-x} & e^{x} & e^{2x} \\ -2e^{-2x} & -e^{-x} & e^{x} & 2e^{2x} \\ 4e^{-2x} & e^{-x} & e^{x} & 4e^{2x} \\ -8e^{-2x} & -e^{-x} & e^{x} & 8e^{2x} \end{vmatrix} \Rightarrow e^{(-2-1+1+2)x} \begin{vmatrix} 1 & 1 & 1 & 1 \\ \overset{\lambda_1}{-2} & \overset{\lambda_2}{-1} & \overset{\lambda_3}{1} & \overset{\lambda_4}{2} \\ \overset{\lambda_1^2}{4} & \overset{\lambda_2^2}{1} & \overset{\lambda_3^2}{1} & \overset{\lambda_4^2}{4} \\ -8 & -1 & 1 & 8 \end{vmatrix}$$

constant الـ فقط ← $\boxed{\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4}$

$$\omega = (-1)^{\frac{n(n-1)}{2}} \cdot V$$

$$-V = -(\overset{\downarrow}{\lambda_1} - \lambda_2)(\overset{\downarrow}{\lambda_1} - \lambda_3)(\overset{\downarrow}{\lambda_1} - \lambda_4)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)$$

$$= -(-2+1)(-2-1)(-1-1)(-1-2)(1-2)$$

$$= -(-1)(-3)(-4)(-2)(-3-1)(-1) = -72 \rightarrow \upsilon = 72$$

$$\rightarrow \frac{(n-1)n}{2} = \frac{4(4-1)}{2} = \frac{12}{2} = 6 \rightarrow (1-)^6 = 1$$

$\therefore \omega = (1)(72) \neq$ zero independance.

Example:-

$$x^3 y''' - 3x^2 y'' + 6x y' - 6y = 0$$

3rd o. Euler eqation homogenous.

and has $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$, check dependance?

2) ما بترتبط ← لانو ليس exp

$$\omega = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = x(12x^2 - 6x^2) - x^2(6x - 0) + x^3(2-0)$$

$$= x(6x^2) - 6x^3 + 2x^3$$

$$= 2x^3 \neq \text{zero}$$

$\therefore y_1, y_2, y_3$ are independent using Gauss

$$W = \left|\begin{array}{ccc} x & x^2 & x^3 \\ 0 & x & 2x^2 \\ 0 & 0 & 2x \end{array}\right| = (x)(x)(2x) = 2x^3 \neq \text{zero}$$

*** How to solve a set of equations:-** (باستخدمها لإيجاد الا constant)

↳ 3, 4, 5, ----

$3 \Rightarrow$ $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ ١

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$ ٢

$a_{33}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$ ٣

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}$$

Cramer's rule to find x_1, x_2, x_3 (unknowns)

1) Find Det. $\begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow$ اي طريقة بتجيب ال

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \Rightarrow \text{find Det}(D_1)$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \Rightarrow \text{find Det}(D_2)$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} \Rightarrow \text{find } Det(D_3)$$

$$x_1 = \frac{Det(D_1)}{D} , \quad x_2 = \frac{Det(D_2)}{D} , \quad x_3 = \frac{Det(D_3)}{D}$$

Example:-

$y_g = c_1 x + c_2 x^2 + c_3 x^3$

$y(1) = 2 , \quad \dot{y}(1) = 1 , \quad \ddot{y}(1) = -4$

$2 = c_1 + c_2 + c_3 \quad \cdots ①$

$\acute{y}_g = c_1 + 2c_2 x + 3c_3 x^2$

$1 = c_1 + 2c_2 + 3c_3 \quad \cdots ②$

$''y_g = 0 + 2c_2 + 6c_3 x$

$-4 = 2c_2 + 6c_3 \quad \cdots ③$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} \begin{vmatrix} c_1 \\ c_2 \\ c_3 \end{vmatrix} = \begin{vmatrix} 2 \\ 1 \\ -4 \end{vmatrix}$$

$Det(D) = 1(12-6) - 1(6-0) + 1(2-0) = 2$

$$D_1 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ -4 & 2 & 6 \end{vmatrix} = 2(12-6) - 1(6+12) + 1(2+8)$$

$= 4$

$c_1 = \frac{4}{2} = 2$

$$D_2 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & -4 & 6 \end{vmatrix} = 1(6+12) - 2(6-0) + 1(-4-0)$$

$$= 2$$

$$c_2 = \frac{2}{2} = 1$$

$$D_3 \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & -4 \end{vmatrix} = 1(-8-2) - 1(-4-0) + 2(2-0)$$

$$= -2$$

$$c_3 = \frac{-2}{2} = -1$$

$$\rightarrow y_g = 2x + x^2 - x^3 \quad \#$$

- check the dependency of y_1, y_2, y_3, y_4 ! 1) $\underset{\sim\sim}{\overset{\equiv}{y} = 0}$ ⇒ $y_1 = 1$, $y_2 = x$, $y_3 = x^2$, $y_4 = x^3$

المعادلة حل

2) $\underset{\sim\sim}{\overset{\equiv}{y} - y = 0}$ ⇒ $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = \cos(x)$, $y_4 = \sin(x)$

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12 \neq \text{Zero}$$

y_1, y_2, y_3, y_4 ⇒ independent

$$= \underset{\boxed{1}}{\begin{vmatrix} 1 & 2x & 3x^2 \\ 0 & 2 & 6x \\ 0 & 0 & 6 \end{vmatrix}} - x \underset{\boxed{2}}{\begin{vmatrix} 0 & 2x & 3x^2 \\ 0 & 2 & 6x \\ 0 & 0 & 6 \end{vmatrix}} + x^2 \underset{\boxed{3} \rightarrow 0}{\begin{vmatrix} 0 & 1 & 3x^2 \\ 0 & 0 & 6x \\ 0 & 0 & 6 \end{vmatrix}} - x^3 \underset{\boxed{4} \rightarrow 0}{\begin{vmatrix} 0 & 1 & 2x \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix}}$$

$\boxed{\text{١}}$ $(12-0) - 2x(0) + 3x^2(0-0) = 12$

$\boxed{\text{٢}}$ $-0 - 2x(0) + 3x^2(0) = 0 \rightarrow x(0) = 0$

2)
$$\begin{vmatrix} e^{x} & e^{-x} & \cos(x) & \sin(x) \\ e^{x} & -e^{-x} & -\sin(x) & \cos(x) \\ e^{x} & e^{-x} & -\cos(x) & -\sin(x) \\ e^{x} & -e^{-x} & \sin(x) & -\cos(x) \end{vmatrix}$$

$\boxed{1}$
$$e^{x}\begin{vmatrix} -e^{-x} & -\sin(x) & \cos(x) \\ e^{-x} & -\cos(x) & -\sin(x) \\ -e^{-x} & \sin(x) & -\cos(x) \end{vmatrix} = \Big(-e^{-x}\left(\cos^2(x)+\sin^2(x)\right) + \sin(x)\left(-\cancel{e^{-x}\cos(x)} - e^{-x}\sin(x)\right) + \cos(x)\left(\cancel{e^{-x}\sin(x)} - e^{-x}\cos(x)\right)\Big)\, e^{x}$$

$$= \left(-e^{-x} - e^{-x}\sin^2(x) - e^{-x}\cos^2(x)\right) e^{x}$$

$$e^{x}\left(-e^{-x}\left(1+\sin^2(x)+\cos^2(x)\right)\right) = \left(-2e^{-x}\cdot e^{x}\right) = -2$$

$\boxed{2}$
$$e^{-x}\begin{vmatrix} e^{x} & -\sin(x) & \cos(x) \\ e^{x} & -\cos(x) & -\sin(x) \\ e^{x} & \sin(x) & -\cos(x) \end{vmatrix} = e^{x}\left(\cos^2(x)+\sin^2(x)\right) + \sin(x)\left(-\cancel{e^{x}\cos(x)} + e^{x}\sin(x)\right) + \cos(x)\left(\cancel{e^{x}\sin(x)} + e^{x}\cos(x)\right)$$

$$= e^{x} + e^{x}\sin^2(x) + e^{x}\cos^2(x)$$

$$= e^{x}(1+1) = 2e^{x} \rightarrow 2e^{x}\cdot e^{-x} = 2$$

$\boxed{3}$
$$\cos(x)\begin{vmatrix} e^{x} & -e^{-x} & \cos(x) \\ e^{x} & e^{-x} & -\sin(x) \\ e^{x} & -e^{-x} & -\cos(x) \end{vmatrix} = e^{x}\left(-e^{-x}\cos(x) - \cancel{e^{-x}\sin(x)}\right) + e^{-x}\left(-e^{x}\cos(x) + \cancel{e^{x}\sin(x)}\right) + \cos(x)\left(-\dot{e} - \dot{e}\right)$$

$$= -\cos(x) - \cos(x) - 2\cos(x)$$

$$= -4\cos(x) * \cos(x) = -4\cos^2(x)$$

$$\sin(x)\begin{vmatrix} e^{x} & -e^{-x} & -\sin(x) \\ e^{x} & e^{-x} & -\cos(x) \\ e^{x} & -e^{-x} & \sin(x) \end{vmatrix} = e^{x}\left(e^{-x}\sin(x) - \cancel{e^{-x}\cos(x)}\right) + e^{-x}\left(e^{x}\sin(x) + \cancel{e^{x}\cos(x)}\right) - \sin(x)\left(-\dot{e} - \dot{e}\right) = 2\sin(x) + 2\sin(x)$$

$$= 4\sin(x) * \sin(x)$$

$$4\sin^2(x)$$

dependency ⇒

$-2-(-2)+(-4\cos^2(x))-(4\sin^2(x))$

$= 0 \quad -4(\cos^2(x)+\sin(x)) = -4 \neq$ zero

$y_1, y_2, y_3, y_4 \Rightarrow$ independent.

Higher O.D.E (linear)

$$y'''' + P(x)\,y''' + q(x)\,y'' + m(x)\,y' + n(x)\,y = r(x) \qquad \text{(linear)}$$

1) constant coefficient

$P(x)$, $q(x)$, $m(x)$, $n(x)$ constant.

2) Euler equation

$$x^4 y'''' + a x^3 y''' + b x^2 y'' + c x y' + d y = r(x) \qquad \text{non-hom}$$

الحل $y_g = y_h + y_p$

$y_h \Rightarrow y'''' + P(x)\,y''' + q(x)\,y'' + m(x)\,y' + n(x)\,y = 0$

y_p :- ↳ hom. part.

1) undetermined coefficients

2) Variation of parameters

Solving higher D.E

linear, homo or non hom

↳ with constant coefficients.

$$(1) \leftarrow y''' + \underline{a} y'' + \underline{b} y' + \underline{c} y = r(x)$$

↳ constant.

$y_g = y_h + y_p$

For $y_h \Rightarrow y''' + a y'' + b y' + c y = 0$

let $y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

$y''' = \lambda^3 e^{\lambda x}$

$$\lambda^3 e^{\lambda x} + a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + c e^{\lambda x} = 0$$

$$[\lambda^3 + a\lambda^2 + b\lambda + c]\, e^{\lambda x} = 0 \rightarrow$$ رح نحاول نجرب رقم ، نطلع اصفار المعادلة ، نقسم قسمة طويلة

$$\underset{\text{zero}}{\downarrow} \qquad \underset{\neq \text{zero}}{\downarrow}$$

↳ in general $\Rightarrow \lambda_1, \lambda_2, \lambda_3$ — real ... 1, 2, 3 / repeated .. 1, 2, 3 / complex ... 1, 2, 3

لانه من الدرجة ٣ ، ٣ حلول .

Example:-

$$\dddot{y} - 3\ddot{y} + 3\dot{y} - y = 30e^{x}$$

$$y_g = y_h + y_p$$

$$y_h \Rightarrow \dddot{y} - 3\ddot{y} + 3\dot{y} - y = 0$$

let $y = e^{\lambda x}$

$$\rightarrow (\lambda^3 - 3\lambda^2 + 3\lambda - \overset{1}{\cancel{\lambda}})\, e^{\lambda x} = 0$$

افرض $\rightarrow \lambda_1 = 1$

$$(1) - 3 + 3 - 1 = 0 \rightarrow ✓ \longrightarrow (\lambda, -1)$$

$$\begin{array}{r|l}
 & \lambda^2 - 2\lambda + 1 \\
\hline
\lambda, -1 & \cancel{\lambda^3} - 3\lambda^2 + 3\lambda - \lambda \\
 & \underline{\cancel{-\lambda^3} \pm \lambda^2} \\
 & -\cancel{2\lambda^2} + 3\lambda - \lambda \\
 & \underline{\cancel{\pm 2\lambda^2} \mp 2\lambda} \\
 & \cancel{\lambda} - 1 \\
 & \underline{\cancel{-\lambda} \pm 1} \\
 & \text{zero} \leftarrow \text{بالزم}
\end{array}$$

$$\rightarrow (\lambda^2 - 2\lambda + 1)(\lambda - 1) = 0$$

$\lambda^2 - 2\lambda + 1 = 0$ $\qquad \lambda = \dfrac{2 \mp \sqrt{4-4}}{2} \rightarrow \lambda_2 = \lambda_3 = 1$

we have 3 repeted roots.

$$\left.\begin{array}{l} y_1 = e^{\lambda_1 x} = e^x \\ y_2 = x e^x \\ y_3 = x^2 e^x \end{array}\right] \quad y_h = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

For $y_p = Ae^x \rightarrow$ مُكرر

$y_p = A x^3 e^x$

$y_p' = \underline{A x^3 e^x} + \underline{3x^2 A}\, e^x$

$y_p'' = Ax^3e^x + 3Ax^2e^x + 3Ax^2e^x + 6Axe^x$

$= \underline{Ax^3e^x} + \underline{6Ax^2e^x} + \underline{6Axe^x}$

$y_p''' = Ax^3e^x + 3Ax^2e^x + 6Ax^2e^x + 12Axe^x + 6Axe^x + 6Ae^x$

$= Ax^3e^x + 9Ax^2e^x + 18Axe^x + 6Ae^x$

$y''' - 3y'' + 3y' - y = \overset{30e^x}{\cancel{\text{Zero}}}$

$A\cancel{x^3}e^x + \underline{9Ax^2e^x} + 18A\cancel{x}e^x + 6Ae^x - 3A\cancel{x}^3e^x - \underline{18Ax^2e^x} - 18\cancel{A}xe^x$
$+ 3A\cancel{x}^3e^x + \underline{9x^2Ae^x} + A\cancel{x}^3e^x = 30e^x$

$6Ae^x = 30e^x \rightarrow A = \dfrac{30}{6} = 5$

$y_p = 5x^3e^x$

$\Rightarrow y_g = c_1e^x + c_2xe^x + c_3x^2e^x + 5x^3e^x$ #

$y_g = (c_1e^x + c_2e^x + c_3x^2 + 5x^3)e^x$

Example:-

$$\overset{'''}{y} + \overset{''}{y} - 2y = 2x^2 + 2x$$

$$y_p = y_h + y_p$$

$$y_h \Rightarrow \quad \overset{'''}{y} + \overset{'}{y} - 2y = 0 \qquad \text{let } y = e^{\lambda x}$$

$$(\lambda^3 + \lambda^2 - 2)e^{\lambda x} = 0$$

$$\quad \hookrightarrow = \text{zero}$$

$$\lambda = 1 \rightarrow 1 + 1 + 2 = 0 \checkmark \rightarrow (\lambda, -1)$$

$$\begin{array}{r|l} & \lambda^2 + 2\lambda + 2 \\ \hline \lambda - 1 & \cancel{\lambda^3} + \lambda^2 - 2 \\ & -\cancel{\lambda^3} \overset{+}{-} \lambda^2 \\ \hline & \cancel{2\lambda^2} - 2 \\ & -\cancel{2\lambda^2} \overset{+}{-} 2\lambda \\ \hline & \cancel{2\lambda} - \cancel{2} \\ & -\cancel{2\lambda} \overset{+}{-} \cancel{2} \\ \hline & \text{zero} \end{array}$$

$$\rightarrow (\lambda - 1)(\lambda^2 + 2\lambda + 2) = 0$$

$$\lambda = \frac{-2 \mp \sqrt{4 - 8}}{2} \rightarrow \frac{-a}{2} = -1 \; , \; \omega = \sqrt{\frac{8 - 4}{4}} = 1$$

$$\therefore y_h = a e^{x} + e^{-x}\left[c_2 \cos(x) + c_3 \sin(x)\right]$$

$$y_p \Rightarrow \quad Ax^2 + Bx + D$$

$$\overset{'}{y_p} = 2Ax + B$$

$$\overset{''}{y_p} = 2A$$

$$\overset{'''}{y_p} = 0$$

$$\overset{'''}{y} + \overset{''}{y} - 2y = 2x^2 + 2x$$

$$0 + 2A - 2Ax^2 - 2Bx - 2D = 2x^2 + 2x$$

$-2Ax^2 = 2x^2$

$\rightarrow A = -1$

$-2Bx = 2x \quad \rightarrow \quad B = -1$

$2A - 2D = 0 \quad \rightarrow \quad -2 - 2D \rightarrow \quad D = -1$

$\therefore y_p = -x^2 - x - 1$

$y_g = c_1 e^{x} + e^{-x}\left[c_2\cos(x) + c_3\sin(x)\right] - x^2 - x - 1$

Example:-

$y''' + 3y'' + 3y' + y = 16e^{x} + x + 3$

$y_p = y_h + y_D$

$y_h \Rightarrow \quad y''' + 3y'' + 3y' + y = 0 \quad$ let $y = e^{\lambda x}$

$(\lambda^3 + 3\lambda^2 + 3\lambda + 1)e^{x} = 0$

↳ $= \text{zero}$.

$\lambda = -1 \rightarrow (-1)^3 + (3)(-1)^2 + 3(-1) + 1 = 0$ ✓ $\quad (\lambda+1)$

$$
\begin{array}{r|l}
 & \lambda^2 + 2\lambda + 1 \\
\hline
\lambda + 1 & \lambda^3 + 3\lambda^2 + 3\lambda + 1 \\
 & \cancel{-\lambda^3} \mp \lambda^2 \\
\hline
 & 2\cancel{\lambda^2} + 3\lambda + 1 \\
 & \cancel{-2\lambda^2} \mp 2\lambda \\
\hline
 & \cancel{\lambda} \cancel{+1} \\
 & \cancel{-\lambda} \mp 1 \\
\hline
 & \text{zero}
\end{array}
$$

$\rightarrow (\lambda+1)(\lambda^2 + 2\lambda + 1) = 0$

↳ $\lambda_1 = -1$

$\lambda = \dfrac{-2 \mp \sqrt{4-4}}{2} \rightarrow \lambda_2 = \lambda_3 = -1$

$\therefore y_h = c_1 e^{-x} + (c_2 + x c_3) e^{-x}$

↳ $y_h = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$

No.

for $y_p = Ae^x + Bx + D$

$y_p' = Ae^x + B$

$y_p'' = Ae^x$

$y_p''' = Ae^x$

$y''' + 3y'' + 3y' + y = 16e^x + x + 3$

$Ae^x + 3Ae^x + 3Ae^x + 3B + Ae^x + Bx + D = 16e^x + x + 3$

$\rightarrow 8Ae^x = 16e^x$

$A = 2$

$\rightarrow Bx = 1x$

$B = 1$

$\rightarrow 3B + D = 3 \rightarrow D = 0$

$\therefore y_p = 2e^x + x$

$y_g = (c_1 + c_2x + c_3x^2)e^{-x} + 2e^x + x$

Example:-

$y'''' - 5y'' + 4y = 10\cos(x)$

$y_g = y_h + y_p$

$y_h \Rightarrow y'''' - 5y'' + 4y = 0$ let $y = e^{\lambda x}$

$(\lambda^4 - 5\lambda^2 + 4)e^{\lambda x} = 0$

$(\lambda - 1)^2(\lambda - 4)^2 = 0$ $\lambda_1 = -1$, $\lambda_2 = 1$

$(\lambda - 1)(\lambda + 1)(\lambda - 2)(\lambda + 2) = 0 \rightarrow$ $\lambda_3 = -2$, $\lambda_4 = 2$

$y_h = c_1e^{-x} + c_2e^{x} + c_3e^{-2x} + c_4e^{2x}$

No.

$y_p = A\cos(x) + B\sin(x)$, $y'_p = -A\sin(x) + B\cos(x)$

$y''_p = -A\cos(x) - B\sin(x)$, $y'''_p = A\overset{\sin}{\cancel{\cos}}(x) + B\cos(x)$

$y''''_p = A\cos(x) + B\sin(x)$

$y'''' - 5y'' + 4y = 10\cos(x)$

$\rightarrow \underline{A\cos(x)} + B\sin(x) + 5\underline{A\cos(x)} + 5B\sin(x) + 4\underline{A\cos(x)} + 4B\sin(x)$

$= 10\cos(x)$

$10A\cos(x) = 10\cos(x) \rightarrow A = 1 \ , B = 0$

$y_p = \cos(x)$

$y_g = c_1 e^{-x} + c_2 e^{x} + c_3 e^{-2x} + c_4 e^{2x} + \cos(x)$

$y'''' + 10y'' + 9y = 2\sinh(x)$ constant , non-homo

$y_g = y_h + y_p$

1	0	10	0	9

$y'''' + 10y'' + 9y = 0$

$(\lambda^4 + 10\lambda^2 + 9)e^{\lambda x} = 0$

$(\lambda^2 + 1)(\lambda^2 + 9) = 0$

$\lambda = \sqrt{-1}$ $\lambda = -$

Higher O.O.D.E Variable coefficints

طالما استخدمها من variation لازم اقسمها على x^3

لازم (1) $x^3 \frac{d^3y}{dx^3} + ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = \boxed{r(x)}$

linear - non hom - Huler

$$y_g = y_h + y_p$$

$$y_h \Rightarrow x^3 y''' + ax^2 y'' + bxy' + cy = 0$$

let $y = x^m \rightarrow$

$$y' = m x^{m-1}$$

$$y'' = m(m-1) x^{m-2}$$

$$y''' = m(m-1)(m-2) x^{m-3}$$

$$\rightarrow \cancel{x^3}[m(m-1)(m-2) x^m . \cancel{x^{-3}}] + a\cancel{x^2}[m(m-1) x^m . \cancel{x^{-2}}] + b\cancel{x}$$

$$[m x^m . \cancel{x^{-1}}] + c x^m = 0$$

$$x^m [m(m-1)(m-2) + am(m-1) + bm + c] = 0$$

$$\Rightarrow [m^3 + (a-3)m^2 + (b-a+2)m + c] x^m = 0$$

$\Downarrow$ $= Zero$ $\quad\quad$ $\hookrightarrow \neq Zero$

Examples:-

(1) لازم $4x^3 y''' + 4x^2 y'' - 8xy' + 8y = 4x^{-2}$

من الدرجة الثالثة
non - hom
Variable.

كالقسم
على x^3 بعد
ماشيل y-5
undc على

$$x^3 y''' + x^2 y'' - 2xy' + 2y = x^{-2}$$

$$y_g = y_h + y_p$$

$x^3 y''' + x^2 y'' - 2x y' + 2y = 0$ let $y = x^m$

$a = 1$, $b = -2$, $c = 2$

$\longrightarrow [m^3 + (a-3)m + (b-a+2)m + c]\, x^m = 0$

$\rightarrow [m^3 - 2m + (-2-1+2)m + 2]\, x^m = 0$

$(m^3 - 2m - m + 2)\, x^m = 0$

zero ↲

$m = 1 \rightarrow (1-2-1+2) = 0 \rightarrow$ صفر الاقتران

عند طريقة القسمة ←

$(m-1)(m^2 - m - 2) = 0$

$m_1 = -1$, $m_2 = 1$, $m_3 = 2$

$y_h = a\underbrace{x^{-1}}_{y_1} + c_2 \underbrace{x}_{y_2} + c_3 \underbrace{x^2}_{y_3}$

To find y_p using Variation of Parameters method.

$y''' + \frac{1}{x} y'' - \frac{2}{x^2} y' + \frac{2}{x^3} y = \overset{r(x)}{\frac{1}{x^5}}$

$$W = \begin{vmatrix} x^{-1} & x & x^2 \\ -x^{-2} & 1 & 2x \\ 2x^{-3} & 0 & 2 \end{vmatrix} = \frac{1}{x}(2-0) - x\left(\frac{-2}{x^2} - \frac{4}{x^2}\right) + x^2\left(0 - \frac{2}{x^3}\right)$$

$$= \frac{2}{x} + \frac{6}{x} - \frac{2}{x} = \frac{6}{x}$$

$$w_1 = \begin{vmatrix} 0 & x & x^2 \\ 0 & 1 & 2x \\ 1 & 0 & 2 \end{vmatrix} = x^2$$

آخر واحد ←

بس one

No.

$$\omega_2 = \begin{vmatrix} \frac{1}{x} & 0 & x^2 \\ -\frac{1}{x^2} & 0 & 2x \\ \frac{2}{x^3} & 1 & 2 \end{vmatrix} = -3$$

$$\omega_3 = \begin{vmatrix} 1/x & x & 0 \\ -1/x^2 & 1 & 0 \\ 2/x^3 & 0 & 1 \end{vmatrix} = \frac{2}{x}$$

$$y_p = \frac{1}{x}\int \frac{x^2}{8/x} \cdot \frac{1}{x^5}\, dx + x\int \frac{-3}{6/x} \cdot \frac{1}{x^5}\, dx + x^2 \int \frac{2/x}{6/x} \cdot \frac{1}{x^5}\, dx$$

$$y_p = \frac{1}{6x}\int \frac{1}{x^2}\, dx - \frac{1}{2}x \int \frac{1}{x^4}\, dx + \frac{1}{3}x^2 \int \frac{1}{x^5}\, dx$$

$$\frac{1}{6x} \cdot \left[\frac{-1}{x}\right] - \frac{x}{2}\left[\frac{-1}{3} \cdot \frac{1}{x^3}\right] + \frac{1}{3}x^2\left[-\frac{1}{4}\, \frac{1}{x^4}\right]$$

$$= \frac{-1}{6x2} + \frac{1}{6x^2} - \frac{1}{12x2} = \frac{-1}{12x^2}$$

$$y_g = c_1 x^{-1} + c_2 x + c_3 x^3 - \frac{1}{12} x^{-2} \quad \#$$

Example:-

$$x^3 y''' + 3x^2 y'' + 6xy' - 6y = \frac{12}{x}$$

$y(1) = 2.5$

$y'(1) = 6.5$

$y''(1) = 5$

$y_h \Rightarrow \quad x^3 y''' + 3x^2 y'' + 6xy' - 6y = 0 \qquad$ let $y = x^m$

$\hookrightarrow (m^3 + (a-3)m^2 + (b-a+2)m + c)\ x^m = 0$

$(m^3 - 6m^2 + 11m - 6)\ x^m = 0$

$m=1 \Rightarrow (m-1)(m^2-5m+6)=0$

$m=1 \; , m=2 , m=3$

$y_h = c_1 x + c_2 x^2 + c_3 x^3$

For $y_p \Rightarrow$ $$w = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

$$w_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4 \; , \quad w_2 \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3$$

$$w_3 \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

$$y_p = x \int \frac{x^4}{2x3} \cdot \frac{12}{x^9} \, dx + x^2 \int \frac{-2x^3}{2x^3} \cdot \frac{12}{x^4} \, dx + x^3 \int \frac{x^2}{2x3} \cdot \frac{12}{x^4} \, dx$$

$$y_p = 6x \int \frac{1}{x^3} \, dx - 12x^2 \int \frac{1}{x^4} \, dx + 6x^3 \int \frac{1}{x^5} \, dx$$

$$y_p = \left(-\frac{1}{2} x^{-2}\right) - 12x^2\left(-\frac{1}{3} x^{-3}\right) + 6x^3\left(-\frac{1}{4} x^{-4}\right)$$

$$= -3x^{-1} + 4x^{-1} - \frac{3}{2} x^{-1} = -0.5 x^{-1}$$

$$y_p = \frac{-0.5}{x}$$

$$y_g = c_1 x + c_2 x^2 + c_3 x^3 - \frac{0.5}{x}$$

$$2.5 = c_1 + c_2 + c_3 - 0.5 \text{ ①} \rightarrow c_1 + c_2 + c_3 = 3$$

$$y'_p = c_1 + 2c_2 x + 3c_3 x^2 + \frac{0.5}{x^2}$$

$$6.5 = c_1 + 2c_2 + 3c_3 + 0.5 \text{ ---- ②} \rightarrow c_1 + 2c_2 + 3c_3 = 6$$

$$y''_g = 0 + 2c_2 + 6c_3 x - \frac{1}{x^3}$$

$$5 = 0 + 2c_2 + 6c_3 - 1 \text{ ---- ③} \rightarrow 0 + 2c_2 + 6c_3 = 6$$

$$\Rightarrow c_1 = 0 \ , c_2 = 3 \ , c_3 = 0$$

$$y_g = 3x^2 - \frac{0.5}{x}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} \begin{vmatrix} c_1 \\ c_2 \\ c_3 \end{vmatrix} = \begin{vmatrix} 3 \\ 6 \\ 6 \end{vmatrix}$$

$$\text{Det}[D] = (12-6) - (6-0) + (2-0) = 2$$

$$D_1 = \begin{vmatrix} 3 & 1 & 1 \\ 6 & 2 & 3 \\ 6 & 2 & 6 \end{vmatrix} = 3(12-6) - (36-18) + (\overset{0}{\cancel{12-12}})$$

$$= 18 - 18 = \text{صفر}$$

$$D_2 = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 6 & 3 \\ 0 & 6 & 6 \end{vmatrix} = (36-18) - 3(6-0) + (6)$$

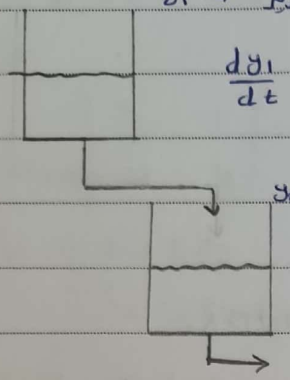
$$= 18 - 18 = 6 \Rightarrow \frac{6}{2} = 3$$

$$D_3 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 6 \\ 0 & 2 & 6 \end{vmatrix} = (12-12) - (6) + 3(2) = 0$$

فقط

System of 1st o.o.D.E

$y_1 \rightarrow$ التركيز



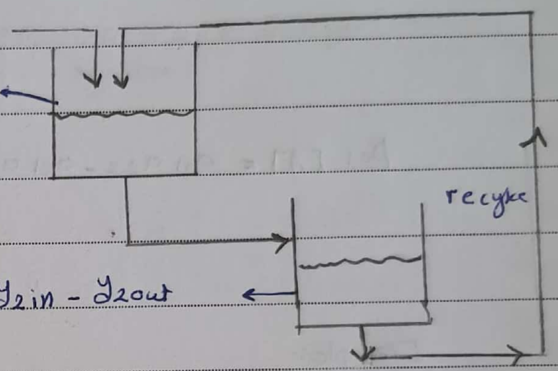
$$\frac{dy_1}{dt} = y_{in} - y_{1out}$$

$$\frac{dy_2}{dt} = y_{2in} - y_{2out}$$

نستطيع حل المعادلة الاولى لوحدها
لكن الثانية لازم بحل الاول.
← لذلك لا يعد system لانو يمكن حل كل
معادلة لوحدها

(لاتشغل لوحدها) $\frac{dy_1}{dt} = y_{in} + y_{2out}$
$- y_{1\,out}$

[system]



$$\frac{d_2}{dr} = y_{2in} - y_{2out}$$

System of 2 equation:-

$$\frac{dy_1}{dt} = \underset{\sim}{a_{11}} y_1 + \underset{\dots}{a_{12}} y_2 + \underset{\dots\dots}{f_1(t)}$$ $\rightarrow$ any function of independent variable (t)

اي معادلة اذا كانت فقط معادلتين حتى يصير non-hom

$$\frac{dy_2}{dt} = \underset{\sim}{a_{21}} y_1 + \underset{\dots}{a_{22}} y_2 + \underset{\dots\dots}{f_2(t)}$$

$\searrow \swarrow$
Coefficient

uler لا يوجد
on-linear و

Case 1: System of 1st O.D.E linear with constant coefficients and homogenous. $f(t) = 0$

case 2: system of 1st O.D.E linear with constant coefficients and non-homogenous. $f(t) \neq 0$

Matrix notation:-

$$\begin{array}{ccc} & c_1 & c_2 \\ R_1 \rightarrow & a_{11} & a_{12} \\ R_2 \rightarrow & a_{21} & a_{22} \end{array} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix}$$

هيك بتقرأ.

[A] matrix $m \overset{2\times2}{} n$ → Rows, colums

Vector Vector

احنا بندرس فقط لما
n = m
Squar matrix

⇒ if m = n then we have a square matrix.

عدد ال variable = عدد ال equation

← لانو لما يكون عندي 2variable
I need 2equation to solve

Det [A] = $a_{11} a_{22} - a_{21} a_{12}$ → (Det = صفر) singular ممنوع تكون

لان المعادلات رح تصير dependent equation

← يعني المعادلة نفسها لكنه مضروبة بـ constant

Example:-

$2x_1 + 4x_2 = 5$

$8x_1 + 16x_2 = 20$

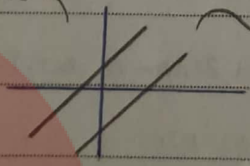
نفسها $4\,(2x_1 + x_2)$

$$\begin{array}{ccc} & c_1 & c_2 \\ R_1 \rightarrow & 2 & 4 \\ R_2 \rightarrow & 8 & 16 \end{array} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 5 \\ 20 \end{vmatrix}$$

Det[A] = 32 − 32 = 0 → dependent equation

if Det[A] = zero then we have a singular matrix.

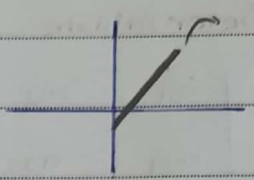
because:- ما فيه تقاطع ما فيه حلول.



→ no cross, no solution

$2x_1 + 4x_2 = 5$

$2x_1 + 4x_2 = 5$



تطابق

infint solution

$$\begin{vmatrix} 2 & 4 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 5 \\ 5 \end{vmatrix}$$

Det [A] = 8 - 8 = Zero → singular matrix

↳ infinit number of solution

we have to avoid this system. ← singular $\left\langle \begin{matrix} \text{infint} \\ \text{no solution} \end{matrix} \right.$

Differentiation:-

الاشتقاق عادي

ما بأثر عل ال matrix

$$y = \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} e^{-2t} \\ \sin(t) \end{vmatrix} \quad و \quad \acute{y} = \begin{vmatrix} \acute{y}_1 \\ \acute{y}_2 \end{vmatrix} = \begin{vmatrix} -2e^{-2t} \\ \cos(t) \end{vmatrix}$$

Transposition:-

هو عكس ال rows لـ colums

و ال colums لـ Rows

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3\times3} \Rightarrow [A]^{T} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}_{3\times3}$$

main diagram

unit matrix

→ اي matrix فيها ال main = 1

or identity matrix

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

square matrix contains zero every were except main diagram.

Determinate Det[A] ⇒

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \Rightarrow \quad \text{Det}[A] = a_{11}\,a_{22} - a_{21}\,a_{12}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow$$

$$\text{Det} \Rightarrow a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Inverse of a matrix المقلوب

$$[A] \Rightarrow [A]^{-1}$$

اذا بدين المطلوب لمعرفيها constants

لطلع معي ال slouhion directly

$$[A][x] \neq [c]$$

$$[x] = [A]^{-1}[c]$$

How to find inverse:-

$$[A^{-1}] = \frac{1}{\text{Det}[A]} * \begin{vmatrix} a_{22} & -a_{12} \\ -a_{2} & a_{11} \end{vmatrix}$$

↗ لازم تكون squre عشان تطلع معي ال inverse

other conditions of inverse of any matrix → Just inverse of square matrix, no inverse for non square matrix.

$$[A][B] \neq [B][A] \curvearrowright$$ مابصير

but if matrix inverse → $[A][A]^{-1} = [A]^{-1}[A] = [I]$

↲ عملية تبادلية.

real
complex ← ارقام

Eigen values and Eigen vectors

اقيم ذاتيه

For amatrix [A] :-

$$[A][x] = \lambda [x]$$

Vector ↲ ↳ Eignvectors

احنا بدنا نطلعها

λ : is called an eigenvalue, (real or complex) must be determined

x : is called an eigenvector of [A].

to find valuse ⇒ eigenvaluses :-

لا نستطيع نطرح اقيم من matrix

$$\therefore [A][x] - \lambda[x] = 0 \Rightarrow [\,[A] - \boxed{\lambda}\,]\,[x] = 0$$

رح نحولها إلى matrix

identy

$$\Rightarrow [\,[A] - \lambda [I]\,]\,[x] = 0$$

identity

$$\left[\underbrace{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{\text{new matrix}}\right]_{2\times 2} \underbrace{\begin{vmatrix} x_1 \\ x_2 \end{vmatrix}}_{\text{Eigenvector}} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} - 0 \\ a_{21} - 0 & a_{22} - \lambda \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

↳ eigenvector

λ ⇒ eignvalues that makes Det [A] = 0

قيم λ :.

$$\text{Det} \Rightarrow (a_{11} - \lambda)(a_{22} - \lambda) - (a_{21})(a_{12}) = 0$$

↖ to makes [A] singular

$$a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{12}a_{21} = 0$$

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

$$\lambda = \frac{(a_{11} + a_{22}) \mp \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}$$

$$\lambda = \frac{a_{11} + a_{22}}{2} \mp \sqrt{\frac{(a_{12} + a_{22})^2 - 4(a_{12}a_{22} - a_{12}a_{22})}{4}}$$

① +ve root ⇒ 2 real distanct roots (λ_1, λ_2)

② zero ⇒ repeated roots

③ -ve root ⇒ complex roots

Example:-

$$[A] = \begin{vmatrix} -5 & 2 \\ 2 & -2 \end{vmatrix}$$

Find the eignvalues and eigenvectors

$[A] \Rightarrow \text{Det}[A] = (-5)(-2) - (2)(2) = 10 - 4 = 6 \neq$ Zero (non singular)

↲ نستطيع حلاً هذا ال system

ولكننا نستثني متى يكون zero

منه final Solution ⇐ لذلك بنطلع eigens

$$\Rightarrow \big[[A] - \lambda[I]\big][x] = [0]$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

To find values of λ that converts the determiate to zero

$$(-5-\lambda)(-2-\lambda) - 4 = 0$$

$$10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + 7\lambda + 6 = 0$$

$$(\lambda + 1)(\lambda + 6) = 0$$

نطلع λ (Eigenvalues) ⟵ نطلع eigenvectors

No.

$\lambda_1 = -6$
$\lambda_2 = -1$ } eigenvalus.

$\lambda_1 = -6$

$$\begin{vmatrix} -5+6 & 2 \\ 2 & -2+6 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \Rightarrow Det = 4-4 = \underbrace{0}_{\text{singular matrix.}}$$

To find the igenvectors ، نحل المعادلات،

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

For the $\lambda_1 = -6$

singluar system.

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

اختار معادلة وحدة فقط

نفسها

$$\left.\begin{array}{l} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{array}\right\} \quad \begin{array}{l} x_1 + 2x_2 = 0 \\ x_1 = -2x_2 \end{array}$$

two unknows

احنا ما بنحل المعادلات x

بنطلع النسبة بين المتغيرات بس

افرض x_1 اي رقم بدك اياه
(يفضل تفرض المعامل)

if we assume avalue ofor x_1 the x_2 can be obtained

→ $\textcircled{+1}x_1 = -2x_2$

كلهم بحققوا المعادلة

if $x_1 = -2$, then x_2 will equal $(+1)$

$x_1 = -4$, then $x_2 \to +2$

$x_1 = -1$, the $x_2 = \frac{1}{2}$

اي رقم عادي

→ $2x_1 + 4x_2 = 0 \Rightarrow x_1 = -2x_2$

∴ for $\lambda_1 = -6$

igen vector $x = \begin{vmatrix} -2 \\ 1 \end{vmatrix}, \begin{vmatrix} -4 \\ 2 \end{vmatrix}, \begin{vmatrix} -1 \\ \frac{1}{2} \end{vmatrix}$

For $\lambda_2 = -1$

$$\begin{vmatrix} -5+1 & 2 \\ 2 & -2+1 \end{vmatrix} \begin{vmatrix} \overset{\textcircled{2}}{x_1} \\ \overset{\textcircled{2}}{x_2} \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \Rightarrow \begin{vmatrix} -4 & 2 \\ 2 & -1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$(-5+1)(-2+1) - 4 = 0 \rightarrow$ singular.

$\rightarrow -4x_1 + 2x_2 = 0$

$2x_1 - x_2 = 0$

$\rightarrow -4x_1 = -2x_2$

$2x_1 = x_2$

ايه داه

if
$x_1 = 1 \rightarrow x_2 = 2$
$x_1 = -2 \rightarrow x_2 = -4$
$x_1 = 0.5 \rightarrow x_2 = 1$

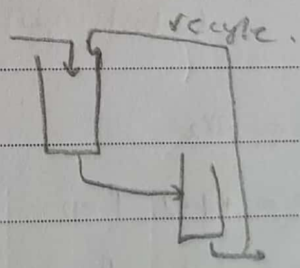
For $\lambda_2 = -1$, the eigenvector

$$\begin{matrix} \overset{\textcircled{2}}{x_1} \\ \overset{\textcircled{2}}{x_2} \end{matrix} = \begin{vmatrix} 1 \\ 2 \end{vmatrix} , \begin{vmatrix} -2 \\ -4 \end{vmatrix} , \begin{vmatrix} 0.5 \\ 1 \end{vmatrix}$$

Homogeneous linear system with constant cofficients.

$$\frac{dy_1}{dt} = a_{11} y_1 + a_{12} y_2$$

$$\frac{dy_2}{dt} = a_{21} y_1 + a_{22} y_2$$



where $a_{11}, a_{12}, a_{21}, a_{22} \Rightarrow$ real constant.

For a system of linear equation we will try to find a solution by assuming $y_1 = Ae^{\lambda t}$, $y_2 = Be^{\lambda t}$

where $A, B, \lambda \Rightarrow$ are constants.

$$\acute{y}_1 = A \lambda e^{\lambda t}$$

$$\acute{y}_2 = B \lambda e^{\lambda t}$$

عوض $\rightarrow$ $A \lambda e^{\lambda t} = a_{11} (A e^{\lambda t}) + a_{12} (B e^{\lambda t})$

$$B \lambda e^{\lambda t} = a_{21} (A e^{\lambda t}) + a_{22} (B e^{\lambda t})$$

$$\Rightarrow (a_{11} A e + a_{12} B - A\lambda) e^{\lambda t} = 0$$

$$(a_{21} A + a_{22} B - B\lambda) e^{\lambda t} = 0$$

$e^{\lambda t} \neq$ Zero

$$a_{11} A + a_{12} B - A\lambda = 0$$

$$a_{21} A + a_{22} B - B\lambda = 0$$

where A, B are unknows [eigenvalues]

λ is the eigenvectors

No.

General Form:- (فقط معادلتين)

$$\begin{aligned} y'_1 &= a_{11} y_1 + a_{12} y_2 \\ y'_2 &= a_{21} y_1 + a_{22} y_2 \end{aligned}\Bigg]$$

لا نستطيع حل y لحالها ، لأن المعادلتين

يعتمدوا على بعض

(طريقة الحل ٤ خطوات):-

١) أكتب الـ system على شكل ⇐ matrix

٢) 3 cases ⟵ الـ eigenvalus

٣) نعوض في الـ system ⟵ نطلع معي eigen Vectors

④ ↙ معادلتين ⇐ 2 arbitiy constant

* Not: Singular لتعرف order

Step 1 (system ⇒ matrix)

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} y'_1 \\ y'_2 \end{vmatrix}$$

matrix

non singular ⇒ Det [A] ≠ zero

بدنا نطلع متى تكون non singular مشان نشيلها من الحل النهائي

Step 2 (([matrix] - λ [unit matrix]) [eigenvector] = [0])

[matrix] (singular) ⇒ non singular ↩

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\underbrace{\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}}_{Z} \begin{vmatrix} A \\ B \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

Det [Z] = 0 ⟶ Singular

$$\text{Det}[z] :\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$$\lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) = 0$$

ⓒ

لـ عبارة تربيعية

$$\lambda = \frac{(a_{11}+a_{22}) \mp \sqrt{\underbrace{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{12})}_{ⓒ}}}{2}$$

if $c > \text{zero}$, λ_1 and λ_2 real distinct roots.

if $c = \text{zero}$, $\lambda_1 = \lambda_2 = \frac{(a_{11}+a_{22})}{2}$, repeated roots.

if $c < \text{zero}$, λ_1 and λ_2 are complex

Step 3

substitute in the general system of D.E by λ_1 and λ_2 to obatin the eigen vector.

λ_1:-

$$\begin{vmatrix} a_{11}-\lambda_1 & a_{12} \\ a_{21} & a_{22}-\lambda_1 \end{vmatrix} \begin{vmatrix} A_1 \\ B_1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

→ بنحل معادلتين بمجهولين

وبيطلع $\begin{vmatrix} A_1 \\ B_1 \end{vmatrix}$

λ_2:-

$$\begin{vmatrix} a_{11}-\lambda_2 & a_{12} \\ a_{21} & a_{21}-\lambda_2 \end{vmatrix} \begin{vmatrix} A_2 \\ B_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \rightarrow \begin{vmatrix} A_2 \\ B_2 \end{vmatrix}$$

Step 4

the general solution will be

$$\left.\begin{aligned} y_1 &= c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 x} \\ y_2 &= c_1 B_1 e^{\lambda_1 x} + c_2 B_2 e^{\lambda_2 x} \end{aligned}\right] \quad y_g = c_1 \begin{vmatrix} A_1 \\ B_1 \end{vmatrix} e^{\lambda_1 x} + c_2 \begin{vmatrix} A_2 \\ B_2 \end{vmatrix} e^{\lambda_2 x}$$

Case 1 : The roots of the caracheristic equation : are real and distinct ⇒ $\lambda_1 \neq \lambda_2$ and not complex.

Then the system has two linarly independent solutions of the Form :-

هاد بطلع منه

$$y_1 = A_1 e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = A_2 e^{\lambda_2 x}$$

$$y_2 = B_1 e^{\lambda_1 x} \qquad \qquad y_2 = B_2 e^{\lambda_2 x}$$

where A_1, A_2, B_1, B_2 are definite constants

The general solution of system :-

$$y_{1g} = c_1 A_1 e^{\lambda_1 x} + c_2 A_2 e^{\lambda_2 x}$$

$$y_{2g} = c_1 B_1 e^{\lambda_2 x} + c_2 B_2 e^{\lambda_2 x}$$

A, B → ارقام

eigen vector من

c_1, c_2 → condition.

Where c_1, c_2 are the arbitrary constant

Example :-

$$\bar{y}_1 = 6y_1 - 3y_2$$

$$\bar{y}_2 = 2y_1 + y_2$$

steps

1) $$\begin{vmatrix} 6 & -3 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} \bar{y}_1 \\ \bar{y}_2 \end{vmatrix}$$

ح نشتغل ع

هاي بس

2) $\begin{vmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$(6-\lambda)(1-\lambda)+6 = \text{Zero}$ (Det)

$6-6\lambda-\lambda+\lambda^2+6=0$

$\lambda^2-7\lambda+12=0$

$\lambda = \frac{7 \mp \sqrt{49-48}}{2} = \frac{7 \mp 1}{2} \Rightarrow \left.\begin{matrix}\lambda_1 = 4 \\ \lambda_2 = 3\end{matrix}\right\}$ two real distinct roots.

3) for $\lambda_1 = 4$

$\begin{vmatrix} 6-4 & -3 \\ 2 & 1-4 \end{vmatrix} \begin{vmatrix} A_1 \\ B_1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \rightarrow \begin{vmatrix} 2 & -3 \\ 2 & -3 \end{vmatrix} \begin{vmatrix} A_1 \\ B_1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$\rightarrow \left.\begin{matrix} 2A_1 - 3B_1 = 0 \\ 2A_1 - 3B_1 = 0 \end{matrix}\right\}$ They are dependent

→ اختار اي وحدة عادي

$\rightarrow 2A_1 = 3B_1$

if $A_1 = 3$ then $B_1 = 2$ (نقدر، نفرض اي رقم، لكننا نختار العوامل عكس)

$\therefore$ the first Eigenvector $= \begin{vmatrix} A_1 \\ B_1 \end{vmatrix} = \begin{vmatrix} 3 \\ 2 \end{vmatrix}$

$\rightarrow \left.\begin{matrix} y_1 = c_1 (A_1) e^{2x} = c_1 (3) e^{4x} \\ y_2 = c_2 (2) e^{4x} \end{matrix}\right\}$ this is part of solution

For $\lambda_2 = 3$

$\begin{vmatrix} 6-3 & -3 \\ 2 & 1-3 \end{vmatrix} \begin{vmatrix} A_2 \\ B_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \rightarrow \begin{vmatrix} 3 & -3 \\ 2 & -2 \end{vmatrix} \begin{vmatrix} A_2 \\ B_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

اي معادلة عادي

$$\rightarrow \begin{cases} 3A_2 - 3B_2 = 0 \\ 2A_2 - 2B_2 = 0 \end{cases}$$

$$3A_2 = 3B_2$$

$A_2 = B_2 \Rightarrow$ if $A_2 = 1$ then $B_2 = 1$

the second eigen vector $= \begin{matrix} A_2 \\ B_2 \end{matrix} = \begin{matrix} 1 \\ 1 \end{matrix}$

$$y_1 = c_2 A_2 e^{\lambda_2 x} = c_2 (1) e^{3x}$$

$$y_2 = c_2 (1) e^{3x}$$

4) the general solution

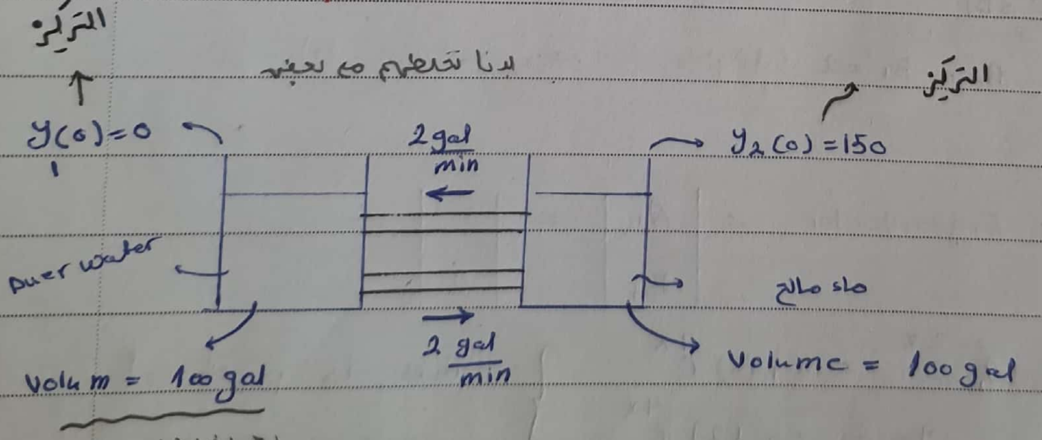
$$y_g = c_1 \begin{vmatrix} 3 \\ 2 \end{vmatrix} e^{4x} + c_2 \begin{vmatrix} 1 \\ 1 \end{vmatrix} e^{3x}$$

$$y_{1g} = 3c_1 e^{4x} + c_2 e^{3x}$$

$$y_{2g} = 2c_1 e^{4x} + c_2 e^{3x}$$

$\Rightarrow$ Apply the conditions here to slouLion c_1 and c_2

Example



رح يضل ثابت

بطلع 2 و دبدخل 2 .

التركيز = الداخل - الخارج

at the stady state

كم الـConcentration ؟

في tank 1 & 2

← في نهاية العملية رح تتساوى التركيز لكل tank 75

ولكن احنا بدنا عند any time

$$\frac{dy_1}{dt} = \underbrace{\overset{\text{out}}{\frac{2}{100}}}_{\text{volume}} \underbrace{y_2}_{\text{concentration}} - \overset{\text{in}}{\frac{2}{100}} y_1$$

Conditions:-

$y_1(0) = 0$, $y_2(0) = 150$

$$\frac{dy_2}{dt} = \frac{2}{100} y_1 - \frac{2}{100} y_2$$

Case 2

if the roots of the characeristic equation:-

$\lambda_1 = \lambda_2$ are real and equal then system has two einearly independent Solution of the Form:-

$$y_1 = A_1 e^{\lambda x}$$

$$y_2 = B_2 e^{\lambda}$$

تكرار

$$y_1 = (A_2 \underline{\underline{x}} + A_3) e^{\lambda x} \rightarrow A_2 x e^{\lambda x} + A_3 e^{\lambda x}$$

$$y_2 = (B_2 \underline{\underline{x}} + B_3) e^{\lambda x} \rightarrow B_2 x e^{\lambda x} + B_3 e^{\lambda x}$$

اذا كانت B_3 ، A_3 تختلف عند قيم A_1 و B_1 لا تجمعهم مع بعض [فليهم زي ماهما

و اذا A_1, B_1 $\xrightarrow{\text{نفس قيم}}$ A_3 و B_3 اجمعهم مع بعض

$$y_g = c_1 \begin{vmatrix} A_1 \\ B_1 \end{vmatrix} e^{\lambda x} + c_2 \left[\begin{vmatrix} A_2 \\ B_2 \end{vmatrix} x + \begin{vmatrix} A_3 \\ B_3 \end{vmatrix} \right] e^{\lambda x}$$

Example:-

$\acute{y}_1 = 4y_1 - y_2$ → linear , homo, constant.

$\acute{y}_2 = y_1 + 2y_2$

$$\begin{vmatrix} 4 & -1 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} \acute{y}_1 \\ y_2 \end{vmatrix}$$

Step 1: Find eigen value ⇒

$$\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$(4-\lambda)(2-\lambda) + 1 = 0 \Rightarrow 8 - 4\lambda - 2\lambda + \lambda^2 + 1 = 0$

$\lambda^2 - 6\lambda + 9 = 0$

$\lambda = \dfrac{6 \mp \sqrt{36-36}}{2}$, $\lambda_1 = \lambda_2 = 3$

↲ تكرار repeated دفعلع للاول

زي مانعودنا / الثاني بتعين الفرضه

For $\lambda = 3$:-

$$\begin{vmatrix} 4-3 & -1 \\ 1 & 2-3 \end{vmatrix} \begin{vmatrix} A_1 \\ B_1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} A_1 \\ B_1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

↳ Det = 0

$A_1 - B_1 = 0 \rightarrow A_1 = B_1$ if $A_1 = 1$, $B_1 = 1$

eigen vector

♡ $\overset{(1)}{y}_1 = (1)\, e^{3x}$

$\overset{(1)}{y}_2 = (1)\, e^{3x}$ $\Rightarrow \overset{(1)}{y} = \begin{matrix} 1 \\ 1 \end{matrix} \; e^{3x}$

Secand part of solution :- ماضي λ_2 عشان نحل زي الاول بنفرض

$\overset{(2)}{y}_1 = (A_2 x + A_3)\, e^{3x}$

$\overset{(2)}{y}_2 = (\Lambda_2 x + B_3)\, e^{3x}$

} take these and substitute in the original system of 1.O.D.E

System:- $y_1' = 4y_1 - y_2$

$y_2' = y_1 + 2y_2$

Su $\overset{(2)}{y}_1{}' = A_2 e^{3x} + 3(A_2 x + A_3)\, e^{3x}$ → عوضنا في y_1'

$\overset{(2)}{y}_2{}' = B_2 e^{3x} + 3(B_2 x + B_2)\, e^{3x}$ → ⊠ y_2' لعوضها

→ $A_2 e^{3x} + 3(A_2 x + A_3)\, e^{3x} = 4(A_2 x + A_3) - (B_2 x + B_3)\, e^{3x}$

$$\boxed{*} \rightarrow A_2 \underline{e^{3x}} + 3A_2 x \cancel{e^{3x}} + 3A_3 \cancel{e^{3x}} = 4A_2 x \cancel{e^{3x}} + 4A_3 \cancel{e^{3x}} - B_2 x \cancel{e^{3x}} - B_3 \cancel{e^{3x}}$$

$$(A_2 + 3A_3) + (3A_2 x) = 4A_3 - B_3 + (4A_2 - B_2)x$$

$$3A_2 x = (4A_2 - B_2)x$$

$$(A_2 - B_2) = \text{Zero}$$

$$B_2 = A_2$$

$$A_2 + 3A_3 = 4A_3 - B_3$$

$$A_2 + B_3 = A_3$$

$\boxed{*}$

$$B_2 \cancel{e^{3x}} + 3B_2 x \cancel{e^{3x}} + 3B_3 \cancel{e^{3x}} = A_2 x \cancel{e^{3x}} + A_3 \cancel{e^{3x}} + 2B_2 x \cancel{e^{3x}} + 2B_3 \cancel{e^{3x}}$$

$$(B_2 + 3B_3) + 3B_2 x = (A_3 + 2B_3) + (2B_3 + A_2)x$$

$$3B_2 x = (A_2 + 2B_2)x$$

ما رح يفيد في الحل ابداً. ← $B_2 = A_2$

اختار معادلتين من ٤ دله

$$B_2 + 3B_3 = A_3 + 2B_3$$

$$B_3 = A_3 - B_2$$

no need for this

$\boxed{*} \rightarrow$ if $B_2 = A_2 \rightarrow A_2 = 1 ,\ B_2 = 1$

افرض اي رقم
ما بأثر بح

$A_2 + B_3 = A_3 \rightarrow 1 + B_3 = A_3$

زي ما بدك افرض

$\Rightarrow$ if $A_3 = 1 \rightarrow B_3 = \text{Zero}$

♡ $\overset{②}{y_1} = ((1) + x(1))\, e^{3x}$

$\overset{③}{y_2} = ((1)x + \text{zero})\, e^{3x}$

ما بدمج للانهم مش زي بعض

$$y_g = c_1 \begin{vmatrix} 1 \\ 1 \end{vmatrix} e^{3x} + c_2 \left[\begin{vmatrix} 1 \\ 1 \end{vmatrix} x + \begin{vmatrix} 1 \\ 0 \end{vmatrix} \right] e^{3x}$$

$$y_{1g} = c_1 e^{3x} + c_2 x e^{3x} + c_2 e^{3x} \rightarrow (c_1 + c_2) e^{3x} + c_2 e^{3x} x$$

$$y_{2g} = c_1 e^{3x} + c_2 x e^{3x}$$

$y_{1g} = c_3 e^{3x} + c_2 x e^{3x}$

$y_{2g} = c_1 e^{3x} + c_2 x c^{3x}$ $\Rightarrow$ Appy the condition here

بعوض فيهم الثنتين

Case 3 :-

The roots of the characteristic equation are complex

$\lambda_1 = a + bj$, $\lambda_2 = a - bj$

if the roots λ_1 and λ_2 are both complex number $(a \mp bi)$, then the solution will based on one of the two roots λ_1 or λ_2 and will be the same.

واحدة باخذ منهم فقط

For $\lambda_1 = a + bi$, $y_1 = Ae^{(a+bi)x} = Ae^{ax} \cdot e^{bix}$

$y_2 = Be^{(a+bi)x} = Be^{ax} \cdot e^{bix}$

$e^{bix} = \cos(bx) + i \sin(bx)$

$\therefore y_1 = Ae^{ax} [\cos(bx) + i\sin(bx)]$

$y_2 = Be^{ax} [\cos(bx) + i\sin(bx)]$

each y_1 and y_2 are cansist of two parts one real and another complex. (imaginary)

$$y_1 = \underbrace{Ae^{ax} \cos(bx)}_{\text{real}} + \underbrace{Aie^{ax} \sin(bx)}_{\text{complex}}$$

$$y_2 = \underbrace{Be^{ax} \cos(bx)}_{\text{real}} + \underbrace{Bie^{ax}\sin(bx)}_{\text{complex}}$$

$$y_g = \underbrace{c_1 \left| \begin{matrix} A \\ B \end{matrix} \right| e^{ax} \cos(bx)}_{\text{real}} + \underbrace{c_2 \left| \begin{matrix} A \\ B \end{matrix} \right| i e^{ax} \sin(bx)}_{\text{Imaginary}}$$

بدنا نشيلها

Example:-

$\acute{y}_1 = 3y_1 + 2y_2$

$\acute{y}_2 = -5y_1 + y_2$

$$\begin{vmatrix} 3 & 2 \\ -5 & 1 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} \acute{y}_1 \\ \acute{y}_2 \end{vmatrix} \Rightarrow$$

$$\text{Det} \begin{vmatrix} 3-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(1-\lambda) + 10 = 0$$

$$3 - 3\lambda - \lambda + \lambda^2 + 10 = 0$$

$$\lambda^2 - 4\lambda + 13 = 0$$

$$\lambda = \frac{4 \mp \sqrt{16-52}}{2}$$

$$= \frac{4 \mp \sqrt{6i}}{2} \Rightarrow \lambda_1 = 2 + 3i \quad \text{و} \quad \lambda_2 = 2 - 3i$$

لبنأخذ وحدة فقط

For $\lambda_1 = 2 + 3i$

$$\begin{vmatrix} 3-2-3i & 2 \\ -5 & 1-2-3i \end{vmatrix} \begin{vmatrix} A_1 \\ B_1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \Rightarrow \begin{vmatrix} 1-3i & 2 \\ -5 & -1-3i \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

باخذ صف واحد

$(1-3i)(-1-3i) + 10 \overset{?}{=} 0 \Rightarrow \text{Det} = 0$ ✓

$-1 - 3i + 3i + 9 + 10 = 0$

⊛ هسا بدنا نشوف هل المصفوفة sing ولا

اذا sing ← يعني المعادلات تعتمد على بعضها (نشطبو بعضهم)

فبوخذ بس صف واحد من المصفوفة بنفرضو ب A وB ونطلع القيم

$(1-3i)A + 2B = 0 \longrightarrow 2B = (3i-1)A$ ✓ اختار راسة

$-5A + (-1-3i)B = 0 \rightarrow -3iB = 1 + 5A$

⊛ اذا non ← singular

المصفوفة لا تعتمد على بعضها وبذلك

لازم اطلع معادلتين وحدة في الصف الاول

ووحدة في الصف الثاني واطلع المطلوب قيمة A وB

✓ if $A = 2$ then $B = 3i - 1$

No.

$$y = A e^{(a+bi)x} = 2 e^{(2+3i)x} = 2 \cdot e^{2x} \cdot e^{3ix} = 2e^{2x}[\cos(3x) + i \sin(x)]$$

$$y_2 = B e^{(a+bi)x} = (3i-1) e^{(2+3i)x} = (3i-1) e^{2x} [\cos(3x) + i \sin(x)]$$

$$\rightarrow y_1 = \underbrace{2e^{2x} \cos(3x)}_{\text{real}} + \underbrace{2e^{2x} i \sin(3x)}_{\text{imaj}}$$

$$y_1 (\text{real}) \Rightarrow 2e^{2x} \cos(3x), \quad y_1 (\text{im}) = 2e^{2x} \sin(3x)$$

$$\rightarrow y_2 = 3i e^{2x} \cos(3x) - 3e^{2x} \sin(3x) - e^{2x} \cos(3x) - ie^{2x} \sin(x)$$

$$y_2 (\text{Real}) \Rightarrow -e^{2x} [3 \sin(3x) + \cos(3x)]$$

$$y_2 (\text{ima}) \Rightarrow e^{2x} [3 \cos(3x) - (1) \sin(3x)]$$

$$y_g = c_1 \begin{bmatrix} 2e^{2x} \cos(3x) + 0 \sin(3x) \\ -2e^{2x} [3\sin(3x) + \cos(3x)] \end{bmatrix}^{21} + c_2$$

No.

example:-

$y_1 = 0y_1 - 2y_2$

$y_2 = 2y_1 - 0y_2$

$$\begin{vmatrix} 0-\lambda & -2 \\ 2 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 4 = 0 \qquad \lambda = 0 \mp 2i$$

$\lambda_1 = 2i$

$$\begin{vmatrix} -2i & -2 \\ 2 & -2i \end{vmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\alpha (-2i)(-2i) + 4 = 0 \rightarrow 4$

$-2iA - 2B = 0 \rightarrow -2iA = 2B \Rightarrow -iA = B$

$A = 1 \rightarrow B = -i$

$y = Ae^{(0+2i)} = e^{0x}[\cos(2x) + i\sin(2x)]$

$y_2 = Be^{(0+2i)} = -ie^{0x}[\cos(2x) + i\sin(2x)]$

$= -ie^{0x}\cos(2x) + e^{0x}\sin(2x)$

$y_g = a \Big[\cos(2x)$

Example:-

$y'''' + y''' - 2y'' = -4x^2 + 18$ → constant , non-Homo

$y(1) = \frac{-3}{2}$, $y'(1) = \frac{-10}{3}$, $y''(1) = -2$, $y'''(1) = 6$

$\Rightarrow y_g = y_h + y_p$

for $y_h \Rightarrow y'''' + y''' - 2y'' = 0$

let $y = e^{\lambda x}$

$(\lambda^4 + \lambda^3 - 2\lambda^2) e^{\lambda x} = 0$

$\lambda^2 (\lambda^2 + \lambda - 2) e^{\lambda x} = 0$

$\neq$ Zero ↲ ↳ = Zero

وحمة رح → $\lambda_1 = \lambda_2 = 0$

نضربها بـ(x) .

$\rightarrow \lambda_{3,4} = \frac{-1 \mp 1+8}{2}$ ↱ $\lambda_3 = -2$

$\lambda_4 = 1$

$\therefore y_h = c_1 e^{0x} + c_2 e^{0x} + c_3 e^{-2x} + c_4 e^{x}$

$y_h = c_1 + \underline{c_2 x} + c_3 e^{-2x} + c_4 e^{x}$

لأنه مكرر

$y_p \Rightarrow r(x) = -4x^2 + 18$ تكرار

$y_p = Ax^2 + \underline{Bx} + Z$

$y_p = Ax^3 + Bx^2 + Zx$ تكرار

$y_p = Ax^4 + Bx^3 + Zx^2$

$y_p' = 4Ax^3 + 3Bx^2 + 2Zx$

$y_p'' = 12Ax^2 + 6Bx + 2Z$

$y_p''' = 24Ax + 6B$

$y_p'''' = 24A$

No.

$$y'''' + y''' - 2y'' = -4x^2 + 18$$

$$\underline{24A} + \underline{24Ax} + \underline{6B} - \underline{24Ax^2} - \underline{12Bx} - \underline{4Z} = \underline{-4x^2} + \underline{18}$$

$$-24A\,\cancel{x^2} = -4\,\cancel{x^2}$$

$$\rightarrow A = \frac{1}{6}$$

$$\overset{4}{\cancel{24}} \times \frac{1}{\cancel{6}}x - 12Bx = 0x$$

$$4x - 12Bx = 0x \Rightarrow x(4 - 12B) = 0x$$

$$\rightarrow B = \frac{4}{12} = \frac{1}{3}$$

$$24\left(\frac{1}{6}\right) + 6\left(\frac{1}{3}\right) - 4Z = 18$$

$$4 + 2 - 4Z = 18 \Rightarrow -4Z = 12$$

$$\rightarrow Z = -3$$

$$y_p = \frac{1}{6}x^4 + \frac{1}{3}x^3 - 3x^2$$

$$y_g = c_1 + c_2x - 3x^2 + \frac{1}{3}x^3 + \frac{1}{6}x^4 + c_3e^{-2x} + c_4e^{x}$$

$$y_g' = c_2 - 6x + x^2 + \frac{2}{3}x^3 - 2c_3e^{-2x} + c_4e^{x}$$

$$y_g'' = -6 + 2x + 2x^2 + 4c_3e^{-2x} + c_4e^{x}$$

$$y_g''' = 2 + 4x - 8c_3e^{-2x} + c_4e^{x}$$

$$y_g'''' = 4 + 16c_3e^{-2x} + c_4e^{x}$$

$$c_1 + c_2 - 3 + \frac{1}{3} + \frac{1}{6} + c_3e^{-2} + c_4e = -\frac{3}{2}$$

$$c_1 + c_2 + c_3e^{-2} + c_4e = 1 \quad \text{-----} ①$$

$$0 + c_2 - 2c_3e^{-2} + c_4e' = 1' \quad \text{---} ②$$

$$0 + 0 + 4c_3e^{-2} + c_4e = 0 \quad \text{----} ③$$

$$0 + 0 + 4c_3e^{-2} + c_4e = 2 \quad \text{---} ④$$

حل لحالك زهقت

$$c_1 = 0,\ c_2 = 1,\ c_3 = 0,\ c_4 = 0$$

$$y_p = x + \frac{1}{6}x^4 + \frac{1}{3}x^3 - 3x^2$$

No.

→ using variation method

$$y''' - 6y'' + 11y' - 6y = e^{2x}\cos(x)$$

$$y = y_h + y_p$$

$$y_h \Rightarrow \quad y''' - 6y'' + 11y' - 6y = 0$$

let $y = e^{\lambda t} \Rightarrow$ (constant cofficient)

$$(\lambda^3 - 6\lambda^2 + 11\lambda - 6)e^{\lambda x} = 0$$

$$(\lambda_1 - 1)(\lambda^2 - 5\lambda + 6) = 0$$

$$\lambda_{2,3} = \frac{5 \mp \sqrt{25-24}}{2}$$

$$\lambda_3 = 3 \quad , \lambda_2 = 2$$

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ 1 & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$y_h \Rightarrow \quad y_h = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

$y_p \Rightarrow$

$$W = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{x}(18e^{5x} - 12e^{5x}) - e^{2x}(9e^{4x} - 3e^{4x}) + e^{3x}(4e^{3x} - 2e^{3x})$$

$$= e^{x}(6e^{5x}) - e^{2x}(6e^{4x}) + e^{3x}(2e^{3x})$$

$$= \cancel{6e^{6x}} - \cancel{6e^{6x}} + 2e^{6x} = 2e^{6x}$$

$$W_1 = \begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ 1 & 4e^{2x} & 9e^{3x} \end{vmatrix} = 0 - e^{2x}(0 - 3e^{3x}) + e^{3x}(0 - 2e^{2x})$$

$$= 3e^{5x} - 2e^{5x} = e^{5x}$$

$$W_2 = \begin{vmatrix} e^{x} & 0 & e^{3x} \\ e^{x} & 0 & 3e^{3x} \\ e^{x} & 1 & 9e^{3x} \end{vmatrix} = e^{x}(0 - 3e^{3x}) - 0 + e^{3x}(e^{x} - 0)$$

$$= -3e^{4x} + e^{4x} = -2e^{4x}$$

$$\omega_3 = \begin{vmatrix} e^x & e^{2x} & 0 \\ e^x & 2e^{2x} & 0 \\ e^x & 4e^{2x} & 1 \end{vmatrix} = e^x(2e^{2x}-0) - e^{2x}(e^x)$$

$$= 2e^{3x} - e^{3x} = e^{3x}$$

$$y_p = y_1 \int \frac{\omega_1}{\omega}\, dx + y_2 \int \frac{\omega_2}{\omega} r(x)\, dx + y_3 \int \frac{\omega_3}{\omega} r(x)\, dx$$

$$e^x \int \frac{e^{5x}}{2e^{6x}} \cdot \cancel{2} e^{2(x)} \cos(x)\, dx + e^{2x} \int \frac{-\cancel{2}e^{4x}}{\cancel{2}e^{6x}} e^{2x} \cos(x)\, dx + e^{3x} \int \frac{e^{3x}}{2e^{6x}} e^{2x} \cos(x)$$

$$= \frac{1}{2} e^x \int e^x \cos(x)\, dx - e^{2x} \int \cos(x)\, dx + \frac{1}{2} e^{3x} \int e^{-x} \cos(x)\, dx$$

$\Rightarrow \int e^x \cos(x)\, dx \qquad u = e^x \qquad dV = \cos(x)\, dx$

$du = e^x \qquad V = \sin(x)\, dx$

$e^x \sin(x) - \int e^x \cos(x)\, dx$

Hind : $\int \cos(ax)\, e^{bx}\, .dx \Rightarrow \frac{e^{bx}}{a^2+b^2} \left(b\sin(ax) + a\cos(ax) \right)$

$\rightarrow a = 1 \; , b = 1 \Rightarrow \frac{1}{2} e^x \left[\frac{e^x}{2} \left(\sin(x) + \cos(x) \right) = \frac{1}{4} e^{2x} \left(\sin(x) + \cos(x) \right) \right.$

$\rightarrow -e^{2x} \int \cos(x)\, dx \Rightarrow -e^{2x} \sin(x)$

$\rightarrow a = 1 \; , b = -1 \Rightarrow \frac{1}{2} e^{3x} \left[\frac{e^{-x}}{2} \left(-\sin(x) + \cos(x) \right) \right] = \frac{1}{4} e^{2x} \left(-\sin(x) + \cos(x) \right)$

$$y_p = \frac{1}{4} e^{2x} (\sin(x) + \cos(x)) - e^{2x} \sin(x) + \frac{1}{4} e^{2x} (-\sin(x) + \cos(x))$$

$$= \frac{1}{4} e^{2x} \sin(x) + \frac{1}{4} e^{2x} \underline{\cos(x)} - \frac{4}{4} e^{2x} \underline{\sin(x)} + -\frac{1}{4} e^{2x} \underline{\sin(x)} + \frac{1}{4} e^{2x} \underline{\cos(x)}$$

$$y_p = -e^{2x} \sin(x) + \frac{1}{2} e^{2x} \cos(x)$$

$y_g =$

Example:-

$$\acute{y}_1 = y_1 - 5y_2 \qquad y_1(0) = 1 \ , \ y_2(0) = 1$$
$$\acute{y}_2 = y_1 - 3y_2$$

$$\begin{vmatrix} 1 & -5 \\ 1 & -3 \end{vmatrix} \begin{vmatrix} y_1 \\ y_3 \end{vmatrix} = \begin{vmatrix} \acute{y}_1 \\ \acute{y}_2 \end{vmatrix}$$

Eign Values: $\begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

Det $\begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = 0$ $\qquad (1-\lambda)(-3-\lambda) + 5 = 0$

$$\lambda^2 + 2\lambda + 2 = 0$$

خذي الكومبة دائماً دونيي أجرد ←

$$\therefore \lambda = \frac{-2 \mp \sqrt{4-8}}{2} = \frac{-2 \mp 2i}{2}$$

بري اعوضه بـ matrix for $\lambda_1 = -1 + \cancel{2}i$

بحوله الثانية رح يطلع زيرو. $\lambda_1 = -1 + i$

$\lambda_2 = -1 - i$

$$\begin{vmatrix} 1-(-1+i) & -5 \\ 1 & -3-(-1+i) \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

باخذ معادلة وحدة ← $\begin{vmatrix} 2-i & -5 \\ 1 & -2-i \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} \begin{vmatrix} 0 \\ 0 \end{vmatrix}$ لعوضها الحل

$$(2-i)A - 5B = 0$$

$$(2-i) = 5B \Rightarrow \text{if } A = 5 \therefore B = 2-i$$

$$y_1 = Ae^{\lambda x} = 5e^{(-1+i)x} = 5 \cdot e^{-x} \cdot e^{ix} = 5e^{-x}[\cos(x) + i\sin(x)]$$
$$y_2 = Be^{\lambda x} = (2-i)e^{(-1+i)x} = (2-i)e^{-x} \cdot e^{ix} = (2-i)e^{-x}[\cos(x) + i\sin(x)]$$

خلص نشيلها $\cancel{x}$

$$y_1 = 5e^{-x}\cos(x) + 5\textcircled{i}e^{-x}\sin(x)$$

$$* \ y = \textcircled{c1}[5e^{-x}\cos(x)] + \textcircled{c2}[5e^{-x}\sin(x)]$$

No.

$\nearrow \quad -i \times i = 1$

$y_2 = e^{-x} [\ 2\cos(x) + 2i\sin(x) - i\cos(x) + (\cdot 1)\sin(x)\ ]$

$y_2 = e^{-x} 2\cos(x) + e^{-x}\sin(x) \quad + \quad [\ 2\sin(x) - \cos(x)\ ]\ \textcircled{i}\ e^{-x}$.

خلص بنشيلها .

$y_2 = c_1 [\ 2e^{-x}\cos(x) + e^{-x}\sin(x)\ ] + c_2 [\ 2e^{-x}\sin(x) - e^{-x}\cos(x)\ ]$

ما إلها حومة الحارة .

$$y_g = c_1 \left| \begin{Bmatrix} 5 \\ 2 \end{Bmatrix} e^{-x}\cos(x) + \begin{Bmatrix} \textcircled{0} \\ 1 \end{Bmatrix} e^{-x}\sin(x) \right|$$

$$+ c_2 \left| \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{-x}\cos(x) + \begin{Bmatrix} 5 \\ 2 \end{Bmatrix} e^{-x}\sin(x) \right|$$

$* \; y_1(0) = 1 \rightarrow \quad 1 = c_1 [\ 5\dot{e}\cos(0)\ ] + c_2 [\ 5\dot{e}\sin(0)\ ]$

$1 = c_1\ [\ 5 * (1)\ ] + 0 \quad \rightarrow \quad c_1 = \frac{1}{5}$

$y_2(0) = 1 \rightarrow \quad 1 = \frac{1}{5}\ [\ 2\dot{e}\cos(0) + 0\ ] + c_2\ [\ 0 - 1\ ]$

~ بعوض c_1 و c_2 $\quad 1 = \frac{2}{5} + {}^{-}c_2 \longrightarrow c_2 = -\frac{3}{5}$

No.

و اسم عالم اشتغل
على function مستخدمها
في حل ال D.E

Laplace Transformation (هو function لحل المعادلات التفاضلية)

الشروط :- To use Laplace is solving differential equation

1) linear
2) constant coficients — Huler da lo
3) Initial valus problems at $x=0$ — لازم كل كل ال conditi عند $x=0$ فقط اذا لا ما بقدر استخدم الجادول بدي احول ان لنهاية عند 0 بالسالب

homo ما بترق اذا ال γ

} There is no need to Find y_h and y_p also isnt need to calculate the arbitrary constant

← كل ال function ميلاتها .

consider the function $f(t)$. The Laplace transform $F(s)$ of the functions $F(t)$ is defind as Follows

$$\underset{\text{transfar function}}{F(s)} = \mathcal{L}(f(t)) = \int_{0}^{\infty} f(t) \cdot e^{-st}\, dt$$

دائماً ما تتغير إلا اذا كانت initial Value $\neq$ zero

اذا كانت ← 1

} لايجاد ال Laplce لا كل function نستخدم هاد

* نستخدم الجدول لتحول f و F

كل قيم الجدول محسوبة من $\int_{0}^{\infty}$

Example:-

Find the Laplace transform of $f(t)=1$

$$\mathcal{L}(f(t)) = \int_{d}^{\infty} f(t) \cdot e^{-st}\, ds = \int_{0}^{\infty} (1)\, e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st} \Big]_{0}^{\infty} \Rightarrow -\frac{1}{s} e^{-\infty} + \frac{1}{s} \cancel{e^{(0)s}}_{(1)}$$

$e^{\infty} = \infty$

$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

$$-\frac{1}{s}\left[\frac{1}{\infty}\right] + \frac{1}{s} = \frac{1}{s}$$

هاي لاي constant

$$2(f(t)) = F(s) = \frac{1}{s}$$

$$\mathcal{L}(4) = 4 \cdot \mathcal{L}(1) = \frac{4}{s}$$

ال constant بطلع برا.

No.

Example

$f(t) = e^{at}$ Form table $\frac{1}{s-a}$

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at} \cdot e^{-st}\, dt$$

$$= \int_0^\infty e^{(a-s)t}\, dt = \frac{1}{a-s} e^{(a-s)t} \Big]_0^\infty$$

$$= \frac{1}{a-s} \left[e^{(a-s)\infty} - \cancel{e^{(a-s)(0)}} \right]$$

$$= \frac{1}{a-s} \left[\frac{1}{e^{(s-a)\infty}} - 1 \right] = \frac{1}{a-s} * (0-1) = \frac{1}{s-a}$$

خاصية الـ Laplace

(linearity of the Laplace) → بقدر اوزع الـ Laplaceعلى
ما بين ++

$$\mathcal{L}[a f(t) + b g(t)] = \mathcal{L}(a f(t)) + \mathcal{L}(b g(t))$$

$$= a \mathcal{L}(f(t)) + b \mathcal{L}(g(t))$$

where $a, b \Rightarrow$ any constant.

Example

$f(t) = \cosh(at)$ Form table :- $\frac{s}{s^2 - a^2}$

$$\cosh(at) = \frac{1}{2}(e^{at} + e^{-at})$$

$$\Rightarrow \mathcal{L}\left(\frac{1}{2} e^{at} + \frac{1}{2} e^{-at}\right) = \frac{1}{2} \mathcal{L}(e^{at}) + \frac{1}{2} \mathcal{L}(e^{-at})$$

$$\left.\begin{aligned} \mathcal{L}(e^{at}) &= \frac{1}{s-a} \\ \mathcal{L}(e^{-at}) &= \frac{1}{s+a} \end{aligned}\right\} \text{From the tably.}$$

$$= \frac{1}{2} \cdot \frac{1}{s-a} + \frac{1}{2} \cdot \frac{1}{s+a}$$

$$\rightarrow \frac{1}{2} \frac{(s+a)+(s-a)}{(s-a)(s+a)} = \frac{1}{\cancel{2}} * \frac{\cancel{2}s}{s^2-a^2} = \frac{s}{s^2-a^2}$$

Example

$f(t) = t \Rightarrow \mathcal{L}(t) = \frac{1}{s^2}$ (from table)

$$\mathcal{L}(t) = \int_0^\infty t \cdot e^{-st}\, dt \qquad \text{let } u = t \qquad dv = e^{-st}\, dt$$

$$du = dt \qquad v = -\frac{1}{s} e^{-st}$$

$$= -\frac{t}{s} e^{-st}\Big]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= \cancel{-\frac{\infty}{s} e^{-\infty}} + \cancel{0} + \frac{1}{s}\left[-\frac{1}{s} e^{-st}\right]_0^\infty$$

$$= \frac{-1}{s^2}\left[\cancel{e^{-\infty}} - e^0\right] = \frac{1}{s^2}$$

Example:-

$$f(t) = \sin(\omega t) \qquad F(s) = \frac{\omega}{s^2+\omega^2}$$

$$\rightarrow e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

$$\rightarrow e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$$

بحل معادلتين :-

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

$$\Rightarrow \mathcal{L}(\sin(\omega t)) = \mathcal{L}\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right) = \int_0^\infty \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \cdot e^{-st}\, dt$$

$$\Rightarrow \frac{1}{2i}\int_0^\infty \left(e^{i\omega t} - e^{-i\omega t}\right) e^{-st}\, dt = \frac{1}{2i}\left[\int_0^\infty \left(\overset{(i\omega - s)t}{e^{i\omega t} \cdot e^{-st}}\right) dt + \int_0^\infty \overset{(-i\omega - s)t}{e^{-i\omega t} \cdot e^{-st}}\, dt\right.$$

$$\Rightarrow \frac{1}{2i}\left[\int_0^\infty e^{-(-i\omega+s)t}\, dt + \int_0^\infty e^{-(i\omega+s)t}\, dt\right.$$

$$\frac{1}{2i}\left(-\frac{1}{-i\omega+s}(0-1)+\frac{1}{i\omega+s}(0-1)\right)$$

$$\frac{1}{2i}\left(\frac{1}{s-i\omega}-\frac{1}{i\omega+s}\right)=\frac{1}{2i}\left(\frac{i\omega+\cancel{s}-\cancel{s}+i\omega}{(s-i\omega)(i\omega+s)}\right)$$

$$=\frac{1}{\cancel{2i}}*\frac{\cancel{2i}\omega}{(s-i\omega)(i\omega+s)}=\frac{\omega}{\cancel{s\omega i}+s^2+\omega^2-\cancel{si\omega}}=\frac{\omega}{s^2+\omega^2}$$

Laplace transform of derivatives f(t)

if f(t) is a continuous of all $t \geq 0$ and has a derivatie of $f'(t)$ then:-

$$\mathcal{L}(f'(t))=\int_0^\infty f'(t)\cdot e^{-st}\,dt$$

Let $u=e^{-st}$ $\qquad dy = f'\,dt$

$du=-s\,e^{-st}\,dt$ $\qquad v = f(t)$

$$f(t)\,e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}\cdot f(t)\,dt$$

$$\cancel{f(\infty)}\,e^{-\infty}-f(0)\,e^{0}+sF(S)$$

$$\mathcal{L}(f'(t)) = sF(S)-f(0)$$

zero

$$\mathcal{L}(f''(t)) = s^{\textcircled{2}}F(S)-sf(0)-f'(0)$$

$$\mathcal{L}(f'''(t)) = s^{\textcircled{3}}F(s)-s^{\textcircled{2}}f(0)-s^{\textcircled{1}}f'(0)-{}^{\textcircled{0}}f''(0)$$

$$\mathcal{L}(f''''(t)) = s^4F(s)-s^3f(0)-s^2f'(0)-s\,f''(0)-f'''(0)$$

No.

← كيف سي استخدم $F(s)$ في Function :-

$\Rightarrow y'' + a y' + b y = r(t)$

↲ ناخذ Laplace لكل الاطراف

$\longrightarrow \mathcal{L}(y'') + a\,\mathcal{L}(y') + b\,\mathcal{L}(y) = \mathcal{L}(r(t))$

$$\left(s^2 Y(0) - s y(0) - y'(0)\right) + a\left(s Y(0) + y(0)\right) + b\,Y(s) = \mathcal{L}(r(t))$$

← بحل non Hom بخطوة وحدة.

general ما يعطي ⇄ **Laplace Transformation**

partical يعطي
~~solution~~

Inversion of the Laplace

هنا نرجع Laplace ⇐ Function

الجية العكسية

Laplace صار ← $F(s) = \frac{1}{s^2}$ $\Rightarrow$ Inversion $\Rightarrow$ $f(t) = \overset{\text{العكس}}{\mathcal{L}^{\textcircled{-1}}}\left(\frac{1}{s^2}\right) = t$

← رح ناخذ عدة طرق للتحويل العكسي

رح نستخدم الجدول بشكل غير مباشر

Example:-

$$F(s) = \frac{s}{s^2+1}$$

$$\mathcal{L}^{\textcircled{-1}}(F(s)) = \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \mathcal{L}^{-1}\ \frac{s}{s^2+(1)^2} \Rightarrow f(t) = \cos(t)\ \#$$

from the table

$f(t)$	$F(t)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^s}$
$\cos(1t)$	$\frac{s}{s^2+(1)^2}$

Example:-

$$F(s) = \frac{2s-3}{s^2+1}$$

مش موجوه بالجدول. لحادل اخليها شكلها زي يلي بالجدول

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{2s-3}{s^2+1}\right)$$

$$f(t) = \mathcal{L}^{-1}\frac{2s}{s^2+1} - \mathcal{L}^{-1}\frac{3}{s^2+1}$$

$$f(t) = 2\mathcal{L}^{-1}\frac{s}{s^2+1} - 3\mathcal{L}^{-1}\frac{1}{s^2+1}$$

$$\frac{1}{s^2+\omega^2}$$

$$f(t) = 2\cos(t) - 3\sin(t)$$

→ Solving D.E using Laplace:-

حول كل شي ال Laplace (الثوابت ، المعادلات ... كل شي)

1) take the Laplace transforms of both sides of the D.E

بوقف ال place ضع المعادلة الاصلية.

2) solve the algebric equation obtained to determine $Y(s)$ (transform function)

لم كل ال $Y(s)$ على طرف ، وكل شي ثابت على الطرف الآخر (اخذها عامل مشترك)

inversion اعمل لل $Y(s)$

3) Having found $T(s)$, employ the table of transforms ~~the~~ to to determine the solution $y(t) = \mathcal{L}^{-1} Y(s)$ of the initial value problem

خذ ال Laplce :-

$$y(t) = \mathcal{L}\, y(t) = Y(s)$$

$$\mathcal{L}^{-1} Y(s) = y(t)$$

دح يكون independent value ⇐ هو (t)

Examples:-

$$\ddot{y} + y = 16\cos(t) \qquad y(0) = 0 \;,\; \dot{y}(0) = 0$$

No.

اذا تغيرت condition هذا الحد يلي رح يختلف

ناخذ ال Laplace لكل الأطراف :-

- $\mathcal{L}(y'') = s^2 Y(s) - s\,\overset{zero}{\cancel{y(0)}} - \overset{zero}{\cancel{y'(0)}} = \Rightarrow s^2 Y(s)$

$\mathcal{L}(y) = Y(s)$

$\mathcal{L}(16\cos(t)) = 16\,\mathcal{L}(\cos(t)) = \dfrac{16\ s}{s^2+1}$

- $y'' + y = 16\cos(t)$ ال $Y(s)$ في طرف والباقي على طرف آخر.

$s^2 Y(s) + Y(s) = \dfrac{16\,s}{s^2+1} \Rightarrow Y(s)\,(s^2+1) = \dfrac{16s}{s^2+1}$

$\Rightarrow Y(s) = \dfrac{16\,s}{(s^2+1)(s^2+1)}$

From the table:-

$f(t)$	$F(s)$
$\dfrac{t}{2\omega}\sin(\omega t)$	$\dfrac{s}{(s^2+\omega^2)^2}$

- $\rightarrow \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\dfrac{16s}{(s^2+\textcircled{1})^2}\right)$, $\omega = 1$

$\downarrow$

$y(t) = 16 \cdot \dfrac{t}{2\cdot 1}\sin(1\cdot t) \Rightarrow 16\,\dfrac{t}{2}\sin(t)$ #

Example:-

$y'' - 3y' + 2y = 12e^{4t}$ constant co / 2nd non Hom., $y(0) = 1$, $y'(0) = 0$

- $\mathcal{L}(y'') = s^2 Y(s) - s\,y(0) - \overset{0}{\cancel{y'(0)}} \Rightarrow s^2 Y(s) - s$

$\mathcal{L}(y') = s\,Y(s) - y(0) \Rightarrow s\,Y(s) - 1$

$\mathcal{L}(y) = Y(s) \Rightarrow Y(s)$

$\mathcal{L}(12e^{4t}) = 12\,\mathcal{L}(e^{4t}) = 12 * \dfrac{1}{s-4} \Rightarrow \dfrac{12}{s-4}$

$y'' - 3y' + 2y = 12e^{4t}$

$s^2 Y(s) - s - 3(s\,Y(s) - 1) + 2Y(s) = \dfrac{12}{s-4}$

$s^2 Y(s) - s - 3s\,Y(s) + 3 + 2Y(s) = \dfrac{12}{s-4}$

$(s^2 - 3s + 2)\,Y(s) = \dfrac{12}{s-4} + (s-3)$

No.

$$(s^2-3s+2)\,Y(s) = \frac{12+s^2-3s-4s+12}{s-4} \Rightarrow \frac{s^2-7s+24}{s-4}$$

الهدف: لازم اسهله عشان اجده في الجدول

- $$Y(s) = \frac{s^2-7s+24}{(s-4)(s^2-3s+2)} \Rightarrow \frac{s^2-7s+24}{(s-4)(s-1)(s-2)}$$

حلها $\frac{a \mp \sqrt{a^2-4b}}{2}$

دائماً حاول نبسط المقام

"غير موجود في الجدول >> الحل بطريقة الاجزاء (طويلةةةةة)

نستخدم طريقة Partial الحل بطريقة limit ٢٢

لـ عندما يكون المقام مجزء

- $$\frac{s^2-7s+24}{(s-4)(s-1)(s-2)} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{s-4}$$

A, B, C ⇒ constant. لكم بعير e.

$$A = \lim_{s\to2}\left(\cancel{(s-2)} \cdot \frac{s^2-7s+24}{(s-4)\cancel{(s-2)}(s-1)}\right)$$

الاقتران الاصلي

عكان المقام يبطل (ب)

اختصر ثم عوض النهاية.

$$A = \frac{14}{-2} = -7$$

$$B = \lim_{s\to1}\left(\cancel{(s-1)} \cdot \frac{s^2-7s+24}{\cancel{(s-1)}(s-2)(s-4)}\right) = 6$$

$$C = \lim_{s\to4}\left(\cancel{(s-4)} \cdot \frac{s^2-7s+24}{\cancel{(s-4)}(s-2)(s-1)}\right) = 2$$

Laplace $\leadsto Y(s) = \frac{-7}{s-2} + \frac{6}{s-1} + \frac{2}{s-4}$

↓ inverion

احنا بدنا $y(t)$ وليس $Y(s)$

$$\mathcal{L}^{-1}Y(s) = \mathcal{L}^{-1}\left(\frac{-7}{s-2}\right) + \mathcal{L}^{-1}\left(\frac{6}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{2}{s-4}\right)$$

$$y(t) = -7e^{2t} + 6e^{t} + \underbrace{2e^{4t}}_{y_p} \quad \#$$

c_2 c_1

هاد ال final solutions

لو كان رقم غير (0) ← هدا

الحل يتغير كله

order P(s) اقل من order المقام No.

→ In general, it we have a transfor function F(s) then:-

اي اقتران بلاس ومش قادر افكه واختصره مع المقام ،

$$\underset{\text{Laplace}}{\bar{F}(s)} = \frac{P(s)}{q(s)} = \frac{P(s)}{(s-q_1)(s-q_2)\cdots(s-\boxed{q_n})}$$ اصفار المقام

→ $q_1, q_2, \ldots q_n$ ⇒ are the zero's of the denominater

then using partial fractions:-

$$F(s) = \frac{A}{s-q_1} + \frac{B}{s-q_2} + \frac{C}{s-q_3} \;-----$$

1) The numerator values (A, B, C ----) can be valuated as Follows:-

ما يكون المقام linear ،

ماضي تكرار ،

→ $A = \lim_{s \to q_1} \left((s-q_1) * \frac{P(s)}{q(s)} \right)$

→ $B = \lim_{s \to q_2} \left((s-q_2) * \frac{P(s)}{q(s)} \right)$

2) → if there are same repeated roots in the denominater we must expand F(s) as sam of terms

اذا ال const ← 2 power

constant ← 3 power

$$F(s) = \frac{P(s)}{\underbrace{(s-q_1)^{②}}_{\text{في تكرار}}(s-q_2)(s-q_m)} = \frac{A}{(s-q_1)^{②}} + \frac{B}{(s-q_1)^{①}} + \frac{C}{s-q_2} \;----$$

→ $A = \lim_{s \to q_1} \left(\cancel{(s-q_1)} * \frac{P(s)}{\cancel{(s-q_1)}\underbrace{(s-q_2)(s-q_n)}_{\text{Remainder}}} \right)$

→ $B = \lim_{s \to q_1} \left(\frac{d}{ds} (\text{Remainder}) \right)$

لو كان التكرار ٣ مرات

$$\frac{A}{(s-q_1)^{\textcircled{3}}} + \frac{B}{(s-q_1)^{\textcircled{2}}} + \frac{C}{(s-q_1)^{\textcircled{1}}}$$

$$A = \lim_{s \to q_1} \left(\cancel{(s-q_1)} * \frac{P(s)}{\cancel{(s-q_1)}\,(s-q_3)\,\cdots} \right) \quad \xrightarrow{} \text{Remainder}$$

$$B = \lim_{s \to q_1} \left(\frac{d}{ds} \left(\text{Remainder} \right) \right)$$

$$C = \lim_{s \to q_1} \left(\frac{d^2}{ds^2} \left(\text{Remainder} \right) \right) * \frac{1}{2!} \longrightarrow \frac{1}{d!}$$

factorial
derivative

in general $\Rightarrow \dfrac{A_1}{(s-q_1)^{\textcircled{3}}} + \dfrac{A_2}{(s-q_1)^{\textcircled{2}}} + \dfrac{A}{(s-q_1)^{\textcircled{1}}}$

(الحد الاول او الثاني وهكذا) القوى.

$$A_m = \lim_{s \to q_1} \left(\frac{d^{m-1}}{ds^{m-1}} \left((s-q_1)^{\textcircled{n}} * F(s) \right) \right) * \frac{1}{(m-1)!}$$

رقم الحد.

$$A_1 = \lim_{s \to q_1} \left(\frac{d^0}{ds^0} \left((s-\cancel{q_1})^3 * \frac{P(s)}{(s-\cancel{q_1})^3\,(s-q_2)\cdots} \right) \right) ✓$$

$$A_2 = \lim_{s \to q_1} \left(\frac{d^1}{ds} \left((s-q_1)^3 * \frac{P(s)}{(s-q_1)^3 (s-q_2)\cdots} \right) \right) * \frac{1}{1!} ✓$$

$$A_3 = \lim_{s \to q_1} \left(\frac{d^2}{ds^2} \left((s-q_1)^3 * \frac{P(s)}{(s-q_1)^3\cdots} \right) \right) * \frac{1}{2!} ✓$$

Example:- $\ddot{y} - 2\dot{y} - 8y = 0$ $y(0) = 3$, $\dot{y}(0) = 6$

$\mathcal{L}(\ddot{y}) = s^2 Y(s) - s\,\overset{\nearrow 3}{y(0)} - \overset{\nearrow 6}{\dot{y}(0)} = s^2 Y(s) - 3s - 6$

$\mathcal{L}(\dot{y}) = s\,Y(s) - 3$

$\mathcal{L}(y) = Y(s)$

$s^2 Y(s) - 3s \cancel{-6} - 2s\,Y(s) \cancel{+6} - 8\,Y(s) = 0$

$(s^2 - 2s - 8)\,Y(s) = 3s$

$Y(s) = \dfrac{3s}{(s^2 - 2s - 8)} = \dfrac{3s}{(s-4)(s+2)} = \dfrac{A}{s-4} + \dfrac{B}{s+2}$

نزلهم لل first order

$A = \lim\limits_{s \to 4} \left(\cancel{(s-4)} \cdot \dfrac{3s}{\cancel{(s-4)}(s+2)} \right) = \dfrac{12}{6} = 2$

$B = \lim\limits_{s \to -2} \left(\cancel{(s+2)} \cdot \dfrac{3s}{\cancel{(s+2)}(s-4)} \right) = \dfrac{-6}{-6} = 1$

$Y(s) = \dfrac{2}{s-4} + \dfrac{1}{s+2} \Rightarrow \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\dfrac{2}{s-4}\right) + \mathcal{L}^{-1}\left(\dfrac{1}{s+2}\right)$

من الجدول

$y(t) = 2e^{4t} + e^{-2t}$ #

Example

$\ddot{y} - 2\dot{y} + y = e^{t} + t$ $y(0) = 1$, $\dot{y}(0) = 0$

$\mathcal{L}(\ddot{y}) = s^2 Y(s) - s(1) - 0$

$\mathcal{L}(\dot{y}) = s\,Y(s) - 1$

$\mathcal{L}(y) = Y(s)$

$\mathcal{L}(e^{t}) = \dfrac{1}{s-1}$

$\mathcal{L}(t) = \dfrac{1}{s^2}$

$\ddot{y} - 2\dot{y} + y = e^{t} + t$

$\underline{s^2 Y(s)} - s - \underline{2s\,Y(s)} + 2 + \underline{Y(s)} = \frac{1}{s-1} + \frac{1}{s^2}$

$(s^2 - 2s + 1)\,Y(s) = \frac{1}{s-1} + \frac{1}{s^2} + (s-2)$

$\rightarrow (s^2-2s+1)\,Y(s) = \frac{s^2+s-1}{s^2(s-1)} + (s-2)$

$(s-1)^2 \hookleftarrow$

$Y(s) = \frac{s^2+s-1}{s^2(s-1)^3} + \frac{s-2}{(s-1)^2}$ ~ الأفضل يفضلوا هيك لا يتجمع.

$Y(s) = \frac{\cancel{s^2}}{\cancel{s^2}(s-1)^3} + \frac{(s-1)^{\cancel{0}}}{s^2(s-1)^{\cancel{3}\,②}} + \frac{s-2}{(s-1)^2}$

$Y(s) = \frac{1}{(s-1)^3} + \frac{1}{s^2(s-1)^2} + \frac{s-2}{(s-1)^2}$

لاحظ هنا انه ليس اقتران

$\underbrace{\frac{1}{(s-1)^3}}_{①}$ منذ الجدول ｜ $\frac{1}{s^2(s-1)^2}$ ③ فيه تكرار ｜ $\underbrace{\frac{s-2}{(s-1)^2}}_{②}$ فيه تكرار

منذ الجدول القانون

① $\mathcal{L}^{-1}\left(\frac{1}{(s-1)^3}\right) = \frac{1}{(3-1)!}\, t^2 \cdot e^{t} \Rightarrow \frac{1}{2} t^2 e^{t}$

② $\frac{s-2}{(s-1)^2} = \frac{A}{(s-1)^{\boxed{2}}} + \frac{B}{(s-1)}$

تكرار

Remaindar

$A = \lim_{s\to 1} \left(\cancel{(s-1)} \cdot \frac{(s-2)}{\cancel{(s-1)^2}} \right) = -1$

$B = \lim_{s\to 1} \left(\frac{d}{ds}(s-2) \right) \Rightarrow 1$

$\Rightarrow \mathcal{L}^{-1} \frac{-1}{(s-1)^2} + \mathcal{L}^{-1} \frac{1}{s-1} = e^{t} - te^{t}$

③ $\frac{1}{s^2(s-1)^2} = \frac{c_1}{s^2} + \frac{c_2}{s} + \frac{c_3}{(s-1)^2} + \frac{c_4}{s-1}$

$$c_1 = \lim_{s\to 0}\left(\cancel{s^2}\cdot\frac{1}{\cancel{s^2}(s-1)^2}\right) = 1$$

$$c_2 = \lim_{s\to 0}\left(\frac{d}{ds}\cdot\left(\frac{1}{(s-1)^2}\right)\right) = 2$$

$$c_3 = \lim_{s\to 1}\left(\cancel{(s-1)^2}\cdot\frac{1}{s^2\cancel{(s-1)^2}}\right) = 1$$

$$c_4 = \lim_{s\to 1}\left(\frac{d}{ds}\cdot\left(\frac{1}{s^2}\right)\right) = -2$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + 2\,\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) + \mathcal{L}^{-1}\left(\frac{-2}{s-1}\right)$$

$$t + 2 + te^{t} + 2e^{t} = te^{t} - 2e^{t} + 2$$

$$y(t) = \frac{1}{2}t^2e^{t} + e^{t} - \cancel{te^{t}} + \cancel{te^{t}} - 2e^{t} + t + 2$$

$$\therefore y(t) = \frac{1}{2}t^2e^{t} + \cancel{e^{t}} - e^{t} + t + 2$$

....

3) complex roots

طريقتين:-

السابقة new

أو هذه الطريقة

$$Y(s) = \frac{F(s)}{(s+\underline{k_1+ik_2})(s+\underline{k_1-ik_2})} \rightarrow \text{root : } k_1 \pm ik_2$$

if F(s) any function of (s) with real roots and $k_1 \pm ik_2$ are complex roots, then Y(s) can be written as

A,B ⇒ complex umber.

$$Y(s) = F_1(s) + \frac{A}{s+k_1+ik_2} + \frac{B}{s+k_1-ik_2}$$

emil له عدة طريقة

لهم طرق خاصة فيهم .

$s \to -2$

positive

negative

No.

Where A, B complex number and can be obtain by limit theory:

* اذا اخذت positive i ضيف الحدود مثلهم negative i

معناها b بتكون negative

$$A = a_1 + b_1 i \Rightarrow \text{Positive } i$$

هون بيكون ←
$$B = a_1 - b_1 i \Rightarrow \text{negative } i$$

$$\mathcal{L}^{-}(Y(s)) = \mathcal{L}^{-}(F(s)) + \mathcal{L}^{-1}\left(\frac{\overset{\text{constant}}{a_1 + b_1 i}}{s + k_1 + ik_2}\right) + \mathcal{L}^{-1}\left(\frac{a_1 - b_1 i}{s + k_1 - ik_2}\right)$$

roots

$$= \mathcal{L}^{-}(F(s)) + (a_1 + b_1 i)\,\mathcal{L}^{-1}\left(\frac{1}{s + \underbrace{k_1 + ik_2}_{\text{root}}}\right) + (a_1 - b_1 i)\left(\frac{1}{s + \underbrace{k_1 - ik_2}_{\text{root}}}\right)$$

(constant) (constant)

$$= \mathcal{L}^{-1}(F(s)) + (a_1 + b_1 i)\,e^{-(k_1 + ik_2)t} + (a_1 - b_1 i)\,e^{-(k_1 - ik_2)t}$$

$$= \mathcal{L}^{-1}(F(s)) + (a_1 + b_1 i)\,e^{-k_1 t} \cdot e^{-ik_2 t} + (a + b_1 i)\,e^{-k_1 t} \cdot e^{ik_2 t}$$

$$= \mathcal{L}^{-1}(F(s)) + (a_1 + b_1 i)\,e^{-k_1 t}\left[\cos(k_2 t) - i\sin(k_2 t)\right] + (a_1 - b_1 i)\,e^{-k_1 t}\left[\cos(k_2 t) + i\sin(\right.$$

$$= \mathcal{L}^{-1}(F(s) + e^{-k_1 t}\Big[a_1\cos(k_2 t) - a_1 i\cancel{\sin}(k_2 t) + b_1 i\cancel{\cos}(k_2 t) + b_1\sin(k_2 t)$$
$$+ a_1\cos(k_2 t) + a_1 \cancel{i}\sin(k_2 t) - b_1 \cancel{i}\cos(k_2 t) + b_1 \sin k_2 t\Big]$$

$$= \mathcal{L}^{-1}(F(s)) + e^{-kt}\left[2a_1\cos(k_2 t) + 2b_1\sin(k_2 t)\right]$$

حفظ
$$= \mathcal{L}^{-1}(F(s)) + 2e^{-kt}\left[a_1\cos(k_2 t) + b_1\sin(k_2 t)\right]$$

For any complex number

$a_1, b_1 \to$ limit theory → compelty → real → ... / complex

$k_1, k_2 \to$ root of complex root.

Example:-

$$\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = 10\cos(t) \qquad y(0)=0,\ \dot{y}(0)=0,\ \ddot{y}(0)=3$$

$$\mathcal{L}(\dddot{y}) = s^3 Y(s) - s^2 y(0) - s\dot{y}(0) - \ddot{y}(0) \Rightarrow s^3 Y(s) - 3$$

$$\mathcal{L}(\ddot{y}) = s^2 Y(s) - s y(0) - \dot{y}(0) \Rightarrow s^2 Y(s)$$

$$\mathcal{L}(\dot{y}) = s Y(s) - y(0) \Rightarrow s Y(s)$$

$$\mathcal{L}(y) \Rightarrow Y(s)$$

$$\mathcal{L}(\cos(t)) \Rightarrow \frac{s}{s^2+1}$$

$$s^3 Y(s) - 3 + 4(s^2 Y(s)) + 5s Y(s) + 2Y(s) = \frac{10s}{s^2+1}$$

$$(s^3+4s^2+5s+2)Y(s) = \frac{10s}{s^2+1} + 3$$

$$(s^3+4s^2+5s+2)Y(s) = \frac{10s+3s^2+3}{s^2+1}$$

$$Y(s) = \frac{3s^2+10s+3}{(s^2+1)(s+2)(s+1)^2}$$

$$\sqrt{\frac{4}{4}} = 1$$

$$s = \frac{0 \mp \sqrt{0-4}}{2} = \underset{k_1}{0} \mp \underset{k_2}{1i}$$

D, E → complex

$$Y(s) = \frac{3s^2+10s+3}{(s^2+1)(s+2)(s+1)^2} = \frac{A}{s+2} + \frac{B}{(s+1)^2} + \frac{C}{s+1} + \frac{D}{s+i} + \frac{E}{s-i}$$

(0 + i) (0 − i)

$$A = \lim_{s\to-2}\left(\frac{\cancel{(s+2)}(3s^2+10s+3)}{\cancel{(s+2)}(s^2+1)(s+1)^2}\right) = \frac{-5}{5} = -1$$

$$C = \lim_{s\to-1}\frac{d}{ds}\left(\frac{3s^2+10s+3}{(s^2+1)(s+2)}\right) = +2$$

$$B = \lim_{s\to-1}\left(\cancel{(s+1)^2}\cdot\frac{3s^2+10s+3}{\cancel{(s+1)^2}(s^2+1)(s+2)}\right) = \frac{-4}{2} = -2$$

No.

$$D = \lim_{s \to i} \left(\cancel{(s+i)} \cdot \frac{3s^2 + 10s + 3}{\cancel{(s+i)}(s-i)(s+2)(s+1)} \right) =$$

$i \times i = -1$
$i \times -i = 1$
$-i \times -i = -1$

$$= \frac{3(-i)^2 + 10(-i) + 3}{(-i-i)(-i+2)\underset{\sim}{(-i+1)^2}} \qquad (-i+1)(-i+1) = -1 - i - i + 1 = -2i$$

$$= \frac{\cancel{-3} - 10i \cancel{+3}}{(2-i)(-2i)(-2i)} = \frac{-10i}{-4(2-i)} = \frac{-10i}{4(i-2)} \quad * \quad \frac{-i-2}{-i-2}$$

* نضرب بمعكوس اشارة i *

$-10i * -i$

$-10 * -1 = 10$
$i * i = -1$ > -10

$-2 * -2 = 4$
$i \times i = -1$ > -4

لازم التخلص من (i) من المقام.

$$\Rightarrow \frac{-10 + 20i}{4(1 \cancel{-2i} \cancel{+2i} + 4)} = \frac{-10+20i}{20} = \underset{a_1}{\underset{\sim}{-\frac{1}{2}}} + \underset{b_1}{\underset{\sim}{(1)}} i \quad \boxed{+}$$

دايما بنخليه $\boxed{+}$

الاثنين نفس الاشارة ← 1

اذا عكس ال b ⇐ سالب

$$y(t) = \mathcal{L}^{-1}\left(\frac{-1}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{-2}{(s+1)^2}\right) + \mathcal{L}^{-1}\left(\frac{2}{s+1}\right) + \text{complex}$$

$$= -e^{-2t} - 2te^{-t} + 2e^{-t} + 2e^{0t}\left[-\frac{1}{2}\cos(1)t) + (1)\sin(1t)\right]$$

$$= -e^{-2t} - 2te^{-t} + 2e^{-t} - \cos(t) + 2\sin(t) \quad \#$$

Example:-

$y'' - 4y' + 3y = 0 \qquad y(0) = 3, \quad y(0.89) = 1$

$y'(0) = ?$

Let $y'(0) = c$

$\mathcal{L}(y'') = s^2 Y(s) - s(3) - c$

$\mathcal{L}(y') = sY(s) - 3$

$\mathcal{L}(y) = Y(s)$

$$\rightarrow s^2 Y(s) - 3s - c - 4(sY(s) - 3) + 3Y(s) = 0$$

$$(s^2 - 4s + 3)\,Y(s) = 3s + c - 12$$

$$Y(s) = \frac{3s + c - 12}{s^2 - 4s + 3} = \frac{3s + c - 12}{(s-1)(s-3)}$$

$$Y(s) = \frac{A}{s-1} + \frac{B}{s-3}$$

$$A = \lim_{s \to 1} \cancel{(s-1)} \frac{3s + c - 12}{\cancel{(s-1)}(s-3)} = \frac{3 + c - 12}{-2} = \frac{9-c}{2} \rightarrow \text{constant}$$

$$B = \lim_{s \to 3} \cancel{(s-3)} \frac{3s + c - 12}{\cancel{(s-3)}(s-1)} = \frac{9 + c - 12}{2} = \frac{c-3}{2} \rightarrow \text{constant}$$

$$\mathcal{L}^{-1} Y(s) = \frac{9-c}{2} \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{c-3}{2} \mathcal{L}^{-1}\left[\frac{1}{s-3}\right]$$

$$y(t) = \frac{9-c}{2} e^{t} + \frac{c-3}{2} e^{3t}$$

$$y(-0.79) = 1 \Rightarrow \frac{9-c}{2} e^{-.79} + \frac{c-3}{2} e^{(3 * -0.79)}$$

$$\frac{9-c}{2} * 0.453 + \frac{c-3}{2}(.09348) = 1 \rightarrow c = 5$$

$$y(t) = \frac{4}{2} e^{t} + \frac{2}{2} e^{3t} = 2e^{t} + e^{3t}$$

Example:-

$$y''' - 5y'' + 7y' - 3y = 20 \sin(t)$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -2$$

قسمة تركيبة $(s^3 - 5s^2 + 7s - 3)\, Y(s) = \frac{20}{s^2+1} - 2$

$(s-1)^2(s-3) \sim$

$$Y(s) = \frac{20}{(s^2+1)(s-1)^2(s-3)} - \frac{2}{(s-1)^2(s-3)}$$

$$* \quad \frac{-2}{(s-1)^2(s-3)} = \frac{A}{(s-1)^2} + \frac{B}{(s-1)} + \frac{C}{(s-3)}$$

$$A = \lim_{s\to 1} (s-1) \cdot \frac{-2}{(s-1)^2(s-3)} = 1$$

$$B = \lim_{s\to 1} \frac{d}{ds}\left(\frac{-2}{s-3}\right) = \lim_{s\to 1} \frac{+2(1)}{(s-3)^2} = \frac{1}{2}$$

$$C = \lim_{s\to 3} (s-3) \cdot \frac{-2}{(s-1)^2(s-3)} = \frac{-2}{4} = \frac{-1}{2}$$

$$\Rightarrow \quad \frac{-2}{(s-1)^2(s-3)} = \frac{1}{(s-1)^2} + \frac{1/2}{(s-1)} + \frac{-1/2}{(s-3)}$$

$$* \quad \frac{20}{(s^2+1)(s-1)^2(s-3)} = \frac{A}{s-3} + \frac{E}{(s-1)^2} + \frac{F}{s-1} + \frac{J}{s+i} + \frac{H}{s-i}$$

$$D = \lim_{s\to 3} (s-3) * \frac{20}{(s^2+1)(s-1)^2(s-3)} = \frac{20}{40} = \frac{1}{2}$$

$$E = \lim_{s\to 1} (s-1)^2 * \frac{20}{(s^2+1)(s-1)^2(s-3)} = \frac{20}{-4} = -5$$

$$F = \lim_{s\to 1} \frac{d}{ds} \frac{20}{(s^2+1)(s-3)} = \frac{20}{8} = \frac{5}{2}$$

$$J = \lim_{s \to -i} (s+i) * \frac{20}{(s-1)^2(s-3)(s+i)(s-1)} = \frac{20}{-4i-12} * \frac{4i-12}{4i-12}$$

$$= \frac{80i-240}{16-48i+48i+144} = \frac{80i-240}{160} = \frac{-240}{180} + \frac{80i}{160}$$

من التحليل → $0 \pm 1i$

$$a = -\frac{3}{2} \qquad b = \frac{1}{2}$$

$$y(t) = \mathcal{L}^{-1} \frac{1}{(s-1)^2} + \mathcal{L}^{-1} \frac{1/2}{s-1} + \mathcal{L}^{-1} \frac{-1/2}{s-3} + \mathcal{L}^{-1} \frac{1/2}{s-3}$$

$$+ \mathcal{L}^{-} \frac{-5}{(s-1)^2} + \mathcal{L}^{-1} \frac{5/2}{s-1} + 2e^{-t}\left[\frac{-3}{2}\cos(t) + \frac{1}{2}\sin(t)\right]$$

$$y(t) = -4te^{t} + 3e^{t} - 3\cos(t) + \sin(t)$$

Example1-

$$(s^2 - 10s + 9)Y(s) = \frac{5}{s^2} - s + 12 \Rightarrow \frac{5 - s^3 + 12s^2}{s^2}$$

$$Y(s) = \frac{5 - s^3 + 12s^2}{s^2(s^2 - 10s + 9)} = \frac{5 - s^3 + 12s^2}{s^2(s-9)(s-1)}$$

$$Y(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-9} + \frac{D}{s-1}$$

$$A = \lim_{s \to 0} s^2 \cdot \frac{5 - s^3 + 12s^2}{s^2(s^2 - 10s + 9)} = \frac{5}{9}$$

$$B = \lim_{s \to 0} \frac{d}{ds}\left(\frac{5 - s^3 + 12s^2}{s^2 - 10 + 9}\right) = \frac{50}{81}$$

No.

$$C = \lim_{s \to 9} (s-9) \frac{5 - s^3 + 12s^2}{s^2(s-9)(s-1)} = 0.382716$$

$$D = \lim_{s \to 1} (s-1) \frac{5 - s^3 + 12s^2}{s^2(s-9)(s-1)} = -2$$

$$y(t) = 9\mathcal{L}^{-1} \frac{1}{s^2} + \frac{50}{81} \mathcal{L}^{-1} \frac{1}{s} + .3827 \mathcal{L}^{-1} \frac{1}{s-9} - 2\mathcal{L}^{-1} \frac{1}{s-1}$$

$$y(t) = \frac{5}{9} t + \frac{50}{81} + 0.382 e^{9t} - 2e^{t}.$$

No.

بتحل المعادلات اللي مستحيل نحلها
في التشابتر السابقة (تشتغل عند هذه النقطة)

Serises solution of D.E

هذه الحالة ما بتمشي $y'' + e^{x}y = \sin(x)$ → linear

المعادلات الأحرناها

$y'' + ay' + by = 0 \Rightarrow$ let $y = e^{\lambda x}$

$x^2 y'' + axy' + by = 0 \Rightarrow$ let $y = x^m$

صار الفرض لحل هذه المعادلات → solution: $\sum_{j=0}^{\infty} aj\,(x - x_0)^j$

Serises ⇐ تجمع لاكثر من حد

if $y = u_0 + u_1 + u_2 \text{ ------ } u_\infty$ → هاد solution حولنا إلى serise

لقدر الحوضه عنهم بصيغة أخرى terms

$$y = \sum_{j=0}^{\infty} u_j$$

upper limit

lower limit

j: sumation index

elearing ⇐ series انواع من

EX: e^x

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \text{ ------} = \sum_{j=0}^{\infty} \frac{x^n}{j!}$$ ← حفظ الدكتور ما يعطني اياه

EX: $\sin x$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \text{ -------} = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!}$$

Types of serises :- انواع :-

1) Taylor series

$$\sum_{j=0}^{\infty} a_j\,(x - x_0)^j = a_0(x - x_0)^0 + a_1(x - x_0)^1 + a_2(x - x_0)^2 \text{----}$$

- x_0 : is ordinary point. معلوما

↳ بنعمل expansion حول هذه النقطة

2) Maclourin series

$x_0=0$ taylar هو

$x_o = 0$

$$\sum_{j=0}^{\infty} a_j x^j = a_0 x^0 + a_1 x + a_2 x^2 + \cdots$$

where $a_0, a_1, a_2 \Rightarrow$ are all real numbers called coffIcients

$x \Rightarrow$ is the variable

$x_0 \Rightarrow$ is the centre of serise. $\Rightarrow$ مركزها

properties of series

1) addition

$$A = \sum_{j=0}^{\infty} a_j (x - x_0)^{j} \rightarrow \text{power}$$

حتى اجمع لازم يكون متشابهين :-

the same ordinary ① point ② , power ③ Lower Limit.

$$B = \sum_{j=0}^{\infty} b_j (x - x_0)^{j}$$

$$\rightarrow A + B = \sum_{j=0}^{\infty} (a_j + b_j)(x - x_0)^j$$

expansion of A:-

$$A = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

$$B = b_0 + b_1 \overset{\text{عوامل مشتركة}}{(x - x_0)} + b_2 (x - x_0)^2 + \cdots$$

$$\rightarrow A + B = (a_0 + b_0) + (a_1 + b)(x - x) + (a_2 + b_2)(x - x_0)^2 \cdots$$

2) Shiftings Rule

اذا اختلفت الـ Power

$$A = \sum_{j=0}^{\infty} j(j-1) a_j x^{\boxed{j-2}}$$ not $x_0 = 0$ ✓ The same

$$B = \sum_{j=0} b_j x^{\boxed{j}}$$ similear $x_0 = 0$ ✓

No.

← يا بنعوض +2 لكل شي الـ j في ① ✓

او بنطلع 2 ع في ②

$$A = \sum^{\infty}_{j+2=0} (j+2)(j-1+2)\, a_{j+2}\, X^{\textcircled{j}}$$

$j+2=0$

$\boxed{j=-2}$ ← لازم $j=0$ بنحل $j=-2$ و $j=-1$

$$A = \sum^{\infty}_{j=-2} (-2+2)\cdots + \sum^{\infty}_{j=-1} (-1+2)(-1-1+2)\cdots + \sum^{\infty}_{j=0} (j+2)(j+1)a_{j+2}\, X^{j}$$

Zero Zero

$$A = 0+\sum^{\infty}_{j=0} (j+2)(j+1)\, a_{j+2}\, x^{j}$$

$$B = \sum^{\infty}_{j=0} b_j\, x^{j}$$

$$\rightarrow A+B = \sum^{\infty}_{j=0} \left[b_j + (j+2)(j+1)\, a_{j+2} \right] x^{j}$$

3) Multiplication by number

هنا نضرب بـ (رقم)

بنوزعها بكل tearm

$$\alpha \sum^{\infty}_{j=0} (x-x_0)^{j} = \sum^{\infty}_{j=0} \alpha (x-x_0)^{j}$$

$$= \alpha (x-x_0)^{0} + \alpha (x-x_0)^{1} + \alpha (x-x_0)^{2} + \cdots\cdots$$

4) Differentiation

هدا فقط في taylor series

$$y = \sum^{\infty}_{j=0} a_j (x-x_0)^{j}$$

نشتق بالطريقة العادية

$$y' = \sum^{\infty}_{\boxed{j=0}} a_j * j (x-x_0)^{j-1}$$

$j=1$ ⟶ لان عند $j=0$ (عوض) كل series بتروح ⇐ $=0$

$$y'' = \sum^{\infty}_{j=2} j(j-1)\, a_j (x-x_0)^{j-2}$$

5) Multipication of series

القسمة فقط ارجعيلها لنوتة تيجر سالب

$$A = \sum^{\infty}_{j=0} a_j (X-X_0)^{j} \quad , \quad B = \sum^{\infty}_{j=0} b_j (X-X_0)^{j}$$

only : the same ordinary point.

No.

$A = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots\ldots$

$B = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \ldots$

$\rightarrow A * B = a_0(b_0 + b_1(x-x_0) + b_2(x-x_0)^2 \ldots) + a_1(x-x_0)(b_0 + b_1(x-x_0) + b_2(x-x_0)^2) + \ldots\ldots$

$x_0 \leftarrow$

Ordinary and singular points

متى تكون ordinary ومتى لا ؟

→ consider the secand order homegenous linear D.E

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0$$ → not normallized

نقسم على المعامل

$$\frac{d^2y}{dx^2} + \underbrace{\frac{a_1(x)}{a_0(x)}}_{P_1(x)}\frac{dy}{dx} + \underbrace{\frac{a_2(x)}{a_0(x)}}_{P_2(x)}\,y = 0$$

(اذا فيه اختصارات عادي اختصر

any point at which denminator of $P_1(x)$ or $P_2(x)$ = 0 is called singular point all other points are called ordinary points and the function is called analytic.

← مشان نجد singular اولاً بنعمل

normallition / نختصر / القيم يلي بتخلي

المقام = صفر (singular) / اذا ما فيه

ولا قيمة (no singular) لكل القيم

ordinary ما بتفرق اذا كان فيه

P_1 او P_2

Example:-

$$y'' + x\,y' + (x^2+2)\,y = 0$$ normallized ✓

$P_1(x) \quad P_2(x)$

$P_1(x) = x$

$P_2(x) = x^2 + 2$

there are no singular points

No.

→ all other points $(-\infty, \infty)$ are ordinary points

↳ we can a series solution around any points

↲ يعني اي نقطة - نقدر نحل expansion.

Example:-

$(x-1)y'' + xy' + \frac{1}{x}y = 0$ → not normaliz

$$y'' + \frac{x}{x-1}y' + \frac{1}{x(x-1)}y = 0$$

$P_1(x) = \frac{x}{x-1}$ → $x=1$ singular point. $P_2(x) = \frac{1}{x(x-1)}$ → $x=0,1$ are singualr

→ let $y = \sum_{j=0} a_j(x-x_0)^j$ → هذه الفرضية المستخدم في حل هذه المعادلات

Note:-

it given conditions $y(0)=2$, $y'(0)=-4$

في هذه الحالة لا نستطيع ايجاد constant من الحل

بينما

$y(2)=2$, $y'(2)=-4$ ⇝ نستطيع ايجاد ال constant لنحل expansion

$$y = \sum_{j=0}^{\infty} a_j(x-\underline{x_0})^j$$

↳ 2 (ordinary point)

$$y = a_0 + a_1(\cancel{x}-2) + a_2(\cancel{x}-2)^2 + a_3(\cancel{x}-2)^3 + \cdots$$

ordinary point ↶ $x=2$ ↲ رح يلتغى لان condition معطى عند $x=2$ ← عوض $x=2$

لو عند $x=1$ ← ال terms ما بنلغى و بصير الحل الى مالانهاية (∞)

Note:- x_0 ⇒ بحددها حسب ال condition

Note:- لو حدد في السؤال $y(1)=2$, $y'(1)=-4$ > بنحل اختبار لتحديد نوع ال singularity

regular / irregular

regular ↲ نقدر احلها بـ series

irregular ↳ مستحيل تنحل

No.

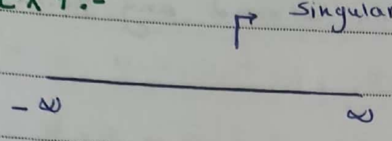
solution ـه < con / div معرفة متى يصير لنا

Radius of convergence

R : radius of conergence is the distance from the point x_0 to the closest singular poi

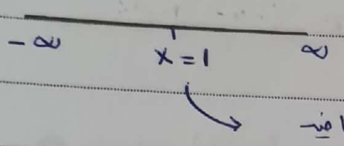
Ex 1 :-

ماعنا singular point



Solve the D.E around $x_0 = 1$?

لـ نستطيع الحل



ماعنا closest singular point

→ Radius of convergence for this solution :

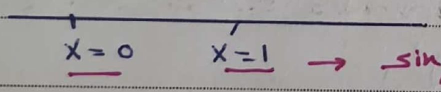
$-\infty < R < \infty$ $\quad x_0 = 1$

اذا عوضت في final solution اي رقم

→ it will conerger ← يعني الحل تكون

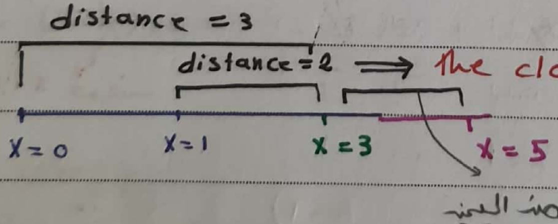
اهم واضح .

Ex 2 :-



$x = 0$ $\quad x = 1$ → singular points

Solve around $x = 3$?



distance = 2 ⟹ the closest .

اذا عوضنا في Final Solution

$x = 2$ → it converge

$x = 6$ → it div

→ the radius of convergence in this cause while → equal the distance (left and right) between this ordinary point and the closest singular point.

في هذا المثال 2 distance ← $x = 1$ هي الاقرب ← distance = 2

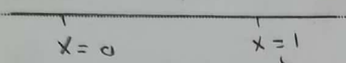
$x = 1$, $x = 0$

$1 < R < 5$ ← من اليمين واليسار

No.

Note:- if we expand on the singular point → نستطيع حلها باستخدام For benius metod

→ the Radius of convergence :-



singular point — بدنا نحل عندها distance $= 0 \rightarrow R = 0$

→ the solution will only converge for that particular $x = 1$

range ماضي — فقط يطلع حل يطبق عند الواحد فقط

اذا تغيرت القيم ما يطلع converge ,

Series Solution

* The series $\sum_{j=0}^{\infty} a_j (x - x_0)^j$ always converges at $x = x_0$ و

because then all its terms except for first , a_0 , are zero

* if there are further valus of x for which the series converges, these values from an interval called convergence interval, and the series converges for all x, such that $|x - x_0| < R$, and diverges for all x , such that $|x - x_0| > R$

غير x_0 / قيم أخرى

المسية x_0

لو هاي x بدو يصير عنده converges

اذا فيه singular point ال R تكون صفر

اذا ما فيه تكون ∞

عند طريقة الحل

من معرفة singular point

* R can be obtained using

$$R = \frac{1}{\lim_{j\to\infty} |a_{j+1}/a_j|}$$

⇒ if the limit is (0) then $R = \frac{1}{0} = \infty$, then the series will converge for all x .

⇒ if the limit is (∞) , then $R = \frac{1}{\infty} = 0$, then the series will converge only at $x = x_0$

div for other values

Example:-

افرضناه هاد الـ solution علانه معينه

$$\sum_{j=0}^{\infty} J!\, x^j \quad , \quad x_0 = 0$$ هل هذا series تكون converges عند x_0 ولا عند كل القيم ؟

قيم aj $\quad \underline{\underline{0!}}\, x^0 + \underline{\underline{1!}}\, x + \underline{\underline{2!}}\, x^2 + \underline{\underline{3!}}\, x^3 - - - - -$

$\Rightarrow a_j = J!$

$$\sum_{\substack{j+1=0 \\ j=-1}}^{\infty} (j+1)!\; x^{j+1} \quad , \quad 0 + (0+1)!\, x + (1+1)! + (2+1)!\, x^3 - - -$$

$\Rightarrow a_{j+1} = (J+1)!$

$$R = \frac{1}{\lim_{j=\infty} \left| \frac{(J+1)!}{J!} \right|} = \frac{1}{\lim_{J\to\infty} (j+1)} = \frac{1}{\infty} = 0$$

if $J = 3 \to a_j = 3! \;/\; a_{j+1} = 4!$ $\quad \lim_{J\to\infty} \frac{4!}{3!} = \lim \left| \frac{4 * \cancel{3!}}{\cancel{3!}} \right| = \lim_{J\to\infty} |J+1|$

$\Rightarrow$ The series will converge any for $x = 0$.

لو اخذنا $X=1$ دعوضنا في الحل بعطي diverges

مش مسموح أي قيمة

$x_0 = 0$

Example:-

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{j=0}^{\infty} x^j$$

$\therefore a_j = 1$

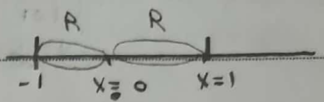
$$\sum_{\substack{j+1=0 \\ j=-1}}^{\infty} x^{j+1} \quad , \quad 0 + x + x^2 + x^3 - - - - - -$$

$\therefore a_{j+1} = 1$

$$\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j\to\infty} \frac{1}{1} = 1 \Rightarrow R = \frac{1}{1} = 1$$

No.

→ the series will converge only for $| x + x_0 | < \overset{R}{1}$



Example:-

e^x , $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = \sum_{j=0}^{\infty} \frac{\overbrace{x^j}}{j!}$ حفظ :)

$\sum_{j=0}^{\infty} \frac{x^j}{j!} = \frac{1 x^0}{0!} + \frac{1 x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$

aj

$aj = \frac{1}{J!}$

$\sum_{J=-1}^{\infty} \frac{x^{j+1}}{(j+1)!} \Rightarrow aj+1 = \frac{1}{(J+1)!}$

$\lim_{j \to \infty} \frac{1/J!}{1/(J+1)!} = \lim \frac{J!}{(J+1)!} = \lim_{J \to \infty} \frac{1}{(J+1)} = \lim \frac{1}{\infty + 1} = 0$

$R = \frac{1}{0} = \infty$, conerge for all number

Solution by series:-

Solve $y' - y = 0$ linear, constant coff, Homc let $y = e^{\lambda x}$

$y(0) = 3$

$e^{\lambda x}(\lambda - 1) = 0$

$\underset{\neq 0}{e^{\lambda x}}$ $\underset{= zero}{(\lambda - 1)}$

$\boxed{\lambda = 1}$

$\boxed{y_g = c_1 e^x}$

→ to solve by series let:- $x_0 = 0$

$y = \sum_{j=0}^{\infty} aj \, x^j$, $x_0 = 0$

$y' = \sum_{j=1}^{\infty} j * aj \, x^{j-1}$

$\sum_{j=1}^{\infty} j * aj \, x^{j-1} - \sum_{j=0}^{\infty} aj \, x^j = 0$

مختلفات

↳ shifting Rule

خطوات الحل على clearing .

No.

$$\sum_{\substack{j+1=1 \\ j=0}}^{\infty} (j+1)\, a_{j+1}\, x^{j} - \sum_{j=0}^{\infty} a_j\, x^{j} = 0$$

$$\rightarrow \sum_{j=0}^{\infty} \left[\underbrace{(j+1)\, a_{j+1} - a_j}_{= Zero} \right] \underbrace{x^{j}}_{\neq Zero} = 0$$

لان فرضنا solution

$$(j+1)\, a_{j+1} - a_j = 0 \ , \ for \ j = 0, 1, 2, 3, -----$$

$$a_{j+1} = \frac{a_j}{j+1} \ , \quad j=0 \Rightarrow a_1 = \frac{a_0}{1} = a_0$$

$$j=1 \Rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2}$$

$$j=2 \Rightarrow a_3 = \frac{a_2}{3} = \frac{a_0}{2*3}$$

$$j=3 \Rightarrow a_4 = \frac{a_3}{4} = \frac{a_0}{2*3*4}$$

ال solution اللي فرضته $y = \Sigma a_j x^j$

$$y = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 \ ....$$

عوضهم ☺ ☺ ☺ ☺

$$= a_0 + a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{6} x^3 + \frac{a_0}{2*3*4} x^4 \ .....$$

$$= a_0 \left[1 + x + \frac{x^2}{2} + \frac{x^3}{2*3} + \frac{x^4}{4*3*2} + --- \right]$$

$$= a_0 \left[\frac{1}{0!} + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 \right]$$

$$y = a_0 e^{x} \Rightarrow \text{conerge for all valus of } x .$$

الخطوات ← clearing

١) فرضنا solution $y = \Sigma a_j x^j$

٢) اشتقنا/ عوضناهم في المعادلة الأصلية

٣) جمعنا او طرحنا حتى نطلع one comen series

٤) طلعنا معادلة (لازم اطلعها مشان اكون احل)

٥) طلعنا constant ← بتيجي بسهولة ردة

٦) بعوض في ال solution

٧) نحاول ادور ع متسلسلة

No.

Example:-

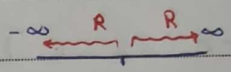
$\ddot{y} + x\dot{y} + (x^2+2)y = 0$ $y(0)=2$, $\acute{y}(0) = 5$ → $x_\circ = 0$

→ not constant coff. , not Huler

بدنا نحل expansion حوليه الصفر عشان
اصلا يطلع converges

→ No singular point.

$|x - x_\circ| < R$, $x_\circ = 0$ ⇒ $|x| < R$



No singular $x_\circ=0$

→ $|x| < \mp\infty$ the obtained solution will converge for all values of x.

ال R ما عليها قيود
مش بس عند صفر لا كل القيم

let $y = \sum_{j=0}^{\infty} a_j x^j$ → $x_0=0$

→ $\acute{y} = \sum_{j=1}^{\infty} j a_j x^{j-1}$, → $\ddot{y} = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2}$

* substitute in the D.E

كل شي بدا series بدنا ندخله

$$\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} + x \sum_{j=1}^{\infty} j a_j x^{j-1} + (x^2+2) \sum_{j=0}^{\infty} a_j x^j = 0$$

بدنا نحاول نوحد power
j=1 بدنا نغير ٣ مرات
j=0 نغير مرتين ✓
ل الاقل بنائد)

$$= \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} + \sum_{j=1}^{\infty} j a_j x^{\cancel{j-1}} \cdot \cancel{x}^{1} + \sum_{j=0}^{\infty} a_j x^j \cdot \cancel{x}^{2} + \sum_{j=0}^{\infty} 2 a_j x^j = 0$$

$(j+2)$

$$= \sum_{\substack{j+2=2 \\ j=0}}^{\infty} (j+2)(j-1+2) a_{j+2} x^{ⓙ} + \sum_{j=\cancel{1}1}^{\infty} j a_j x^{ⓙ} + \sum_{\substack{j-2=0 \\ j=2}} a_{j-2} x^{ⓙ} + \sum_{j=0}^{\infty} 2 a_j x^{ⓙ} = 0$$ ✓

بدنا خليهم ← j=0
كلهم j=2

$$\Rightarrow (0+2)(0+1) a_2 x^0 + (1+2)(1+1) a_3 x^1 + \sum_{\boxed{j=2}}^{\infty} (j+2)(j+1) a_{j+2} x^j + (1) a_1 x^1$$

$$+ \sum_{\boxed{j=2}} j a_j x^j + \sum_{\boxed{j=2}} a_{j-2} x^j + 2a_1 x^1 + \sum_{\boxed{j=2}} 2 a_j x^j = 0$$ ✓

↳ $+ 2a_0 x^0$

جمع الحدود

$\Rightarrow 2a_2 + 6a_3 x + a_1 x + 2a_0 x^0 + 2a_1 x + \text{series}$

→ خلينا ...

No.

اجمعهم

$$+\sum_{j=2}^{\infty}\left[(j+2)(j+1)a_{j+2} + j a_j + a_{j-2} + 2a_j\right]x^j = 0$$

$$\Rightarrow 2(a_2+a_0) + (3a_1+6a_3)x + \sum_{j=2}^{\infty}\left[(j+2)(j+1)a_{j+2} + (j+2)a_j + a_{j-2}\right]x^j = 0$$

كل الحدود تساوي صفر

$\rightarrow 2(a_2+a_0)=0$ ---① $\quad \bullet\, a_2 = -a_0$

$3a_1 + 6a_3 = 0$ ---② $\quad \bullet\, a_3 = -\frac{a_1}{2}$

Recurrence formula:-

$$(j+2)(j+1)a_{j+2} + (j+2)a_j + a_{j-2} = 0$$

نكتبها بدلالة الـ lower

$j+2$

$$a_{j+2} = -\frac{(j+2)a_j - a_{j-2}}{(j+2)(j+1)} \quad \text{و} \quad j = 2,3,4,5,6 -----$$

مشان نعرف اذا بنحل مع a_4 لازم لانو فوق معنا a_0, a_1, a_2, a_3

For $j=2 \Rightarrow a_4 = \frac{-(4)a_2 - a_0}{(4)(3)} = \frac{-4}{12}a_2 - \frac{a_0}{12} = \frac{-1}{3}\overset{-a_0}{a_2} - \frac{a_0}{12}$

For $j=3 \Rightarrow a_5 = \frac{-(5)a_3 - a_1}{(5)(4)} = -\frac{1}{4}\overset{-\frac{a_1}{2}}{a_3} - \frac{1}{20}a_1$

For $j=4 \Rightarrow a_6 = \frac{-(6)a_4 - a_2}{(6)(5)} = \frac{-\overset{j=2}{a_4}}{5} \frac{-1\overset{-a_0}{a_2}}{30}$

$\rightarrow -\frac{(+\frac{1}{3}\cdot\frac{1}{12})a_0}{5} + \frac{a_0}{30}$

- $j=2 \Rightarrow (+\frac{1}{3} - \frac{1}{12})a_0 = \frac{3}{12}a_0 = \frac{1}{4}a_0$
- $j=3 \Rightarrow (\frac{1}{8} - \frac{1}{20})a_1 = \frac{12}{160}a_1 = \frac{3}{40}a_1$

$j=4 \Rightarrow \frac{\frac{1}{4}a_0}{5} + \frac{a_0}{30} = \frac{1}{20}a_0 + \frac{1}{30}a_0 = \frac{50}{600}a_0 = \frac{5}{60}a_0 = \frac{1}{12}a_0$

$a_2 = -a_0,\ a_3 = -\frac{a_1}{2},\ a_4 = \frac{1}{4}a_0,\ a_5 = \frac{3}{40}a_1$

عوضنا }

فكينا

لانه فرضنا بالبداية

$$y = \sum_{j=0}^{\infty} a_j x^j = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 -----$$

No.

$$\therefore y = \underline{a_0} + \underline{a_1 x} - \underline{a_0 x^2} - \frac{a_1}{2} x^3 + \frac{1}{4} a_0 x^4 + \frac{3}{40} a_1 x^5 + \cdots\cdots$$

لتتأكد انك تحل مع المعادلة تانت

2order ⇒ y''

2constant فيه

٢ ⇒ a_1, a_2

$$y = \underline{a_0}\left(1 - x^2 + \frac{1}{4}x^4 + \cdots\right) + \underline{a_1}\left(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 \cdots\right)$$

$a_0, a_1 \Rightarrow$ given conditions من $\qquad y(0)=2, \; y'(0)=5$

$$2 = a_0(1-0+0+\cdots) + a_1(0-0+0\cdots) \Rightarrow \boxed{a_0 = 2}$$

$$y' \quad = 2(-2x + x^3 + \cdots) + a_1\left(1 - \frac{3}{2}x^2 + \frac{15}{40}x^4 \cdots\right)$$

$$5 = 2(0) + a_1(1-0)$$

$$\boxed{a_1 = 5}$$

$$\rightarrow y = 2\left(1 - x^2 + \frac{1}{4}x^4 + \cdots\right) + 5\left(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \cdots\right)$$ الحل النهائي

Example:-

$$y'' - xy = 0 \qquad y(1) = 3, \; y'(1) = -2$$

let $y = \sum_{j=0}^{\infty} a_j (x-1)^j$ → من المعادلة ⇒ No singular points

تأكد انه sing اردة

all points are ordinary

$-\infty < R < +\infty$

$$y' = \sum_{j=1}^{\infty} j * a_j (x-1)^{j-1}, \quad y'' = \sum_{j=2}^{\infty} j*(j-1)*a_j \, x^{j-2}$$

$$\rightarrow \sum_{j=2} j(j-1)a_j(x-1)^{j-2} - x\sum_{j=0} a_j(x-1)^j = 0$$

$$\rightarrow \sum_{j=2} j(j-1)a_j(x-1)^{j-2} - (x-1+1)\sum_{j=0} a_j(x-1)^j = 0$$

ماتقدر ادخلها جوا القوس (x-1) لانه مش نفس ال

عادي لانه ما معه x

$$\rightarrow \sum_{j=2} j(j-1)a_j(x-1)^{j-2} - \sum_{j=0} a_j(x-1)^{j+1} - (1)\sum_{j=0} a_j(x-1)^j = 0$$

No.

$\rightarrow \sum_{j+2=2}^{\infty} (j+2)(j-1+2)\, a_{j+2} (x-1)^{j-2+2} - \sum_{j=1}^{\infty} a_{j-1} (x-1)^{j+1-1} - \sum_{j=0}^{\infty} a_j (x-1)^j = 0$

$\sum_{j=0}^{\infty} (j+2)(j+1)\, a_{j+2} (x-1)^{ⓙ} + - \sum_{j=+1}^{\infty} a_{j-1} (x-1)^{ⓙ} - \sum_{j=0}^{\infty} a_j (x-1)^{ⓙ} = 0$ ✓

$\Rightarrow (0+2)(0+1)\, a_2 (x-1)^0 + \sum_{\boxed{j=1}}^{\infty} (j+2)(j+1)\, a_{j+2} (x-1)^j - \sum_{\boxed{j=1}}^{\infty} a_{j-1} (x-1)^j$

$- a_0 (x-1)^0 - \sum_{\boxed{j=1}}^{\infty} a_j (x-1)^j = 0$ ✓

$2a_2 - a_0 + \sum_{j=1}^{\infty} \left((j+2)(j+1)\, a_{j+2} - a_{j-1} - a_j \right) x^j = 0$

$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{1}{2} a_0$

علاقة الأثر. $\underline{\underline{a_{j+2}}} = \frac{a_j + a_{j-1}}{(j+2)(j+1)}$ و $j = 1, 2, 3, 4 \, \ldots$

expansion استخدمنا a_2 هنا

$j = 1 \Rightarrow \boxed{a_3} = \frac{a_1 + a_0}{(3)(2)} = \frac{1}{6} a_1 + \frac{1}{6} a_0$

$j = 2 \Rightarrow a_4 = \frac{a_2 + a_1}{(4)(3)} = \frac{1}{12} a_2 + \frac{1}{12} a_1 = \frac{1}{24} a_0 + \frac{1}{12} a_1$

$j = 3 \Rightarrow a_5 = \frac{a_3 + a_2}{(5)(4)} = \frac{a_3}{20} + \frac{1}{20} a_2 = \frac{1}{6 \times 20} a_1 + \frac{1}{6 \times 20} a_0 + \frac{1}{40} a_0$

$= \frac{1}{120} a_0 + \frac{1}{120} a_1 + \frac{1}{40} a_0$

$\rightarrow y = \sum_{j=0}^{\infty} a_j (x-1)^j$

$= \frac{1}{30} a_0 + \frac{1}{40} a_1$

expan:

$y = a_0 + a_1 (x-1) + a_2 (x-1)^2 + a_3 (x-1)^3 + a_4 (x-1)^4 + a_5 (x-1)^5$

[illegible]

$= a_0 + a_1 (x-1) + \frac{1}{2} a_0 (x-1)^2 + \left(\frac{1}{6} a_0 + \frac{1}{6} a_1\right)(x-1)^3 + \left(\frac{1}{24} a_0 + \frac{1}{12} a_1\right)(x-1)^4$

$+ \left(\frac{1}{30} a_0 + \frac{1}{120} a_1\right)(x-$

No.

$$y = a_0\left[1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \frac{1}{30}(x-1)^5 \cdots\right] + a_1\left[(x-1) + \frac{1}{6}(x-1)^3 + \frac{1}{12}a_1(x-1)^4 + \frac{1}{120}(x-1)^5 \cdots\right]$$

$y(1) = 3 \ , \ y'(1) = -2$

$y(1) = a_0[1 + 0] + 0 \rightarrow 3 = a_0$

$y' \rightarrow a_0[0 + 1(x-1) \cdots] + a_2\left[1 + \frac{1}{2}(x-1)^2 \cdots\right]$

$-2 = a_0 * 0 + a_2(1) \Rightarrow a_2 = -2$

$$y = 3\left[1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \frac{1}{30}(x-1)^5 \cdots\right] + -2\left[x-1 + \frac{1}{6}\cdots\right]$$

particular solution.

Example:-

$(x-2)y'' + 3y' - xy = 0 \qquad y(\underline{2}) = 3 \ , \ y'(2) = -2$

(2 مولية epp لانها) x_0 not Euler, not constant coffieint

لذلك نفكر بـ series

$y'' + \frac{3}{x-2}y' - \frac{x}{x-2}y = 0$, $x = 2$ is a singular point All other points are ordinary.

ما نقدر نحل ها

$y = \sum_{j0} a_j (x-2)^j$

فيه هذه الحالة لنحل ها —— بنحل ع تغير الـ power

0 + constant

Frobenius Method

برمز مثل كل sing هاي الطريقة بتحل

Applies when we have x_0 is a singular point. The singular point must be regular singular, IF the point is Irregular then Frobenius method will not be able to find a solution.

let $p(x)y'' + q(x)y' + r(x)y = 0$

$y'' + \frac{q(x)}{p(x)}y' + \frac{r(x)}{p(x)}y = 0$

if $x=a$ is a singular point, then if

$$\lim_{x\to a}\left((x-a)\frac{q(x)}{p(x)}\right) \text{ and } \lim_{x\to a}\left((x-a)^2\frac{r(x)}{p(x)}\right)$$

→ both exist, then $\underline{\underline{a}}$ is singular regular point.

→ if any of the limits does not exist, ∴ $x=a$ is an irregular singular point.

ex: $\overset{''}{y} + \frac{3}{x-2}\overset{'}{y} - \frac{x}{x-2}y = 0$ $x=2$ singular point

→ $\lim_{x\to 2}\left((x-2)\cdot\frac{3}{(x-2)}\right) = 3$

both lim are exist, ∴ $x=2$

$\lim_{x\to 2}\left((x-2)\frac{(-x)}{(x-2)}\right) = -2$ is a regular singular point.

Example:-

$$x^3(1-x)\overset{''}{y} + (3x+2)\overset{'}{y} + x^4 y = 0$$

$$\overset{''}{y} + \frac{3x+2}{x^3(1-x)} + \frac{\cancel{x^4}\, x^1}{\cancel{x^3}(1-x)}y = 0$$

$x=1$ and $x=0$ are singular points.

for $x=1$

$\lim_{x\to 1}\left((x-1)\cdot\frac{3x+2}{x^3(x-1)}\right) = \frac{5}{1} = 5$

both limits are exist ∴ $x=1$

$\lim_{x\to 1}\left((x-1)^2\cdot\frac{x}{(x-1)}\right) = 0$ is a regular singular point

for $x=0$

$\lim_{x\to 0}\left((x)\frac{3x+2}{x^3(1-x)}\right)$ = does not exist

No.

$x=0$ is an irregular singular.

To solve a differential equation about a regular singular point,

let:-

$y=\sum_{j=0}^{\infty} a_j (x-a)^{j+r}$, where a is regular singular point and r is a constant to be determined the factor $(x-a)^r$ converages for all x such that $|(x-a)|<R$, R is the distance from $x=a$ to the nearest singularity (other than a).

If a is a regular singelar point:-

1) let $\sum a_j (x-a)^{j+r}$

2) substitute y, y', y'' into the D.E, and gets the indicial eq. and the recurrence eq.

r_1 and r_2 will be obtained from the indicial eq.

3) if $r_1-r_2 \neq$ zero and $r_1-r_2 \neq$ integer then

$y_1|_{r_1} = \sum a_j (x-a)^{j+r_1}$ and $y_2|_{r_2} = \sum b_j (x-a)^{j+r_2}$

Then the general solution will be obatined by merging these two solution as

فقط حالتين

diff ضف

- not integer

- not zero

* بس نعمل ✓ لـ sing اللي موجودة

$y_g = y_1|_{r_1} + y_2|_{r_2} = c_1 y_1|_{r_1} + c_2 y_2|_{r_2}$

ضف ال condition / التعريف عو ال reg

reg ← for نتحقق ir ← مابلها

بعوض

نطلع 2 forml

وبعدة من selve

ونبدأ expansion

integ ماشي أكمل الحل

نفحص r ← fraction نعوض r في recurreng ، نطلع الاول solution

~ الثاني ~ ~ r_2

نجمع مع بعض

Example:-

$2x^2 y'' - xy' + (x-5)y = 0 \qquad y(0) = 2 \,,\; y'(0) = 5$

$y'' - \frac{1}{2x} y' + \frac{x-5}{2x^2} y = 0 \quad , \; x = 0$ is a singular point

Frobenius:-

$\lim_{x\to 0} \left(x \cdot \frac{-1}{2x} \right) = -\frac{1}{2}$ exist.

both limits are exist

$\lim_{x\to 0} \left(x^2 \cdot \frac{x-5}{2x^2} \right) = \frac{-5}{2}$ exist

$\therefore x = 0$ is a regular singular point.

let $y = \sum_{j=0}^{\infty} a_j (x)^{j+r}$

$y' = \sum_{j=0}^{\infty} (j+r)\, a_j\, (x)^{j+r-1} \quad , \quad y'' = \sum_{j=0}^{\infty} (j+r)(j+r-1)\, a_j\, x^{j+r-2}$

لـ بتطلع 0 لانه ال power $(j+r)$

$\to (x)^{0+r-1}$ =

$\to 2x^2 y'' - xy' + (x-5)y = 0$

كل شي برا دخلو جوا

$2\cancel{x^2} \sum_{j=0} (j+r)(j+r-1) a_j\, x^{j+r-2} - \cancel{x} \sum_{j=0} (j+r)\, a_j\, (x)^{j+r-1} + x - 5 \sum_{j=0} a_j (x)^{j+r} = 0$

$x^{j+r} \cdot \cancel{x^{-2}} \qquad x^{j+r} \cdot \cancel{x^{-1}}$

$\sum_{j=0} 2(j+r)(j+r-1)\, a_j\, x^{\boxed{j+r}} - \sum_{j=0} (j+r)\, a_j\, x^{\boxed{j+r}} + \sum_{j=0} a_j\, x^{j+r+1} - \sum_{j=0} 5 a_j\, x^{\boxed{j+r}} = 0$

the same power

$\sum_{j=0} 2(j+r)(j+r-1)\, a_j\, x^{j+r} - \sum_{j=0} (j+r)\, a_j\, x^{j+r} + \sum_{j=1} a_{j-1}\, x^{j+r} - \sum_{j=0} 5 a_j\, x^{j+r} = 0$

بدنا نقسم دايم

$(2)(r)(r-1)\, a_0 x^r + \sum_{j=1} 2(j+r)(j+r-1)\, a_j\, x^{j+r} - (r)\, a_0 x^r - \sum_{j=1} (j+r)\, a_j\, x^{j+r} + \sum_{j=1} a_{j-1}\, x^{j+r}$

$- 5 a_0 x^r - \sum_{j=1} 5 a_j\, x^{j+r} = 0$

$\rightarrow [2(r)(r-1) - (r) - 5]\underline{a_0 X^r} + \sum_{j=1}^{\infty} [2(j+r)(j+r-1)a_j - (j+r)a_j + a_{j-1} - 5a_j] X^{j+r} = 0$

"in dicial equ." $\quad \hookrightarrow \neq zero$

لملايجاد آيم (r)

$2(r-1)r - r - 5 = 0 \Rightarrow 2r^2 - 2r - r - 5 = 0 \Rightarrow 2r^2 - 3r - 5 = 0$

$r^2 - \frac{3}{2}r - \frac{5}{2} = 0 \Rightarrow r = \frac{3/2 \mp \sqrt{9/4 + 4(5/2)}}{2}$

$r_1 = 5/2 \ , \ r_2 = -1$

$\rightarrow r_1 - r_2 = \frac{5}{2} + 1 = \frac{7}{2} \neq zero \neq \text{Integer} \rightarrow$ طلع كسر وهو ابسط الحالات

باخذ r بعوضه في recurrence وبطلع 2solution ونجمعهم .

for r_1 :-

$r_1 = \frac{5}{2}$ $\qquad a_j > a_{j-1}$

$a_j = \frac{-a_{j-1}}{2(j+\frac{5}{2})(j+\frac{5}{2}-1) - (j+\frac{5}{2}) - 5}$, $j = 1, 2, 3 \ldots\ldots$

$\hookrightarrow (2j+5)(j+\frac{5}{2}-\frac{2}{2}) - j - \frac{5}{2} - \frac{10}{2} =$

$2j^2 + 3j + 5j + \cancel{\frac{15}{2}} - j - \cancel{\frac{15}{2}} =$

$2j^2 + 7j = j(2j+7)$

$a_j = \frac{-a_{j-1}}{j(2j+7)}$

$j=1 \Rightarrow a_1 = \frac{-a_0}{1(2+7)} = \frac{-a_0}{9}$

$j=2 \Rightarrow a_2 = \frac{-a_1}{22} = \frac{a_0}{198}$

$j=3 \Rightarrow a_3 = \frac{-a_2}{39} = \frac{-a_0}{7722}$

$\rightarrow y = \sum_{j=0} a_j (x)^{j+r}$

$y = a_0 X^r + a_1 X^{1+r} + a_2 X^{2+r} + a_3 X^{3+r}$

$\therefore y_1|_{r_1 = \frac{5}{2}} = a_0 X^{\frac{5}{2}} + a_1 X^{1+\frac{5}{2}} + a_2 X^{2+\frac{5}{2}} + a_3 X^{3+\frac{5}{2}}$

$$= [a_0 x^0 + \underline{\underline{a_1}} x^1 + \underline{\underline{a_2}} x^2 + \underline{\underline{a_3}} x^3]\ x^{\frac{5}{2}}$$

$$= [a_0 + \frac{-1}{9} a_0 x + \frac{1}{198} a_0 x^2 - \frac{1}{7722} a_0 x^3]\ x^{\frac{5}{2}}$$

$$y \Rightarrow a_0 \ [1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722}]\ x^{\frac{5}{2}}$$

$\uparrow$ arbitrary constant

For r_2:- $r = -1$

$$a_j = \frac{-g_{j-1}}{\quad} \rightarrow (2)(j+r-1)(j+r) - (j+r) - 5$$

$$a_j = \frac{-b_{j-1}}{2(j-1-1)(j-1) - (j-1) - 5} = \frac{-b_{j-1}}{j(2j-7)}$$

$$j=1 \Rightarrow b_1 = \frac{-b_0}{-5} = \frac{b_0}{5}$$

$$j=2 \Rightarrow b_2 = \frac{-b_1}{-6} = \frac{b_1}{6} = \frac{b_0}{30}$$

$$j=3 \Rightarrow b_3 = \frac{-b_2}{-3} = \frac{b_2}{3} = \frac{b_0}{90}$$

$$y_2|_{r_2} = b_0 x^{-1} + b_1 x^{1-1} + b_2 x^{2-1} + b_3 x^{3-1}$$

$$x^{-1}[b_0 + \frac{b_0}{5} x^1 + \frac{b_0}{30} x^2 + \frac{b_0 x^3}{90} + \cdots] = b_0 x^{-1} [1 + \frac{1}{5} x + \frac{1}{30} x^2 + \frac{1}{90} x^3 + \cdots]$$

$$y_g = y_1|_{r_1} + y_2|_{r_2}$$

$$y_g = a_0 x^{\frac{5}{2}} [1 - \frac{1}{9} x + \frac{1}{198} x^2 - \frac{1}{7722} x^3 \cdots] + b_0 x^{-1} [1 + \frac{1}{5} x + \frac{1}{30} x^2 + \cdots]$$

$\downarrow$

مثر عند X=0

هاد الحل بكون

converges.

Example Solve $2x^2 y'' + xy' + (x^2-3)y = 0 \qquad y(0) = 2, \ y'(0) = 4$

Example Find at least the 4th approximation for the following D.E at $x_0 = 0$

$$\frac{dy}{dx} - y = \underline{\underline{e^x}} \quad \rightarrow 1^{st} \text{ - linear - constant - non Hom.}$$

لما في حد هو function ومش مضروب بالـ x هون الحالة
بـ x بعوض مكان الـ function بـ الـ Series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\ldots$$

$$y' - y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots$$

4^{th} approxim
الـ ٤ حدود

→ The D.E has No singular point. $-\infty < \cancel{x} < \infty$ $\quad x_0 = 0$ ordinary point

let $y = \sum_{j=0}^{\infty} a_j (x - x_0)^j = \sum_{j=0}^{\infty} a_j x^j$

$$y' = \sum_{j=1}^{\infty} j a_j (x)^{j-1}$$

$$y' - y = e^x$$

$$\sum_{\underset{\approx}{j=1}}^{\infty} j a_j (x)^{\underset{\approx}{j-1}} - \sum_{j=0}^{\infty} a_j x^j = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \ldots\ldots$$

the same power, limit

$$\sum_{j=0}^{\infty} (j+1) a_{j+1} x^j - \sum_{j=0}^{\infty} a_j x^j = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$\sum_{j=0}^{\infty} [(j+1) a_{j+1} - a_j] x^j = (x^j) 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

رح نساوي كل حد مع الي بشبهه ع

↓
expansion

$$[(0+1)a_{0+1}-a_0]x^0+[2a_2-a_1]x^1+[3a_3-a_2]x^2+[4a_4-a_3]x^3$$

$$+[5a_5-a_4]x^4+\cdots = 1x^0+x^1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4\cdots$$

$$a_1-a_0=1 \Rightarrow a_1=1+a_0$$

$$2a_2-a_1=1 \Rightarrow a_2=[1+a_1]\frac{1}{2}=[2+a_0]\frac{1}{2}=1+\frac{1}{2}a_0$$

$$3a_3-a_2=\frac{1}{2} \Rightarrow a_3=[\frac{1}{2}+a_2]\frac{1}{3}=\frac{1}{3}[\frac{1}{2}+1+\frac{1}{2}a_0]$$

$$a_4= \qquad =\frac{1}{3}[\frac{3}{2}+\frac{1}{2}a_0]=\frac{1}{2}+\frac{1}{6}a_0$$

$$4a_4-a_3=\frac{1}{6} \Rightarrow \frac{1}{4}[\frac{1}{6}+a_3]=\frac{1}{4}[\frac{1}{6}+\frac{1}{2}+\frac{1}{6}a_0]=\frac{1}{6}+\frac{1}{36}a_0$$

$$\sum_{j=0}^{\infty} a_j x^j = a_0x^0+a_1x+a_2x^2+a_3x^3+a_4x^4$$

$$y=a_0+[1+a_0]x+[1+\frac{1}{2}a_0]x^2+[\frac{1}{2}+\frac{1}{6}a_0]x^3+[\frac{1}{6}+\frac{1}{36}a_0]x^4$$

$$y=a_0+x+a_0x+x^2+\frac{x^2}{2}a_0+\frac{1}{2}x^3+\frac{1}{6}a_0x^3+\frac{1}{6}x^4+\frac{1}{36}a_0x^4$$

$$y=a_0[1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{36}x^4]+x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4$$

$$y(0)=0$$

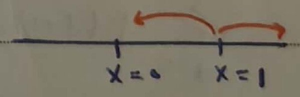
$$0=a_0[1+0]+0+0 \Rightarrow a_1=1$$

$$y_p=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{36}x^4\cdots x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4$$

Example:-

$$x\acute{y}=3y+3 \qquad y(1)=2 \xrightarrow{} x_0=1$$

$$x\acute{y}-3y=3 \Rightarrow \acute{y}-\frac{3}{x}y=3 \qquad \therefore x=0 \text{ is a singular point.}$$



$$\text{let } y=\sum_{j=0}^{\infty} a_j(x-1)^j$$

$0<x_0<2$ اذن مافي داعي لفحص لسيري

$$\acute{y} = \sum_{j=1}^{\infty} j a_j (x-1)^{j-1}$$

$$\Rightarrow x \sum_{j=1}^{\infty} j a_j (x-1)^{j-1} - 3 \sum_{j=0} a_j (x-1)^{j} = 3(x-1)^0 + 0(x-1) + 0(x-1)^2$$

$$(x-1+1) \sum_{j=1} j a_j (x-1)^{j-1} - 3 \sum_{j=0} a_j (x-1)^{j} = 3(x-1)^0$$

$$\sum_{j=1} j a_j (x-1)^{\boxed{j}} + \sum_{j=1} j a_j (x-1)^{\boxed{j-1}} - 3 \sum_{j=0} a_j (x-1)^{\boxed{j}} = 3(x-1)^0$$

$$\sum_{\boxed{j=1}} j a_j (x-1)^{j} + \sum_{\boxed{j=0}} (j+1) a_{j+1} (x-1)^{j} - \sum_{\boxed{j=0}} 3 a_j (x-1)^{j} = 3(x-1)^0$$

$$\sum_{j=1} j a_j (x-1)^{j} + a_1 (x-1)^0 - 3 a_0 (x-1)^0 + \sum_{j=1} (j+1) a_{j+1} (x-1)^{j} - \sum_{j=1} 3 a_j (x-1$$

$$= 3(x-1)^0$$

$$\Rightarrow [a_1 - 3a_0](x-1)^0 + \sum_{j=1}^{\infty} [j a_j + (j+1) a_j - 3 a_j](x-1)^{j} = 3(x-1)^0$$

$$\rightarrow a_1 - 3a_0 = 3 \Rightarrow a_1 = 3[1 + a_0]$$

$$a_{j+1} = \frac{3a_j - j a_j}{j+1} = \frac{(3-j) a_j}{j+1} \quad , \quad j = 1, 2, 3, 4 \ \ldots$$

$$j=1 \Rightarrow a_2 = \frac{2a_1}{2} = a_1 = 3[1 + a_0]$$

$$j=2 \Rightarrow a_3 = \frac{a_2}{3} = \frac{1}{3} a_1 = \frac{1}{3} \times 3 [1 + a_0] = 1 + a_0$$

$$j=3 \Rightarrow a_4 = 0$$

$$j=4 \Rightarrow a_5 = \frac{-a_4}{5} = 0 ,$$

No.

$$\sum_{j=0}^{\infty} a_j (x-1)^j = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$$

$$y = \underset{\substack{\downarrow \\ -1+1}}{a_0} + 3(1+a_0)(x-1) + 3(1+a_0)(x-1)^2 + (1+a_0)(x-1)^3$$

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$$y = a_0 + 1 - 1 + 3(1+a_0)(x-1) + 3(1+a_0)(x-1)^2 + (1+a_0)(x-1)^3$$

$$= a_0 + 1 \left[1 + 3(x-1) + 3(x-1)^2 + 1(x-1)^3 \right] - 1$$