



Mathematical

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* Differential equations and its applications in engineering :-

A differential equation is an equation that contains an unknown function and its derivatives. The order of the highest derivative that occurs in the equation.

المعادلة التفاضلية الحل
سبغها يكون امتزان وليس
رقم

بدي لوجد العلاقة بين y و x
كيف شكلها

الامتزان $F(x)$ يكون حل للمعادلة
التفاضلية في حال اذا اعمتنا $F(x)$
كان الشق الاثنان يساوي الايسر

→ y → dependent

→ x → independent

differential equation :-

المعادلة يلى بحتوي على ال
dependent (y) ومشتقة



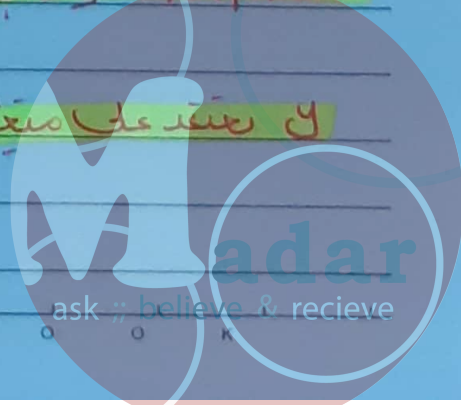
Ordinary differential equation (ODE) :- is an equation that contains one or several derivatives of an unknown function, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t).

The equation may also contain y itself, known functions of x (or t) and constant s .

المعادلة تكون تحتوي على ال depend و ال

independent وثابت

ان يعتمد على متغير واحد



Examples :-

first order
diff. eqn.

$$1 - \frac{dy}{dx} = 2x + 3$$

لو x مش ظاهرة
بالعادة ، نعتبر معادلة
تفاضلية .

second order
diff. eqn.

$$2 - \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + ay = 0$$

third order
diff. eqn.

$$3 - \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$$

* Partial differential equation :-

Examples :-

partial
diff. eqn.
على متغيرين

$$1 - \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$x, y \rightarrow$ independent
 $u \rightarrow$ dependent

$$2 - \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial t^4} = 0$$

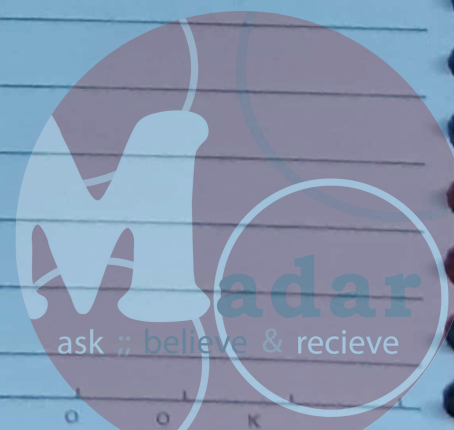
$$3 - \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}$$

$x, t \rightarrow$ independent
 $u \rightarrow$ dependent

* Order of differential equation :-

is the order of the highest derivative in differential equation.

مرتبة أعلى مشتقة في المعادلة .



Degree of differential equation :-

is power of the highest order derivative term.

هـوة أكبر مشتقة في المعادلة

1- $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + ay = 0$ 1

2- $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$ 1

3- $\left(\frac{d^3y}{dx^3}\right)^3 + \left(\frac{dy}{dx}\right)^5 + 3 = 0$ 3

→ third order ordinary differential equation of degree 3

Linear differential equation :-

A differential equation is linear, if :-

1- dependent variable and its derivatives are degree one.

2- no product of dependent variables and /or derivatives occur.

3- no transcendental function of the dependent variable or its derivatives occur.

إذا ما الها دخل في تحديد إذا كانت المعادلة التفاضلية خطية أم لا

→ transcendental functions $\Rightarrow \sin, \cos, \tan, \ln \dots$



Examples :-

1- $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0$

linear

2- $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$

non-linear

3- $x^2 \frac{d^2y}{dx^2} + y \frac{dy}{dx} = x^3$

non-linear

4- $\frac{dy}{dx} = \sin y$

non-linear

$\rightarrow \sin y = y - \frac{y^3}{3!} + \dots$

← $\frac{dy}{dx}$ مرتبة 4 = قوة

← $\frac{dy}{dx}$ في جيبين y دمجها

← $\sin y$ في



No	Differential equation	Ordinary or partial	linear or non linear	Order	degree	Independent variables	dependent variables
1	$y' = x + 6y$	Ordinary	linear	1	1	x	y
2	$y'' = 4y + y^3$	ordinary	non linear	2	1	أعزى أي مستوي عربي x	y
3	$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} - 2y = x^3$	ordinary	non linear	2	3	x	y
4	$y'' + 2xy' + 4y = \cos 2x$	Ordinary	linear	2	1	x	y
5	$\frac{dy}{dx} = \frac{x^2 - 1}{y + 4}$	Ordinary	non linear	1	1	x	y
6	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$	partial		1	1	x, y, z	u, v, w
7	$\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}$	partial	linear	2	1	x, t	u

1st - order differential equation :-

1- Derivative Form

تكون ظاهرة المعادلة $\frac{dy}{dx} + a_1(x) \frac{dy}{dx} + a_2(x)(y) = g(x)$

2- differential Form

تكون $dx/dy + M(x,y) dx + N(x,y) dy = 0$
معشولات على بعض

3- General Form

→ explicit Form

أفضل المستحقة في $\frac{dy}{dx} = f(x,y)$ or $f(x, y, \frac{dy}{dx}) = 0$
طرف رياضي الأسياء على الطرف الثاني $\rightarrow$ implicit Form
نفس الجهة

Second order differential equations:-

explicit لها نفس المتغيرات بتعب ال 1st مع الأسياء أنه في حالة ال

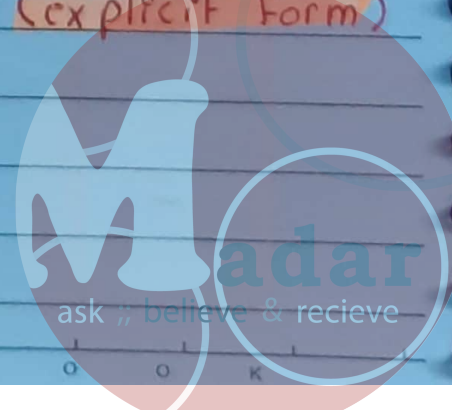
$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$

تكون على

* solution of differential equations

$x^{-3}y' - 4y^2 = 0$ (implicit Form)

$\frac{y'}{x^3} - 4y^2 = 0 \rightarrow \frac{y'}{x^3} \times 4y^2 \rightarrow y' = 4y^2 x^3$
(explicit Form)



A Function

$$y = h(x)$$

is called a solution of given ODE ($F(x, y, \dot{y}) = 0$) on some open interval $a < x < b$

لو أخذت الاختيار واستقيته دعو ضمة المعادلة (ODE) داح قطع العزف التآقي يساوي صفر.

if $h(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity if y and $\dot{y}$ are replaced with h and h' , respectively

The curve (the graph) of h is called a solution curve.

Here, open interval means that the end points a and b are not regarded as points belonging to the interval.

Also, $a < x < b$ includes infinite intervals $-\infty < x < b$, $a < x < \infty$, $-\infty < x < \infty$ (the real line) as special cases.

Origin of differential equations solution :-

1- Geometric origin

1- for the family of straight lines

$$y = c_1 x + c_2 \quad \text{the differential equation} \quad \frac{d^2 y}{dx^2} = 0$$

slope intercept

د² x² إلى الكل بعتها هو الخط

2- for the family of curves

$$A. y = c e^{x^2/2} \quad \text{the differential equation is} \quad \frac{dy}{dx} = xy$$

$$B. y = c_1 e^{2x} + c_2 e^{-3x} \quad \text{the differential equation is}$$

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

عشان اتحقق إذا هيا المعادلة أم لا اشتق بشرط لازم يكون التفاضل متساويين

Physical origin :-

1- free falling stone

$$\frac{d^2 s}{dt^2} = -g$$

distance or height $\rightarrow$ acceleration due to gravity

2- spring vertical displacement

$$m \frac{d^2 y}{dt^2} = -ky$$

mass $\rightarrow$ spring constant $\rightarrow$ displacement

3- RLC - circuit, Kirchhoff's second law

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = E$$

inductance $\rightarrow$ charge on capacitor $\rightarrow$ resistance $\rightarrow$ capacitance $\rightarrow$ voltage

physical origin :-

1- Newton's law of cooling

$$\frac{dT}{dt} = K(T - T_s)$$

rate of cooling of the liquid $\rightarrow$ temp difference between the liquid T and its surrounding T_s

2- Growth and Decay

$$\frac{dy}{dt} = ky$$

quantity present at any time

* Example 2 :-

$$y' = dy/dx = \cos x$$

$$\int dy = \int \cos x dx$$

$$y = \sin x + C$$

arbitrary constant $\rightarrow$ family of solution

* تغير أمت حيث مختلفة حل
C حيث قطع حل ليس يغير
عندى شئت للعقل

* Initial value problem :-

In most cases the unique solution of a given problem, hence a particular solution, is obtained from a general solution by an initial condition $y(x_0) = y_0$, with given values x_0 and y_0 , that is used to determine a value of arbitrary constant c .

Geometrically this condition means that the solution curve should pass through the point (x_0, y_0) in the x - y plane.

An ODE together with an initial condition is called an initial value problem (IVP). Thus, if ODE is explicit, $\dot{y} = F(x, y)$, the (IVP) is of the form, $\dot{y} = F(x, y)$, $y(x_0) = y_0$.

* Example 5 :-

Given an amount of a radioactive substance, say 0.5 g, Find the amount of present at any later time.?"

$$k = 1.4 \times 10^{-11} \text{ s}^{-1}$$

المسألة
الأساسية

$$\frac{dy}{dx} = ky$$

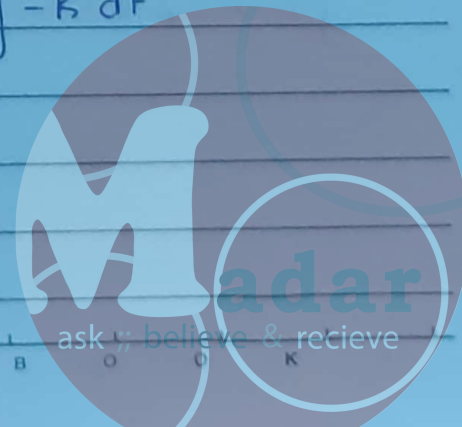
المعادلة التفاضلية
من الدرجة الأولى

The time rate of change $\dot{y}(t) = dy/dt$ is proportional to $y(t)$

Initial condition $y(0) = 0.5$

$$\frac{dy}{dx} = -ky \rightarrow \frac{dy}{dt} = -ky \rightarrow \int \frac{dy}{y} = \int -k dt$$

$$\ln y = -kt + c \Rightarrow y(t) = c e^{-kt}$$



$$y(0) = 0.5$$

$$y(0) = c e^{-k(0)} = 0.5$$

$$c = 0.5$$

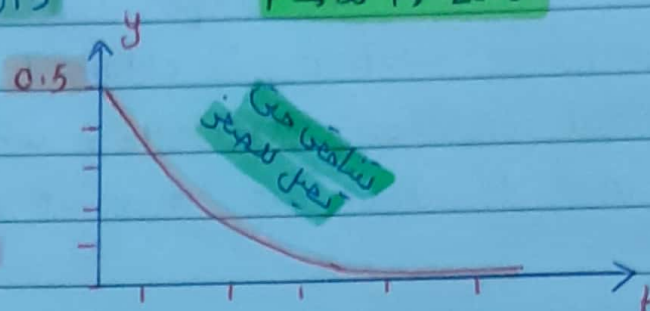
The limit of y as

$t \rightarrow \infty$ is zero.

$$y(t) = 0.5 e^{-kt}$$

إذا ما حل

لازم كفيق المعادلة التفاضلية



check result

$$y(t) = 0.5 e^{-kt}$$

$$\frac{dy}{dt} = -0.5k e^{-kt} \rightarrow \frac{dy}{dt} = -ky$$

نفس المعادلة الأصلية

* Example 8-

$y = 2x + c$ is solution of 1st order differential equation

$$\frac{dy}{dx} = 2 \rightarrow \text{arbitrary constant}$$

→ As is solution of the differential equation for every value of c , hence it is known as general solution (// lines).

كل الخطوط تكون متوازية لكن معقول إليها لإراحة معينة

* Examples :-

$$1 - \int \dot{y} = \int \sin(x)$$

$$y = -\cos(x) + c \rightarrow 1 \text{ parameter}$$

$$2 - \int \ddot{y} = \int 6x + e^x$$

$$\int \dot{y} = \int 3x^2 + e^x + c_1$$

$$y = x^3 + e^x + c_1 x + c_2$$

→ 2 parameters



* Separable equations :- $\rightarrow$ أمثل عن x

Not all equations have an explicit formula for a solution. But some types of equation can be solved explicitly. Among others separable equations is one type.

A separable equation is a first order differential equation in which the expression for dy/dx can be factored into the product of a function of x and a function of y . That is, a separable equation can be written in the form $\frac{dy}{dx} = f(x)g(y)$ #

$$\frac{dy}{dx} = f(x)g(y) \rightarrow \text{أمثل عن } \frac{dx}{dy}$$

separate and integrate

$$\int \frac{dy}{g(y)} = \int f(x) dx + c$$

* Examples :-

1. $\frac{dy}{dx} = \frac{y}{x} \rightarrow \frac{dy}{y} = \frac{dx}{x}$ (separable equation)

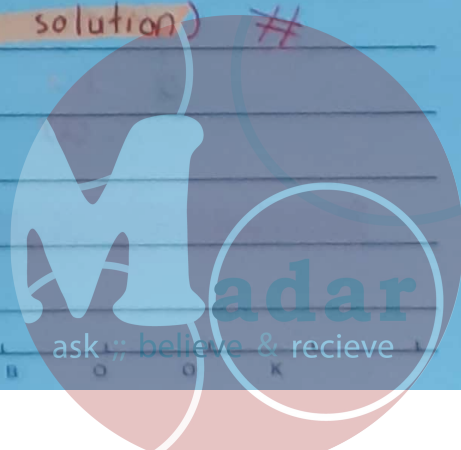
$$\int \frac{dy}{y} = \int \frac{dx}{x} + c \rightarrow \ln y = \ln x + c$$

$$\ln y - \ln x = c \rightarrow \ln \frac{y}{x} = c \rightarrow \frac{y}{x} = e^c$$

لازم أمثل
 $y = f(x)$

$$y = x e^c \rightarrow y = c_1 x \text{ (general solution) } \#$$

$\hookrightarrow$ constant c_1



2. Find the general solution of

$$\frac{dy}{dx} = \frac{y^2}{x} \rightarrow \frac{dy}{y^2} = \frac{dx}{x}$$

$$\int \frac{dy}{y^2} = \int \frac{dx}{x} + C \rightarrow \frac{-1}{y} \ln x + C \text{ (implicit solution)}$$

$$y(\ln x + C) = -1 \rightarrow y = \frac{1}{-C - \ln x} \text{ (general solution)}$$

explicit solution

$$C - \ln x \neq 0$$

$$x > 0$$

interval of validity

and find particular solution for which $y = 2$ when $x = 1$

$$y(1) = 2 \rightarrow \text{initial condition}$$

$$y(1) = \frac{1}{-C - \ln(1)} = 2 \Rightarrow C = -\frac{1}{2}$$

المطلوب إيجاد قيمة

$$y(x) = 0$$

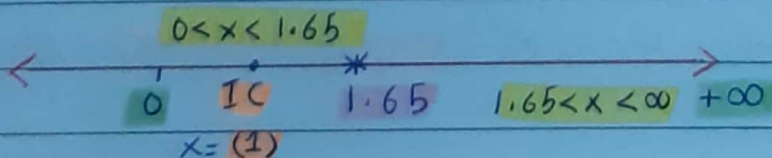
$$y(x) = 0 = \frac{1}{+0.5 - \ln(x)}$$

$$0.5 - \ln x = 0$$

$$0.5 = \ln x$$

$$x = \sqrt{e} = \pm 1.65$$

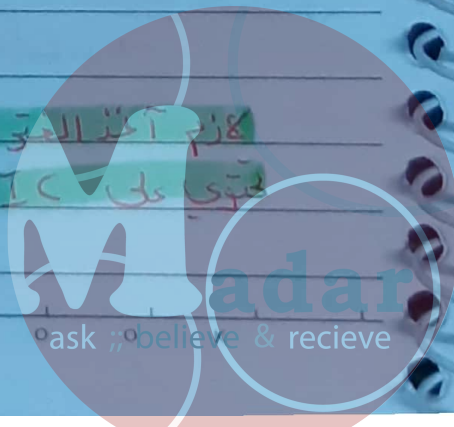
$$(x > 0) \rightarrow \text{so } x = +1.65$$



$$\text{Interval of validity} = 0 < x < 1.65$$

لازم أخذ الحصة التي

تحتوي على IC



$$3- x \frac{dy}{dx} - y = 2x^2y \rightarrow x \frac{dy}{dx} = 2x^2y + y$$

$$\frac{dy}{dx} = \frac{2x^2y + y}{x} \rightarrow \frac{dy}{dx} = \underbrace{y}_{g(y)} \left(\underbrace{\frac{2x^2 + 1}{x}}_{f(x)} \right) \rightarrow \text{separable}$$

$$\int \frac{dy}{y} = \int \frac{2x^2 + 1}{x} dx + C$$

$$\int \frac{dy}{y} = \int \frac{2x^2}{x} + \frac{1}{x} dx + C$$

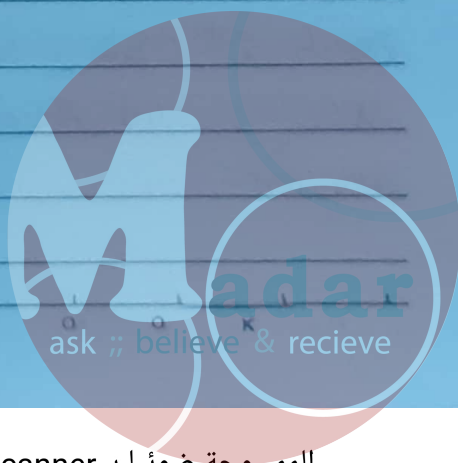
$$\ln y = x^2 + \ln x + C \rightarrow \text{(implicit solution)}$$

$$\ln y - \ln x = x^2 + C \rightarrow \ln \frac{y}{x} = x^2 + C$$

$$\frac{y}{x} = e^{x^2} \boxed{e^C} \rightarrow y = c_1 x e^{x^2} \text{ (explicit solution) } \#$$

$$(-\infty < x < +\infty)$$

Interval of validity
المساحة التي يكون فيها الحل صالحاً



* Reduction of 1st ODE to separable form :-

1st ODE is reduced to separable form if it composed of homogeneous function.

→ homogeneous function =>

$f(x, y)$ is homogeneous function of degree n if there exists a constant n such that for every parameter λ :

$$f(x\lambda, y\lambda) = \lambda^n f(x, y)$$

* Examples :-

1- $f(x, y) = x^2 + xy \rightarrow f(x\lambda, y\lambda) \rightarrow$

لازم أن يكون لها داخل
المعادلة ويطرح عندي
مثل فوقه ↑

$$f(x\lambda, y\lambda) = x^2\lambda^2 + \lambda x \lambda y$$

$$f(x\lambda, y\lambda) = \lambda^2 (x^2 + xy)$$

$f(x, y)$

$$f(x\lambda, y\lambda) = \lambda^2 f(x, y) \rightarrow \text{homogeneous function with degree 2} \rightarrow$$

ماد الكلي
للاختيار الأعمى

2- $f(x, y) = \tan(x/y) = \frac{\sin(x/y)}{\cos(x/y)}$

$$f(\lambda x, \lambda y) = \tan\left(\frac{\lambda x}{\lambda y}\right)$$

$$f(\lambda x, \lambda y) = f(x, y) \lambda^0 \rightarrow \text{degree} = 0$$

* A 1st ODE, can be reduced to a separable form if both $M(x, y)$ and $N(x, y)$ are homogeneous function of the same degree.

$$u(x,y) dx + v(x,y) dy = 0 \rightarrow \text{differential form}$$

* Example 8-

$$y^2 + x^2 \frac{dy}{dx} - xy \frac{dy}{dx} = 0 \rightarrow y^2 + (x^2 - xy) \frac{dy}{dx} = 0$$

$$y^2 dx + (x^2 - xy) dy = 0 \rightarrow \text{differential form}$$

$\downarrow$ $\downarrow$
 $u(x,y)$ $v(x,y)$

$$u(x,y) = y^2$$

$$u(\lambda x, \lambda y) = \lambda^2 y^2 \rightarrow u(x,y)$$

$\rightarrow$ homogeneous function with degree 2

$$v(x,y) = x^2 - xy$$

$$v(\lambda x, \lambda y) = \lambda^2 x^2 - \lambda x \lambda y$$

$$= \lambda^2 (x^2 - xy)$$

$$\rightarrow v(x,y)$$

$\rightarrow$ homogeneous function with degree 2

$\therefore u(x,y)$ & $v(x,y) \rightarrow$ homogeneous function with same degree 2

To solve $\rightarrow$ let $v = \frac{y}{x} \rightarrow y = vx$

$\rightarrow$ متغير

من ثابت

$$y = vx \rightarrow \text{استخدم مشتقة متغير}$$

$$\frac{dy}{dx} = \hat{v}x + v$$

الآن نفوض بالمعادلة
الاحتمالية



Subject :

Chapter 1

6/3/2021

$$y^2 + (x^2 - xy) \frac{dy}{dx} = 0 \rightarrow (x^2 - xy) \frac{dy}{dx} = -y^2$$

$$\frac{dy}{dx} = \frac{-y^2}{x^2 - xy} \rightarrow \text{عوضنا ٨ س٩٩}$$

بالي حو٩٩٩٩

$$= \sqrt{x} + v = \frac{v^2 x^2}{x^2 v x - x^2} \rightarrow \sqrt{x} + v = \frac{x^2}{x^2} \frac{v^2}{v-1}$$

$$\sqrt{x} = \frac{v^2}{v-1} - v \rightarrow \sqrt{x} = \frac{v}{v-1}$$

توحيد مقامات

$$\sqrt{x} = \frac{1}{x} \left(\frac{v}{v-1} \right) \rightarrow \text{separable}$$

$$\int \left(\frac{v-1}{v} \right) dv = \int \frac{dx}{x} + C$$

$$= \int \frac{x^1}{x} - \frac{1}{v} dv = \int \frac{dx}{x} + C$$

$$v - \ln v = \ln x + C \rightarrow v - C = \ln x + \ln v$$

$$\frac{\ln(xv)}{e} = \frac{v-C}{e} \rightarrow xv = e^v \cdot e^{-C} \rightarrow C_1$$

$$\frac{xv}{x} = \frac{e^v C_1}{x} \rightarrow v = \frac{e^v C_1}{x} \rightarrow v = y/x$$

$$\frac{y}{x} = \frac{e^{y/x} C_1}{x} \rightarrow y = e^{y/x} C_1 \#$$

implicit solution

بأن ما قدرت أمحل
نكتبها على طرف x على طرف
شأن

ask :: believe & recieve

Exact 1st ODE :-

$$\frac{dy}{dx} = -kx \rightarrow \int dy = \int -kx dx$$

$$y = -\frac{kx^2}{2} + c \rightarrow c = y + \frac{kx^2}{2} = f(x, y)$$

constant
y → x

$$\frac{dy}{dx} = -ky \rightarrow \int \frac{dy}{y} = \int -k dx$$

$$\ln y = -kx + c \rightarrow y = e^{-kx+c} = e^{-kx} \cdot e^c$$

↪ c1

$$y = c_1 e^{-kx} \rightarrow c_1 = \frac{y}{e^{-kx}} \rightarrow c_1 = y e^{kx} = f(x, y)$$

f(x, y(x))

→ Differentiate :-

$$\left(0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} \right) * \partial x$$

$$\left(\frac{\partial F}{\partial x} \right) \frac{\partial x}{dx} + \left(\frac{\partial F}{\partial y} \right) \frac{\partial y}{dy} = 0 \rightarrow \text{exact differential}$$

* Note :-

Any 1st ODE can be written in differential form

$$\rightarrow \left(\frac{dy}{dx} = -ky \right) * dx \rightarrow dy + (ky) dx = 0$$

$$(ky) dx + (1) dy = 0$$

$u(x, y)$ $v(x, y)$

يدخل تكون هاي الكمالية
الكمالية يلي طرقت معنا فوق

$$\frac{\partial F}{\partial x} = u(x, y) \rightarrow \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial x} \right) = \frac{\partial u}{\partial y}$$

$$\frac{\partial F}{\partial y} = v(x, y) \rightarrow \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y} \right) = \frac{\partial v}{\partial x}$$

* Note :

$$\frac{\partial}{\partial y} \left(\frac{\partial F}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y} \right)$$

To be exact necessary and sufficient condition

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \rightarrow \begin{array}{l} \text{عشان يكون} \\ \text{exact لازم يكونوا} \\ \text{متساويين} \end{array}$$

* Example :

$$1- y^2 dx + (x^2 - xy) dy = 0$$

Test for exactness :-

$$u(x, y) = y^2 \rightarrow \frac{\partial u}{\partial y} = 2y$$

$$v(x, y) = x^2 - xy \rightarrow \frac{\partial v}{\partial x} = 2x - y$$

$$\left(\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x} \right)$$

it is not exact ODE

$$2- (2xy^2 + 2) dx + (2x^2y + 4y) dy = 0$$

$u(x, y)$

$v(x, y)$

$$\frac{\partial F}{\partial x} = u(x, y)$$

$$\frac{\partial F}{\partial y} = v(x, y)$$

integrate



$$F = \int u(x, y) dx + c(y)$$

مضت الـ F

عن الـ x والـ y بعين

$$F = \int v(x, y) dy + c(x)$$

كاملنا

$$\int u(x, y) dx + c(y) = \int v(x, y) dy + c(x)$$

→ Test for exactness:-

$$u(x, y) = 2xy^2 + 2 \rightarrow \frac{\partial u}{\partial y} = 4xy$$

$$v(x, y) = 2x^2y + 4y \rightarrow \frac{\partial v}{\partial x} = 4xy$$

$$\left(\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \right)$$

exact

$$F = \int (2xy^2 + 2) dx + c(y)$$

$$F = x^2y^2 + 2x + c(y)$$

$$F = \int (2x^2y + 4y) dy + c(x)$$

$$F = x^2y^2 + 2y^2 + c(x)$$

$$\cancel{x^2y^2} + 2x + c(y) = \cancel{x^2y^2} + 2y^2 + c(x)$$

$$2x + \underset{2y^2}{c(y)} = \underset{2x}{2y^2} + c(x)$$

عشان يكونوا متساويين

$$c(x) = 2x$$

لازم

$$c(y) = 2y^2$$

$$F = x^2y^2 + 2y^2 + 2x = c$$

$$x^2y^2 + 2y^2 = c - 2x$$

$$y^2(x^2 + 2) = c - 2x$$



Subject: Chapter 1

7/3/2021

$$y^2 = \frac{c-2x}{x^2+2} \rightarrow y = \pm \sqrt{\frac{c-2x}{x^2+2}} \quad x^2+2 > 0$$

OR

$$V = x^2y^2 + 2x + c(y)$$

$$\frac{\partial F}{\partial y} = 2y \cancel{x^2} + c'(y) = v(x,y) = \cancel{2x^2y} + 4y$$

$$\frac{dc}{dy} = \int c'(y) = \int 4y$$

$$c(y) = 2y^2 + c_2$$

$$F = x^2y^2 + 2x + 2y^2 + c_2 = c_3$$

$$x^2y^2 + 2x + 2y^2 = c_3 - c_2$$

$\uparrow \quad \uparrow$
 c

$$x^2y^2 + 2y^2 = c - 2x$$

$$y^2(x^2+2) = c-2x$$

$$y^2 = \frac{c-2x}{x^2+2} \rightarrow y = \pm \sqrt{\frac{c-2x}{x^2+2}} \quad x^2+2 > 0$$



Subject :

Chapter 1

7, 3, 2020

* Examples:

$$\frac{dy}{dx} = \frac{x-y}{x+y} \rightarrow \underbrace{(x-y)}_{M(x,y)} dx - \underbrace{(x+y)}_{N(x,y)} dy = 0$$

$$M(x,y) = x-y \rightarrow \partial M / \partial y = -1$$

exact

$$N(x,y) = -x-y \rightarrow \partial N / \partial x = -1$$

$$\frac{\partial F}{\partial x} = M(x,y) = x-y$$

$$\frac{\partial F}{\partial y} = N(x,y) = -x-y$$

$$F = \int (x-y) dx + c(y)$$

$$F = \frac{x^2}{2} - yx + c(y)$$

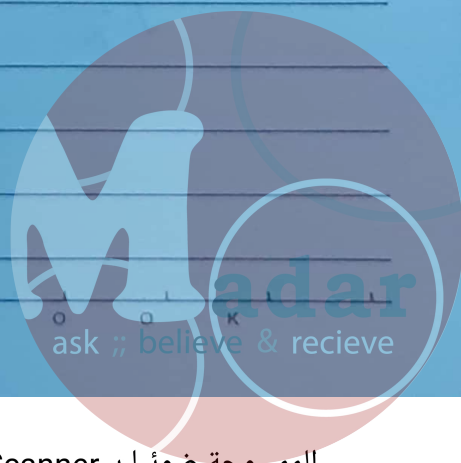
$$\frac{\partial F}{\partial y} = \cancel{-x} + c'(y) = \cancel{-x-y}$$

$$\int c'(y) = \int -y \rightarrow c(y) = \frac{-y^2}{2} + c_1$$

$$\left(F = \frac{x^2}{2} - yx - \frac{y^2}{2} + c_1 = c_2 \right) * 2$$

$$x^2 - 2yx - y^2 = \underbrace{2(c_2 - c_1)}_{c_3}$$

$$y^2 + 2xy - x^2 = c_3 \quad \text{Implicit solution}$$



Subject:

Chapter 1

7/3/2021

* Notes:

$$ax^2 + bx + c = 0$$

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-2x \pm \sqrt{(2x)^2 - 4(-x^2 - 3)}}{2} \rightarrow (2x)^2 + 4x^2 + 4(3) \geq 0$$

solve as homogeneous functions of u & v of degree 1

$$v = \frac{y}{x} \rightarrow y = xv \rightarrow \dot{y} = x\dot{v} + v$$

$$\frac{dy}{dx} = \frac{x-y}{x+y} = \frac{x-xv}{x+xv}$$

$$\downarrow$$
$$x\dot{v} + v$$

$$x\dot{v} + v = \frac{x(1-v)}{x(1+v)} \rightarrow x\dot{v} + v = \frac{1-v}{1+v}$$

$$x \frac{dv}{dx} = \frac{1-v}{1+v} - v$$

$$x \frac{dv}{dx} = \frac{1-2v-v^2}{1+v} \rightarrow \int \frac{dx}{x} = \int \frac{1+v}{1-2v-v^2} dv$$

$$\text{let } u = 1-2v-v^2$$

$$du = -2-2v dv$$

$$du = -2(1+v) dv$$

$$dv = \frac{-du}{2(1+v)}$$

$$\ln x = \int \frac{dx}{x}$$

ask :: believe & recieve

$$\int \frac{(1+v)}{(1-2v-v^2)} dv = - \int \frac{(1+v)}{u} \frac{du}{2(1+v)}$$

$$= -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln u$$

$$\ln x = -\frac{1}{2} \ln u + c$$

$$2 \ln x + \ln u = 2c$$

$$\ln x^2 u = 2c \rightarrow x^2 u = e^{2c}$$

$$e \quad e \rightarrow c_1 \quad u = 1 - 2v - v^2$$

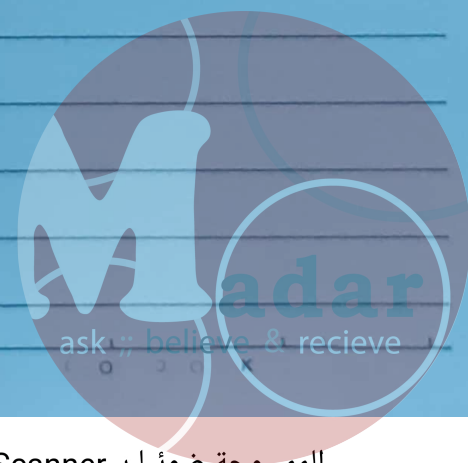
$$x^2 (1 - 2v - v^2) = c_2$$

$$v = \frac{y}{x}$$

$$x^2 (1 - 2y/x - (y/x)^2) = c_2$$

$$x^2 - 2xy - y^2 = c_2 \rightarrow y^2 + 2xy - x^2 = c_3$$

$\rightarrow$ Implicit solution



Nonexact ODE :-

$$M(x,y) dx + N(x,y) dy = 0$$

معامل x الـ

معامل y الـ

sufficient

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exactness n & s condition
necessary

إذا كانو متساويين
يكونوا exact

→ if not exact

بجرب الحادلة بـ $\mu(x,y)$
→ integrating factor

$$\mu(x,y) * M(x,y) dx + \mu(x,y) * N(x,y) dy = 0$$

معامل x الـ μM

معامل y الـ μN

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \rightarrow \text{كأنى بحسب انهم exact}$$

$$= \mu \frac{\partial M}{\partial y} + \frac{\partial \mu}{\partial y} \cdot M = \mu \frac{\partial N}{\partial x} + \frac{\partial \mu}{\partial x} \cdot N$$



$$\frac{\partial \mu}{\partial y} = \mu_y$$

حسبة μ نسبة للتغير

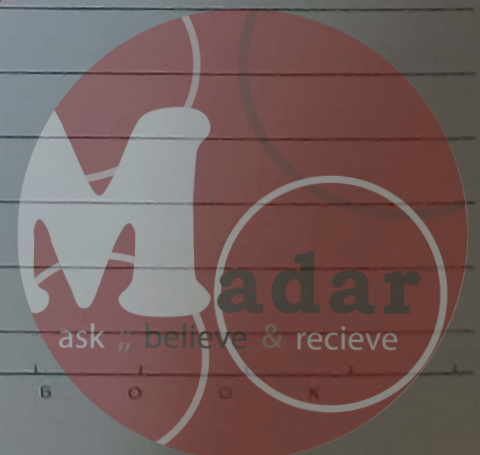
$$= \mu M_y + \mu_y M = \mu N_x + \mu_x N \neq$$

$$\Rightarrow \text{if } \mu(x,y) = \mu(x) \quad (\mu_y = 0)$$

لا μ_y

إذا كان μ اختزان بدلالة x

بس كيتكون مشتقة μ نسبة الى x
متساوي بغير



$$\mu M_y - \mu N_x = \mu \mu_x$$

$$\leftarrow \mu_y = 0$$

$$\frac{\mu(M_y - N_x)}{\mu} = \frac{\mu \mu_x}{\mu} \rightarrow \frac{1}{\mu} (M_y - N_x) = \frac{\mu_x}{\mu}$$

$$\Rightarrow \frac{1}{\mu} \left(\frac{\partial \mu}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{\mu} \frac{\partial \mu}{\partial x}$$

* Note :-

$$\frac{d}{dx} (\ln \mu(x)) = \frac{1}{\mu} \cdot \frac{\partial \mu}{\partial x}$$

أول شيء يوجد مشتقة $\ln \mu$
يجب أن مشتقة μ نسبة لـ x

لـ مشتقة $\ln \mu$

مشتقة $\mu(x)$

$$\frac{d}{dx} (\ln \mu(x)) = \frac{1}{\mu} \left(\frac{\partial \mu}{\partial y} - \frac{\partial N}{\partial x} \right) = R(x, y)$$

هذا الطرف
سيفيه $R(x, y)$

$$\left(\frac{d}{dx} (\ln \mu(x)) = R(x, y) \right) * dx$$

$$\int d(\ln \mu(x)) = \int R(x, y) dx$$

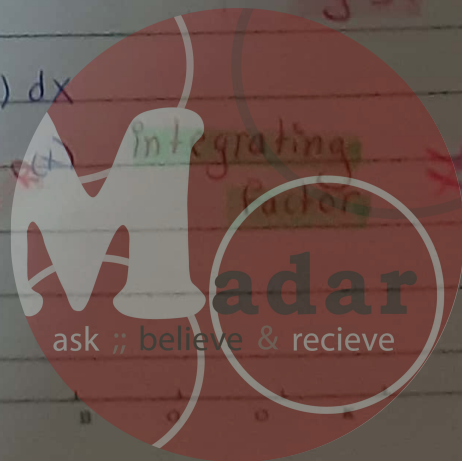
$$\ln \mu(x) = \int R(x, y) dx \rightarrow \mu(x) = e^{\int R(x, y) dx}$$

ما يقدر أكامل
منطقة كالم
أفحق R
اختار بـ x
أد y

$$\text{if } R(x, y) = R(x) \Rightarrow \mu(x) = e^{\int R(x) dx}$$

إذا كانت R فقط
المتغير x فقط

Integrating
factor



$\Rightarrow$ if $\mu = \mu(y)$ ($\mu_x = 0$)

إذا كانت μ اختزان
للدالة y فقط

$$\mu M_y + \mu_y \mu = \mu N_x \rightarrow \mu_y \mu = \mu N_x - \mu M_y$$

$$= \frac{\mu_y \mu}{\cancel{\mu} \cancel{\mu}} = \frac{\mu (N_x - M_y)}{\cancel{\mu} \cancel{\mu}} \rightarrow \frac{1}{\mu} \mu_y = \frac{1}{\mu} (N_x - M_y)$$

$$= \frac{1}{\mu} \frac{\partial \mu}{\partial y} = \frac{-1}{\mu} (\mu_y - N_x) = R^*(x, y)$$

if $M_y = N_x \rightarrow$ exact

if $M_y \neq N_x \rightarrow$ deviation of exact

$$\downarrow$$

$$\frac{d}{dy} (\ln \mu(y)) = \frac{-1}{\mu} (\mu_y - N_x) = R^*(x, y)$$

if $R^*(x, y) = R^*(y) \Rightarrow \mu(y) = e^{\int R^*(y) dy}$

In summary :-

$$R(x, y) = \frac{1}{\mu} (\mu_y - N_x)$$

$$(R(x, y) = R(x))$$

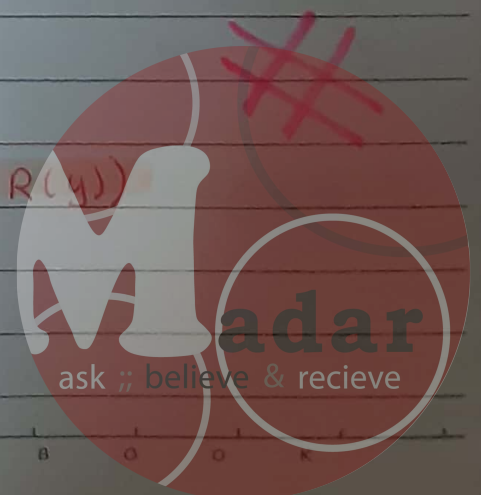
$$\mu(x) = e^{\int R(x) dx}$$

$$R^*(x, y) = \frac{-1}{\mu} (\mu_y - N_x)$$

$$(R^*(x, y) = R(y))$$

$$\mu(y) = e^{\int R^*(y) dy}$$

* بالكتاب :-
 $\mu(x, y) = P(x, y)$
 $N(x, y) = Q(x, y)$
 $\mu(x, y) = F(x, y)$



* Example :-

$$(2y^2 + 3x) dx + (2xy) dy = 0 \quad (\text{nonlinear 1st ODE})$$

$\downarrow$ $\downarrow$
 $M(x,y)$ $N(x,y)$

$$M(x,y) = 2y^2 + 3x \rightarrow \frac{\partial M}{\partial y} = 4y$$

$$N(x,y) = 2xy \rightarrow \frac{\partial N}{\partial x} = 2y \quad \left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \right] \quad \text{not exact}$$

To make it exact

$$R(x,y) = R(x) \quad \text{or} \quad R^*(x,y) = R(y)$$

بدي أتكيد إذا R
اختار أن بدالة X أو Y

$$R(x,y) = \frac{1}{N} (M_y - N_x)$$

$$= \frac{1}{2xy} (4y - 2y) = \frac{1}{x} = R(x) \quad (\text{Function of } x \text{ only})$$

$$\mu(x) = e^{\int 1/x dx} = e^{\ln x} = x$$

$$\mu(x) = x$$

يعني لو ضربت المعادلة بـ $\mu(x)$
لازم نحصل exact

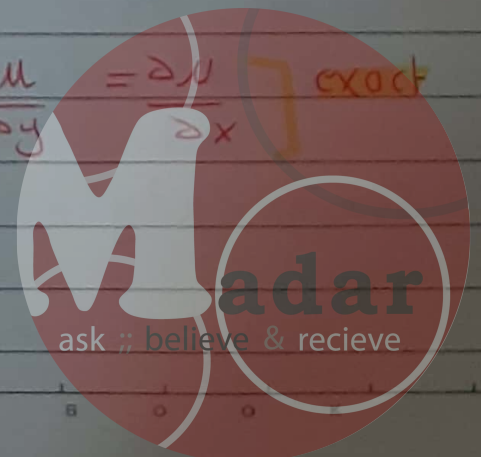
⇒ Multiply by $\mu(x) = x$:-

$$(2xy^2 + 3x^2) dx + (2x^2y) dy = 0$$

$$M(x,y) = 2xy^2 + 3x^2 \rightarrow \frac{\partial M}{\partial y} = 4yx$$

$$N(x,y) = 2x^2y \rightarrow \frac{\partial N}{\partial x} = 4yx$$

$$\left[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \right] \quad \text{exact}$$



$$F(x, y) = k \rightarrow \text{constant}$$

$$\frac{\partial F}{\partial x} = M(x, y) = 2xy^2 + 3x^2$$

$$\frac{\partial F}{\partial y} = N(x, y) = 2x^2y$$

هيسل
بناخذهم بعد
ما صار
exact.

→ اختيار الأسهل

إذا اخترت التكامل بدلالة y

عشان أعمل التكامل (الثاني)

يكون الثابت بدلالة x والعكس

أسهل لأنني حذواحد

مصحح

$$F = \int N(x, y) dy + c(x) \rightarrow F = \int 2x^2y dy + c(x)$$

$$F(x, y) = x^2y^2 + c(x)$$

$$\frac{2x^2y^2}{2}$$

بما أني بدلالة x

بدا

بنسقي الاختران بدلالة

نوجد هيا؟

x

$$\frac{\partial F}{\partial x} = 2xy^2 + c'(x)$$

$$= 2xy^2 + 3x^2$$

$$\rightarrow 3x^2$$

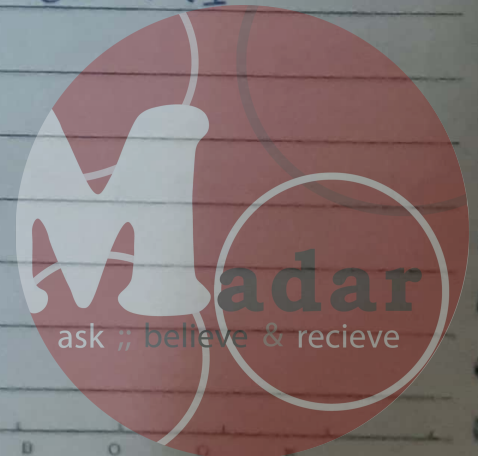
لما كيف عرفنا؟ عشان لازم يكون مساوية

لـ $M(x, y)$ يعني هون.

$$\Rightarrow c'(x) = \frac{dc}{dx} = 3x^2 \xrightarrow{\text{تكامل}} c = \int 3x^2 dx + c_1$$

$$c = x^3 + c_1$$

$$F(x, y) = x^2y^2 + x^3 + c_1 = k$$



$$= x^2 y^2 + x^3 = k - (1 = k_1)$$

$$y^2 = \frac{k_1 - x^3}{x^2} \rightarrow y = \pm \sqrt{\frac{k_1 - x^3}{x^2}} \rightarrow \text{ما داخل الجذر أكبر من أو يساوي صفر}$$

زبطنا الشكل

راكمام لا يساوي صفر

$$y = \pm \sqrt{k_1 x^{-2} - x}$$

special case :- (For linear 1st ODE)

$$\frac{dy}{dx} + P(x)y = q(x)$$

$$P(x, y) = P(x) \text{ only}$$

لازم يكون
معاملها 1
لازم يكون
بداية x لو كان
بداية y يكون
non linear

$$\left(\frac{dy}{dx} + P(x)y = q(x) \right) dx$$

إعادة ترتيب

$$dy + P(x)y dx = q(x) dx \rightarrow (q(x) - P(x)y) dx = dy$$

$$(P(x)y - q(x)) dx + 1 dy = 0$$

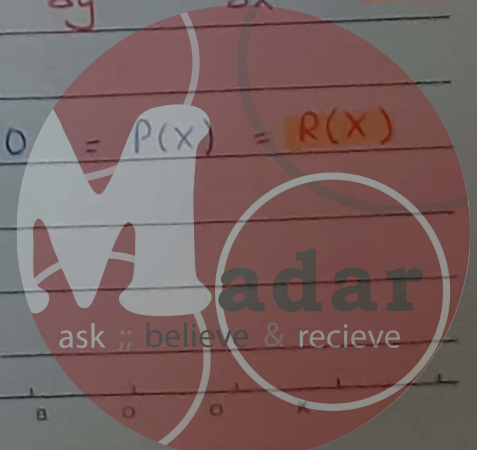
هنا
المعادلة
بسالبة

$$M(x, y) = P(x)y - q(x) \rightarrow \frac{\partial M}{\partial y} = P(x)$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ not exact}$$

$$N(x, y) = 1 \rightarrow \frac{\partial N}{\partial x} = 0$$

$$R(x, y) = \frac{1}{N} (M_y - N_x) = \frac{1}{1} (P(x) - 0) = P(x) = R(x)$$



$$\mu(x) = e^{\int R(x) dx} \Rightarrow \mu(x) = e^{\int P(x) dx}$$

تقبل معامل y في

linear 1st ODE

integrating factor

* Example :-

$$\frac{dy}{dx} - y = e^{2x}$$

(linear)

↓

$$P(x) = -1$$

تقبل عليها

الشروط

$$\mu(x) = e^{\int P(x) dx} = e^{\int -1 dx} \rightarrow \mu(x) = e^{-x}$$

Multiply by $\mu(x)$:-

$$e^{-x} \frac{dy}{dx} - e^{-x} y = e^{-x} e^{2x}$$

$\rightarrow e^x$

* Note :-

$$\frac{d}{dx} (\mu y) = \mu \frac{dy}{dx} + y \frac{\partial \mu}{\partial x}$$

مشتقة
حزب

$$\frac{d}{dx} (e^{-x} y) = e^{-x} \dot{y} + y (-e^{-x})$$

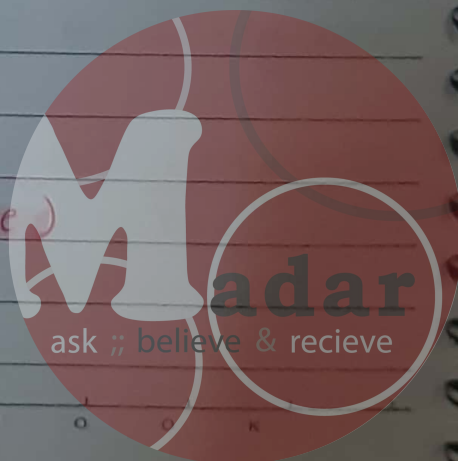
مشتقة
على

نلاحظ أن هذا الشيء

مقابل المعادلة التي في

$$e^x =$$

$$\frac{d}{dx} (e^{-x} y) = e^x \quad (\text{separate \& integrate})$$



$$\int d(e^{-x}y) = \int e^x dx + c$$

$$\frac{(e^{-x}y)}{e^{-x}} = \frac{e^x}{e^{-x}} + \frac{c}{e^{-x}} \rightarrow y = e^{2x} + ce^x$$

* Example :-

$$\frac{dy}{dx} + y = x$$

$$y(0) = 2$$

Initial value

problem (IVP)

(linear)

$$P(x) = 1$$

$$\mu(x) = e^{\int P(x) dx} = e^{\int 1 dx} = e^x \rightarrow \mu(x) = e^x$$

$$e^x \frac{dy}{dx} + e^x y = x e^x$$

$$\frac{d(e^x y)}{dx} = e^x \dot{y} + e^x y = x e^x$$

$$\int d(e^x y) = \int x e^x dx \rightarrow e^x y = \int x e^x dx + c$$

تكاملي الأجزاء
(by part)

$$\# \int x e^x dx$$

$$u = x$$

$$\int dv = \int e^x dx$$

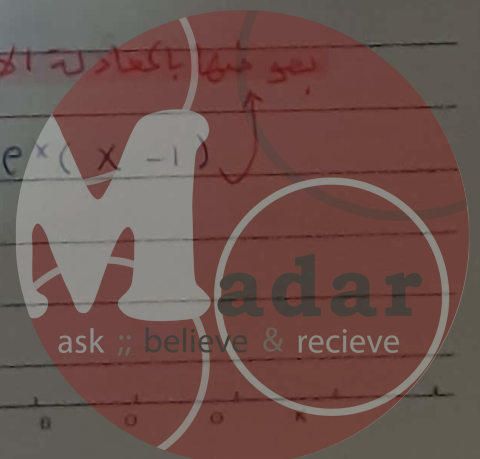
$$du = dx$$

$$v = e^x$$

$$\hookrightarrow \left(\int u dv = uv - \int v du \right)$$

$$x e^x - \int e^x dx = x e^x - e^x \rightarrow e^x (x - 1)$$

بعضها بالعادة الأمثلة



$$\cancel{e^x} y = \cancel{e^x} (x-1) + \frac{c}{\cancel{e^x}}$$

$$y = (x-1) + ce^{-x} \quad (\text{general equation (solution)})$$

→ Particular solution :- $I.C \rightarrow y(0) = 2$

$$y(0) = (0-1) + ce^0 = 2$$

$$= -1 + c = 2 \rightarrow (c = 3)$$

$$y(x) = (x-1) + 3e^{-x} \quad \text{X}$$

linear 1st ODE $\Rightarrow$

$$\frac{dy}{dx} + p(x)y = q(x)$$

if $q(x)$

$= 0$ homogeneous equ.

$\neq 0$ non homogeneous equ.

to make it exact

$$\mu(x) \dot{y} + \mu(x) p(x) y = \mu(x) q(x)$$

magic condition ∇

$$\text{let } \mu'(x) = \mu(x) p(x)$$

unknown

نفوض

$$\mu(x) \dot{y} + \mu(x) y = \mu(x) q(x)$$



Subject:

Chapter 1

20/3/2021

$$\frac{d}{dx} (\mu(x) y) = \mu(x) q(x) \quad \left(\frac{d}{dx} (\mu(x) y) = \mu' y + \mu y' \right)$$

$$d(\mu(x) y) = \int \mu(x) q(x) dx + c$$

$$= \frac{\mu(x)}{\mu} y = \frac{1}{\mu} \int \mu(x) q(x) dx + c$$

$$y = \frac{1}{\mu} \left(\int \mu q(x) dx + c \right)$$

How to find $\mu(x)$?

$$\mu'(x) = \mu(x) P(x) \rightarrow P(x) = \frac{\mu'(x)}{\mu(x)} = \frac{1}{\mu(x)} \frac{d\mu}{dx}$$

من الشروط
بلي فو حصة

(من الشروط ادرج
(م

مشتقة
 $\ln \mu(x)$

$$\Rightarrow P(x) = \frac{d}{dx} (\ln \mu(x))$$

$$\int d(\ln \mu(x)) = \int P(x) dx \rightarrow \ln(\mu(x)) = \int P(x) dx$$

$$\mu(x) = e^{\int P(x) dx} \rightarrow \text{integration factor}$$



2nd ODE :-

→ General 2nd ODE (linear)

$$P(x) \ddot{y} + q(x) \dot{y} + r(x)y = g(x)$$

كل المعادلات
اختراعات بسلامة
X

variable coefficients

→ linear with constant coefficients :-

حالة لها يكونوا المعاملات
أعداد ثوابت

$$a \ddot{y} + b \dot{y} + cy = g(x)$$

= 0

homogeneous

≠ 0

→ non homogeneous

→ Homogeneous constant coefficient 2nd :-

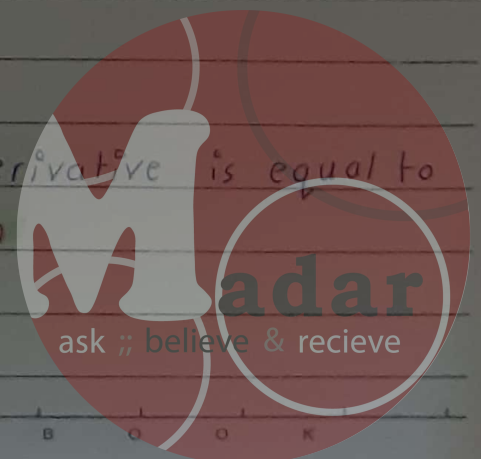
$$a \ddot{y} + b \dot{y} + cy = 0$$

* Example :-

Determine some solution to

$$y'' - 9y = 0 \rightarrow \ddot{y} = 9y$$

→ look for some functions, where 2nd derivative is equal to
(9 * function) ⇒ exponential function



look for exp.

$$y_1(x) = e^{3x}, \quad y_2(x) = e^{-3x}$$

$$\dot{y}_1(x) = 3e^{3x}$$

$$\ddot{y}_1(x) = 9e^{3x}$$

$$9y_1(x)$$

$$\dot{y}_2(x) = -3e^{-3x}$$

$$\ddot{y}_2(x) = 9e^{-3x}$$

$$9y_2(x)$$

* بيدي أوجد اختزان
تكون المسئلة الثانية له
يساري 9 أمثال الاختزان
الأصلي.

* Also a linear combination of (e^{3x}, e^{-3x}) is also a solution.

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{3x} + c_2 e^{-3x}$$

حاصل جمع الاختزانين
معاً يعتبر حل

$$\dot{y}(x) = -3c_1 e^{-3x} + 3c_2 e^{3x}$$

$$\ddot{y}(x) = 9c_1 e^{-3x} + 9c_2 e^{3x} \rightarrow 9(c_1 e^{-3x} + c_2 e^{3x})$$

$$9y(x)$$

assume $\rightarrow$

$$\ddot{y} - 9y = 0$$

$$y = e^{rx}$$

$$= r e^{rx}$$

$$= r^2 e^{rx}$$

$$\rightarrow r^2 e^{rx} - 9 e^{rx} = 0$$

$$e^{rx} (r^2 - 9) = 0$$

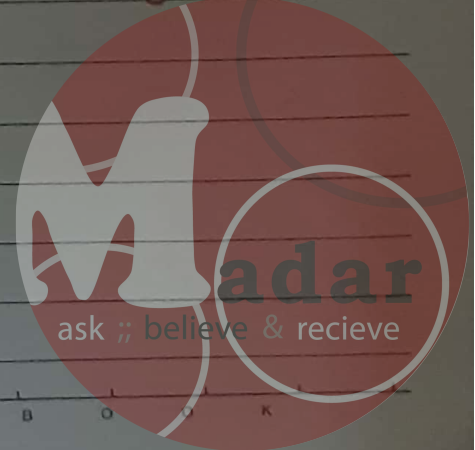
$$e^{rx} \neq 0$$

$$r^2 - 9 = 0$$

$$\sqrt{r^2} = \sqrt{9} \quad r = \pm 3$$

$$y_1 = e^{3x}$$

$$y_2 = e^{-3x}$$



Principle of superposition (for homogeneous ODE or linearity principle)

→ if $y_1(x)$ & $y_2(x)$ are two solutions to a linear, homogeneous equation then

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad (\text{general solution})$$

↓ For any homogeneous differential equation.

• We have c_1 & c_2 two constants :-

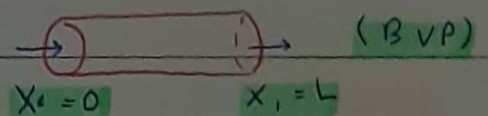
we need two equations or conditions to find them

عناطينة تحتاج معادلتين

لأيجاد الثوابت

One way :-

specify $y(x)$ at two distinct points



$$y(x_0) = y_0$$

$$\& \quad y(x_1) = y_1$$

→ Boundary value problem (BVP)

الاهتزازات

الاهتزازات

معروف على نقطة معينة

معروف على نقطة أخرى

Or :-

specify $y(x)$ and its derivative at a particular solution (point)

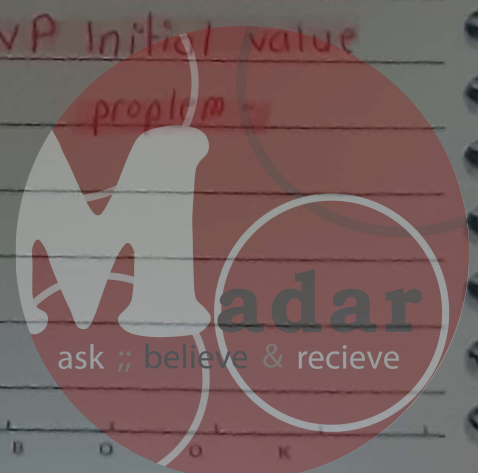
$$y(x_0) = y_0$$

$$\& \quad y'(x_0) = y'_0$$

→ IVP Initial value problem

initial condition

كلنا نعرف
التي عندي نقطة
على مستوى معين نقطة
لهذه النقطة نفس
المستوى



Example:-

solve the following IVP :-

$$\ddot{y} - 9y = 0$$

$$y(0) = 2$$

$$\dot{y}(0) = -1$$

general soln $\rightarrow y_1 = e^{-3x}$
 $y_2 = e^{3x}$

$$y = c_1 e^{-3x} + c_2 e^{3x}$$

بدنا في حد هتية

 c_2, c_1

$$y = c_1 e^{-3x} + c_2 e^{3x}$$

$$\dot{y} = -3c_1 e^{-3x} + 3c_2 e^{3x}$$

2 equations

2 unknown (c_1, c_2)at $x = 0$

$$y(0) = c_1 e^0 + c_2 e^0 = 2 \rightarrow c_1 + c_2 = 2 \rightarrow \textcircled{1}$$

$$\dot{y}(0) = -3c_1 e^0 + 3c_2 e^0 = -1 \rightarrow -3c_1 + 3c_2 = -1 \rightarrow \textcircled{2}$$

$$(c_1 + c_2 = 2) \times 3 \rightarrow 3c_1 + 3c_2 = 6$$

$$-3c_1 + 3c_2 = -1$$

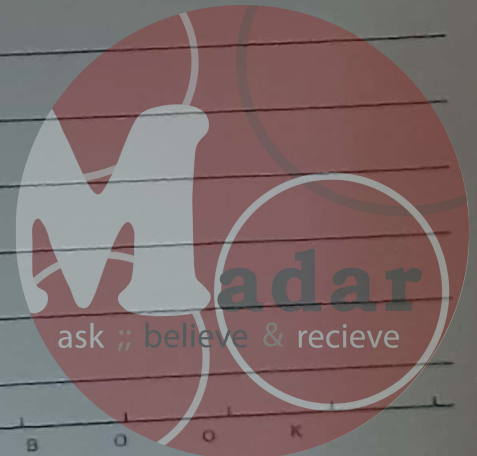
$$-3c_1 + 3c_2 = -1$$

$$6c_2 = 5 \rightarrow c_2 = 5/6$$

$$c_1 = 7/6$$

لـ لازم تكون القيمتين

مختلفات



* Notes:-

All solutions to

$$ay'' + by' + cy = 0$$

$$\begin{cases} y(x) = e^{rx} \\ y'(x) = re^{rx} \\ y''(x) = r^2e^{rx} \end{cases}$$

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$e^{rx}(ar^2 + br + c) = 0 \quad e^{rx} \neq 0$$

$$(ar^2 + br + c = 0) \rightarrow \text{Characteristic Equation}$$

معادلة تربيعية

has 2 roots r_1 & r_2

$\rightarrow$ we have 2 soln.

له حلين أو كل حل
يطلع يكون مختلفين
الثنائي.

$$y_1(x) = e^{r_1x}$$

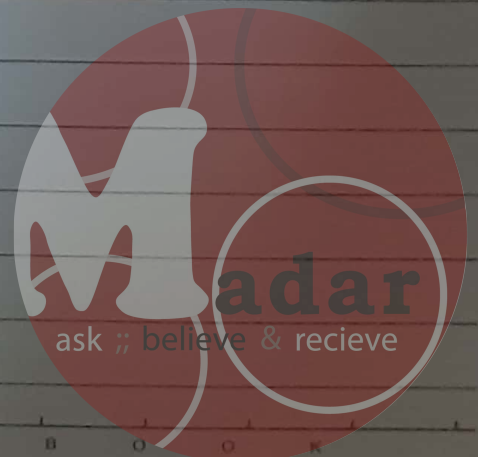
$$y_2(x) = e^{r_2x}$$

* roots of the characteristic equation could be:

1- Real and distinct roots, $r_1 \neq r_2$ (double real)

2- Complex roots, $r_{1,2} = \lambda \pm \beta j$
من القانون العام $\rightarrow$ real part $\rightarrow$ imaginary part

3- Repeated roots, $r_1 = r_2 = r$ (double real)



$$ay'' + by' + cy = 0$$

$$ar^2 + br + c = 0 \quad (\text{characteristic equation})$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow b^2 - 4ac$$

$\rightarrow$ any number (+) $\rightarrow$ حقيقيين
 $\rightarrow = 0 \rightarrow$ حقيقة واحدة
 $\rightarrow (-)$ number (complex) $\rightarrow$ صافين
 $\rightarrow$ ما تحت الجذر سالبي

Complex Numbers :-

$\rightarrow$ is a combination of a Real Number and an Imaginary Number

$$i = \sqrt{-1}$$

$$-1 \rightarrow 1 * -1$$

imaginary factor i

$$\sqrt{i^2} = \pm i$$

imaginary Number like $3i$

$$i^2 = -1$$

$$-1.04i \leftarrow \text{أي رقم مضروب بـ } i$$

$$a + bi$$

Real

x-axis

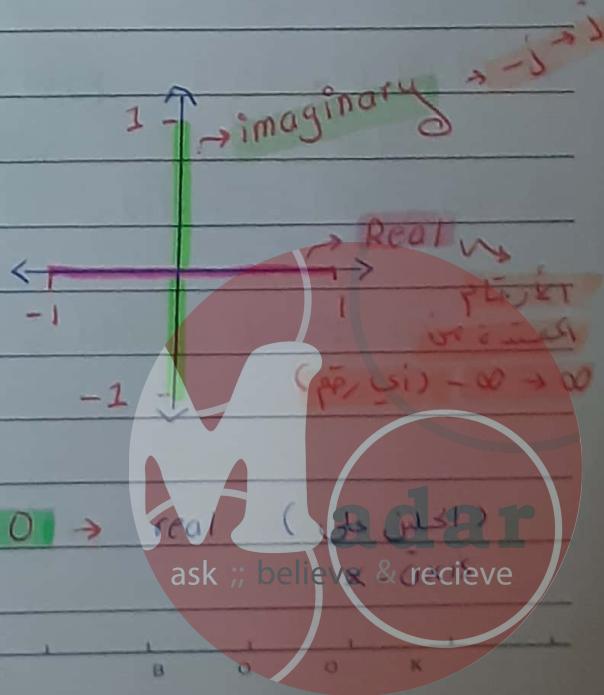
$$\sqrt{-1}$$

imaginary

y-axis

$$\text{if } a = 0 \rightarrow \text{الكل على y-axis}$$

$$bi = 0 \rightarrow \text{real}$$



Complex
Number

Real Port

Imaginary
Port

$$3 + 2i$$

3

2

5

المحور
الافقي
x-axis

5

0

Purely Real

-6i

المحور
العمودي
y-axis

0

-6

Purely imaginary

To add two complex number :-

$$(a + bi) + (c + di) = (a + b) + (b + d)i$$

real

imaginary

يجمع الـ real مع بعض

والـ imaginary مع بعض

* Example :-

$$(3 + 2i) + (1 + 7i)$$

$$= (3 + 1) + (2 + 7)i$$

$$= 4 + 9i$$

Multiplying :-

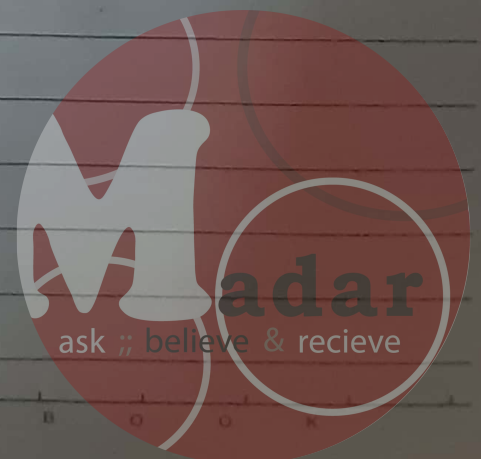
حزب عادي بوزع

القوس الأولى على

الثاني

$$(a + bi)(c + di)$$

$$= ac + adi + cb + bdi^2$$



* Conjugates :-

conjugate is where we change the sign in the middle.

$$\begin{array}{ccc} & a + bi & \\ \text{conjugate} \swarrow & & \searrow \text{conjugate} \\ & a - bi & \end{array}$$

مشتريين بال real
ومختلفين بال imaginary
عكس الإشارة

* Example :-

$$5 - 3i = 5 + 3i$$

ال Conj

* إذا كان الكمية سالبة (complex)

راح يكون r_2, r_1 conjugates لأن

حسب القانون العام راح يكونوا

$$-b \pm \sqrt{b^2 - 4ac}$$

مشتريين
real part

* Case 1 :-

r_1, r_2 real & distinct

$$y_1 = e^{r_1 x}$$

$$y_2 = e^{r_2 x}$$

general solution

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

* Example :-

$$\ddot{y} + 11\dot{y} + 24y = 0$$

$$y(0) = 0$$

$$\dot{y}(0) = -7$$

$$r^2 + 11r + 24 = 0$$

$$(r + 3)(r + 8) = 0$$

$$r_1 = -3 \rightarrow y_1 = e^{-3x}$$

$$r_2 = -8 \rightarrow y_2 = e^{-8x}$$



$$y(x) = c_1 e^{-3x} + c_2 e^{-8x}$$

$$\dot{y}(x) = -3c_1 e^{-3x} - 8c_2 e^{-8x}$$

initial condition , c_1 & c_2 $y(0) = 0$ $\dot{y}(0) = -7$

$$y(0) = c_1 + c_2 = 0 \rightarrow c_1 = -c_2 \rightarrow (1)$$

$$\dot{y}(0) = -3c_1 - 8c_2 = -7 \rightarrow (2)$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ -3 & -8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \end{bmatrix}$$

matrix form

نقدر نكتب المعادلات بهذا الشكل

$$-3(-c_2) - 8c_2 = -7$$

$$3c_2 - 8c_2 = -7$$

$$\frac{-5c_2}{-5} = \frac{-7}{-5}$$

$$\rightarrow c_2 = \frac{7}{5} \quad c_1 = -\frac{7}{5}$$

$$y(x) = -\frac{7}{5} e^{-3x} + \frac{7}{5} e^{-8x} \rightarrow \frac{7}{5} (e^{-8x} - e^{-3x}) \quad \#$$

Cramers Rule :-

طريقة ثانية عشان نخلص

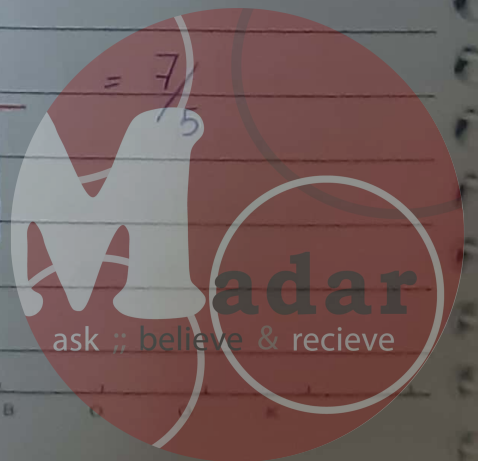
الثوابت باستخدام

matrix

$$c_1 = \frac{\begin{vmatrix} 0 & 1 \\ -7 & -8 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & -8 \end{vmatrix}} = \frac{-7}{5}$$

$$c_2 = \frac{\begin{vmatrix} 1 & 0 \\ -3 & -7 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & -8 \end{vmatrix}} = \frac{7}{5}$$

القام
دا الخ
المطوية التي
تحتل معادلات
المعادلات



#Case 2 :-

Roots are complex conjugate ($r_{1,2} = \lambda \pm \mu j$)

2 solution

$$\hookrightarrow j^2 = -1 \rightarrow j = \sqrt{-1}$$

$$y_1 = e^{(\lambda + \mu j)x}$$

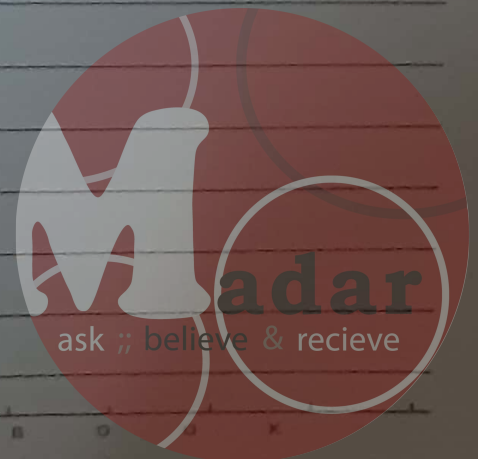
$$y_2 = e^{(\lambda - \mu j)x}$$

→

← مشترکین r_2, r_1

بال real و مختلفین بال

imaginary



Case 2:

Roots are complex conjugate

$$(r_{1,2} = \lambda \pm \mu j)$$

Real ← Imaginary

2 solution

$$\hookrightarrow j^2 = -1 \rightarrow j = \sqrt{-1}$$

$$y_1 = e^{(\lambda + \mu j)x}$$

$$y_2 = e^{(\lambda - \mu j)x}$$

→ مشتركين r_2, r_1
بال real ومختلفين بال
imaginary

* Euler's Formula:

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

$$y_1(x) = e^{(\lambda + \mu j)x} = e^{\lambda x} \cdot e^{(\mu x)j} = e^{\lambda x} (\cos \mu x + j \sin \mu x)$$

$$y_2(x) = e^{(\lambda - \mu j)x} = e^{\lambda x} \cdot e^{(-\mu x)j} = e^{\lambda x} (\cos \mu x - j \sin \mu x)$$

General solution $\Rightarrow$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

بنا فقط real
عن ال imag

$$y(x) = c_1 e^{\lambda x} (\cos \mu x + j \sin \mu x) + c_2 e^{\lambda x} (\cos \mu x - j \sin \mu x)$$

$$= e^{\lambda x} \left((c_1 \cos \mu x + c_1 j \sin \mu x) + (c_2 \cos \mu x - c_2 j \sin \mu x) \right)$$

$$= e^{\lambda x} \left((c_1 + c_2) \cos \mu x + j(c_1 - c_2) \sin \mu x \right)$$

$\downarrow A$ $\downarrow B$

$$A = c_1 + c_2$$

$$B = j(c_1 - c_2)$$

$$j = \sqrt{-1}$$

$$\Rightarrow y(x) = e^{\lambda x} (A \cos \mu x + B \sin \mu x)$$



add $y_1(x) + y_2(x)$

$$y_1(x) + y_2(x) = e^{\lambda x} (\cancel{\cos \mu x} + j \cancel{\sin \mu x} + \cancel{\cos \mu x} - j \cancel{\sin \mu x})$$

$$\frac{y_1(x)}{2} + \frac{y_2(x)}{2} = \frac{2 e^{\lambda x} \cos \mu x}{2}$$

$$\frac{1}{2} y_1(x) + \frac{1}{2} y_2(x) = e^{\lambda x} \cos \mu x \quad \Leftrightarrow \quad c_1 = c_2 = \frac{1}{2}$$

subtract $y_1(x) - y_2(x)$

$$y_1(x) - y_2(x) = e^{\lambda x} (\cancel{\cos \mu x} + j \sin \mu x - \cancel{\cos \mu x} + j \sin \mu x)$$

$$\frac{y_1(x)}{2j} - \frac{y_2(x)}{2j} = \frac{2j e^{\lambda x} \sin \mu x}{2j}$$

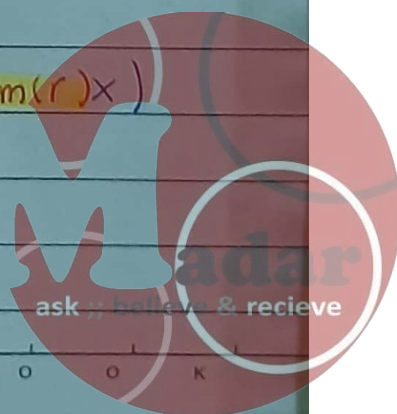
$$\frac{y_1(x)}{2j} - \frac{y_2(x)}{2j} = e^{\lambda x} \sin \mu x \quad \Leftrightarrow \quad \begin{aligned} c_1 &= 1/2j \\ c_2 &= -1/2j \end{aligned}$$

$\Rightarrow$ General solution :-

$$y(x) = c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x$$

$$y(x) = e^{\lambda x} (c_1 \cos \mu x + c_2 \sin \mu x)$$

$$\therefore y(x) = e^{\operatorname{Re}(r)x} (c_1 \cos(\operatorname{Im}(r)x) + c_2 \sin(\operatorname{Im}(r)x))$$



* Example 8 -

complex roots, solve the following IVP :-

To remember $\begin{cases} \ddot{y} \rightarrow r^2 \\ \dot{y} \rightarrow r \\ y \rightarrow r^0 = 1 \end{cases}$

$$\ddot{y} - 4\dot{y} + 9y = 0 \quad y(0) = 0$$

$$\dot{y}(0) = -8$$

$$r^2 - 4r + 9 = 0$$

$$r_{1,2} = \frac{4 \pm \sqrt{16 - 4 \times 9}}{2} = \frac{4 \pm \sqrt{-20}}{2} \rightarrow \text{complex} \quad \sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$$

$$r_{1,2} = \frac{4 \pm 2\sqrt{5}}{2} \rightarrow \frac{4}{2} \pm \frac{2\sqrt{5}}{2} \rightarrow 2 \pm \sqrt{5} = 2 \pm \sqrt{5}j^2$$

$$r_{1,2} = 2 \pm (\sqrt{5})j \Rightarrow \begin{cases} \text{Real part} \rightarrow 2 \rightarrow \lambda \\ \text{Im part} \rightarrow \sqrt{5} \rightarrow \mu \end{cases}$$

$$\rightarrow y(x) = e^{\lambda x} (c_1 \cos(\mu x) + c_2 \sin(\mu x))$$

Re part λ $\rightarrow$ Im (μ)

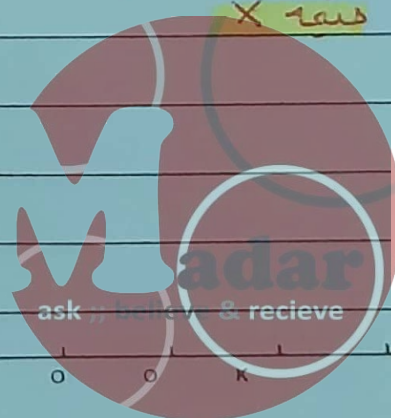
$$y(x) = e^{2x} (c_1 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x))$$

$$y(0) = 0 = e^0 (c_1 \cos(0) + c_2 \sin(0))$$

* نلاحظ ان sine وال cosine تزداد بزيادة قيمة x

$$c_1 = 0$$

$$y(x) = e^{2x} (0 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x))$$



$$y(x) = e^{2x} c_2 \sin(\sqrt{5} x)$$

$$\dot{y}(x) = 2e^{2x} c_2 \sin(\sqrt{5} x) + e^{2x} c_2 \sqrt{5} \cos(\sqrt{5} x)$$

$$\dot{y}(x) = 2c_2 e^{2x} \sin(\sqrt{5} x) + c_2 e^{2x} \cos(\sqrt{5} x) * \sqrt{5}$$

$$\dot{y}(0) = -8 = 2(c_2 e^0 \sin(0)) + c_2 e^{(2 \times 0)} \cos(0) * \sqrt{5}$$

$\swarrow \quad \searrow$
 $1 \quad 1$

$$-8 = \sqrt{5} c_2 \rightarrow c_2 = -\frac{8}{\sqrt{5}}$$

General solution :- (Particular solution)

$$y(x) = \frac{-8}{\sqrt{5}} e^{2x} \sin(\sqrt{5} x)$$

~~XXXX~~

~~Case 3 :-~~

Repeated roots

* Example :-

$$y'' - 4y' + 4y = 0$$

$$y(0) = 12$$

$$y'(0) = -3$$

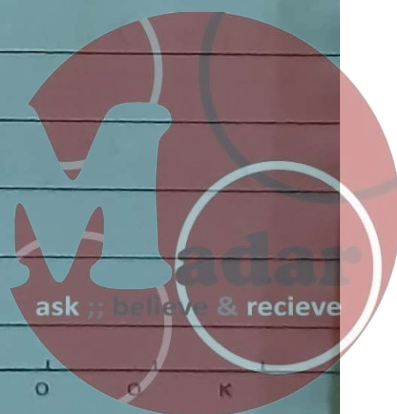
$$r^2 - 4r + 4 = 0$$

$$(r-2)(r-2) = 0$$

$$r_{1,2} = 2$$

$$y_1 = e^{2x}$$

$$y_2 = e^{2x}$$



General solution :-

$$y(x) = c_1 e^{rx} + c_2 x e^{rx} \rightarrow e^{rx} (c_1 + c_2 x)$$

$$y(x) = c_1 e^{2x} + c_2 x e^{2x}$$

هون عني

اختراين خيشتيق

مستكة
عنر

لضرب باختراي

لانه جب ان يكون هناك

2 roots

$$y'(x) = 2c_1 e^{2x} + c_2 e^{2x} + c_2 x \cdot 2e^{2x}$$

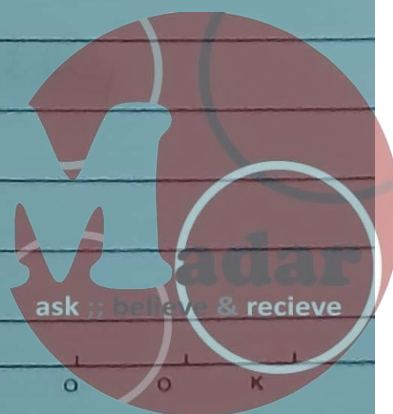
$$y'(x) = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x}$$

$$y(0) = 12 = c_1 e^0 + c_2 e^0 \cdot 0 \rightarrow \boxed{c_1 = 12}$$

$$y'(0) = -3 = 2 \cdot 12 \cdot e^0 + c_2 e^0 + 2c_2 \cdot 0 \cdot e^0$$

$$24 + c_2 = -3 \rightarrow \boxed{c_2 = -27}$$

$$\therefore y(x) = (12 - 27x) e^{2x} \rightarrow \text{Particular solution}$$



Reduction of order method :-

unknown function

$$y_2(x) = \underline{r(x)} \times y_1(x) \rightarrow y_2(x) = r(x)e^{rx}$$

$$\dot{y}_2(x) = \dot{r}(x)e^{rx} + r(x)re^{rx}$$

$$\ddot{y}_2(x) = \ddot{r}(x)e^{rx} + \underbrace{\dot{r}(x)re^{rx}} + \underbrace{\dot{r}(x)re^{rx}} + r(x)r^2e^{rx}$$

$$\ddot{y}_2(x) = \ddot{r}(x)e^{rx} + 2\dot{r}(x)re^{rx} + r(x)r^2e^{rx}$$

$$\Leftrightarrow b^2 - 4ac = 0 \quad r_1 = r_2 \rightarrow r_{1,2} = \frac{-b}{2a} = r$$

من القانون العام

$$\therefore y_1(x) = e^{(-b/2a)x}$$

1st solution

$$\begin{aligned} b &= -2ar \\ b + 2ar &= 0 \end{aligned}$$

$$\begin{array}{c} \ddot{y} \quad \dot{y} \quad y \\ ar^2 + br + c \end{array}$$

طرح

$$a(\ddot{r}e^{rx} + 2\dot{r}(x)re^{rx} + r(x)r^2e^{rx}) + b(\dot{r}e^{rx} + rre^{rx})$$

$$+ c r(x)e^{rx}$$

إعادة ترتيب

$$= e^{rx} (a\ddot{r} + (2ar + b)\dot{r} + (r^2a + br + c)r) = 0$$

zero

من القانون العام

characteristic equation

zero

$$= e^{rx} (a \ddot{r}) = 0 \rightarrow a e^{rx} \ddot{r} = 0$$

$$e^{rx} \neq 0 \rightarrow a \ddot{r} = 0 \rightarrow \ddot{r} = 0$$

$$\frac{d}{dx}(\dot{r}) = 0 \rightarrow \int d(\dot{r}) = c_1 \quad = \dot{r} = c_1 \rightarrow \frac{dv}{dx} = c_1$$

$$\int dv = \int c_1 dx + c_2 \rightarrow v = c_1 x + c_2$$

if $c_1 = 1, c_2 = 0$

$$v = x \rightarrow y_2(x) = x e^{rx}$$

$$y_1(x) = e^{rx}$$

∴ 2nd solution

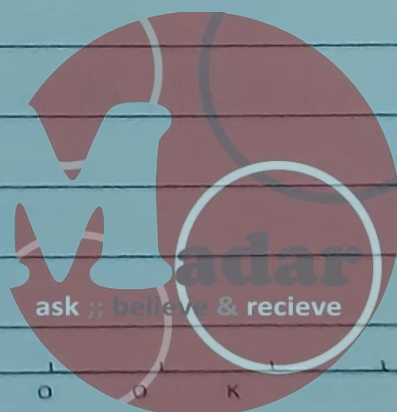
$$y_2(x) = v(x) e^{rx} = (c_1 x + c_2) e^{rx}$$

حالة خاصة

Real & repeated roots

$$\cancel{\#} y(x) = \underbrace{c_1 e^{rx}}_{y_1(x)} + \underbrace{c_2 x e^{rx}}_{y_2(x)} \rightarrow e^{rx} (c_1 + c_2 x)$$

General solution



*** Reduction of order :-**

if we know one solution then a 2nd linearly independent solution can be obtained by solving a 1st ODE.

*** Example :-**

شرح بشكل عام

Find a basis of solutions of the ODE :-

$$(x^2 - x)y'' - xy' + y = 0 \rightarrow \text{variable coefficient}$$

Given $\rightarrow y_1 = x$

$y_1' = 1$ ← النتيجة طلع

$$\rightarrow -x + x = 0$$

$y_1'' = 0$ أو x حل

$$\rightarrow x = x$$

$$y_2 = v(x) y_1(x)$$

$$\rightarrow y_2 = v(x) x$$

$$y_1 = x$$

← unknown function

$$y_2' = v' y_1 + v y_1'$$

$$\rightarrow y_2' = v' x + v$$

$$y_2'' = v'' y_1 + v' y_1' + v' y_1' + v y_1''$$

$$y_2'' = v'' x + 2v' y_1' + v y_1''$$

↓

$$y_2 = v'' x + 2v' \quad (\text{عومتنا } y_1', y_1, y_1'')$$

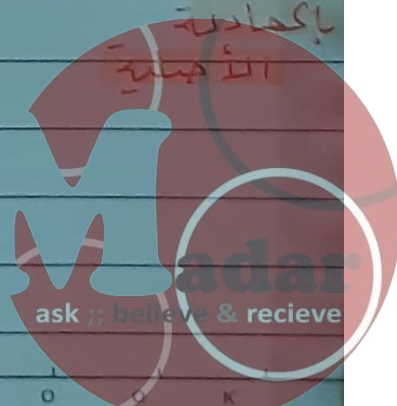
من فوق

$$(x^2 - x)(v'' x + 2v') - x^2 v' - x v + v x = 0 \rightarrow$$

عومتنا

بالحدادية
الأمثلة

$$(x^2 - x)(v'' x + 2v') - x^2 v' = 0$$



$$(x^2 - x) \ddot{v} + (x^2 - x) 2 \dot{v} - x^2 \dot{v} = 0$$

$$(x^2 - x) \ddot{v} + \cancel{x^2 \dot{v}} + \cancel{x^2 \dot{v}} - 2x \dot{v} - \cancel{x^2 \dot{v}} = 0$$

$$(x^2 - x) \ddot{v} - 2x \dot{v} + x^2 \dot{v} = 0$$

بدي أصول المعادلة من 2nd إلى 1st

$$(x^2 - x) \ddot{v} + (x - 2) \dot{v} = 0 \rightarrow \text{linear ODE}$$

$$\text{if } u = \dot{v}$$

$$\dot{u} = \ddot{v}$$

لن نفصل بسيط حتى نحصل على هذه المعادلة

$$(x^2 - x) \dot{u} + (x - 2) u = 0 \rightarrow 1^{\text{st}} \text{ ODE}$$

↓
separate & integrate

$$(x^2 - x) du + (x - 2) u dx = 0$$

$$\frac{du}{u} = \frac{(2 - x)}{(x^2 - x)} dx \rightarrow \int \frac{du}{u} = \ln u$$

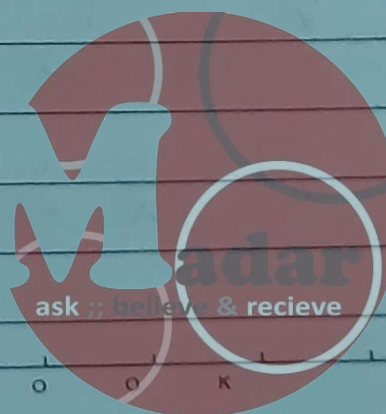
$$\rightarrow \frac{2 - x}{x^2 - x} dx \rightarrow \frac{2 - x}{x(x - 1)} dx \rightarrow \text{Partial Fraction}$$

$$\frac{2 - x}{x^2 - x} = \frac{2 - x}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1}$$

$$\hookrightarrow x = 0$$

$$x = -1$$

$$2 - x = A(x - 1) + Bx$$



if $x = 1$

$$2 - 1 = A(1 - 1) + B(1)$$

$B = 1$

if $x = 0$

$$2 - 0 = A(0 - 1) + B(0)$$

$A = -2$

$$\frac{2-x}{x(x-1)} = \frac{-2}{x} + \frac{1}{x-1}$$

$$\int \frac{2-x}{x(x-1)} dx = \int \frac{-2}{x} dx + \int \frac{1}{x-1} dx$$

$\downarrow$ $\ln x$ $\downarrow$ $\ln(x-1)$
 $\ln x^{-2}$

$$\int \frac{2-x}{x(x-1)} dx = \ln(x-1) - 2\ln x = \ln(x-1) - \ln x^2$$

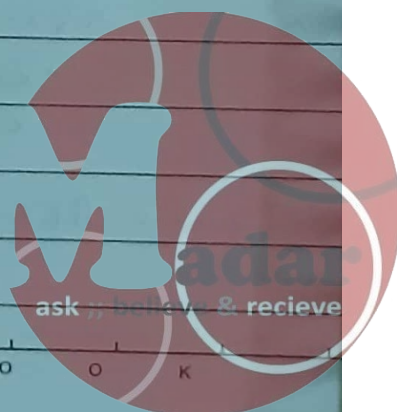
$\hookrightarrow \ln(x-1) / x^2$

~~$\ln u = \ln x-1$~~ $\rightarrow u = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}$

since $\rightarrow \dot{v} = u$

$$\dot{v} = \frac{1}{x} - \frac{1}{x^2} \rightarrow \frac{dv}{dx} = \frac{1}{x} - \frac{1}{x^2}$$

$$\int dv = \int \frac{1}{x} - \frac{1}{x^2} dx$$



Subject: Chapter 2

28 / 3 / 2021

$$v = \ln x + x^{-1} \rightarrow v = \ln x + \frac{1}{x}$$

2nd solution →

$$y_2 = v(x) y_1 = \left(\ln x + \frac{1}{x} \right) x \rightarrow y_2 = x \ln x + 1$$

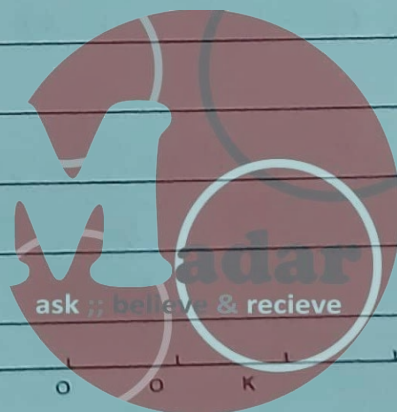
$\swarrow$ $\searrow$
 $v(x)$ y_1

$y_1(x)$ } linearly independent
 $y_2(x)$ }

General solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$y(x) = c_1 x + c_2 (x \ln x + 1) \quad (x > 0)$$



Non homogeneous 2nd order linear ODE :-

$$\ddot{y} + P(x)\dot{y} + q(x)y = g(x)$$

$\hookrightarrow \neq 0$

→ Suppose that $Y_1(x)$ & $Y_2(x)$ are solution to non homo...

$$\ddot{y} + P(x)\dot{y} + q(x)y = g(x)$$

$\hookrightarrow = 0$

→ $y_1(x)$ & $y_2(x)$ are a fundamental set of solution to homo...

* Note :-

difference $Y_1(x) - Y_2(x)$ is a solution to the homogeneous equation, and can be written as $\Rightarrow$

$$Y_1(x) - Y_2(x) = c_1 y_1(x) + c_2 y_2(x)$$

if $Y_1(x)$ & $Y_2(x)$ are soln. :-

$$\ddot{Y}_1 + P(x)\dot{Y}_1 + q(x)Y_1 = g(x)$$

$$\ddot{Y}_2 + P(x)\dot{Y}_2 + q(x)Y_2 = g(x)$$

نظري معادلة
① من ②

$$(\ddot{Y}_1 - \ddot{Y}_2) + P(x)(\dot{Y}_1 - \dot{Y}_2) + q(x)(Y_1 - Y_2) = 0$$

↓

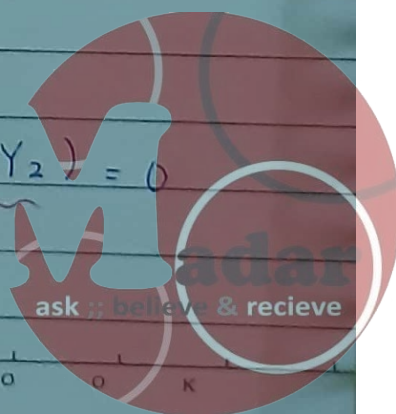
$$(\ddot{Y}_1 + P(x)\dot{Y}_1 + q(x)Y_1) - (\ddot{Y}_2 + P(x)\dot{Y}_2 + q(x)Y_2) = 0$$

↓

$g(x)$

↓

$g(x)$



$\Rightarrow Y_1 - Y_2 = c_1 y_1(x) + c_2 y_2(x) \rightarrow$ solution of homogeneous equation

$\Rightarrow$ let $y(t)$ general solution to non-homo., and $y_p(t)$ a particular solution to non-homo.

$y(t) - y_p(t) = c_1 y_1 + c_2 y_2 \rightarrow$ الفرق بينهم حل لا particular solution to non homo homogeneous.



$y(t) = (c_1 y_1 + c_2 y_2) + y_p(t) \Rightarrow y_c + y_p$ (complementary solution) of homog..)

*Example :-

$\ddot{y} - 4\dot{y} - 12y = 3e^{5x}$

$\hookrightarrow g(x) \neq 0$

Find a particular solution :-

$y(0) = 18 \quad \dot{y}(0) = -1/7$

$y(t) = y_c + y_p$

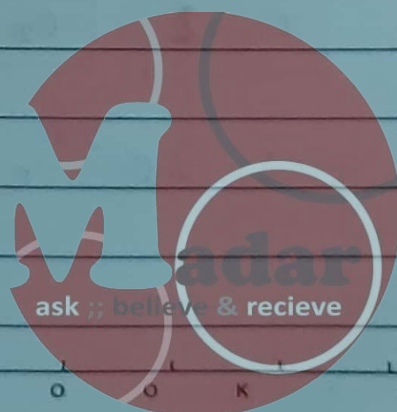
$(y_c) \rightarrow$ homogeneous :-

$r^2 - 4r - 12 = 0$

$(r + 2)(r - 6) = 0 \rightarrow r_1 = -2$

$r_2 = 6$

$y_c(x) = c_1 e^{-2x} + c_2 e^{6x}$



$(y_p) \rightarrow$ non homogeneous

$$g(x) = 3e^{5x}$$

assume $\rightarrow y_p = Ae^{5x}$

$\rightarrow g(x)$

لنفس القوة ما بقدرها بس
بغير الثابت

* لازم نوجد حينا
A

$$y_p' = 5Ae^{5x}$$

$$y_p'' = 25Ae^{5x}$$

نفسهم
بالقوة
الاحدية

method of undetermined
coefficient

$$25Ae^{5x} - 20Ae^{5x} - 12Ae^{5x} = 3e^{5x}$$

$$Ae^{5x}(25 - 20 - 12) = 3e^{5x}$$

$$-7Ae^{5x} = 3e^{5x} \rightarrow -7A = 3 \rightarrow A = -\frac{3}{7}$$

$$\therefore y_p = -\frac{3}{7}e^{5x}$$

$$y(x) = c_1e^{-2x} + c_2e^{6x} - \frac{3}{7}e^{5x}$$

$\rightarrow$ general solution

$$y(0) = \frac{18}{7} = c_1e^0 + c_2e^0 - \frac{3}{7}e^0 \rightarrow c_1 + c_2 = 3$$

$$g(x) = -2c_1e^{-2x} + 6c_2e^{6x} - \frac{15}{7}e^{5x} \rightarrow -c_1 + 3c_2 = 1$$

$$c_1 = 2$$

$$c_2 = 1$$

$$y(x) = 2e^{-2x} + e^{6x} - \frac{3}{7}e^{5x}$$

Method of undetermined coefficients :-

$$\ddot{y}(x) + p(x)\dot{y} + q(x)y = g(x)$$

$p(x)$ & $q(x) \rightarrow \text{constant}$

• useful for constant coefficient ODE, look at $g(x)$ to form y_p leaving the coefficients undetermined. plug y_p in ODE & determine value of coefficient

* Example :-

Find particular solution :-

$$\ddot{y} - 4\dot{y} - 12y = \sin(2x)$$

$y_c \rightarrow$

$$r^2 - 4r - 12 = 0$$

$$(r - 6)(r + 2) = 0 \rightarrow \begin{cases} r_1 = -2 \\ r_2 = 6 \end{cases}$$

$$y_c(x) = c_1 e^{-2x} + c_2 e^{6x}$$

$y_p \rightarrow$

$$g(x) = \sin(2x)$$

$$y_p = A \sin(2x)$$

$$\dot{y}_p = 2A \cos(2x)$$

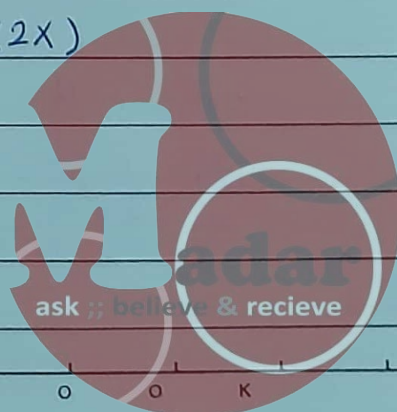
$$\ddot{y}_p = -4A \sin(2x)$$

نضوض
بإعدادية
اللا متغيرة

$$-4A \sin(2x) - 8A \cos(2x) - 12A \sin(2x) = \sin(2x)$$

$$-16A \sin(2x) - 8A \cos(2x) = \sin(2x)$$

بإستخدام كل جزء
من المعادلة
التي نشتقها من
المعادلة



$$-16 A \sin(2x) = \sin(2x)$$

$$-16 A = 1 \rightarrow A = 1 / -16$$

2 solution

$$-8A \cos(2x) = 0 \rightarrow A = 0$$

المعروف من الجواب

عندئذ متعة واحدة لـ

A هو من الجواب عنا حقيقين

يعني الـ 16 يلي اختاراه

عالم

* Guess :-

فرض من عنا

$$y_p(x) = A \cos(2x) + B \sin(2x)$$

$$y_p'(x) = -2A \sin(2x) + 2B \cos(2x)$$

نقوم من
بالمعادلة
اللا خطية

$$y_p''(x) = -4A \cos(2x) - 4B \sin(2x)$$

$$= -4A \cos(2x) - 4B \sin(2x) + 8A \sin(2x) - 8B \cos(2x) - 12A \cos(2x) + 12B \sin(2x) = \sin(2x)$$

$$-16A \cos(2x) - 8B \cos(2x) + 8A \sin(2x) - 16B \sin(2x) = \sin(2x)$$

$$\rightarrow \frac{-16A \cos(2x)}{8} = \frac{8B \cos(2x)}{8} \rightarrow A = -\frac{1}{2} B \rightarrow (1)$$

$$\rightarrow 8A \sin(2x) - 16B \sin(2x) = \sin(2x)$$

$$\sin(2x) (8A - 16B) = \sin(2x)$$

$$8A - 16B = 1 \rightarrow (2)$$

$$4 \cdot \frac{-1}{2} B - 16B = 1 \rightarrow -4B - 16B = 1$$

$$-20B = 1 \rightarrow B = -\frac{1}{20}$$

$$A = -\frac{1}{2} \cdot -\frac{1}{20} = \frac{1}{40}$$

$$y_p(x) = \frac{1}{40} \cos(2x) - \frac{1}{20} \sin(2x)$$

*Example :-

$$y'' - 4y' - 12y = 2x^3 - x + 3$$

$y_c \rightarrow$

$$r^2 - 4r - 12 = 0$$

$$(r - 6)(r + 2) = 0 \quad \begin{matrix} \rightarrow r = -2 \\ \rightarrow r = 6 \end{matrix}$$

$$y_c(x) = c_1 e^{-2x} + c_2 e^{6x} \quad \#$$

$y_p \rightarrow$ (Polynomial of degree 3)

$$P_3(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$
A **B** **C** **D**

$$y_p(x) = Ax^3 + Bx^2 + Cx + D \quad \rightarrow \quad \begin{matrix} \text{عندى اربع ثوابت} \\ \text{بى اربع مجهول} \end{matrix}$$

$$y_p'(x) = 3Ax^2 + 2Bx + C$$

$$y_p''(x) = 6Ax + 2B$$

$$(6Ax + 2B) + (-12Ax^2 - 8Bx - 4C) - 12Ax^3 - 12Bx^2 - 12Cx - 12D = 2x^3 - x + 3$$

$$-12Ax^3 + (-12A - 12B)x^2 + (6A - 8B - 12C)x + (2B - 4C - 12D) = 2x^3 + x + 3$$

$$\rightarrow -12A x^3 = 2x^3 \rightarrow -12A = 2 \rightarrow A = -\frac{1}{6}$$

$$\rightarrow -12A - 12B x^2 = 0x^2 \rightarrow \frac{2}{-12} \cdot -\frac{1}{6} - 12B = 0 \rightarrow B = \frac{1}{6}$$

$$\rightarrow (6A - 8B - 12C)x = -x \rightarrow 6 \cdot -\frac{1}{6} - 8 \cdot \frac{1}{6} - 12C = -1 \rightarrow C = -\frac{1}{9}$$

$$\rightarrow 2B - 4C - 12D = 3 \rightarrow 2 \cdot \frac{1}{6} - 4 \cdot -\frac{1}{9} - 12D = 3 \rightarrow D = -\frac{5}{27}$$

$$y_p(x) = -\frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{1}{9}x - \frac{5}{27}$$

Summary :

$g(x)$

$y_p(x)$

① $a e^{\beta x}$

$A e^{\beta x}$

② $a \cos(\beta x)$

$B \sin(\beta x) + A \cos(\beta x)$

③ $b \sin(\beta x)$

$B \sin(\beta x) + A \cos(\beta x)$

④ $a \cos(\beta x) + b \sin(\beta x)$

$A \cos(\beta x) + B \sin(\beta x)$

⑤ n^{th} degree polynomial

$(A_n x^n + A_{n-1} x^{n-1} + \dots + A_0)$

نفس الطريقة

$g(x)$

⑥ more complicated functions

يعود مكان كل اختبار حسب
العلاقات التي في الأعلى مع
النتيجة للعدد الثابت.

* More complicated functions :-

لما يكون عندى $y(x)$ و أكثر من آخران

* Example :-

مع بعض

Find $y_p(x)$ for

$$\ddot{y} - 4\dot{y} - 12y = xe^{4x}$$

بأخذ كل

* دائماً إذا كان عندنا آخران

آخران كالحالو

أسي بنوخذ بعض

$$y_p(x) = (Ax+B)e^{4x}$$

$$x \rightarrow Ax+B$$

الاعتبار الآخران

$$\dot{y}_p(x) = 4e^{4x}(Ax+B) + Ae^{4x}$$

$$e^{4x} \rightarrow Ce^{4x}$$

الثاني

لما نأخذ داي
لو ما حطينا

$$\ddot{y}_p(x) = 16e^{4x}(Ax+B) + 4Ae^{4x} + 4Ae^{4x}$$

عادي

$$\ddot{y}_p(x) = 16Ax e^{4x} + 16e^{4x}B + 8Ae^{4x}$$

$$(16Ax + 16B + 8A)e^{4x} - 16e^{4x}(Ax+B) - 4Ae^{4x} - 12(Ax+B)e^{4x} = xe^{4x}$$

ننظر
نحل المعادلة

$$-12Ax e^{4x} + (4A - 12B)e^{4x} = xe^{4x}$$

يدي أساري

كل حرف جهة اليسار

$$\rightarrow -12Ax e^{4x} = xe^{4x} \rightarrow -12A = 1 \quad A = -\frac{1}{12}$$

مع الإلبي ليحاسبهمو

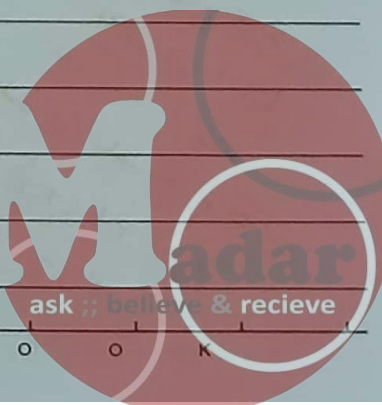
باليسار

$$\rightarrow (4A - 12B)e^{4x} = 0$$

$$e^{4x} \neq 0 \quad , \quad 4A - 12B = 0 \rightarrow 4 \cdot -\frac{1}{12} - 12B = 0 \quad B = -\frac{1}{36}$$

$$y_p(x) = \left(-\frac{1}{12}x + \frac{-1}{36} \right) e^{4x}$$

particular solution



*Example:-

$$\ddot{y} - 4\dot{y} - 12y = e^{6x}$$

 $y_e \rightarrow$

$$r^2 - 4r - 12 = 0$$

$$(r - 6)(r + 2) = 0 \rightarrow \begin{cases} r_1 = -2 \\ r_2 = 6 \end{cases}$$

$$y_c(x) = \underbrace{c_1 e^{-2x}}_{y_1} + \underbrace{c_2 e^{6x}}_{y_2} \quad \#$$

$$y_p \rightarrow g(x) = e^{6x}$$

$$y_p(x) = A e^{6x}$$

$$\dot{y}_p(x) = 6A e^{6x}$$

$$\ddot{y}_p(x) = 36A e^{6x}$$

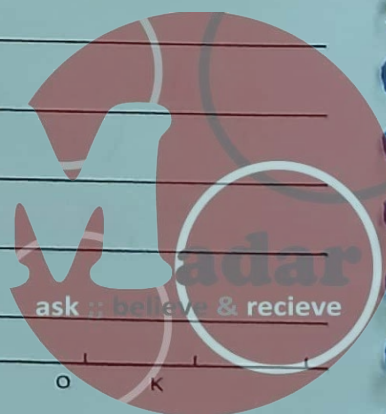
$$36A e^{6x} - 24A e^{6x} - 12A e^{6x} = e^{6x}$$

$$e^{6x} (\underbrace{36A - 24A - 12A}_{\text{zero}}) = e^{6x}$$

$$0 e^{6x} = e^{6x}$$

$$0 \stackrel{?}{=} 1 \rightarrow \text{something wrong}$$

هنا خطأ
يعني صفرنا اقل
لله. homo
الفرق بين غلط
على طريق التوطين
الاصح د آخر الى
بسيط X



⇒ Modify guess:-

$$y_p(x) = A x e^{6x}$$

$$\dot{y}_p(x) = A e^{6x} + 6 A x e^{6x} \rightarrow e^{6x} (A + 6 A x)$$

$$\ddot{y}_p(x) = 6 A e^{6x} + 6 A e^{6x} + 6 A x \cdot 6 e^{6x}$$

$$\ddot{y}_p(x) = 12 A e^{6x} + 36 A x e^{6x} \rightarrow e^{6x} (12 A + 36 A x)$$

$$(12 A + 36 A x) e^{6x} - 4 e^{6x} (A + 6 A x) - 12 A x e^{6x} = e^{6x}$$

$$(\cancel{36 A - 24 A - 12 A}) x e^{6x} + (12 A - 4 A) e^{6x} = e^{6x}$$

zero

$$(12 A - 4 A) e^{6x} = e^{6x}$$

$$\cancel{8 A e^{6x}} = \cancel{e^{6x}} \rightarrow 8 A = 1 \rightarrow A = \frac{1}{8}$$

$$y_p(x) = \frac{1}{8} x e^{6x}$$

* Example:-

$$\ddot{y} + 3 \dot{y} - 28 y = 7x + e^{-7x} - 1$$

$$\hookrightarrow (7x - 1) + e^{-7x}$$

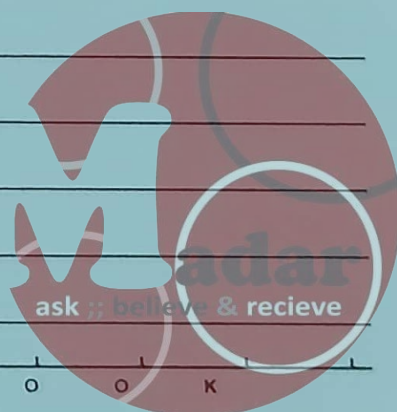
$y_c \rightarrow$

$$r^2 + 3r - 28 = 0$$

$$(r - 4)(r + 7) = 0 \rightarrow r_1 = 4$$

$$\rightarrow r_2 = -7$$

$$y_c(x) = c_1 e^{4x} + c_2 e^{-7x} \quad \#$$



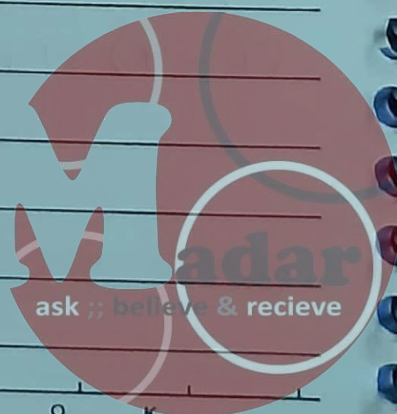
Subject :

Chapter 2

28 / 3 / 2021

$y_p \rightarrow$

$$y_p(x) = (Ax + B) + \boxed{x} e^{-7x}$$



* Variation of Parameters method :-

advantages :-

General method

disadvantages :-

① Complementary solution is absolutely required to do the problem

② solution contains integrals

لأنه مشكل بدرجة كبيرة
أعمل تكامل

$$P(x)y'' + q(x)y' + r(x)y = g(x)$$

$$y_c = c_1 y_1 + c_2 y_2 \quad (\text{complementary solution})$$

$$P(x)y'' + Q(x)y' + r(x)y = 0$$

y_1 & y_2 Form a fundamental set of solution.

Let :- particular solution (y_p) :-

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

where $u_1, u_2 \rightarrow$ unknown function to be determine.

إذا لم نكن نعرفه

نبحث أن نجد الـ

particular soln.

$$\sum_{i=1}^2 u_i(x) y_i$$

بشكل عام

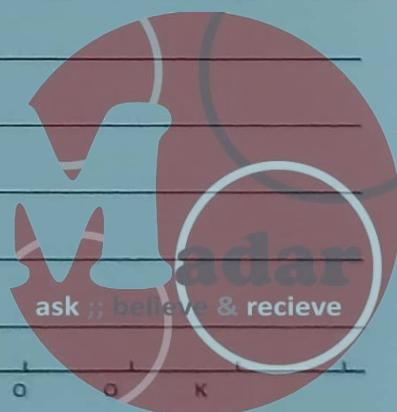
$$\dot{y}_p(x) = \cancel{u_1} y_1 + u_1 \dot{y}_1 + \cancel{u_2} y_2 + u_2 \dot{y}_2$$

assume $\rightarrow$

$$\dot{u}_1 y_1 + \dot{u}_2 y_2 = 0$$

1st condition

$$\therefore \dot{y}_p(x) = u_1 \dot{y}_1 + u_2 \dot{y}_2$$



we get :-

$$y_p'(x) = u_1' y_1 + u_2' y_2$$

$$y_p''(x) = u_1' y_1 + u_1 y_1'' + u_2' y_2 + u_2 y_2''$$

← التوضيح
بالمعادلة
اللا متجانسة

$y_p(x) = u_1 y_1 + u_2 y_2 \rightarrow$ if it is a solution it should satisfy equation 1 $\rightarrow$ non homogeneous.

$$\rightarrow P(x) y'' + Q(x) y' + r(x) y = g(x)$$

$$P(x) (u_1' y_1 + u_1 y_1'' + u_2' y_2 + u_2 y_2'') + Q(x) (u_1 y_1' + u_2 y_2')$$

$$+ r(x) (u_1 y_1 + u_2 y_2) = g(x)$$

المعادلة
ثانية

zero

$$\rightarrow P(x) (u_1' y_1 + u_2' y_2) + u_1 (P(x) y_1'' + Q(x) y_1' + r(x) y_1) + u_2 (P(x) y_2'' + Q(x) y_2' + r(x) y_2) = g(x)$$

zero

known

$$\rightarrow P(x) (u_1' y_1 + u_2' y_2) = g(x) \rightarrow \underbrace{u_1' y_1 + u_2' y_2}_{\text{unknowns}} = \frac{g(x)}{P(x)}$$

$$\text{if } P(x) = 1 \Rightarrow$$

$$\bullet u_1 y_1 + u_2 y_2 = 0$$

$\rightarrow$ linear system in u_1 & u_2

$$\bullet u_1 y_1 + u_2 y_2 = g(x)$$

solve by Cramers rule.



* Cramers Rule :-

→ In matrix Form ⇒

$$\begin{bmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ g(x) \end{bmatrix}$$

$\downarrow$ $\downarrow$
 u_1 u_2

$$u_1 = \frac{\begin{vmatrix} 0 & y_2 \\ g(x) & \dot{y}_2 \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix}} = \frac{-y_2 g(x)}{y_1 \dot{y}_2 - y_2 \dot{y}_1}$$

لو حلينا اعماد كذا
بالحدث والمقويين راح
يطلع عنا نفس النتائج

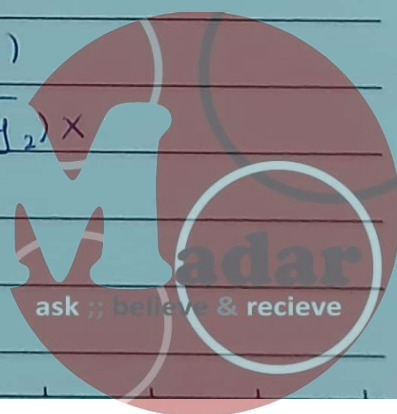
$$u_2 = \frac{\begin{vmatrix} y_1 & 0 \\ \dot{y}_1 & g(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix}} = \frac{y_1 g(x)}{y_1 \dot{y}_2 - y_2 \dot{y}_1}$$

* Note :-

$$w(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix} = y_1 \dot{y}_2 - y_2 \dot{y}_1 \neq 0$$

↙ Wronskian

$$u_1 = - \int \frac{-y_2 g(x)}{w(y_1, y_2) x} dx \quad \& \quad u_2 = \int \frac{y_1 g(x)}{w(y_1, y_2) x} dx$$



⇒ Particular solution :-

$$y_p(x) = y_1 u_1 + y_2 u_2$$

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{w(y_1, y_2)(x)} dx + y_2 \int \frac{y_1 g(x)}{w(y_1, y_2)} dx$$

summary :-

$$\ddot{y} + Q(x)\dot{y} + r(x)y = g(x)$$

standard form

assume →

2nd non homogeneous Eq.

y_1 & y_2 are a fundamental set of solution for →

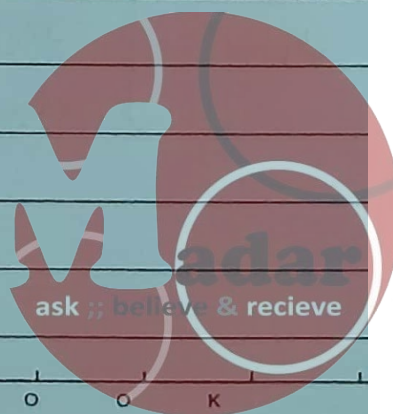
$$\ddot{y}(x) + Q(x)\dot{y} + r(x)y = 0 \quad (\text{homogeneous equation})$$

Then →

a particular solution to the nonhomogeneous 2nd ODE is →

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{w(y_1, y_2)} dx + y_2 \int \frac{y_1 g(x)}{w(y_1, y_2)} dx$$

$$y_p(x) = u_1 y_1 + u_2 y_2$$



*** Example :-**

Find a general solution to :-

$$\frac{2}{2} \ddot{y} + \frac{18}{2} y = \frac{6}{2} \tan(3x)$$

standard Form $\Rightarrow$

$$\ddot{y} + 9y = 3 \tan(3x)$$

$y_c \Rightarrow$

$$r^2 + 9 = 0 \rightarrow \sqrt{r^2 = -9} \rightarrow r_{1/2} = \sqrt{+9j^2} = \pm 3j$$

$$y_c(x) = e^{\underbrace{Rer(x)}_0} \left(c_1 \cos(\underbrace{Im(r)}_3 x) + c_2 \sin(\underbrace{Im(r)}_3 x) \right)$$

$$y_c(x) = c_2 \sin(3x) + c_1 \cos(3x)$$

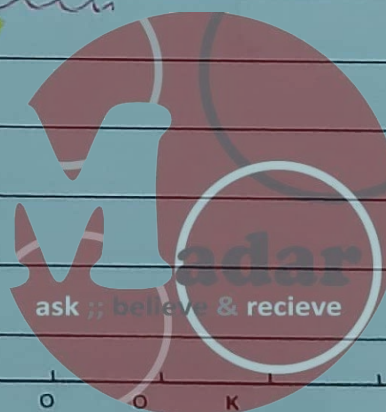
$$y_1(x) = \cos(3x) \rightarrow \dot{y}_1(x) = -3 \sin(3x)$$

$$y_2(x) = \sin(3x) \rightarrow \dot{y}_2(x) = 3 \cos(3x)$$

complementary solution

$$W(y_1, y_2)(x) = \begin{bmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3 \sin(3x) & 3 \cos(3x) \end{bmatrix}$$

$$= 3 \cos^2(3x) + 3 \sin^2(3x) = 3 (\underbrace{\cos^2(3x) + \sin^2(3x)}_1) = 3$$



⇒ Particular solution :-

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{w(y_1, y_2)} dx + y_2 \int \frac{y_1 g(x)}{w(y_1, y_2)} dx$$

$$\alpha = \int \frac{y_2 g(x)}{w(y_1, y_2)} dx = \int \frac{\sin(3x) \cdot \cancel{3} \tan(3x)}{\cancel{3}} dx \quad \begin{matrix} \sin(3x) \\ \cos(3x) \end{matrix}$$

$$= \int \frac{\sin^2(3x)}{\cos(3x)} dx = \int \frac{(1 - \cos^2(3x))}{\cos(3x)} dx$$

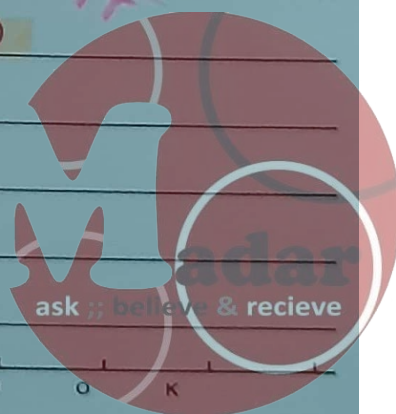
$$= \int \frac{dx}{\cos(3x)} - \int \frac{\cancel{\cos(3x)} \cos^2(3x)}{\cancel{\cos(3x)}} dx$$

$$= \int \sec(3x) dx - \frac{\sin(3x)}{3}$$

$$= \ln(\sec(x) - \tan(x)) - \frac{\sin(3x)}{3} \quad \begin{matrix} * y_1 \\ \downarrow \\ \sin(3x) \\ \cos(3x) \end{matrix} \quad \begin{matrix} \text{Simplified} \\ -(\cos(3x)) \end{matrix}$$

$$\alpha = \int \frac{y_1 g(x)}{w(y_1, y_2)} dx = \int \frac{\cos(3x) \cdot \cancel{3} \tan(3x)}{\cancel{3}} dx \quad \begin{matrix} \sin(3x) \\ \cos(3x) \end{matrix}$$

$$= \int \sin(3x) dx = \frac{-\cos(3x)}{3} \quad \begin{matrix} * y_2 \\ \downarrow \\ \sin(3x) \end{matrix}$$



* Variation of Parameters :-

* Example :-

Find general solution of :-

$$\ddot{y} - 2\dot{y} + y = \frac{e^x}{x^2 + 1}$$

$y_c \rightarrow$

$$r^2 - 2r + 1 = 0$$

$$(r - 1)(r - 1) = 0 \rightarrow r_1 = r_2 = 1 \rightarrow y_1 = e^x, \dot{y}_1 = e^x$$

double
root

$$y_2 = x e^x \rightarrow \dot{y}_2 = e^x + x e^x = e^x(1+x)$$

$y_p \rightarrow$

$$y_p(x) = u_1 y_1 + u_2 y_2$$

$$u_1 = - \int \frac{y_2 g(x)}{w(y_1, y_2)} dx$$

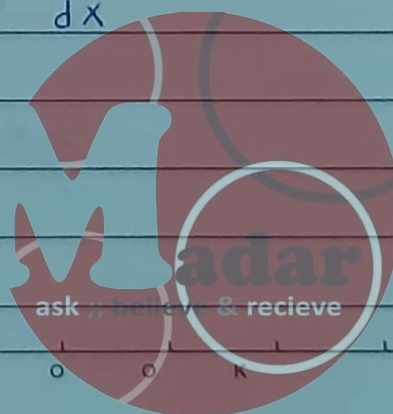
$$u_2 = \int \frac{y_1 g(x)}{w(y_1, y_2)} dx$$

$$g(x) = \frac{e^x}{x^2 + 1}$$

$$w(y_1, y_2)(x) = \begin{vmatrix} e^x & x e^x \\ e^x & e^x(1+x) \end{vmatrix}$$

$$= e^{2x}(1+x) - x e^{2x} = e^{2x} + \cancel{e^{2x}x} - \cancel{x e^{2x}} = e^{2x} \neq 0$$

$$u_1 = - \int \frac{x e^x \cdot (e^x / (x^2 + 1))}{e^{2x}} dx = - \int \frac{x}{x^2 + 1} dx$$



$$\text{let } u = x^2 + 1 \rightarrow du = 2x dx$$

$$x dx = \frac{du}{2}$$

$$= \frac{-1}{2} \int \frac{du}{u} = \frac{-1}{2} \ln u = \frac{-1}{2} \ln(x^2 + 1) * e^x$$

$$u_2 = \int \frac{e^x e^x (x^2 + 1)}{e^{2x}} dx = \int \frac{1}{x^2 + 1} dx = \tan^{-1}(x) * x e^x$$

$$y = y_c + y_p \rightarrow$$

$$c_1 e^x + c_2 x e^x - \frac{e^x}{2} \ln(x^2 + 1) + x e^x \tan^{-1}(x) //$$

* Example :-

$$\cancel{x} \ddot{y} - \frac{(x+1)}{x} \dot{y} + \frac{y}{x} = \frac{x^2}{x}$$

$$y_1 = e^x$$

$$y_2 = x + 1$$

لازم التحقق

أنه سهل

النتيجه ملء فأنا شغلهم

وليس منهم المعادلة الأصلية

لازم نعطينا مبرر

$$\ddot{y} - \frac{(x+1)}{x} \dot{y} + \left(\frac{1}{x}\right) y = x$$

$$w(y_1, y_2)(x) = \begin{bmatrix} e^x & x+1 \\ e^x & 1 \end{bmatrix}$$

$$= \cancel{e^x} - \cancel{e^x x} - \cancel{e^x} = -x e^x$$



$$u_1 = - \int \frac{y_2 g(x)}{w(y_1, y_2)(x)} dx = - \int \frac{(x+1)x}{-x e^x} dx$$

$$= \int (x+1) e^{-x} dx = \int x e^{-x} dx + \int e^{-x} dx$$

$\hookrightarrow -e^{-x}$

$$\rightarrow \int x e^{-x} dx \quad (\text{by parts})$$

$$u = x \quad \int dv = \int e^{-x} dx = -x e^{-x} + \int e^{-x} dx$$

$$du = dx \quad v = -e^{-x}$$

$$= -x e^{-x} - e^{-x} - e^{-x} = -x e^{-x} - 2e^{-x} = -e^{-x}(x+2) \quad (\underbrace{e^x}_{\hookrightarrow y_1})$$

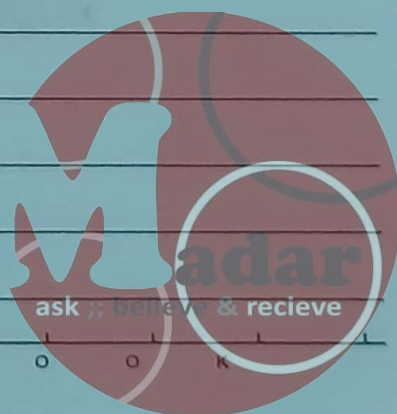
$$u_2 = \int \frac{y_1 g(x)}{w(y_1, y_2)(x)} dx = \int \frac{e^x x}{-e^x x} dx = \int -1 dx = -x - (x+1) \quad (\underbrace{e^x}_{\hookrightarrow y_2})$$

$$y_p(x) = -\cancel{e^x} (\cancel{e^x} (x+2)) + (x+1)(-x)$$

$$= -(x+2) - x(x+1) = -x - 2 - x^2 - x$$

$$= -x^2 - 2x - 2$$

$$y(x) = c_1 e^x + c_2 (x+1) - x^2 - 2x - 2 \quad \#$$



* Euler - Cauchy equations :-

Form $\Rightarrow$

$$(ax^2)y'' + (bx)y' + (c)y = 0 \rightarrow \text{variable coefficients}$$

 $a, b, c \rightarrow \text{constants}$

لو شئنا y'
 نطلع عن معادلة في y
 $ax^2 + bx + c = 0$

let solution $\rightarrow$

$y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2}$

$$(ax^2)(r(r-1))x^{r-2} + (bx)(rx^{r-1}) + c x^r = 0$$

نفترض
 بالحدسية
 إعادة في y

$$x^r (ar(r-1) + br + c) = 0 \rightarrow x^r \neq 0$$

$$\rightarrow ar(r-1) + br + c = 0 \rightarrow \text{characteristic}$$

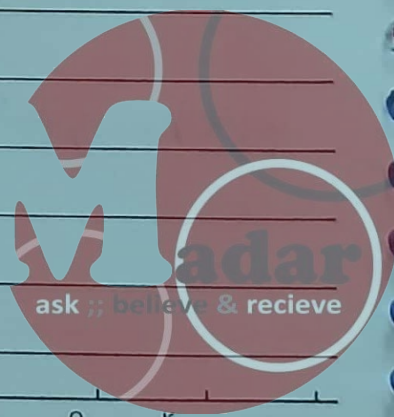
 $\downarrow$ quadratic

equation

equation

$$y = x^r \Rightarrow r \text{ root of characteristic equation}$$

$$ar^2 + (b-a)r + c = 0$$

 $\hookrightarrow 2 \text{ roots}$ 

* Cases :-

① r_1 & r_2 Real & distinct

→ 2 solution =>

$$y_1 = x^{r_1} \quad y_2 = x^{r_2} \rightarrow \text{Basis of solutions}$$

General solution →

$$y = c_1 x^{r_1} + c_2 x^{r_2}$$

② Double real ($r_1 = r_2 = r$)

$$y_1 = x^r \quad y_2 = x^r \ln x \rightarrow \text{reduction of order method}$$

General solution →

$$y = c_1 x^r + c_2 x^r \ln x$$

$$y = x^r (c_1 + c_2 \ln x)$$

③ Complex conjugate roots

$$x^2 y'' + ax'y + by = 0$$

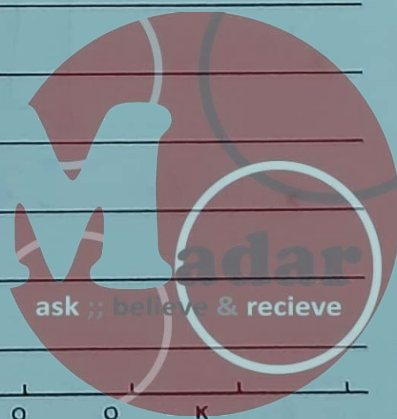
اثبات كين
طريقا معنا
 $y = x^r \ln x$

$$r^2 + (a-1)r + b = 0$$

القانون
المعك

$$r_{1/2} = \frac{1}{2} (1-a) \pm \sqrt{\frac{1}{4}(1-a)^2 - b}$$

$$\frac{1}{4}(1-a)^2 - b = 0 \Rightarrow b = \frac{(1-a)^2}{4}$$



$$\Rightarrow r = r_1 = r_2 = \frac{1}{2}(1-a) \rightarrow \text{double real Root}$$

$$y_1(x) = x^{\frac{1}{2}(1-a)}$$

Reduction of order method :- $y_2 = u(x) y_1$

$$\dot{y}_2 = \dot{u} y_1 + u \dot{y}_1$$

$$\ddot{y}_2 = \ddot{u} y_1 + 2\dot{u} \dot{y}_1 + u \ddot{y}_1$$

$$\Rightarrow \ddot{y} + p(x) \dot{y} + q(x) y = 0 \quad ***$$

$$\ddot{u} y_1 + \dot{u} (2\dot{y}_1 + p y_1) + u (\ddot{y}_1 + p \dot{y}_1 + q y_1) = 0$$

$\rightarrow$ zero, y_1 solution of equation ***

$$\rightarrow \ddot{u} + \left(\frac{2\dot{y}_1 + p y_1}{y_1} \right) \dot{u} = 0$$

$u \rightarrow$ does not appear

let $v = \dot{u} \quad \dot{v} = \ddot{u}$

$$\dot{v} + \left(\frac{2\dot{y}_1 + p y_1}{y_1} \right) v = 0 \rightarrow \text{separate \& integrate}$$

$$\rightarrow \ln v = - \left[\int 2 \left(\frac{\dot{y}_1}{y_1} \right) dx + \int p dx \right] \quad \text{بفتح حير}$$

$$\rightarrow \ln v = - [2 \ln y_1 + \int p dx] \quad -2 \ln y_1 = \ln \left(\frac{1}{y_1^2} \right)$$

$$v = e^{\ln(1/y_1^2) - \int P dx} = e^{\ln(1/y_1^2)} \cdot e^{-\int P dx}$$

$$v = \left(\frac{1}{y_1^2}\right) e^{-\int P dx}$$

since $v = u$

$$u = \int \left[\frac{1}{y_1^2} \cdot e^{-\int P dx} \right] dx$$

بشكل
أكبر

2nd solution :-

$$y_2 = u y_1 = y_1 \int \left[\frac{1}{y_1^2} \cdot e^{-\int P dx} \right] dx$$

* Note that :-

$$x^2 \ddot{y} + a x \dot{y} + \frac{1}{4} (1-a)^2 y = 0$$

$$\Rightarrow \ddot{y} + \left(\frac{a}{x}\right) \dot{y} + \left(\frac{(1-a)^2}{4x^2}\right) y = 0$$

$$\ddot{y} + P(x) \dot{y} + q(x) y = 0$$

$$P(x) = \frac{a}{x}$$

$$q(x) = \frac{(1-a)^2}{4x^2}$$

$$e^{-\int P dx} = e^{-\int \frac{a}{x} dx} = e^{-a \ln x} = e^{\ln(1/x^a)} = \frac{1}{x^a}$$

$$y_1 = x^{\frac{1}{2}(1-a)}$$

$$\frac{1}{y_1^2} = \left(\frac{1}{x^{(1-a)/2}} \right)^2 = \frac{1}{x^{(1-a)}}$$



$$\Rightarrow \frac{1}{y_1^2} \cdot e^{-\int p dx} = \frac{1}{x^{(1-a)}} = \frac{1}{x^a} = \frac{1}{x}$$

Form $\Rightarrow$

$$y_2 = y_1 \int \frac{1}{y_1^2} \cdot e^{-\int p dx} dx$$

$$y_2 = x^{\frac{(1-a)}{2}} \int \frac{1}{x} dx = x^{\frac{(1-a)}{2}} \ln x$$

$$y_2 = y_1 \ln x$$

$$\Rightarrow y_1 = x^{(1-a)/2}, \quad y_2 = x^{(1-a)/2} \ln x$$

$$y = (c_1 + c_2 \ln x) x^{(1-a)/2}$$

$$\frac{1-a}{2} = r$$

repeated
root

$$y_1 = x^r$$

$$y_2 = (c_1 + c_2 \ln x) x^r$$

* Example :-

$$x^2 \ddot{y} + 1.5 x \dot{y} - 0.5 y = 0 \quad \rightarrow y = x^r$$

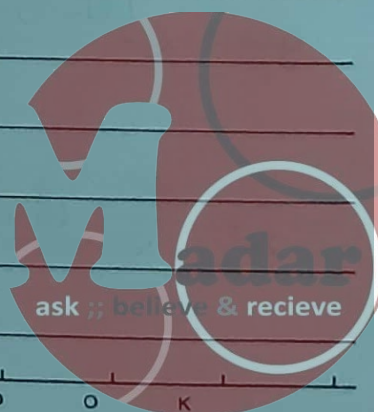
$$a = 1.5$$

$$b = -0.5$$

$$r^2 + (r-1)r + b = 0$$

$$r^2 + 0.5r - 0.5 = 0$$

$$(r+1)(r-0.5) = 0 \quad \rightarrow \begin{cases} r_1 = -1 \\ r_2 = 0.5 \end{cases}$$



$$y_1 = x^{-1}$$

$$y_2 = x^{0.5}$$

$$y = c_1 x^{-1} + c_2 x^{0.5} \rightarrow c_1 \frac{1}{x} + c_2 x^{0.5}$$

$x \rightarrow x \neq 0$

#

* Example 8 -

$$x^2 \ddot{y} - 5x \dot{y} + 9y = 0$$

$$a = -5 \quad b = 9$$

$$r^2 + (a-1)r + b = 0$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0 \rightarrow r_1 = r_2 = 3$$

$\rightarrow$ double

$$y_1 = x^3$$

$$y_2 = x^3 \ln x$$

$$y(x) = (c_1 + c_2 \ln x) x^3$$

#

Case 3 $\rightarrow$ complex conjugate :-

* Example :-

$$x^2 \ddot{y} + 0.6x \dot{y} + 16.04y = 0$$

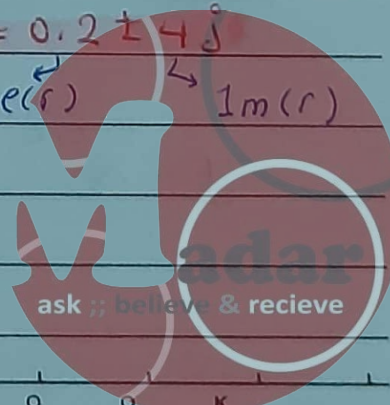
$$a = 0.6 \quad b = 16.04$$

$$r^2 + (a-1)r + b = 0$$

$$r^2 - 0.4r + 16.04 = 0 \rightarrow r_{1,2} = 0.2 \pm 4j$$

$\rightarrow$ $Re(r)$ $\rightarrow$ $Im(r)$

نحلها على الآلة الحاسبة



$$y_1 = x^{r_1}$$

$$y_2 = x^{r_2}$$

$$y_1 = x^{0.2+4j}$$

$$y_2 = x^{0.2-4j}$$

$$y_1 = x^{0.2} \cdot x^{4j} = x^{0.2} \cdot e^{\ln(x^{4j})}$$

بكتابة
بما في الصورة

$$y_2 = x^{0.2} \cdot x^{-4j} = x^{0.2} \cdot e^{\ln(x^{-4j})}$$

$$y_1 = x^{0.2} \cdot e^{4(\ln x)j}$$

euler formula →

$$y_2 = x^{0.2} \cdot e^{-4(\ln x)j}$$

$$e^{\pm \theta j} = \cos \theta \mp j \sin \theta$$

$$y_1 = x^{0.2} (\cos(4 \ln x) + j \sin(4 \ln x))$$

$$y_2 = x^{0.2} (\cos(4 \ln x) - j \sin(4 \ln x))$$

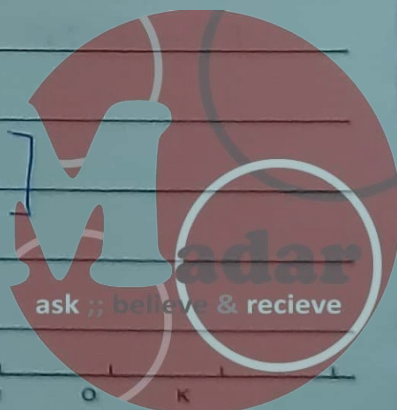
$$y_1 + y_2 = 2x^{0.2} \cos(4 \ln x)$$

$$y_1 - y_2 = 2jx^{0.2} \sin(4 \ln x)$$

$$\Rightarrow \frac{y_1 + y_2}{2} = x^{0.2} \cos(4 \ln x)$$

$$\Rightarrow \frac{y_1 - y_2}{2j} = x^{0.2} \sin(4 \ln x)$$

$$y = x^{0.2} (C_1 \cos(4 \ln x) + C_2 \sin(4 \ln x))$$



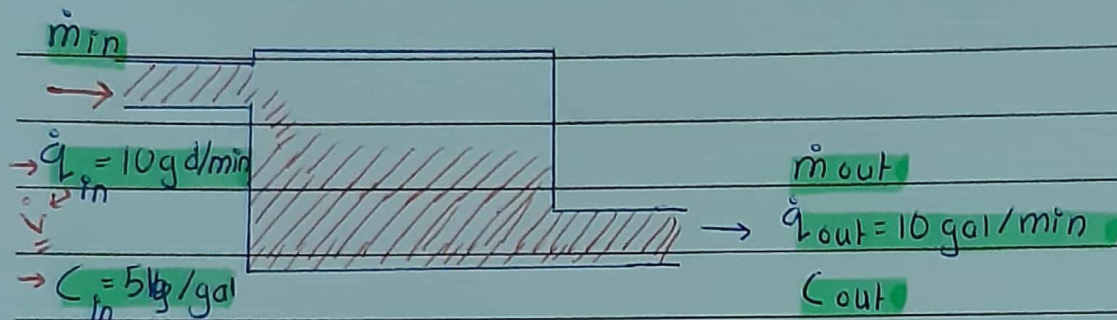
In general :-

$$y = x^{\operatorname{Re}(r)} \left[c_1 \cos(\operatorname{Im}(r) \cdot \ln x) + c_2 \sin(\operatorname{Im}(r) \cdot \ln x) \right]$$



* Mixing Problems.

Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at rate of 10 gal/min, and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time t 366



$$\frac{dm}{dt} = m_{\text{total in}} - m_{\text{total out}}$$

since $q_{\text{in}} = q_{\text{out}} = 10$
 $V_{\text{tank}} = 1000 \text{ gal}$

Brine mass balance

$$\dot{m}_{\text{in}} - \dot{m}_{\text{out}}$$

$$\dot{m} = \underset{\text{volume}}{q} \cdot \underset{\text{concentration}}{C} \rightarrow \left[\frac{\text{lb}}{\text{min}} \right] = \left[\frac{\text{gal}}{\text{min}} \cdot \frac{\text{lb}}{\text{gal}} \right]$$

$$\dot{m}_{\text{in}} = 10 * 5 = 50 \text{ lb/min salt}$$

$$\dot{m}_{\text{out}} = q_{\text{out}} C_{\text{out}} = 10 C_{\text{out}}$$

Perfect mixing

$$C_{out} = C_{Tank} = C$$

$$m_{Tank} = V_{Tank} \cdot C_{Tank} = \left[\frac{gal \cdot lb}{gal} \right] = [16]$$

constant

$$\frac{50 - 10C}{1000} = \frac{d(CV_{Tank})}{dt} = \frac{1000}{1000} \frac{dC}{dt}$$

$$\frac{50}{1000} - \frac{10C(t)}{1000} = \frac{dC(t)}{dt}$$

$$-\frac{1}{100} C(t) + \frac{dC(t)}{dt} = \frac{5}{100}$$

معادلة
3 نيت

→ non homogeneous, linear,
use integrating factor.

$$\mu = e^{\int 1/100 dt} = e^{t/100}$$

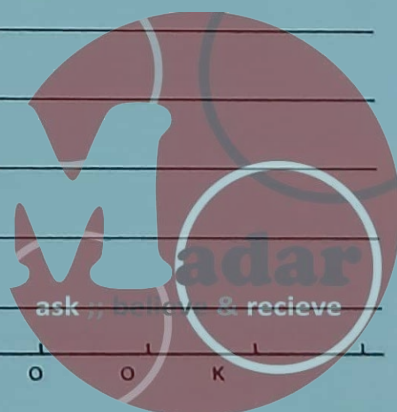
يُضرب المعادلة
بـ $e^{t/100}$

$$e^{t/100} \cdot \frac{1}{100} C(t) + e^{t/100} \cdot \dot{C}(t) = e^{t/100} \cdot \frac{5}{100}$$

$$\frac{d}{dt} (e^{t/100} \cdot C(t)) = \frac{5}{100} e^{t/100}$$

$$\int d(C e^{t/100}) = \frac{5}{100} \int e^{t/100} dt + const$$

$$e^{t/100} \cdot C(t) = 5 e^{t/100} + const$$



$$c(t) = 5 + \text{const} \cdot e^{-t/100}$$

$$c(t) = \frac{m(t)}{V}$$

$$t=0 \rightarrow \frac{1}{10} = 5 + \text{const}$$

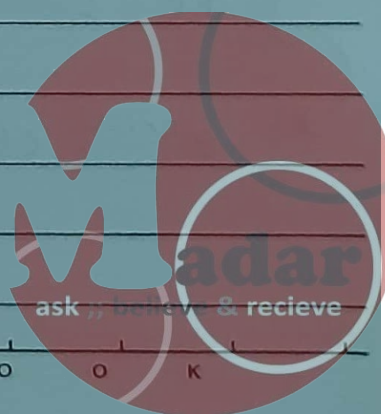
$$c(0) = \frac{100}{1000} = \frac{1}{10}$$

$$\text{const} = -4.9$$

$$c(t) = 5 - 4.9 e^{-t/100} \rightarrow (m(t) = c(t) \cdot V)$$

$$m(t) = 1000 (5 - 4.9 e^{-t/100})$$

$$m(t) = 5000 - 4900 e^{-t/100}$$



Laplace Transforms :-

$$f(t) \text{ (t-domain)} \xrightarrow{L(f(t))} (s\text{-domain}) F(s)$$

$$\xleftarrow{L^{-1}(F(s))}$$

* عندى اختوان $f(t)$ بدى اهلو

ل $F(s)$ اهلو بالعلى عن طريق

Laplace Transforms.

$\Rightarrow f(t)$ continuous over t or piece wise continuous over domains of t .

• Defn :- كيف اهلو وكيف احيب s

$$F(s) = L(f(t)) = \int_{t=0}^{t=\infty} e^{-st} f(t) dt$$

ال s بتغير ثابت

فأ t بتغير

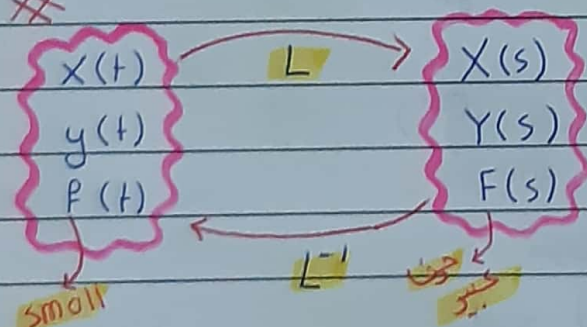
t بتغير بتغير

عندى اختوان بدلا

s

$$\therefore L(f(t)) = F(s)$$

XX



$t \rightarrow s$
real $\rightarrow$ could be complex

* Properties :-

L linear operator

$$a f(t) + b g(t) \rightarrow a F(s) + b G(s)$$

* لو كان عندى اختوان

و جملتهم (a) كمال الجمع

بتساوى (b) لكلى اختوان

بال

$$L(a f(t) + b g(t)) = L(a f(t)) + L(b g(t))$$



$$\rightarrow L(aP(t)) = \int_0^{\infty} aP(t) e^{-st} dt = a \int_0^{\infty} P(t) e^{-st} dt = a F(s) \quad \#$$

$$\rightarrow L(aP(t) + bQ(t)) = aL(P(t)) + bL(Q(t)) = aF(s) + bG(s) \quad \#$$

* Examples:-

$$1- P(t) = 1 \Rightarrow L(1) = \int_0^{\infty} 1 \cdot e^{-st} dt = \left[-\frac{1}{s} e^{-st} \right]_0^{\infty}$$

$$= -\left(\left(\frac{1}{\infty}\right) - \frac{1}{s}\right) \Rightarrow L(1) = \frac{1}{s} \quad \#$$

$$\text{if } P(t) = a \text{ (constant)} \Rightarrow L(a) = \frac{a}{s} \quad \#$$

$$2- P(t) = e^{-at} \quad a > 0 \quad \text{(important function)}$$

$$L(e^{-at}) = \int_0^{\infty} e^{-at} \cdot e^{-st} dt = \int_0^{\infty} e^{-(a+s)t} dt$$

$$= \left[-\frac{1}{(a+s)} e^{-(a+s)t} \right]_0^{\infty} = \frac{1}{a+s} = L(e^{-at}) \quad \#$$

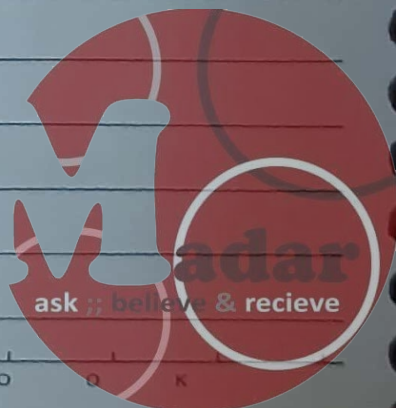
هون سالب
هون موجب

$$\text{if } P(t) = e^{at} \quad a > 0 \Rightarrow L(e^{at}) = \frac{1}{s-a} \quad \#$$

سالب
موجب

$$3- P(t) = t^n \quad , n = 1, 2, 3, \dots$$

$$L(t^n) = \frac{n!}{s^{n+1}} \quad \#$$



Some Functions $F(t)$ and Their Laplace Transforms $L(F)$:

$F(t)$	$L(F)$
1- 1	$1/s$
2- t	$1/s^2$
3- t^2	$2! / s^3$
4- t^n ($n=0,1,\dots$)	$n! / s^{n+1}$
5- t^a (a positive)	$\Gamma(a+1) / s^{a+1}$
6- e^{at}	$1 / s-a$
7- $\cos \omega t$	$s / s^2 + \omega^2$
8- $\sin \omega t$	$\omega / s^2 + \omega^2$
9- $\cosh at$	$s / s^2 - a^2$
10- $\sinh at$	$a / s^2 - a^2$
11- $e^{at} \cos \omega t$	$(s-a) / (s-a)^2 + \omega^2$
12- $e^{at} \sin \omega t$	$\omega / (s-a)^2 + \omega^2$

4- $P(t) = \sin(at)$

$$L(\sin(at)) = \frac{a}{s^2 + a^2} \quad \#$$

5- $F(t) = \cos(at)$

$$L(\cos(at)) = \frac{s}{s^2 + a^2} \quad \#$$

6- $F(t) = \sinh(at)$

$$L(\sinh(at)) = \frac{a}{s^2 - a^2} \quad \#$$

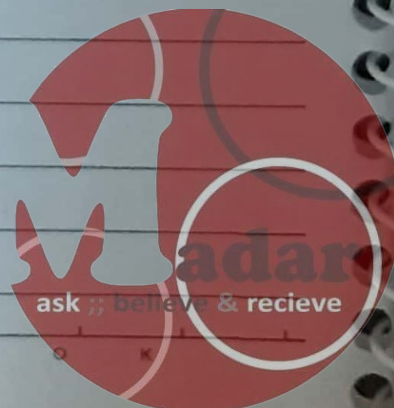
7- $F(t) = \cosh(at)$

$$L(\cosh(at)) = \frac{s}{s^2 - a^2} \quad \#$$

8- $F(t) = e^{at} \sin(bt)$

$$L(e^{at} \sin(bt)) = \frac{b}{(s \pm a)^2 + b^2} \quad \#$$

حسب إشارة a
 دالة b
 موجبة



9- $f(t) = e^{at} \sinh(bt)$

$$L(e^{at} \sinh(bt)) = \frac{b}{(s+a)^2 - b^2}$$

حسب إشارة a دائماً سالبة

10- $f(t) = e^{at} \cos(bt)$

$$L(e^{at} \cos(bt)) = \frac{s+a}{(s+a)^2 + b^2}$$

موجباً

11- $f(t) = e^{at} \cosh(bt)$

$$L(e^{at} \cosh(bt)) = \frac{s-a}{(s-a)^2 - b^2}$$

سالبة

* Example 5-

1- $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

متكافئة من الجيد

$$L(f(t)) = L(6e^{-5t}) + L(e^{3t}) + L(5t^3) - L(9)$$

↓

↓

↓

↓

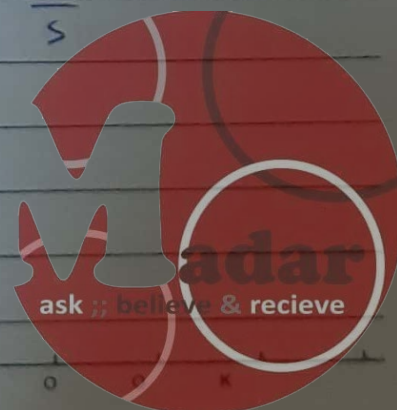
$$\frac{6}{s+5}$$

$$\frac{1}{s-3}$$

$$\frac{5 \cdot 3!}{s^4}$$

$$\frac{9}{s}$$

$$F(s) = \frac{6}{s+5} + \frac{1}{s-3} + \frac{5 \cdot 3!}{s^4} - \frac{9}{s} \quad \#$$



2- $f(t) = 4 \cos(4t) - 9 \sin(4t) + 2 \cos(10t)$

$$F(s) = 4 \left(\frac{s}{s^2 + 4^2} \right) - 9 \left(\frac{4}{s^2 + 4^2} \right) + 2 \left(\frac{s}{s^2 + 10^2} \right) \quad \#$$

3- $F(t) = 3 \sinh(2t) + 3 \sin(2t)$

$$F(s) = 3 \left(\frac{2}{s^2 - 2^2} \right) + 3 \left(\frac{2}{s^2 + 2^2} \right) \quad \#$$

4- $F(t) = t \cosh(3t)$ مش موجود بالجدول

→ From Table $\Rightarrow t^n f(t)$
 $(-1)^n F^{(n)}(s)$

→ nth derivative

نشتق حسب n ، $n=1$ ← مشتقة أولى

$n=2$ ← مشتقة ثانية وهكذا...

$$F(t) = \cosh(3t) \rightarrow F(s) = \frac{s}{s^2 - 3^2} \quad (n=1)$$

$$F'(s) = \frac{-(s^2 + 9)}{s^2 - 9}$$

$$L(t \cosh(3t)) = - \left(\frac{-(s^2 + 9)}{s^2 - 9} \right) = \frac{s^2 + 9}{s^2 - 9} \quad \#$$



Inverse Laplace Transforms :-

$$f(t) = L^{-1}(F(s)) \quad \text{عملية عكسية}$$

$$L^{-1}(aF(s) + bG(s)) = aL^{-1}(F(s)) + bL^{-1}(G(s)) = af(t) + bg(t)$$

* Example :-

$$1- F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}$$

$$f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{6}{s}\right) - L^{-1}\left(\frac{1}{s-8}\right) + L^{-1}\left(\frac{4}{s-3}\right)$$

$$f(t) = 6 - e^{8t} + 4e^{3t} \quad \#$$

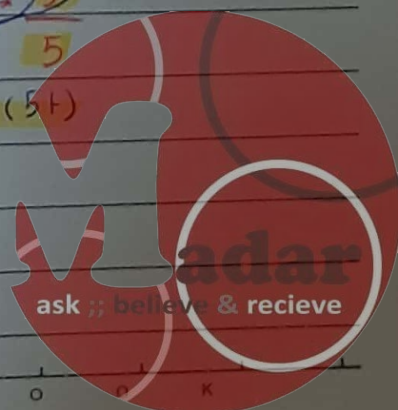
$$2- F(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5} \quad \text{هون 5 اذا عني 4!}$$

$$\begin{aligned} & \downarrow \quad \downarrow \quad \downarrow \\ & 19e^{-2t} \quad \frac{1}{3(s-\frac{5}{3})} \quad \frac{7}{s^5} \rightarrow \left(\frac{7}{4!}\right)\left(\frac{4!}{s^5}\right) \\ & \text{اخرجنا 3 عامل مشترك} \quad \downarrow \quad \downarrow \\ & \frac{1}{3}e^{(5/3)t} \quad \left(\frac{7}{4!}\right)t^4 \end{aligned}$$

$$f(t) = 19e^{-2t} - \frac{1}{3}e^{(5/3)t} + \frac{7}{4!}t^4 \quad \#$$

$$3- F(s) = \frac{6s}{s^2+25} + \frac{3}{s^2+25} \rightarrow \frac{3}{s^2+25} \quad \begin{aligned} & \text{لازم يكون} \\ & \text{عني 5} \end{aligned}$$

$$f(t) = 6 \cos(5t) + \frac{3}{5} \sin(5t) \quad \#$$



$$4- F(s) = \frac{8}{3s^2+12} + \frac{3}{s^2-49}$$

$$\downarrow$$

$$\frac{4 \cdot 8}{3(s^2+4)} \times \frac{2}{2} = \frac{32}{3(s^2+4)}$$

$$\downarrow$$

$$\frac{4}{3} \sin(2t)$$

$$\downarrow$$

$$\frac{3}{s^2-49} \times \frac{7}{7} = \frac{21}{s^2-49}$$

$$\downarrow$$

$$\frac{3}{7} \sinh(7t)$$

$$P(t) = \frac{4}{3} \sin(2t) + \frac{3}{7} \sinh(7t) \quad \#$$

$$5- F(s) = \frac{6s-5}{s^2+7} = \frac{6s}{s^2+7} - \frac{5}{s^2+7}$$

لتفصيل

$$\downarrow$$

$$6 \cos(\sqrt{7}t)$$

$$\downarrow$$

$$\frac{5}{s^2+7} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{5\sqrt{7}}{s^2+7}$$

$$P(t) = 6 \cos(\sqrt{7}t) - \frac{5}{\sqrt{7}} \sin(\sqrt{7}t) \quad \#$$

$$6- F(s) = \frac{1-3s}{s^2+8s+21} \quad (\text{complex roots})$$

$$s^2 + 8s + 21 \rightarrow \frac{d_1^2}{4} = \frac{64}{4} = 16 < d_0 = 21$$

$$s^2 + d_1 s + d_0 = (s^2 + d_1 s + \frac{d_1^2}{4}) + (d_0 - \frac{d_1^2}{4})$$

$$s^2 + d_1 s + d_0 = (s + \frac{d_1}{2})^2 + (d_0 - \frac{d_1^2}{4})$$

$$\frac{d_1^2}{4} < d_0$$

ask, believe & recieve

Subject :

Chapter 6

5 / 4 / 2021

$$s^2 + 8s + 21 = \left(s + \frac{8}{2}\right)^2 + \left(21 - \frac{64}{4}\right) = (s+4)^2 + 5$$

$$\Rightarrow F(s) = \frac{1 - 3s}{(s+4)^2 + 5} = \frac{1 - 12 + 12 - 3s}{(s+4)^2 + 5} = \frac{13 - 12 - 3s}{(s+4)^2 + 5}$$

$$= \frac{13}{(s+4)^2 + 5} - \frac{3(s+4)}{(s+4)^2 + 5}$$

في جمل هو موجود على
اللابرينج

$$P(t) = \frac{13}{\sqrt{5}} e^{-4t} \sin(\sqrt{5}t) - 3 e^{-4t} \cos(\sqrt{5}t)$$

#

$$7- F(s) = \frac{s+7}{(s+2)(s-5)}$$

(Partial Fraction)

$$F(s) = \frac{A}{(s+2)} + \frac{B}{(s-5)} = \frac{s+7}{(s+2)(s-5)}$$

$\hookrightarrow s = -2, s = 5$

$$s+7 = A(s-5) + B(s+2)$$

$$s = -2 \rightarrow A = \frac{-5}{7}$$

$$s = 5 \rightarrow B = \frac{12}{7}$$

$$P(t) = L^{-1}\left(\frac{A}{s+2}\right) + L^{-1}\left(\frac{B}{s-5}\right)$$

$$P(t) = A e^{-2t} + B e^{5t}$$

$$P(t) = \frac{-5}{7} e^{-2t} + \frac{12}{7} e^{5t}$$

#



Partial Fractions:-

الاستاتيكي لي

Factor in denominator

Term in Partial Fractions

① $(ax+b)$

$$\frac{A}{(ax+b)}$$

مخرج القوة

② $(ax+b)^k$

repeated roots

k times

$$\frac{A_1}{(ax+b)^k} + \dots + \frac{A_2}{(ax+b)^2} + \frac{A_1}{(ax+b)}$$

بشكل من القوة ونجعل أول كد

ما أدخل القوة 1

③ (ax^2+bx+c)

complex roots

conjugate

$$\frac{Ax+B}{ax^2+bx+c}$$

④ $(ax^2+bx+c)^k$

complex conjugate

repeated k times

$$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$



* Examples :-

1- $F(s) = \frac{86s - 78}{(s+3)(s-4)(5s-1)}$

→ Partial Fraction

$$F(s) = \frac{A}{s+3} + \frac{B}{s-4} + \frac{C}{5(s-\frac{1}{5})} = \frac{86s - 78}{(s+3)(s-4)(5s-1)}$$

$\hookrightarrow s = -3 \quad \hookrightarrow s = 4 \quad \hookrightarrow s = \frac{1}{5}$

$$f(t) = Ae^{-3t} + Be^{4t} + \frac{C}{5} e^{(1/5)t}$$

$$86s - 78 = A(s-4)(5s-1) + B(s+3)(5s-1) + C(s+3)(s-4)$$

$$s = -3 \rightarrow A = 3, \quad s = 4 \rightarrow B = 2, \quad s = \frac{1}{5} \rightarrow C = 5$$

$$P(t) = 3e^{-3t} + 4e^{4t} + \frac{5}{5} e^{(1/5)t}$$

2- $F(s) = \frac{2-5s}{(s-6)(s^2+11)}$ $= \frac{A}{s-6} + \frac{Bx+c}{s^2+11}$

$\hookrightarrow s = 6$ $\hookrightarrow$ complex

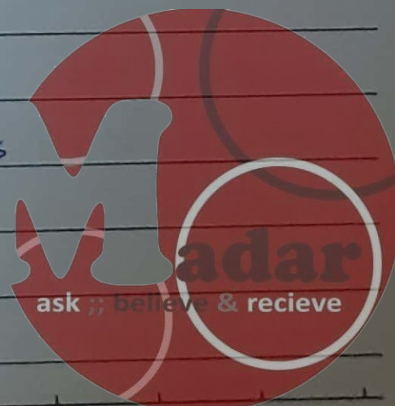
$$2-5s = A(s^2+11) + (Bx+c)(s-6)$$

$$s = 6 \rightarrow A = -\frac{28}{47}$$

نقل الأجزاء
و جمع الأجزاء
المشابهة

$$(A+B)s^2 + (-6B+c)s + (11A-6c) = 2-5s$$

$$s^2 \therefore (A+B) = 0 \rightarrow B = -A = -\left(-\frac{28}{47}\right) = \frac{28}{47}$$



Subject :

Chapter 6

5 / 4 / 2021

$$s:- (-6B + C) = -5 \rightarrow C = -5 + 6 \times \frac{28}{47} = -\frac{67}{47}$$

$$F(s) = \frac{(-28/47)}{s-6} + \frac{((28/47))s - ((67/47))}{s^2 + 11}$$

$$\downarrow \quad \downarrow \text{نحل}$$

$$-\frac{28}{47} e^{6t} \quad \frac{28}{47} \frac{s}{s^2+11} - \frac{67}{47} \frac{1}{s^2+11} = \frac{\sqrt{11}}{\sqrt{11}}$$

$$\downarrow$$

$$\frac{28}{47} \cos(\sqrt{11} t) - \frac{67}{47\sqrt{11}} \sin(\sqrt{11} t)$$

$$f(t) = -\frac{28}{47} e^{6t} + \frac{28}{47} \cos(\sqrt{11} t) - \frac{67}{47\sqrt{11}} \sin \sqrt{11} t$$

3- $F(s) = \frac{25}{s^3(s^2+4s+5)}$

s^3 (repeated (root=0)) $\rightarrow$ complex $4^2 < 5$

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{(s^2+4s+5)}$$

$$f(t) = A + Bt + \left(\frac{C}{2}\right)t^2 + L^{-1}\left(\frac{Bs+E}{s^2+4s+5}\right)$$

$$C = \frac{25}{s^2+4s+5} = 5$$



$$25 = As^2(s^2 + 4s + 5) + Bs(s^2 + 4s + 5) + C(s^2 + 4s + 5) + s^3(Ds + E)$$

$$A = 11/5$$

$$A = -4$$

$$C = 5$$

$$D = -11/5$$

$$E = -24/5$$

$$\Rightarrow L^{-1} \left(\frac{Ds + E}{s^2 + 4s + 5} \right) = \frac{Ds}{s^2 + 4s + 5} + \frac{E}{s^2 + 4s + 5}$$

$$s^2 + 4s + 5 = \left(s + \frac{4}{2} \right)^2 + \left(5 - \frac{4^2}{4} \right) = (s + 2)^2 + 1$$

منعنا دخرجنا 2

$$D \frac{s}{(s+2)^2 + 1} = D \frac{((s+2) - 2)}{(s+2)^2 + 1}$$

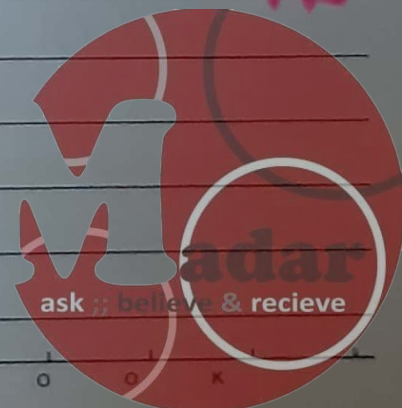
$$= D \frac{s+2}{(s+2)^2 + 1} - \frac{2D}{(s+2)^2 + 1} + \frac{E}{(s+2)^2 + 1}$$

$$\frac{D(s+2)}{(s+2)^2 + 1} + \frac{(E-2D)}{(s+2)^2 + 1}$$

$$D e^{-2t} \cos(t)$$

$$(E-2D) e^{-2t} \sin(t)$$

$$f(t) = \frac{11}{5} - 4t + \frac{5}{2} t^2 - \frac{11}{5} e^{-2t} \cos(t) - \frac{2}{5} e^{-2t} \sin(t) \quad \#$$



Solving IVP with Laplace transforms :-

$$f(t), f'(t), f''(t), \dots$$

$$f(0), f'(0) \rightarrow \text{initial condition}$$

$$L(f'(t)) = s L(f(t)) - f(0)$$

$$\hookrightarrow F(s)$$

$$\Rightarrow L(f'(t)) = s F(s) - f(0)$$

عدد الحدود يساوي رتبة المشتقة

1 +

$$\Rightarrow L(f''(t)) = s^2 F(s) - s f(0) - f'(0)$$

$$\Rightarrow L(f^{(3)}(t)) = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$\circ \circ L(f^{(n)}(t)) = s^n F(s) - s^{(n-1)} f(0) - s^{n-2} f'(0) - \dots -$$

$$s f^{(n-2)}(0) - f^{(n-1)}(0) \quad \text{X}$$

• Laplace transforms :-

$$f(t) \xrightarrow{L} F(s)$$

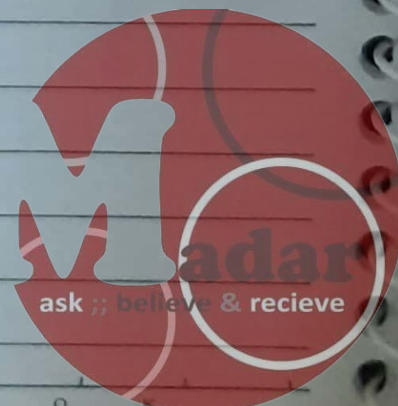
(t-domain)

s-domain

$$\xleftarrow{L^{-1}}$$

(x-domain)

$$L(f(t)) = \int_0^t e^{-st} f(t) dt$$



ODEs :-

$$\ddot{y} + ay + by = r(t)$$

$$y(0) = k.$$

Initial value

$$\dot{y}(0) = k_1$$

Problem

linear & nonhomogeneous

$$L(\ddot{y} + ay + by) = L(r(t))$$

$$L(\ddot{y}) + L(ay) + L(by) = L(r(t))$$

$$s^2 Y(s) - sy(0) - \dot{y}(0) + a(sY(s) - y(0)) + bY(s) = R(s)$$

solve For $Y(s) \Rightarrow$

$$s^2 Y(s) + asY(s) + bY(s) - sy(0) - \dot{y}(0) - ay(0) = R(s)$$

العادة
3 سب

$$Y(s)(s^2 + as + b) - (sy(0) + \dot{y}(0) + ay(0)) = R(s)$$

$$Y(s) = \frac{R(s) + (y(0)(s+a) + \dot{y}(0))}{s^2 + as + b}$$

من مولاتنا
المعادلة
من معادلة سلاسل
س إلى +

if $y(0) = \dot{y}(0) = 0$

$$Y(s) = \frac{R(s)}{s^2 + as + b}$$

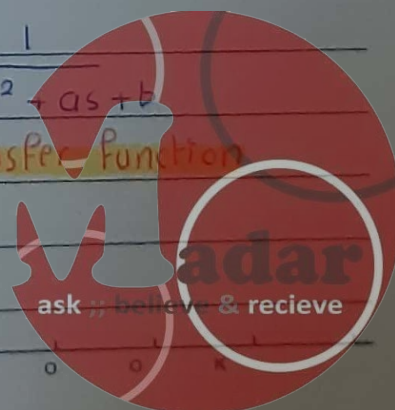
$$\text{let } Q(s) = \frac{1}{s^2 + as + b}$$

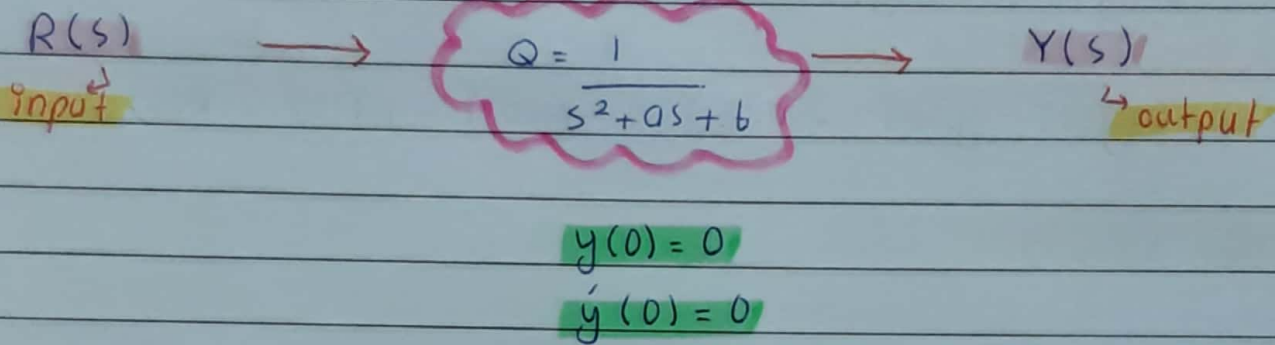
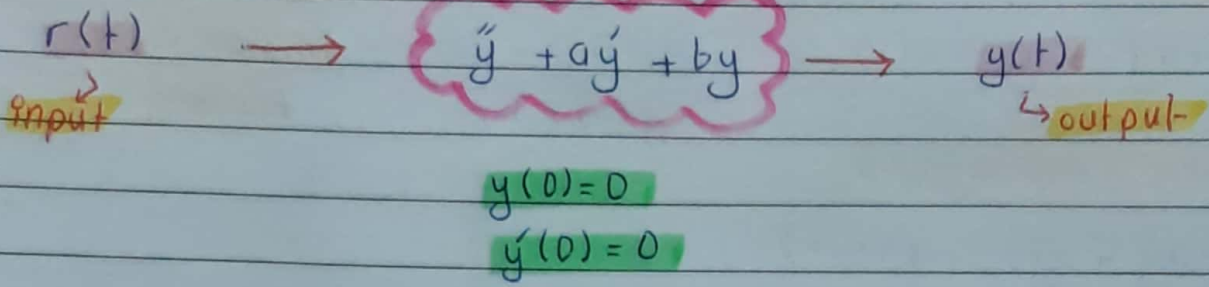
transfer function

$$Y(s) = Q(s) R(s) \rightarrow \frac{Y(s)}{R(s)} = Q(s)$$

input

output





to get $y(t) \Rightarrow$
 $y(t) = \mathcal{L}^{-1}(Y(s))$

$$Y(s) = R(s) Q(s) + \frac{y(0)(s+a)}{s^2+as+b} + \frac{\dot{y}(0)}{s^2+as+b}$$

partial fraction



* Example :-

solve $\ddot{y} - y = t$ $y(0) = 1$ $\dot{y}(0) = 1$

$$L(\ddot{y} - y) = L(t)$$

$$(s^2 Y(s) - s y(0) - \dot{y}(0)) - L(y(t)) = \frac{1}{s^2}$$

$$s^2 Y(s) - s y(0) - \dot{y}(0) - Y(s) = \frac{1}{s^2}$$

إعادة
3

$$Y(s)(s^2 - 1) = s y(0) + \dot{y}(0) + \frac{1}{s^2}$$

↓ ↓
1 1

$$Y(s) = \frac{s + 1 + 1/s^2}{(s^2 - 1)} = \frac{s + 1}{s^2 - 1} + \frac{1}{s^2(s^2 - 1)}$$

$$Y(s) = \frac{(s+1)}{(s+1)(s-1)} + \frac{1}{s^2(s^2-1)}$$

#

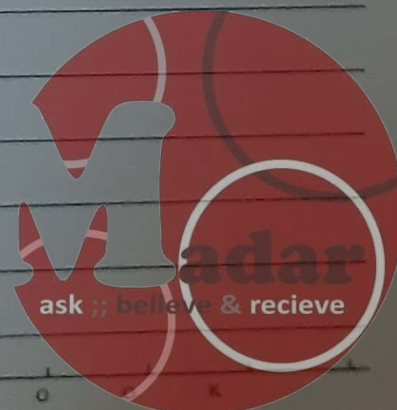
$$\rightarrow \frac{1}{s^2-1} - \frac{1}{s^2}$$

ما استعملنا
partial fraction
لأنه يتكونها

$$L^{-1}(Y(s)) = L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{1}{s^2-1}\right) - L^{-1}\left(\frac{1}{s^2}\right)$$

$$y(t) = e^t + \sinh(t) - t$$

#



Linear Algebra :

Matrix Eigenvalue Problems

→ A matrix eigenvalue problem considers the vector equation :-

$$A x = \lambda x \rightarrow \text{unknown vector}$$

$\swarrow$ given square matrix $\downarrow$ unknown scalar

→ In a matrix eigenvalue problem, the task is to determine λ and x .

الهدف من الـ matrix eigenvalue هو معرفة λ و x .

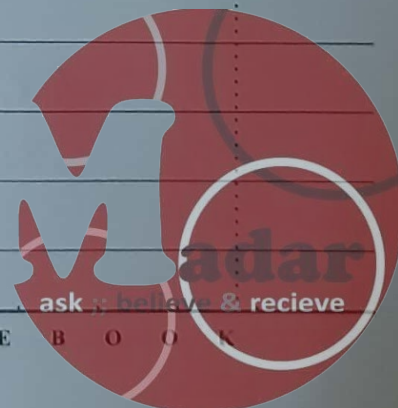
ممكن يكون عندي أكثر من λ أو x .

→ $x=0$ is always a solution for any and thus not interesting, we only admit solutions with $x \neq 0$.

دائمًا هاي المعادلة بتكون satisfy عند $x=0$.

لكن إحنا بنهتم بـ $x \neq 0$.

→ The λ that satisfy $(Ax = \lambda x)$ are called eigenvalues of A , and the corresponding nonzero x that also satisfy $(Ax = \lambda x)$ are called eigenvectors of A .



Determining Eigenvalues and Eigenvectors :-

→ let $A = [a_{jk}]$ be a given nonzero square matrix of dimension $n \times n$

$$Ax = \lambda x$$

* المسألة التي يكون هدفها إيجاد قيم λ وقيم $x \neq 0$ بحيث إنها تحقق $(Ax = \lambda x)$ eigenvalue problem

→ λ و $x \neq 0$ → called an eigenvalue or characteristic value of matrix A .

→ Another term for λ is a latent root.

* Example 1 :-

We illustrate all the steps in terms of the matrix

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \rightarrow Ax = \lambda x$$

$$= \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

بدي اختيار matrix vector

مختار أول صف مع آخره من

ال vector

$$-5x_1 + 2x_2 = \lambda x_1 \rightarrow$$

$$2x_1 - 2x_2 = \lambda x_2$$

نتوقع x_1 مع بعض
يكون معادلة x_2

$$(-5-\lambda)x_1 + 2x_2 = 0$$

$$2x_1 + (-2-\lambda)x_2 = 0$$

} homogeneous linear system

$$\rightarrow Ax = \lambda x$$

$$Ax - \lambda x = 0 \rightarrow \text{zero vector}$$

$$(A - \lambda I)x = 0 \rightarrow \text{عشان تطلع } x \text{ عامل مشترك لازم اشرح}$$

matrix من matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A → matrix

$$\downarrow \text{لو هنربنا}$$

$$x_2$$

$$\lambda \rightarrow \lambda I \rightarrow \text{identity matrix}$$

$$\text{vector ال } 0 \ 0 \ 1$$

$$x_3$$

with diagonal 1 1 1.

$$I \rightarrow x$$

$$\text{يعطيني نفس}$$

$$\text{vector ال}$$

$$(I * x) \Rightarrow$$

$$① 1(x_1) + 0(x_2) + 0(x_3) = x_1$$

$$② 0(x_1) + 1(x_2) + 0(x_3) = x_2$$

$$③ 0(x_1) + 0(x_2) + 1(x_3) = x_3$$

رجع

أعطاني

نفس x

$$(A - \lambda I) \neq 0$$

$$\det(A - \lambda I) = 0$$

→ By Cramer's theorem . it has a nontrivial solution (an eigenvector of A we are looking for) if and only if its coefficient determinant is zero, that is, #

مكونا أعرف λ

نفس x

$$D(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= (-5-\lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0$$

ask ; believe & recieve

$$\lambda^2 + 7\lambda + 6 = 0 \rightarrow (\lambda + 6)(\lambda + 1) = 0$$

$$\lambda_1 = -1$$

$$\lambda_2 = -6$$

eigenvalues of A

① x_1 ← يقابلها λ_1
 x_2 ← يقابلها λ_2

بدلاً من قيم λ بالمعادلة

الأصلية ونوجد قيم x

$$\Rightarrow \lambda_1 :- \lambda_1 = -1$$

$$\rightarrow (-5 - \lambda)x_1 + 2x_2 = 0$$

$$(-5 - (-1))x_1 + 2x_2 = 0 \rightarrow -4x_1 + 2x_2 = 0$$

$$\rightarrow 2x_1 + (-2 - \lambda)x_2 = 0$$

$$2x_1 + (-2 - (-1))x_2 = 0 \rightarrow 2x_1 - x_2 = 0 \rightarrow \text{لو ضربنا}$$

هنا المعادلة بـ 2 راح نحصلها نفس المعادلة الأولى

بالتالي هنا معادلة واحدة راح يكون عندي عدلان نهائي

$$2x_1 = x_2$$

بن الكل

$$x_1 = \frac{x_2}{2} \quad \text{كل ما أختار قيمة لـ } x_1 \text{ يوجد قيم } x_2$$

$$x_2 = 2x_1$$

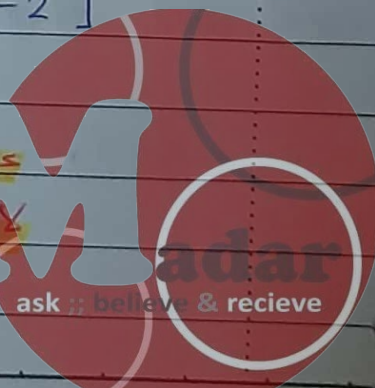
$$x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow Ax_1 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$= \lambda_1 x_1 \quad \neq$$

$$-1$$

عشان يكون الكل صح

لازم نطلع عندي $\lambda_1 x_1$



$$\Rightarrow \lambda_2 = - \lambda_2 = -6$$

$$\Rightarrow (-5 - \lambda) x_1 + 2x_2 = 0$$

$$(-5 - -6) x_1 + 2x_2 = 0 \rightarrow x_1 + 2x_2 = 0$$

$$\Rightarrow 2x_1 + (-2 - \lambda) x_2 = 0$$

$$2x_1 + (-2 - -6) x_2 = 0 \rightarrow 2x_1 + 4x_2 = 0 \rightarrow$$

هون راج بيون عنا عدد لانها بي

من الحلول

$$x_2 = -x_1$$

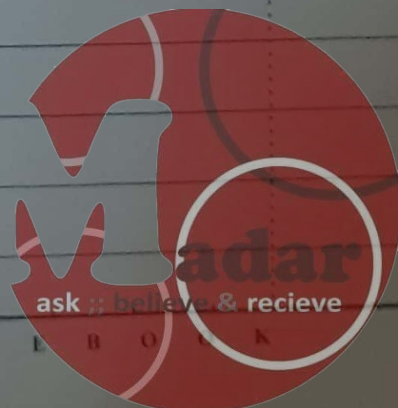
$$\frac{2}{2} x_1$$

$$x_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow Ax_2 = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix}$$

بفضل
اختار قيم
بطلع عنا
أعداد صحيحة
مثل كور

$$= \lambda_2 x_2$$

$$-6$$



homogeneous system of 1st ODEs :-

$$(1) \frac{dy_1}{dt} = 4y_1 - 2y_2 = \dot{y}_1 \rightarrow \text{ما يتقدّم على هاهي المعادلة}$$

بدلالة y_1 بدون ما أعرف y_2

$$(2) \frac{dy_2}{dt} = y_1 + y_2 = \dot{y}_2 \rightarrow \text{ونفس الأسّي للمعادلة الثانية.}$$

(المعادلتين مرتبطتين).

$$\rightarrow \ddot{y}_1 = 4\dot{y}_1 - 2\dot{y}_2 \rightarrow \ddot{y}_1 = 4\dot{y}_1 - 2y_1 - 2y_2$$

من المعادلة الثانية

من المعادلة الأولى

$$\ddot{y}_1 = 4\dot{y}_1 - 2y_1 - 4y_1 + \dot{y}_1$$

$$y_2 = 2y_1 - \frac{1}{2}\dot{y}_1$$

الفكرة التي بدى أني للمعادلتين

$$\ddot{y}_1 = 5\dot{y}_1 - 6y_1 \rightarrow \text{معادلة تفاضلية واحدة داخلها.}$$

مبارت المعادلة

كلها بدلالة y_1

$$\ddot{y}_1 - 5\dot{y}_1 + 6y_1 = 0$$

$$r^2 - 5r + 6 = 0$$

$$(r - 2)(r - 3) = 0$$

$$\rightarrow r_1 = 2 \rightarrow y_1 = e^{2t}$$

$$\rightarrow r_2 = 3 \rightarrow y_2 = e^{3t}$$

$$y_1(t) = c_1 e^{2t} + c_2 e^{3t}$$

$$y_2(t) = 2y_1 - \frac{1}{2}\dot{y}_1$$

$$\rightarrow \dot{y}_1 = 2c_1 e^{2t} + 3c_2 e^{3t}$$

$$y_2(t) = 2c_1 e^{2t} + 2c_2 e^{3t} - c_1 e^{2t} - \frac{3}{2}c_2 e^{3t}$$

$$y_2(t) = c_1 e^{2t} + \frac{1}{2}c_2 e^{3t}$$



$$y_1 = c_1 e^{2t} + c_2 e^{3t}$$

$$y_2 = c_1 e^{2t} + \frac{1}{2} c_2 e^{3t}$$

matrix form

2, 3

eigenvalue

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} e^{3t}$$

* Example:-

$$y_1' = 4y_1 - 2y_2$$

نفس المثال بدنا نحل

$$(\dot{Y} = AY)$$

$$y_2' = y_1 + y_2$$

بطريقة ثانية

↓

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

→ matrix form

↓
 $\dot{Y}$ ↓
 A ↓
 Y

assume solutions

 $x_1, x_2 \rightarrow$ parameter not variable

$$y_1 = x_1 e^{\lambda t} \rightarrow \dot{y}_1 = x_1 \lambda e^{\lambda t}$$

$$y_2 = x_2 e^{\lambda t} \rightarrow \dot{y}_2 = x_2 \lambda e^{\lambda t}$$

→ تحتاج أوجد متبة
 λ دقيم x

$$\rightarrow Ax = \lambda x \Leftrightarrow (A - \lambda I)x = 0$$

$$\Leftrightarrow A - \lambda I \text{ is singular}$$

$$\Leftrightarrow \det(A - \lambda I) = 0$$



$$\Rightarrow (\dot{Y} = AY)$$

↓

$$\begin{bmatrix} x_1 \lambda e^{\lambda t} \\ x_2 \lambda e^{\lambda t} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$

 $\rightarrow y_1, y_2 \rightarrow$

دلي فرستيم

د انهن حلول

لمستفاد اكلول
اي فرستيم

↓

$$\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$

→ vectors

scalar

عشان هين
اخر جانا

فامل مشترك

↓

$$\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

↓

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

→

عشان اطلع x عامل

مشترك لازم نخلي الـ λ

بغيره

Note :-

$$\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

~~~~~

→ I

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

↓

$$Ax = \lambda x$$

→ eigenvalue

problem

I → identity

matrix



$$\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

↓

$$(4-\lambda)(1-\lambda) + 2 = 0$$

$$4 - 5\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0 \rightarrow (\lambda - 2)(\lambda - 3) = 0$$

$$\left. \begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 3 \end{array} \right\} \text{eigenvalues}$$

For  $\lambda_1 = 2$ 

$$\begin{bmatrix} (4-2) & -2 \\ 1 & (1-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 - 2x_2 = 0 \rightarrow x_1 = x_2$$

$$x_1 - x_2 = 0 \rightarrow x_1 = x_2$$

$$\rightarrow x_1 = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \text{eigenvector corresponding to } \lambda_1 = 2$$

لأنه في كل مرة  
لا نهائي من الحلول

 $\lambda_1 = 2$  :-

$$Y_1 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

لأن الحل الأول  
بأنه في كل مرة  
بأنه في كل مرة

For  $\lambda_2 = 3$ 

$$\begin{bmatrix} (4-3) & -2 \\ 1 & (1-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvector  
corresponding  
to  $\lambda_2 = 3$ 

$$x_1 - 2x_2 = 0 \rightarrow x_1 = 2x_2 \rightarrow x_2 = \frac{1}{2}x_1 \rightarrow x_2 = k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x_1 - 2x_2 = 0 \rightarrow x_1 = 2x_2$$

ما عموماً  $x_1 = 1$  عشان  
ما بدنا يكون عينا 0,5

 $\lambda_2 = 3$  :-

$$Y_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}$$

General solution  $\Rightarrow$ 

$$Y = Y_1 + Y_2 = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}$$

#Note :-

homogeneous  $\rightarrow r - k = 0 \rightarrow r = k$  يوجد حل

$$\bullet 1^{st} \text{ ODE} \quad \dot{y} - ky = 0 \rightarrow \frac{dy}{dt} = ky \rightarrow \frac{dy}{y} = k dt$$

تجريد

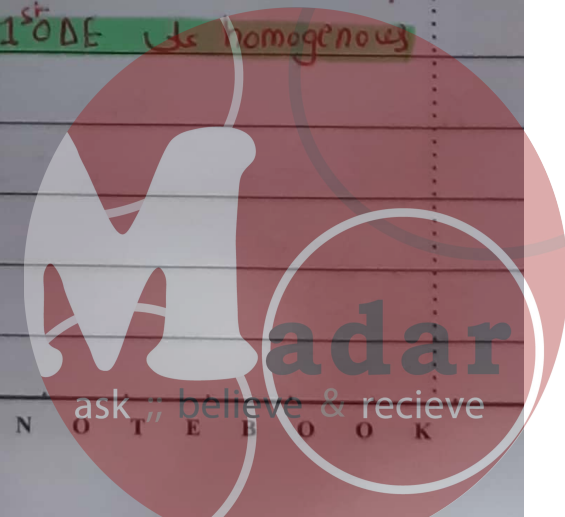
$$\ln y = kt \rightarrow y = e^{kt} \rightarrow y = c_1 e^{kt}$$

نقدر ايجاب مرات حل ال

1<sup>st</sup> ODE على homogeneous

$$\bullet 2^{nd} \text{ ODE} \quad \ddot{y} - b\dot{y} + cy = 0$$

$$y = e^{rt} \rightarrow \dot{y} = re^{rt} \rightarrow \ddot{y} = r^2 e^{rt}$$





$$e^{rt}(r^2 + br + c) = 0 \quad e^{rt} \neq 0$$

$$r^2 + br + c = 0 \rightarrow r_1, r_2$$

$$y_1 = e^{r_1 t}$$

$$y_2 = e^{r_2 t}$$

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

By applying the formula for a determinant, you can verify that  $\det(A - \lambda I)$  is a polynomial of degree  $n$  in  $\lambda$ . It is called the characteristic polynomial of  $A$ , written  $c(\lambda)$  for short.

$$\lambda \text{ is an eigenvalue of } A \Leftrightarrow c(\lambda) = 0$$

$\Leftrightarrow \lambda$  is a root of  $A$ 's characteristic polynomial.

Since  $c(\lambda)$  has degree  $n$ , it can have at most  $n$  roots, so  $A$  has at most  $n$  eigenvalues.

More precisely,  $c(\lambda)$  can be factored as

$$c(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

$A$ 's eigenvalues are  $\lambda_1, \lambda_2, \dots, \lambda_n$ . These may be  $n$  distinct values, or some values may be repeated; thus  $A$  has anywhere from 1 to  $n$  distinct eigenvalues.

Geometric interpretation

note that if  $x$  is an eigenvector associated with  $\lambda$ , so is any multiple of  $x$ . Can think of  $x$  as a special direction that is unchanged by multiplication by  $A$ .

\*Example :-

$$\dot{y}_1 = 6y_1 + 5y_2$$

$$\dot{y}_2 = y_1 + 2y_2$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\downarrow$                        $\downarrow$                        $\downarrow$   
 $\dot{Y}$                        $A$                        $Y$

assume solution :-

$$y_1 = x_1 e^{\lambda t} \rightarrow \dot{y}_1 = x_1 \lambda e^{\lambda t}$$

$$y_2 = x_2 e^{\lambda t} \rightarrow \dot{y}_2 = x_2 \lambda e^{\lambda t}$$

$\downarrow$

$$\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$

$$\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$





$$\begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

↓

$$\begin{bmatrix} 6-\lambda & 5 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(6-\lambda)(2-\lambda) - 5 = 0$$

$$\lambda^2 - 8\lambda + 7 = 0 \rightarrow (\lambda-1)(\lambda-7) = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = 7$$

For  $\lambda_1 = 1$

$$\begin{bmatrix} (6-1) & 5 \\ 1 & (2-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5x_1 + 5x_2 = 0 \rightarrow x_1 = -x_2$$

$$x_1 + x_2 = 0 \rightarrow x_1 = -x_2$$

$$x_1 = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 1 \Rightarrow$$

$$Y_1 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t$$



For  $\lambda_2 = 7$ 

$$\begin{bmatrix} (6-7) & 5 \\ 1 & (2-7) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 5x_2 = 0 \rightarrow x_1 = 5x_2 \rightarrow x_2 = \frac{x_1}{5} \quad x_2 = c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$x_1 - 5x_2 = 0 \rightarrow x_1 = 5x_2$$

 $\lambda_2 = 7 :-$ 

$$Y_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{7t}$$

General solution :-

$$Y = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{7t}$$

#

$$y_1 = -c_1 e^t + 5c_2 e^{7t}$$

$$y_2 = c_1 e^t + c_2 e^{7t}$$

\* Example : (complex eigenvalues)

$$\dot{y}_1 = -3y_1 - 2y_2$$

$$\dot{y}_2 = 5y_1 - y_2$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$





$$Y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$

$$\begin{bmatrix} -3-\lambda & -2 \\ 5 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(-3-\lambda)(-1-\lambda) + 10 = 0$$

$$\lambda^2 + 4\lambda + 13 = 0 \rightarrow \Delta = b^2 - 4ac = 16 - 4 \times 13 < 0 \quad \text{complex conjugate}$$

$$\lambda_1 = -2 + 3j$$

$$\lambda_2 = -2 - 3j$$

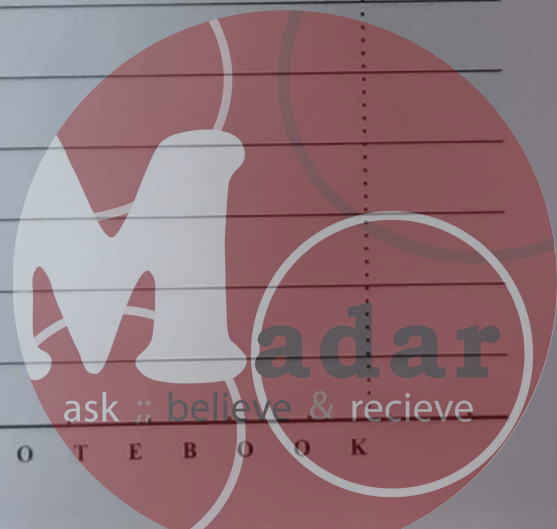
$$\text{for } \lambda_1 = -2 + 3j$$

$$\begin{bmatrix} \underbrace{-3+2}_{-1} - 3j & -2 \\ 5 & \underbrace{-1+2}_{1} - 3j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{(-1 - 3j)}_{2} x_1 = \underbrace{2 x_2}_{-1-3j} \rightarrow x_1 = \begin{bmatrix} 2 \\ -1-3j \end{bmatrix}$$

مشتان يكون العرض  
مستويين

$$Y_1 = \begin{bmatrix} 2 \\ -1-3j \end{bmatrix} e^{(-2+3j)t}$$



use  $\rightarrow$  Euler Formula

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

$$e^{(-2+3j)t} = e^{-2t} \cdot e^{(3t)j}$$

$$e^{(-2+3j)t} = e^{-2t} (\cos(3t) + j \sin(3t))$$

$$Y_1 = \begin{bmatrix} 2 \\ -1-3j \end{bmatrix} e^{-2t} (\cos(3t) + j \sin(3t))$$

$$Y_1 = e^{-2t} \begin{bmatrix} 2 \cos(3t) + 2j \sin(3t) \\ -\cos(3t) - j \sin(3t) - 3j \cos(3t) + 3 \sin(3t) \end{bmatrix}$$

يأتي

أفصل الـ

real عن

الـ imaginary

$$Y_1 = e^{-2t} \left\{ \begin{bmatrix} 2 \cos(3t) \\ -\cos(3t) + 3 \sin(3t) \end{bmatrix} + j \begin{bmatrix} \sin(3t) \\ -\sin(3t) - 3 \cos(3t) \end{bmatrix} \right\}$$

$\downarrow$   $X_1$   $\downarrow$   $X_2$



For  $\lambda_2 = -2 - 3j$

↳ give some answer

$$\begin{array}{l}
 \text{real part} \quad x_1 = \begin{bmatrix} 2\cos(3t) \\ -\cos(3t) + 3\sin(3t) \end{bmatrix} \\
 \text{imaginary part} \quad x_2 = \begin{bmatrix} \sin(3t) \\ -\sin(3t) - 3\cos(3t) \end{bmatrix}
 \end{array}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \overset{\curvearrowright}{x_1} e^{-2t} + c_2 \overset{\curvearrowright}{x_2} e^{-2t}$$

$$Y = c_1 \begin{bmatrix} 2\cos(3t) \\ -\cos(3t) + 3\sin(3t) \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} \sin(3t) \\ -\sin(3t) - 3\cos(3t) \end{bmatrix} e^{-2t}$$

$$y_1 = 2c_1 \cos(3t) e^{-2t} + c_2 \sin(3t) e^{-2t}$$

$$y_2 = -c_1 (\cos(3t) + 3\sin(3t)) e^{-2t} + c_2 (-\sin(3t) - 3\cos(3t)) e^{-2t}$$



**Case :-** Repeated eigenvalues (double)

→ Matrix A has 2 eigenvalues, if repeated we have one eigenvector

**solution ⇒**

$$Y_1 = \bar{x} e^{\lambda t} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$

**to find 2nd solution :-**

$$Y_2 = ?!$$

1st modify  $Y_1$  :  $Y_2 = t Y_1 = t \bar{x} e^{\lambda t}$  unknown function

$$\dot{Y}_2 = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} t e^{\lambda t}$$

$$\cancel{e^{\lambda t}} \cdot t \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cancel{e^{\lambda t}} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} t \cancel{e^{\lambda t}}$$

$$= t (\lambda \bar{x}) + (\bar{x}) = (A \bar{x}) t$$

$t \rightarrow A \bar{x} = \lambda \bar{x}$  eigenvalue problem  $[\bar{x} \neq 0]$

$t^0 \rightarrow \bar{x} = 0$  cannot be  $\rightarrow$  هنا الحل غير ممكن يعني لازم نغير الترمين

**so modify  $Y_2$  :-**

$$Y_2 = t \bar{x} e^{\lambda t} + n e^{\lambda t}$$

$$\dot{Y}_2 = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cancel{e^{\lambda t}} + t \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cancel{e^{\lambda t}} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \lambda \cancel{e^{\lambda t}}$$



$$Y_2 = \bar{x} e^{\lambda t} + t \lambda \bar{x} e^{\lambda t} + \lambda \bar{\eta} e^{\lambda t}$$

↓ نفوض

$$Y_2' = A Y_2$$

$$\bar{x} e^{\lambda t} + t \lambda \bar{x} e^{\lambda t} + \lambda \bar{\eta} e^{\lambda t} = A (t \bar{x} e^{\lambda t} + \bar{\eta} e^{\lambda t})$$

$$t: (\lambda \bar{x}) = (A \bar{x}) \Rightarrow A \bar{x} = \lambda \bar{x} \rightarrow \text{eigenvalue problem}$$

$\bar{x} \rightarrow$  eigenvector

corresponding to the repeated eigenvalue.

$$t^0: \bar{x} + \lambda \bar{\eta} = A \bar{\eta}$$

↓

$$A \bar{\eta} - \lambda \bar{\eta} = \bar{x}$$

$$(A - \lambda I) \bar{\eta} = \bar{x}$$

# procedure :-

• solve eigenvalue problem  $A \bar{x} = \lambda \bar{x}$  For  $\bar{x}$

• Then solve equation  $(A - \lambda I) \bar{\eta} = \bar{x}$  For  $\bar{\eta}$

\* Example :-

$$y_1' = 4y_1 - y_2$$

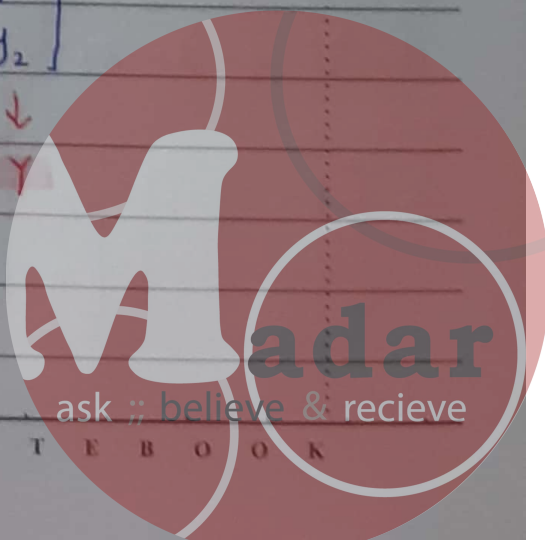
$$y_2' = y_1 - 2y_2$$

$$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

↓  
 $Y'$

↓  
 $A$

↓  
 $Y$



$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix} = (4-\lambda)(2-\lambda) + 1 = 0$$

$$8 - 6\lambda + \lambda^2 + 1 = 0 \Rightarrow \lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)(\lambda - 3) = 0$$

$$\lambda = \lambda_1 = \lambda_2 = 3 \text{ (double)}$$

For  $\lambda = 3$  (double) And eigenvector  $\bar{x} \Rightarrow$

$$(A - \lambda I)\bar{x} = 0$$

$$\begin{bmatrix} 4-3 & -1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow x_1 - x_2 = 0$$

$$x_1 = x_2$$

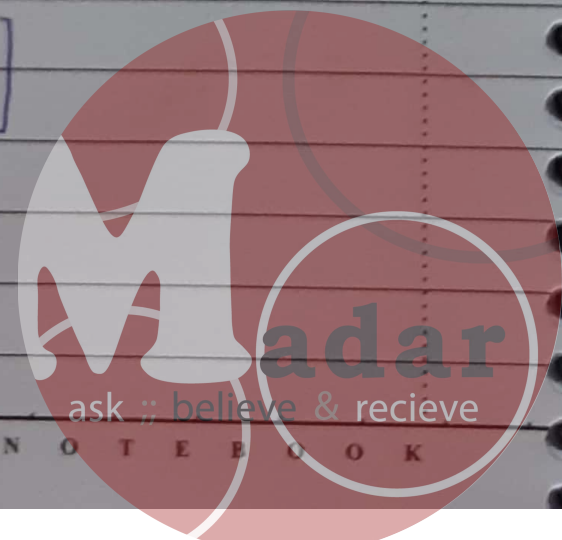
$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

to find  $Y_2 \Rightarrow (A - \lambda I)\bar{x} = \bar{x}$

$$\text{For } \lambda \rightarrow A - \lambda I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \Leftrightarrow \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$





$$\eta_1 - \eta_2 = 1 \rightarrow \eta_2 = \eta_1 - 1$$

$$\bar{\eta} = \begin{bmatrix} \eta_1 \\ \eta_1 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\text{if } \eta_1 = 2$$

بناءً أرقام متتالية

$$Y_2 = t \bar{x} e^{\lambda t} + \eta e^{\lambda t}$$

$$Y_2 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}$$

General solution :-  $Y = c_1 \bar{Y}_1 + c_2 \bar{Y}_2$

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \right\}$$



## # Nonhomogeneous linear system of 1<sup>st</sup> ODEs :-

Consider  $\Rightarrow$

$$Y' = AY + (g(t))$$

General solution:-

$$Y(t) = Y_c(t) + Y_p(t)$$

Method of undetermined coefficients

$$y_1' = \frac{dy_1}{dt} = 2y_1 + y_2$$

$$y_2' = \frac{dy_2}{dt} = 3y_2 + te^t$$

↓

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ te^t \end{bmatrix}$$

$\downarrow$                        $\downarrow$                        $\downarrow$                        $\downarrow$   
 $Y$                        $A$                        $Y$                        $g(t)$

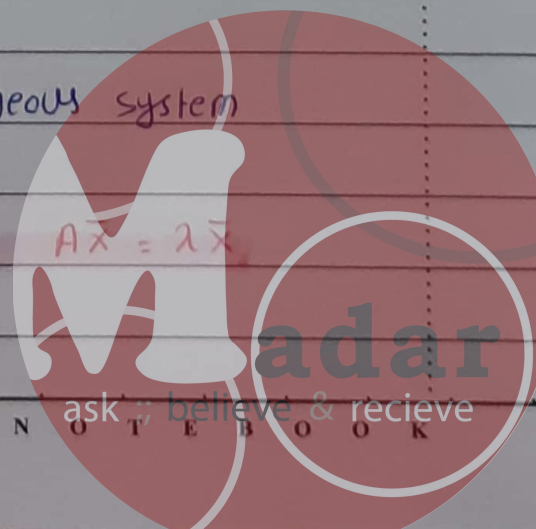
$$Y(t) = Y_c(t) + Y_p(t)$$

$Y_c(t) \Rightarrow$  complementary solution of homogeneous system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

solve

$$A\bar{x} = \lambda\bar{x}$$





$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0 \rightarrow (2-\lambda)(3-\lambda) = 0$$

$$\lambda_1 = 2 \quad \lambda_2 = 3$$

$$Y_1 = \bar{x}_1 e^{2t}$$

$$Y_2 = \bar{x}_2 e^{3t}$$

$$Y_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{2t}$$

$$Y_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{3t}$$

For  $\lambda_1 = 2 \Rightarrow \bar{x}_1 = ?$

$$\begin{bmatrix} \overset{\text{zero}}{2-2} & 1 \\ 0 & \underset{1}{3-2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 0x_1 + x_2 = 0$$

$$x_2 = 0 \rightarrow \text{for any value}$$

كل ما أعني  $0 \neq x_1$

قيمة  $x_1$  لا  $x_2$  أو  $x_2 > x_1$  بتساوي

بشكل

$$\bar{x}_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ if } x_1 = 1$$



For  $\lambda_2 = 3$ 

$$\begin{bmatrix} \underline{2-3} & 1 \\ 0 & \underline{3-3} \\ & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -x_1 + x_2 &= 0 \\ x_1 &= x_2 \end{aligned}$$

$$\bar{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{if } x_1 = 1$$

 $Y_c(t) :-$ 

$$Y_1 = \bar{x}_1 e^{2t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$$

$$\Rightarrow Y_c(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

$$Y_2 = \bar{x}_2 e^{3t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

# For particular solution :-  $Y_p(t)$ 

$$g(t) = \begin{bmatrix} 0 \\ te^t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} e^t \Rightarrow Y_p = \begin{bmatrix} y_{p1} \\ y_{p2} \end{bmatrix}$$

polynomial of zero degree      polynomial of 1<sup>st</sup> degree

$$\begin{bmatrix} y_{p1} \\ y_{p2} \end{bmatrix} = (B + Ct)e^t = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} t e^t$$



$$\begin{bmatrix} y_{p1}' \\ y_{p2}' \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^t + t \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^t + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^t$$

$$\begin{bmatrix} y_{p1}' \\ y_{p2}' \end{bmatrix} = e^t \left\{ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + t \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right\} \rightarrow Y' = AY + g(t)$$

⇓

$$\left( \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + t \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right) e^t = A \left( \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} t \right) e^t$$

↓

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + t \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$$

$$t^0: \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \text{حل های یکماده و یوجد c دیونه ها}$$

$$t^1: \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{با معادله الثانية دیوجد B}$$

$$A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

↓

$$(A - I) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

→ solve for

c

$$(A - I) \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

→ solve for

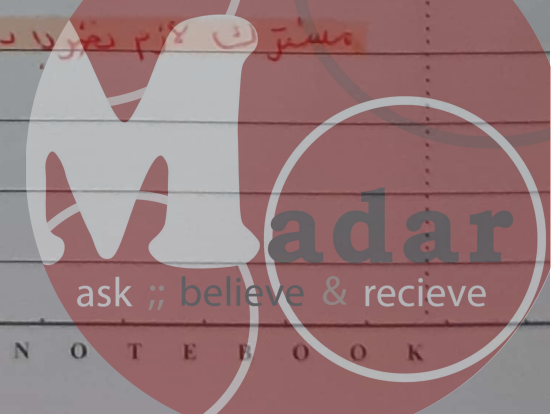
B

\* Note 8-

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = I \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

عشان تطلع c عامل

مستتر لازم بظروا I



$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - I = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \rightarrow$$

نتابع  
عازي على  
مدى الحد في  
بقا بلو بالمتغيرة  
الثنائية.

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow c_1 = -c_2 = \frac{1}{2}$$

$$c_2 = -\frac{1}{2}$$

$$\underline{c} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \#$$

solve  $\Rightarrow (A - I) \underline{B} = \underline{c}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \Rightarrow B_1 = \frac{3}{4}$$

$$B_2 = -\frac{1}{4}$$

$$Y_p(t) = \begin{bmatrix} y_{p1} \\ y_{p2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} t e^t \#$$





**\* Example :-**

$$y_1' = 2y_1 - 4y_2 + 2t^2 + 10t$$

$$y_2' = y_1 - 3y_2 + t^2 + 9t + 3$$

$$\dot{Y} = AY + g(t) \Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$$

$$Y = Y_c(t) + Y_p(t)$$

 **$Y_c(t)$  :-**solve eigenvalue problem  $A\bar{x} = \lambda\bar{x}$ 

$$\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A - \lambda I$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{vmatrix} = (2-\lambda)(-3-\lambda) + 4 = 0$$

$$-6 + 3\lambda + \lambda^2 + 4 = 0 \rightarrow \lambda^2 + 3\lambda - 2 = 0$$

$$(\lambda - 1)(\lambda + 2) = 0$$

For  $\lambda_1 = 1$  $\lambda_1 = 1 \quad \lambda_2 = -2$ 

$$\begin{bmatrix} \overset{\text{↪ 1}}{2-1} & -4 \\ 1 & \overset{\text{↪ -4}}{-3-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 - 4x_2 = 0 \\ x_1 = 4x_2 \end{matrix}$$

$$\bar{x}_1 = \begin{bmatrix} 4x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

if  $x_2 = 1$ for  $\lambda_2 = -2$ 

$$\begin{bmatrix} 2+2 & -4 \\ 1 & -3+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2$$

$\begin{matrix} \text{2+2} \rightarrow 4 \\ \text{-3+2} \rightarrow -1 \end{matrix}$

$$\bar{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\hookrightarrow x_2 = x_1$

$$Y_c(t) = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

 $Y_p(t) =$ 

poly with degree 2

حسب  $g(t)$  لي

إسؤال

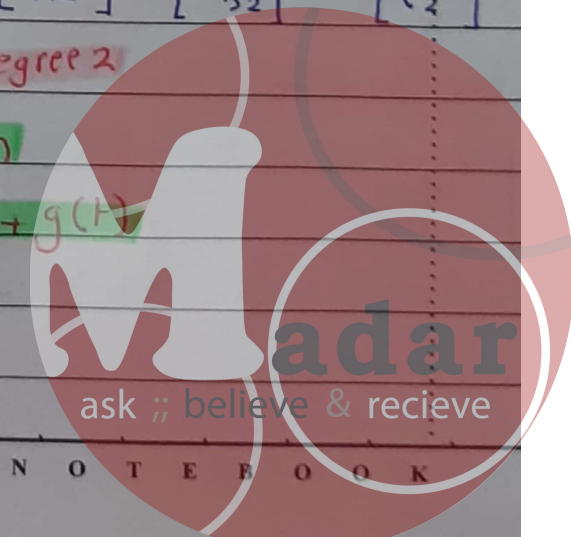
$$Y_p(t) = \begin{bmatrix} y_{p1} \\ y_{p2} \end{bmatrix} = \begin{bmatrix} A_1 + B_1 t + C_1 t^2 \\ A_2 + B_2 t + C_2 t^2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} t + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} t^2$$

$\hookrightarrow$  poly with degree 2

$$Y_p' = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + 2t \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

 $\rightarrow$  substitute in

$$Y_p' = A Y_p + g(t)$$





$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + 2t \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = A \left\{ \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} t + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} t^2 \right\} t^2 + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$$

$$A \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + A \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} t + A \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} t^2 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} t^2 + \begin{bmatrix} 10 \\ 9 \end{bmatrix} t + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + 2t \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$t^2: A \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \text{solve for } C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

كل ما أعرف  
وحدة بوحدها

$$t: A \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 9 \end{bmatrix} = 2 \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \rightarrow \text{solve for } B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

بالتالية  
دعنا

$$t^0: A \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \rightarrow \text{solve for } A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

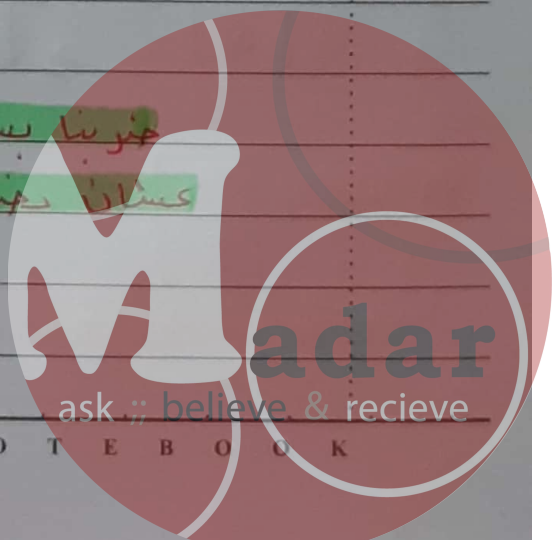
solve for C :-

$$\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = - \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

جربنا بسالب 2

كشأن دهنه C2

\* -2



$$\begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \Rightarrow c_2 = 0, c_1 = -1, \underline{C} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

solve for  $B$ :-

$$\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 10 \\ 9 \end{bmatrix} = \begin{bmatrix} -12 \\ -9 \end{bmatrix}$$

\* -2

$$\begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} \Rightarrow B_2 = 3, B_1 = 0, \underline{B} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

solve for  $A$ :-

$$\begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow A_2 = 0, A_1 = 0$$

$$Y_p(t) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 3 \end{bmatrix} t = \begin{bmatrix} -t^2 \\ 3t \end{bmatrix} \quad \#$$

$$Y(t) = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + \begin{bmatrix} -t^2 \\ 3t \end{bmatrix} \quad \#$$

General solution



**\* Example 8:-**

$$\frac{dy_1}{dt} = \dot{y}_1 = -3y_1 + y_2 - 6e^{-2t}$$

$$\frac{dy_2}{dt} = \dot{y}_2 = y_1 - 3y_2 + 2e^{-2t}$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

 **$Y_c(t)$  :-**

$$(A - \lambda I) = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix}$$

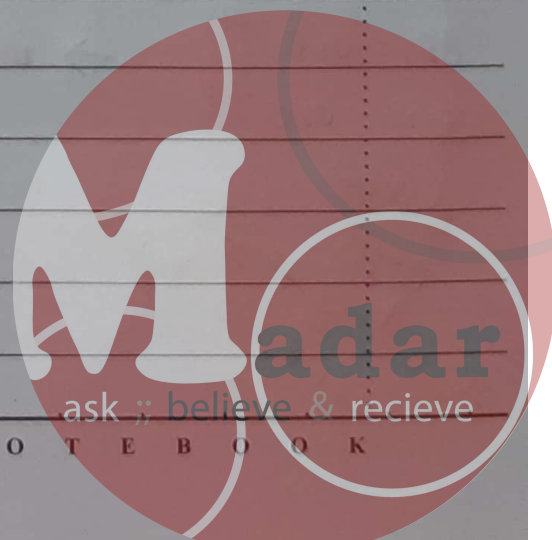
$$(-3-\lambda)^2 - 1 = 0 \rightarrow \lambda^2 + 6\lambda + 8 = 0$$

$$(\lambda + 2)(\lambda + 4) = 0$$

$$\lambda_1 = -2 \quad \lambda_2 = -4$$

$$Y_c = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \bar{x}_1 e^{-2t} + c_2 \bar{x}_2 e^{-4t}$$

Find eigenvectors  $\bar{x}$  :-



For  $\lambda_1 = -2$  :-

$$\begin{bmatrix} -3+2 & 1 \\ 1 & -3+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{matrix} \text{wavy line} \\ \rightarrow -1 \end{matrix}$        $\begin{matrix} \text{wavy line} \\ \rightarrow -1 \end{matrix}$

$$\Rightarrow x_1 = x_2$$

$$\bar{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

if  $x_1 = 1$ For  $\lambda_2 = -4$  :-

$$\begin{bmatrix} -3+4 & 1 \\ 1 & -3+4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{matrix} \text{wavy line} \\ \rightarrow 1 \end{matrix}$        $\begin{matrix} \text{wavy line} \\ \rightarrow 1 \end{matrix}$

$$\Rightarrow x_1 = -x_2$$

$$\bar{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Y_c(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

 $Y_p(t)$  :-

Note :-

$$g(t) = \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

↓

$$y_{p1} = v_1 e^{-2t}$$

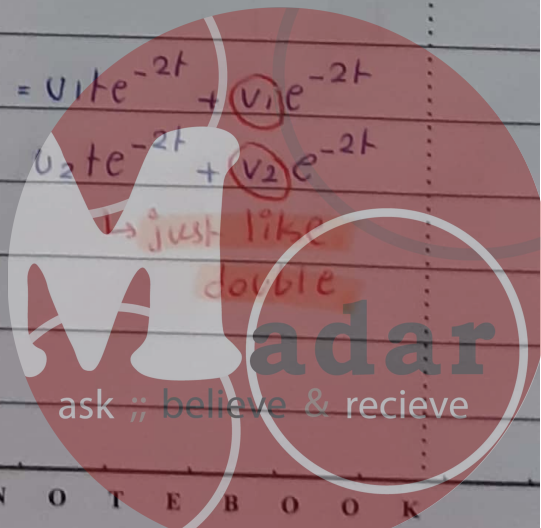
$$y_{p2} = v_2 e^{-2t}$$

$$\rightarrow \text{modify} = v_1 t e^{-2t} = v_2 t e^{-2t}$$

$$= v_1 t e^{-2t} + v_1 e^{-2t} + v_2 t e^{-2t} + v_2 e^{-2t}$$

just like double

لا نحتاج  $e^{-4t}$  لأننا عندي  $e^{-2t}$  homo





$$Y_p = \begin{bmatrix} y_{p1} \\ y_{p2} \end{bmatrix} = t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{-2t}$$

$$Y_p' = \begin{bmatrix} y_{p1}' \\ y_{p2}' \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{-2t} - 2t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{-2t} - 2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{-2t}$$

↓ substitute in  $Y_p' = AY + g(t)$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{-2t} - 2t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{-2t} - 2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{-2t} = A \left\{ t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

↓

$$[u] - 2t[u] - 2[v] = tA[u] + A[v] + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

∴  $-2[u] = A[u] \rightarrow Au = -2u \rightarrow$  eigenvolve problem  
 $u$  eigen vector

$$\bar{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ for } \lambda_1 = -2$$

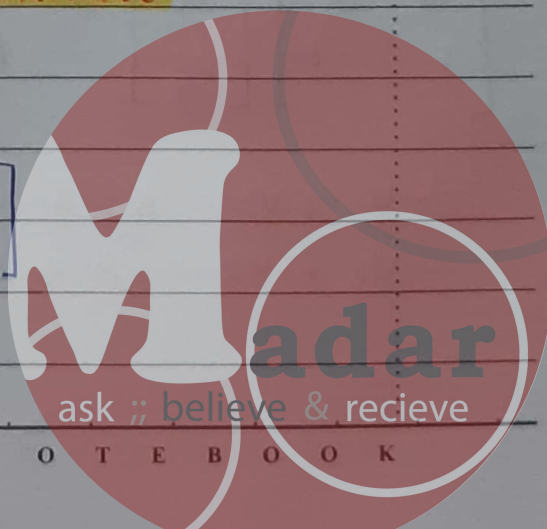
corresponding to  $\lambda_1 = -2$

$$u = \bar{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ if } x_1 = k$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$k$  should be determined

∴  $[u] - 2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$



$$[v] - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

↳ I Identity matrix

$$(A + 2I)[v] = [v] - \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$(A + 2I)v = v - \begin{bmatrix} -6 \\ 2 \end{bmatrix} \quad \text{solve for } v$$

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

$$A + 2I = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A + 2I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\det(A + 2I) = 1 - 1 = 0$$

singular matrix

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

↓

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{matrix} k+6 \\ k-2 \end{matrix}$$

$$\rightarrow -v_1 + v_2 = k+6$$

$$(v_1 - v_2 = k - 2) \times -1$$

$$-v_1 + v_2 = k+6$$

$$\Rightarrow k+6 = -k+2$$

$$-v_1 + v_2 = -k+2$$

$$2k = -4$$

$$k = -2$$





$$\underline{U} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$-v_1 + v_2 = k + 6 \rightarrow \text{if } k = -2$$

$$-v_1 + v_2 = 4 \rightarrow v_2 = v_1 + 4$$

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 + 4 \end{bmatrix} \rightarrow \text{for any value of } v_1$$

$$\underline{Y}_p(t) = \begin{bmatrix} y_{p1} \\ y_{p2} \end{bmatrix} = t \begin{bmatrix} -2 \\ -2 \end{bmatrix} e^{-2t} + \begin{bmatrix} v_1 \\ v_1 + 4 \end{bmatrix} e^{-2t} \quad \#$$

$$\underline{Y}(t) = \underline{Y}_c(t) + \underline{Y}_p(t)$$

$$\underline{Y}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + t \begin{bmatrix} -2 \\ -2 \end{bmatrix} e^{-2t} + \begin{bmatrix} v_1 \\ v_1 + 4 \end{bmatrix} e^{-2t}$$

↪ for any value of  $v_1$

if  $v_1 = 1$

$$\underline{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$



## # variation of parameters method :-

$$\underline{\dot{y}}(t) = A(t) \underline{y}(t) + \underline{g}(t)$$

A(t) → variable

coefficients

general solution

$$\underline{y}(t) = c_1 \underline{\bar{y}}_1(t) + c_2 \underline{\bar{y}}_2(t) + \underline{y}_p(t)$$

→ solutions of homogeneous equation → ( $\underline{\dot{y}}(t) = A(t) \underline{y}(t)$ )

$\underline{\bar{y}}_1(t)$  → solution corresponding to eigenvalue  $\lambda_1$

$\underline{\bar{y}}_2(t)$  → solution corresponding to eigenvalue  $\lambda_2$

$$2 \begin{bmatrix} \underline{\bar{y}}_1(t) \end{bmatrix}, 2 \begin{bmatrix} \underline{\bar{y}}_2(t) \end{bmatrix} \rightarrow \text{vectors}$$

Fundamental matrix  $2 \begin{bmatrix} \underline{\bar{y}} \end{bmatrix} \Rightarrow 2 \begin{bmatrix} \underline{\bar{y}}(t) \end{bmatrix} = \begin{bmatrix} \underline{\bar{y}}_1(t) & \underline{\bar{y}}_2(t) \end{bmatrix}$

\* Particular solution for  $\underline{\dot{y}} = A \underline{y} + \underline{g}(t) \Rightarrow$ 

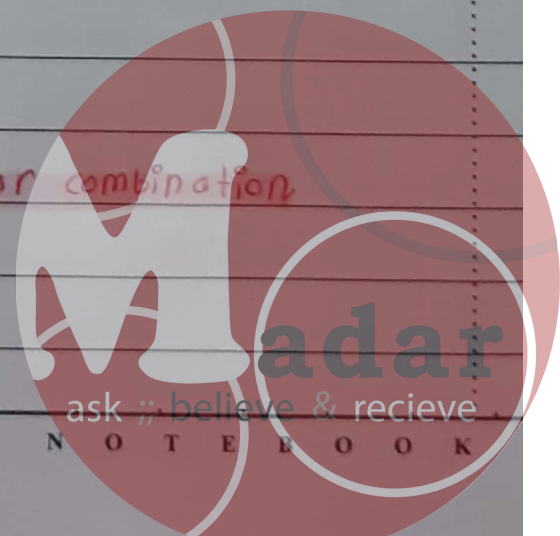
$$\underline{y}_p(t) = \underline{\bar{y}}(t) \int \underline{\bar{y}}^{-1} \cdot \underline{g}(t) dt$$

↳ inverse

Proof ⇒

let  $\Rightarrow \underline{y}_p(t) = u_1 \underline{\bar{y}}_1(t) + u_2 \underline{\bar{y}}_2(t)$

linear combination





$$= \begin{bmatrix} \bar{Y}_1(t) & \bar{Y}_2(t) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\underline{y}_p(t) = \begin{bmatrix} \bar{Y}(t) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \xrightarrow{\text{L}} \underline{v}(t)$$

↓

$$\underline{y}_p(t) = \bar{Y} \underline{v}(t), \text{ for } \underline{v}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \text{ To be determined}$$

↓ if it is a solution then

$$\underline{\dot{y}}_p = A(t) \underline{y}_p + \underline{g}(t)$$

$$\underline{\dot{y}}_p = (\bar{Y} \underline{v})' = \bar{Y}' \underline{v} + \bar{Y} \underline{\dot{v}} \quad \xrightarrow{\text{أخوّن}} \quad \bar{Y}' \underline{v} + \bar{Y} \underline{\dot{v}} = A(\bar{Y} \underline{v}) + \underline{g}(t)$$

$$\begin{aligned} \bar{Y}_1' &= A \bar{Y}_1 \\ \bar{Y}_2' &= A \bar{Y}_2 \end{aligned} \rightarrow \begin{bmatrix} \bar{Y}_1' & \bar{Y}_2' \end{bmatrix} = \begin{bmatrix} \bar{Y}' \end{bmatrix}$$

$$\begin{bmatrix} \bar{Y} \end{bmatrix} = \begin{bmatrix} \bar{Y}_1 & \bar{Y}_2 \end{bmatrix}$$

$$\bar{Y}' = A \bar{Y} \quad , \quad \begin{bmatrix} \bar{Y}' \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \bar{Y} \end{bmatrix}$$

$$\begin{bmatrix} \bar{Y}' \end{bmatrix} \underline{v} + \begin{bmatrix} \bar{Y} \end{bmatrix} \underline{\dot{v}} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \bar{Y} \end{bmatrix} \cdot \underline{v} + \underline{g}(t)$$

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \bar{Y} \end{bmatrix} \underline{v} + \begin{bmatrix} \bar{Y} \end{bmatrix} \underline{\dot{v}} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \bar{Y} \end{bmatrix} \underline{v} + \underline{g}(t)$$

$$\boxed{\bar{Y}} \boxed{\bar{U}} = \boxed{g(t)}$$



$$\bar{U} = \bar{Y}^{-1} g(t)$$

$$\bar{U} = \int \bar{Y}^{-1} g(t) dt$$

$\bar{Y} \rightarrow$  non singular

$$\det \bar{Y} \neq 0$$



$\bar{Y}^{-1}$  exists

↳ inverse

$$y_p(t) = \bar{Y} \bar{U}(t) = \bar{Y}(t) \int \bar{Y}^{-1} g(t) dt$$

### # Procedure s-

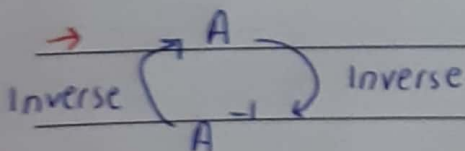
solution of homogeneous system

- Find  $\bar{Y}_1, \bar{Y}_2$  and construct  $\bar{Y} = [\bar{Y}_1 \mid \bar{Y}_2]$ .
- compute  $\bar{Y}^{-1} g(t)$  which is  $\bar{U}(t)$ .
- compute  $\bar{U}(t) = \int \bar{Y}^{-1} g(t) dt$ .
- Find  $y_p(t) = \bar{Y} \bar{U}(t)$ .
- compute  $\bar{Y}(t) = \bar{Y}_c + y_p(t)$ .

يوجد ال inverse

Fundamental matrix

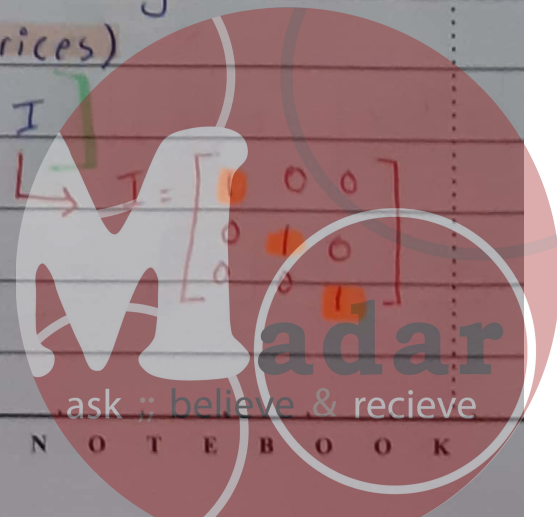
### # Note :-



when we multiply a matrix by its inverse we get the Identity matrix (which like 1 for matrices)

$$[A \times A^{-1} = I]$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





## \* How to calculate the inverse :-

2x2 Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

\* Ex :-

↪ determinant

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4 \times 6 - 7 \times 2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

## \* Example :-

$$\dot{y}_1 = -3y_1 + y_2 - 6e^{-2t}$$

$$\dot{y}_2 = y_1 - 3y_2 + 2e^{-2t}$$

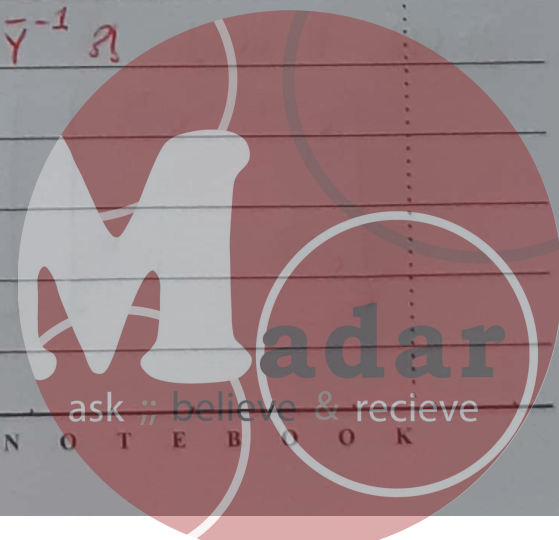
$$\bar{Y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

$$\bar{Y}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$g(t) = \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

له هذان السؤال حلين  
من اجل دالة جيب  
لـ  $y_1, y_2$  بالمتغير

$$\Rightarrow \bar{Y} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}$$

compute  $\bar{Y}^{-1}$  ؟

$$\bar{Y}^{-1} = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}$$

$$\underline{U} = \bar{Y}^{-1} \cdot g(t) = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix} \begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix}$$

$$\underline{U}(t) = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix}$$

$$\underline{U}(t) = \int_0^t \bar{Y}^{-1} g(t) dt = \int_0^t \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix} dt$$

رکابی  
عمل  
کال

$$= \begin{bmatrix} \int_0^t -2 dt \\ \int_0^t -4e^{2t} dt \end{bmatrix} = \begin{bmatrix} -2t \\ -2e^{2t} \end{bmatrix}_0^t$$

$$\underline{U}(t) = \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

$$y_p(t) = \bar{Y} \underline{U}(t)$$

$$= \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$





$$= \begin{bmatrix} -2te^{-2t} + e^{-4t}(-2e^{2t} + 2) \\ -2te^{-2t} - e^{-4t}(-2e^{2t} + 2) \end{bmatrix}$$

$$y_p(t) = \begin{bmatrix} -2t \\ -2t \end{bmatrix} e^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-4t}$$

$$y_p(t) = -2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$y_c(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$y(t) = y_p + \bar{y}_c$$

$$\rightarrow c_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$= -2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + \left\{ 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} \right\} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

$$y(t) = -2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$



## # series solutions of ODE(s) :-

## → Power series method

a power series is an infinite series of the form =

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

$a_m$   $\rightarrow m=0, 1, 2, \dots$  real numbers coefficients  
 $x_0$  variable center of the series

ما دينا  
مبدأ  
→ Taylor series expansion  
of  $x_0$

→ If  $f$  has a power series representation (expansion) at  $a$ , that is, if  $\Rightarrow$

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n \quad |x-a| < R$$

فنتس بي  
موت بس  
غير الموزا بس

then its coefficients are given by the formula

$$C_n = \frac{f^n(a)}{n!}$$

المشتقة  
حسب n

→ if  $x_0 = 0$ , we get a power series in powers of  $x$

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \dots$$

→ Maclouren series at (around)  
 $x_0 = 0$

ask, believe & recieve



It's used to solve ODE(s) with variable coefficients  $\Rightarrow$  solution more complicated  $\Rightarrow$  non elementary Functions

(Legendre polynomials  
Bessel functions)

\* Example 3: (maclouren series)

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \dots$$

expansion of  $\frac{1}{1-x}$  around  $x_0 = 0$

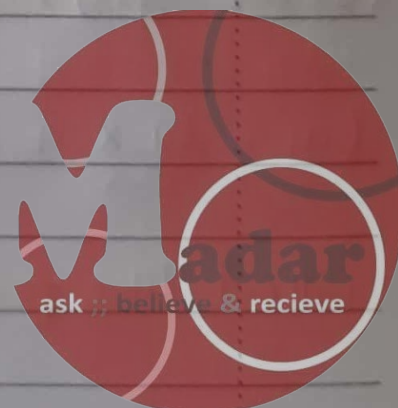
$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

odd power of x

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

even power of x



## # idea &amp; Technique of the Power series :-

## \* Example :-

solve  $\dot{y} - y = 0$  (separate & integrate)

$$\frac{dy}{y} = dx \rightarrow \ln y = x + c \rightarrow y = c_1 e^x$$

or

$$r - 1 = 0 \rightarrow r = 1$$

$$y_1 = e^x \rightarrow y(x) = c e^x$$

↓  
solution using power series  $\Rightarrow$

$$y(x) = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$\dot{y}(x) = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

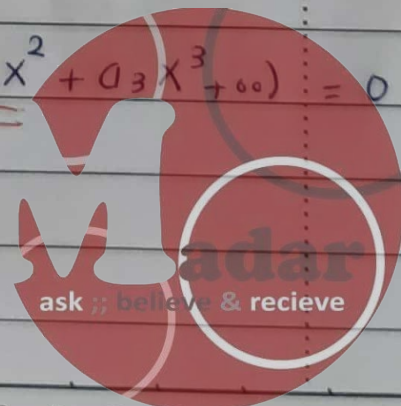
عادي المسمى  
① ② ③

$$\ddot{y}(x) = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + \dots$$

بما أنها المعادلة  $\dot{y} - y = 0$  يعني  $\dot{y}(x) - y(x)$  (تقريباً)

$$(a_1 + 2a_2 x + 3a_3 x^2 + \dots) - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots) = 0$$

collect like powers of  $x \Rightarrow$





$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \dots = 0$$

راج نشاری کل حد قبل انکساره بالصغر

$$x^0: a_1 - a_0 = 0 \rightarrow a_1 = a_0$$

$$x: 2a_2 - a_1 = 0 \rightarrow a_2 = \frac{1}{2} a_1 = \frac{1}{2} a_0 = \frac{1}{2!} a_0$$

$$x^2: 3a_3 - a_2 = 0 \rightarrow a_3 = \frac{a_2}{3} = \frac{1}{3!} a_0 \rightarrow \frac{1}{2} a_0 \cdot \frac{1}{3} \rightarrow \frac{1}{6} a_0 \rightarrow \frac{1}{3!} a_0$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

↓

$$y = a_0 + a_0 x + \frac{1}{2!} a_0 x^2 + \frac{1}{3!} a_0 x^3 + \dots$$

$$y = a_0 \left( 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots \right)$$

های عبارت

$e^x$  expansion عن  
around  $x_0 = 0$

$$y = a_0 e^x \quad \#$$



**\* Example :-**

$$\dot{y} = 2xy \rightarrow \dot{y} - 2xy = 0$$

$$(a_1 + 2a_2x + 3a_3x^2 + \dots) - 2x(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) = 0$$

$$(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots) - (2a_0x + 2a_1x^2 + 2a_2x^3 - 2a_3x^4 - \dots) = 0$$

$$x^0 = a_1 = 0$$

$$x^1 = 2a_2 - 2a_0 = 0 \rightarrow a_2 = a_0$$

$$x^2 = 3a_3 - 2a_1 = 0 \rightarrow a_3 = \frac{2}{3} a_1 = 0$$

$$x^3 = 4a_4 - 2a_2 = 0 \rightarrow a_4 = \frac{1}{2} a_2 = \frac{1}{2} a_0$$

Note :-

Odd coefficients  $\rightarrow$  كلاً بعداد منفر

$$a_1 = 0, a_3 = 0, a_5 = 0, \dots$$

even coefficients

$$a_2 = a_0, a_4 = \frac{1}{2} a_0, a_6 = \frac{1}{3} a_4 = \frac{1}{3!} a_0, \dots$$





$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \dots$$

$\downarrow$   $a_0$        $\downarrow$   $\frac{1}{2!} a_0$        $\downarrow$   $\frac{1}{3!} a_0$

$$y(x) = a_0 + a_0 x^2 + \frac{1}{2!} a_0 x^4 + \frac{1}{3!} a_0 x^6 + \dots$$

$$y(x) = a_0 \left( 1 + x^2 + \frac{1}{2!} x^4 + \frac{1}{3!} x^6 + \frac{1}{4!} x^8 + \dots \right)$$

• Note :-

$$e^{x^2} = 1 + x^2 + \frac{1}{2!} x^4 + \frac{1}{3!} x^6 + \frac{1}{4!} x^8 + \dots$$

$$y(x) = a_0 e^{x^2} \quad \#$$

$$\Rightarrow y' - 2xy = 0 \quad (\text{separate \& integrate})$$

$$\frac{dy}{y} = 2x dx \rightarrow \ln y = x^2 + C \rightarrow y = c_1 e^{x^2} \quad \#$$

# properties of summation :-

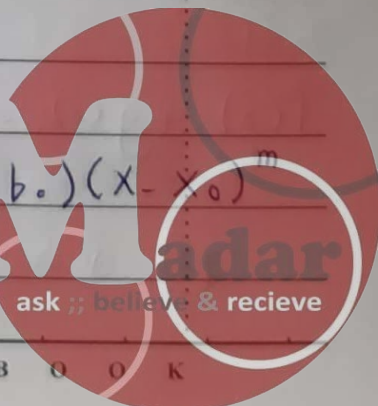
$$f(x) = \sum_{m=0}^{\infty} a_m (x-x_0)^m, \quad g(x) = \sum_{m=0}^{\infty} b_m (x-x_0)^m$$

• Addition  $\Rightarrow$

$$f(x) + g(x) = \sum_{m=0}^{\infty} (a_m + b_m) (x-x_0)^m$$

• multiplication  $\Rightarrow$

$$f(x) * g(x) = \sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \dots + a_m b_0) (x-x_0)^m$$



→ multiplying each term of  $f(x)$  by each term of  $g(x)$  & collecting like power of  $(x-x_0)$

• multiplication by a number  $\alpha \Rightarrow$

$$\alpha f(x) = \alpha \left( \sum_{m=0}^{\infty} a_m (x-x_0)^m \right) = \sum_{m=0}^{\infty} \alpha a_m (x-x_0)^m$$

• Differentiation  $\Rightarrow$

$$f(x) = \sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1 x + a_2 x^2 + \dots$$

$$f'(x) = \sum_{m=0}^{\infty} m a_m (x-x_0)^{m-1} = \sum_{m=1}^{\infty} m a_m (x-x_0)^{m-1}$$

$$f''(x) = \sum_{m=0}^{\infty} m(m-1) a_m (x-x_0)^{m-2} = \sum_{m=2}^{\infty} m(m-1) a_m (x-x_0)^{m-2}$$

• Shifting summation indices  $\Rightarrow$  (very important  $\times$ )

$$f(x) = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} \quad , \quad g(x) = \sum_{m=0}^{\infty} a_m x^m$$

ما جدير أن نلاحظه مبكراً مثل هذا لأن القوة تكملناه

$$f(x) + g(x) = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m$$

shift عشان أولي القوى بين نفس الأس بعدد أربع



let  $n = m - 2$  ( $m = 0, n = -2$ )

$$n+1 = m-1$$

$$n+2 = m$$

الاشهر انو نحلي

بلي موبها اقل مثل

بلي موبها كيرة

$$m(m-1) a_m x^{m-2} = (n+2)(n+1) a_{n+2} x^n$$

$$\sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} = \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

لو بلشنا من  $n=0$  يعني  
الاسي ( $n=-2$ )

$$f(x) + g(x) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{m=0}^{\infty} a_m x^m$$

$n, m$  dummy index  
يعني يقدر أعط  
 $n$  بدل  $m$

$$f(x) + g(x) = \sum_{m=0}^{\infty} ((m+2)(m+1) a_{m+2} + a_m) x^m \quad \#$$

\* Example :-

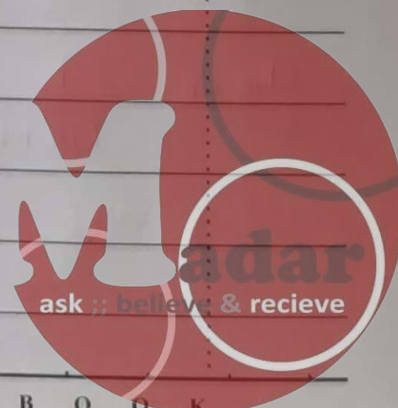
$$\dot{y} - y = 0$$

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

$$\dot{y}(x) = \sum_{m=0}^{\infty} m a_m x^{m-1}$$

$$m=0$$

$$\rightarrow m=1$$



$$\sum_{m=0}^{\infty} m a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^m = 0$$

let  $n = m - 1$   $(m=0) \rightarrow (n=-1)$

$n+1 = m$   $(m=1) \rightarrow (n=0)$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} ((m+1) a_{m+1} x^m - a_m x^m) = 0$$

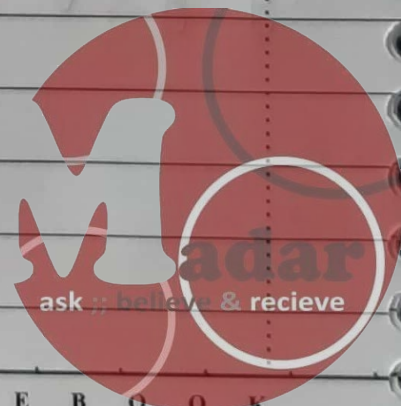
$$\sum_{m=0}^{\infty} ((m+1) a_{m+1} - a_m) x^m = 0$$

$x^m : (m+1) a_{m+1} - a_m = 0$

$m = 0, 1, 2, \dots$

solve for  $a_{m+1} \rightarrow$  Recurrence Relation

$$a_{m+1} = \frac{1}{(m+1)} a_m$$





$$m=0 \rightarrow a_1 = a_0$$

$$m=1 \rightarrow a_2 = \frac{1}{2} a_1 = \frac{1}{2!} a_0$$

$$m=2 \rightarrow a_3 = \frac{1}{3} a_2 = \frac{1}{3 \times 2} a_0 = \frac{1}{3!} a_0$$

$$m=3 \rightarrow a_4 = \frac{1}{4} a_3 = \frac{1}{4 \times 3 \times 2} a_0 = \frac{1}{4!} a_0$$

⋮

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$y(x) = a_0 + a_0 x + \frac{1}{2!} a_0 x^2 + \frac{1}{3!} a_0 x^3 + \frac{1}{4!} a_0 x^4 + \dots$$

$$y(x) = a_0 \left( 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \dots \right)$$

↳  $e^x$

$$y(x) = a_0 e^x \quad \#$$

\* Example 8-

$$\ddot{y} + y = 0$$

↓

$$r^2 + 1 = 0$$

$$r_{1/2} = \pm j$$

$$y(x) = C_1 \cos(x) + C_2 \sin(x)$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\dot{y}(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\ddot{y}(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

let  $m = n - 2$  ( $n = 2, m = 0$ )  
 $\downarrow$   
 $n = m + 2$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} ((n+2)(n+1) a_{n+2} + a_n) x^n = 0$$

$$(n+2)(n+1) a_{n+2} + a_n = 0 \quad (\text{solve for } a_{n+2})$$

$$a_{n+2} = - \frac{a_n}{(n+1)(n+2)} \quad n = 0, 1, \dots$$

→ Recurrence relation

$$n = 0 \rightarrow a_2 = - \frac{a_0}{1 \times 2}$$

$$n = 1 \rightarrow a_3 = - \frac{a_1}{2 \times 3}$$

$$n = 2 \rightarrow a_4 = - \frac{a_2}{3 \times 4} = \frac{a_0}{1 \times 2 \times 3 \times 4} = \frac{a_0}{4!}$$

$$n = 3 \rightarrow a_5 = - \frac{a_3}{4 \times 5} = \frac{a_1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{a_1}{5!}$$





$$n=4 \rightarrow a_6 = -\frac{a_4}{5 \times 6} = -\frac{a_0}{4! \times 5 \times 6} = -\frac{a_0}{6!}$$

even  $\rightarrow$ 

$$a_{2k} = \frac{(-1)^k \times a_0}{(2k)!}$$

 $k = 1, 2, \dots$ odd  $\rightarrow$ 

$$a_{2k+1} = \frac{(-1)^k \times a_1}{(2k+1)!}$$

 $k = 1, 2, \dots$ 

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_{2k} x^{2k} + a_{2k+1} x^{2k+1} + \dots$$

$$y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \dots + \frac{(-1)^k a_0}{(2k)!} x^{2k} + \frac{(-1)^k a_1}{(2k+1)!} x^{2k+1} + \dots$$

$$y = a_0 \left\{ 1 - \frac{1}{2!} x^2 + \dots + \frac{(-1)^k}{(2k)!} x^{2k} + \dots \right\} \rightarrow \text{even}$$

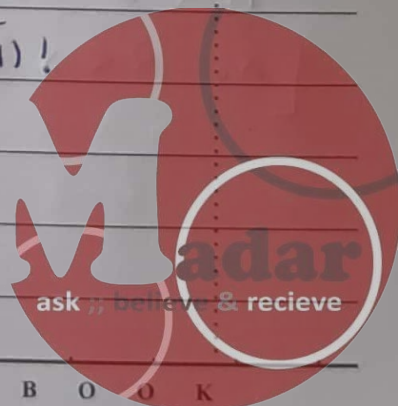
$$y = a_1 \left\{ x - \frac{1}{3!} x^3 + \dots + \frac{(-1)^k}{(2k+1)!} x^{2k+1} + \dots \right\} \rightarrow \text{odd}$$

Note:-

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$y = a_0 \cos(x) + a_1 \sin(x) \quad \#$$



## # Convergence of Power series :-

Given  $P(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$  there exists number  $0 \leq R < \infty$

so that the series  $\Rightarrow$

converge for  $|x-x_0| < R$   $\rightarrow$  Radius of  
and diverge for  $|x-x_0| > R$  convergence

$\rightarrow$  To determine  $R$ , use Ratio test.

$$\text{if } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad (0 \leq L \leq \infty)$$

$$\text{then } |x-x_0| < R = \frac{1}{L}$$

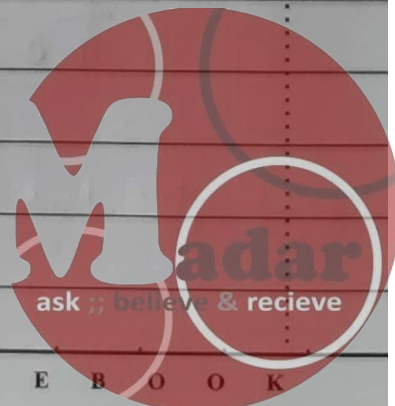
$$R = \begin{cases} \infty & \text{if } L = 0 \\ 0 & \text{if } L = \infty \end{cases}$$

then  $\rightarrow$

$$|x-x_0| < R$$

$$-R < x-x_0 < R$$

$$[x_0 - R < x < x_0 + R] \rightarrow \text{convergence}$$





**\* Example :-**

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{n! 7^{n+1}} (x-5)^n$$

**Ratio test →**

$$a_n = \frac{(-3)^n}{n! 7^{n+1}}, \quad a_{n+1} = \frac{(-3)^{n+1}}{(n+1)! 7^{n+2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{-3}{7} \right| = \frac{3}{7} = L \Rightarrow R = \frac{7}{3}$$

$$|x-5| < \frac{7}{3}$$

$$5 - \frac{7}{3} < x < 5 + \frac{7}{3}$$

**\* Example :-**

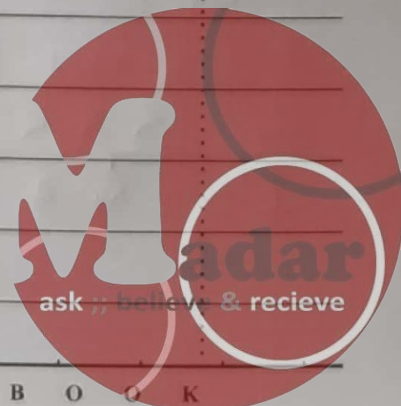
$$e^x = \sum \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \dots$$

$$x_0 = 0 \rightarrow (x-x_0)^m = (x-0)^m = x^m$$

$$|x-0| \stackrel{?}{<} R$$

**Ratio test →**

$$a_n = \frac{x^n}{n!}, \quad a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$



$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{x}{n+1} = x \lim_{n \rightarrow \infty} \frac{1}{n+1} = x \cdot 0 = 0$$

$$L = 0 \Rightarrow R = \frac{1}{L} = \frac{1}{0} = \infty$$

convergence interval

$$-\infty < x < \infty$$

$\infty$  series  $e^x = \sum \frac{x^n}{n!}$  always converges for any value.

\* Example :-

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

$$L = \left| \frac{a_{n+1}}{a_n} \right| = \frac{1^{n+1}}{1^n} = 1 \rightarrow L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

$$R = \frac{1}{L} = 1$$

$$-1 < x < 1$$

$$|x| < 1 \rightarrow -1 < x < 1$$

↳ converge for any  $x \in (-1, 1)$



**\* Example :-**

$$\sum_{m=0}^{\infty} m! x^m = 1 + x + 2x^2 + \dots$$

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{n!} \right| = \lim_{n \rightarrow \infty} (n+1) = \infty$$

$$R = \frac{1}{L} = \frac{1}{\infty} = 0$$

$$x_0 = 0 \quad |x - 0| < 0$$

$$0 < x < 0$$

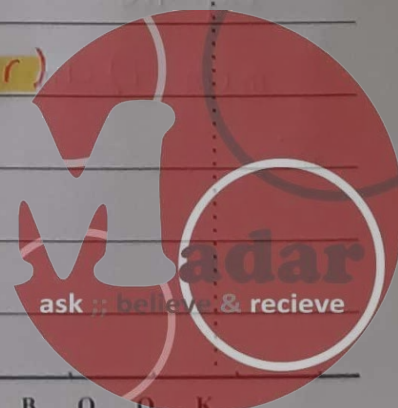
↳ converge only at the center  $x_0 = 0$

**# Summary :-**

$$e^x \rightarrow \begin{array}{c} -\infty < x < \infty \\ -\infty \quad 0 \quad \infty \\ \text{any value of } x \end{array}$$

$$\frac{1}{1-x} = \sum x^n \quad |x| < 1$$

$$\sum_{n=0}^{\infty} n! x^n \quad \begin{array}{c} 0 < x < 0 \\ 0 \\ \text{only around } x=0 \text{ (center)} \end{array}$$



# Legendre's Equation, Legendre polynomials  $P_n(x) \Rightarrow$ 

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$\hookrightarrow$  Legendre's differential equation

$n \rightarrow \text{constant}$

family of equations

عند قيم مختلفة لـ  $n$ ،  
يكون عندنا معادلات كثيرة.

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0$$

analytic around

$$x = x_0 = 0$$

• apply power series method :-

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

$$y'(x) = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$\text{let } k = n(n+1)$$

$$y''(x) = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$\downarrow$

$$(1-x^2) \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - 2x \left( \sum_{m=1}^{\infty} m a_m x^{m-1} \right) + k \left( \sum_{m=0}^{\infty} a_m x^m \right) = 0$$

expand

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0$$

$\hookrightarrow m=0$



$$n = m - 2$$

$$n + 1 = m - 1$$

$$\Rightarrow (n+2)(n+1) a_{n+2} x^n$$

$$n + 2 = m \quad (m = 2 \rightarrow n = 0)$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{m=0}^{\infty} m(m-1) a_m x^m - \sum_{m=0}^{\infty} 2m a_m x^m +$$

$$\sum_{m=0}^{\infty} k a_m x^m = 0$$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - m(m-1) a_m x^m - 2m a_m x^m + k a_m x^m = 0$$

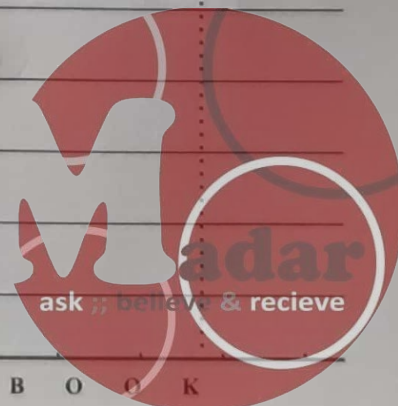
$$(m+2)(m+1) a_{m+2} - m(m-1) a_m - 2m a_m + k a_m = 0$$

$$-m(m-1) - 2m + k = -m^2 + m - 2m + k = -m^2 - m + k$$

$$a_{m+2} = \left( \frac{m^2 + m - k}{(m+2)(m+1)} \right) a_m$$

$$(m+2)(m+1) a_{m+2} = (m(m-1) + 2m - k) a_m$$

$$a_{m+2} = \left( \frac{m^2 - m + 2m - k}{(m+2)(m+1)} \right) a_m$$



$$a_{m+2} = \left( \frac{m^2 + m - n(n+1)}{(m+2)(m+1)} \right) a_m$$

$$K = n(n+1)$$

Note :-

$$\begin{aligned} m^2 + m - n^2 - n &= m^2 - n^2 + (m - n) \\ &= (m - n)(m + n) + (m - n) \\ &= (m - n)(1 + m + n) \end{aligned}$$

$$a_{m+2} = \left( \frac{(m - n)(1 + m + n)}{(m+2)(m+1)} \right) a_m$$

$$a_{m+2} = - \left( \frac{(n - m)(1 + m + n)}{(m+2)(m+1)} \right) a_m$$

أد  
بعض  
بالطريقة  
بلي روش العنوية

$$m = 0, 1, 2, \dots$$

$$m = 0 \rightarrow a_2 = - \frac{n(n+1)}{2} a_0$$

$$m = 1 \rightarrow a_3 = - \left( \frac{(n-1)(2+n)}{3 \cdot 2} \right) a_1 = - \left( \frac{(n-1)(n+2)}{3!} \right) a_1$$

$$m = 2 \rightarrow a_4 = - \left( \frac{(n-2)(3+n)}{4 \times 3} \right) a_2 = \left( \frac{(n-2)(n+3)}{4 \times 3 \times 2} n(n+1) \right) a_0$$

$$a_4 = \frac{(n-2)(n+3)(n)(n+1)}{4!} a_0$$





$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

$$y(x) = \underbrace{a_0}_{y_1} y(x) + \underbrace{a_1}_{y_2} y(x)$$

$$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)(n+3)n(n+1)}{4!} x^4 - \dots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \dots$$

→ Legendre's polynomials

# Ordinary & singular points ⇒

2nd linear

$$a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0$$

if  $a_0(x) \neq 0$

$$y'' + P(x)y' + Q(x)y = 0 \quad (\text{standard form})$$

المقام يساوي 0 ← ∞  
مبتلي

نختصم  
المعادلة  
على  $a_0(x)$

Any point at which the denominator of  $P(x)$  &  $Q(x) = \infty$  is called singular point, the other points are called ordinary point & the function is called analytic. ( $P(x)$  &  $Q(x)$  have Taylor series expansion around  $x = x_0$ )

Legendre's

↓  
linear

↓  
homo

↓  
variable  
coefficient

**\* Example :-**

$$\ddot{y} + x\dot{y} + (x^2 + 2)y = 0$$

$$P(x) = x/1$$

→ ordinary every where

$$Q(x) = (x^2 + 2)/1$$

$P(x)$  &  $Q(x)$  are analytic function

$P(x)$  &  $Q(x)$  have Taylor series expansion

**\* Example :-**

$$\ddot{y} + \left(\frac{x}{x-1}\right)\dot{y} + \left(\frac{1}{x(x-1)}\right)y = 0$$

$$P(x) = \frac{x}{x-1}$$

→ singular at  $x=1$  → analytic every where except at  $x=1$

$$Q(x) = \frac{x}{x(x-1)}$$

→ singular at  $x=0$  &  $x=1$  → analytic every where except at  $x=0$  &  $x=1$

∴  $x=0$  &  $x=1$  are singular points & all other points are ordinary points.

**# series solutions near a Regular singular point ⇒**

$$P(x)\ddot{y} + Q(x)\dot{y} + R(x)y = 0$$

[ solution obtained by power series method when  $P(x_0) \neq 0$





• Defn  $\Rightarrow$

let  $x = x_0$  be a singular point of the ODE  $\rightarrow$

$$P(x)\ddot{y} + Q(x)\dot{y} + R(x)y = 0$$

$$\lim_{x \rightarrow x_0} \frac{(x-x_0) Q(x)}{P(x)} \quad \& \quad \lim_{x \rightarrow x_0} \frac{(x-x_0)^2 R(x)}{P(x)}$$

$\xrightarrow{\text{coeff } \dot{y}}$        $\xrightarrow{\text{coeff } y}$   
 $\xrightarrow{\text{coeff } \ddot{y}}$        $\xrightarrow{\text{coeff } \ddot{y}}$

both exist  $\rightarrow x_0$  is a singular point  
 $\downarrow$   
 regular

otherwise it is called an irregular singular point

\* Example :-

$$(x-2)\ddot{y} + 3\dot{y} - xy = 0 \rightarrow \ddot{y} + \left(\frac{3}{x-2}\right)\dot{y} - \left(\frac{x}{x-2}\right)y = 0$$

$\downarrow$        $\downarrow$        $\downarrow$   
 $P(x)$      $Q(x)$      $R(x)$

$\rightarrow x_0 = 2 \rightarrow$  singular point (لأنها تتخطى المقام 0)

$$P(x) = x-2$$

$$Q(x) = 3$$

$$R(x) = -x$$

$$\lim_{x \rightarrow 2} \frac{(x-2) \cdot 3}{x-2} = 3 \quad \& \quad \lim_{x \rightarrow 2} \frac{(x-2)^2 \cdot (-x)}{(x-2)} = 0$$

$\xrightarrow{x_0}$

both limits are finite  $\rightarrow$  so  $x_0 = 2 \rightarrow$  regular  
 singular point

\* Example :-

$$x^3(1-x) \ddot{y} + (3x+2) \dot{y} + x^4 y = 0$$

↓

$$\ddot{y} + \frac{3x+2}{x^3(1-x)} \dot{y} + \frac{x^4}{x^3(1-x)} y = 0$$

→  $x_0 = 0$  &  $x_0 = 1$  → singular points

$$P(x) = x^3(1-x)$$

$$Q(x) = 3x+2$$

$$R(x) = x^4$$

$$\lim_{x \rightarrow 0} \frac{3x+2}{x^3(1-x)} (x-0) = \lim_{x \rightarrow 0} \frac{(3x+2)}{x^3(1-x)} \quad (\text{does not exist})$$

$$\lim_{x \rightarrow 0} \frac{x^4}{x^3(1-x)} (x-0)^2 = \lim_{x \rightarrow 0} \frac{x^2}{1-x} = 0 \quad (\text{finite})$$

∴  $x=0$  → irregular singular point





# Bessel's equation of order ( $r$ ): (extended power series method)

$$x^2 \ddot{y} + x \dot{y} + (x^2 - r^2) y = 0 \quad (\text{Frobenius method})$$

$$x = x_0 = 0$$

→ singular point

$$\ddot{y} + \left(\frac{1}{x}\right) \dot{y} + \frac{(x^2 - r^2)}{x^2} y = 0$$

كل ما أعطي فيه  $r$  ينتج معادلة جديّة

$$P(x) = 1$$

$$Q(x) = \frac{1}{x}$$

$$R(x) = \frac{x^2 - r^2}{x^2}$$

$$\lim_{x \rightarrow 0} \frac{1}{x} \cdot x = 1$$

$$\lim_{x \rightarrow 0} \frac{x^2 - r^2}{x^2} \cdot x^2 = -r^2$$

2 limits exists (finite) → so  $x_0 = 0$  Regular singular point.

solution of the form →

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$$

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$r$  → complex or real

$$\dot{y}(x) = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} = x^{r-1} \sum_{m=0}^{\infty} a_m (m+r) x^m$$

$$\ddot{y}(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} = x^{r-2} \sum_{m=0}^{\infty} a_m (m+r)(m+r-2) x^m$$

$$x^2 y'' + x y' + (x^2 - r^2) y = 0$$

$$x^2 \cdot x^{r-2} \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^m + x \cdot x^{r-1} \sum_{m=0}^{\infty} a_m (m+r) x^m + (x^2 - r^2) x^r \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r} + \sum_{m=0}^{\infty} a_m (m+r) x^{m+r} +$$

$$-r^2 \sum_{m=0}^{\infty} a_m x^{m+r} + x^{2+r} \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} \{ a_m ((m+r)(m+r-1) + (m+r) - r^2) \} x^{m+r}$$

$$+ \sum_{m=0}^{\infty} a_m x^{m+r+2}$$

let  $n = m+2$

$m = n-2$  ( $m=0, n=2$ )

$$\sum_{m=0}^{\infty} \{ a_m ((m+r)(m+r-1) + (m+r) - r^2) \} x^{m+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$a_0 (r(r-1) + r - r^2) x^r + a_1 ((r+1)r + (r+1) - r^2) x^{r+1} + \sum_{m=2}^{\infty} a_m ((m+r)(m+r-1) + (m+r) - r^2) x^{m+r} + \sum_{m=2}^{\infty} a_{m-2} x^{m+r} = 0$$



$$a_0 (r^2 - v^2) x^r + a_1 ((r+1)^2 - v^2) x^{r+1}$$

$$+ \sum_{m=2}^{\infty} \{ a_m ((m+r)^2 - v^2) + a_{m-2} \} x^{m+r} = 0$$

$$x^r = a_0 (r^2 - v^2) = 0 \rightarrow a_0 \neq 0 \rightarrow r = \pm v$$

$$x^{r+1} = a_1 (r^2 + 2r + 1 - v^2) = 0 \rightarrow a_1 = 0$$

$\rightarrow \neq 0$

$$x^{m+r} = a_m ((m+r)^2 - v^2) + a_{m-2} = 0 \quad m = 2, 3, 4, \dots, \infty$$

$$a_m = - \left( \frac{a_{m-2}}{(m+r)^2 - v^2} \right)$$

$\rightarrow$  (cases  $\Rightarrow$ )

$$r = +v$$

$$a_m = - \frac{a_{m-2}}{(m+r)^2 - v^2} = - \frac{a_{m-2}}{m^2 + 2mr} = - \frac{a_{m-2}}{m(m+2r)} \quad m \geq 2$$

$$m=2 \rightarrow a_2 = - \frac{a_0}{2(2+2v)} = - \frac{a_0}{2^2(1+v)}$$

$$m=3 \rightarrow a_3 = - \frac{a_1}{m(m+2r)} \xrightarrow{\text{zero}} = 0$$



$$m=4 \rightarrow a_4 = \frac{-a_2}{2^2(4+2r)} = -\frac{a_2}{2 \cdot 2^2(r+2)} = \frac{a_0}{2^2 \cdot 2 \cdot 2^2(1+r)(2+r)}$$

$$a_4 = \frac{a_0}{2 \cdot 2^4 \cdot (r+2)(r+1)}$$

real

$$a_{2m} = \frac{(-1)^m a_0}{m! \cdot 2^{2m} (r+1) \dots (r+m)}$$

For even

For odd = 0  $a_3 = 0$   $a_1 = 0$  ...

$$y = x^r \sum_{m=0}^{\infty} a_m x^m$$

$$a_0 = \frac{1}{2^r r!}$$

if  $r$  integer  $r=n$

$$r! (r+1)(r+2) \dots (r+m) = (r+m)!$$

$$a_{2m} = \frac{(-1)^m}{m! \cdot 2^{2m} (r+m)!}$$

$m = 1, 2, \dots$

$$a_{2m} = \frac{(-1)^m}{m! \cdot 2^{2m+n} (n+m)!}$$

#Note →

$$n! (n+1) \dots (n+m) = (n+m)!$$





$$y(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m! 2^{2m+n} (n+m)!}$$

↓

$J_n(m)$  Bessel function of 1<sup>st</sup> kind of order n

$$\rightarrow y_1(x) = J_n(m) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m! 2^{2m+n} (n+m)!}$$

$n=0$

$$J_0(m) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2} = 1 - \frac{x^2}{2^2 \cdot (1!)^2} + \frac{x^4}{2^4 (2!)^2} + \frac{-x^6}{2^6 (3!)^2} + \dots$$

↓

Bessel function of  
order zero

similar to cosine

$n=1$

$$J_{(1)}(m) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m} m! (m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1! 2!} + \frac{x^5}{2^5 2! 3!} - \frac{x^7}{2^7 3! 4!} + \dots$$

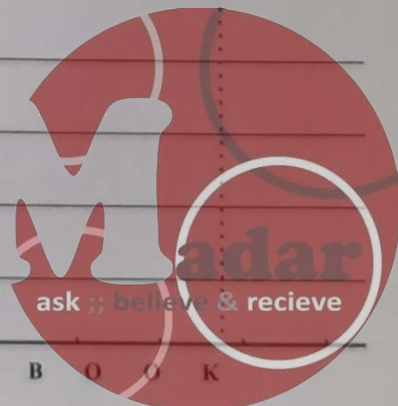
↓

Bessel function of  
order one

similar to sine

For any value  $x \geq 0$

$$a_0 = \frac{1}{2^n n!}$$



$$a_0 = \frac{1}{2^r \Gamma(r+1)}$$

$$\Gamma(r+1) = \int_0^{\infty} e^{-t} t^r dt \quad r > -1$$

$$\Gamma(r+1) = r \Gamma(r)$$

$$\Gamma(n+1) = n! \quad n=0,1,\dots$$

$$y(x) = c_1 J_r(x) + c_2 J_{-r}(x)$$

$r \neq \text{integer}$

$r \rightarrow \text{integer}$

$$y(x) = c_1 J_n(x)$$

