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Mathematical

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$y = f(x)$ → independent varial

dependent varial

$y = ax + b$, $y = ax^2 + bx + c$ ⇒ algebric

y'

deffrential

solution is → unknown fun ⇒ $y(x)$

Eg: ⇒ $y(x)$ ordenary deff. eq ⇒ ODE

⇒ $y(x, z)$

y_x y_z

partial derivative

Ex: $y' = 2x + y$ $\square \overset{?}{=} \square$

1^{st} order dif. eq ⇒ sol: find $y(x)$ and $y'(x)$

$\frac{dy}{dx} = Kx$ ⇒ $\int dy = \int Kx \, dx$

$\boxed{y = \frac{Kx^2}{2} + c}$ the sol, $y' = Kx$

$Kx = Kx$ ✓ ← subsitute it

$y = ax^2 + bx + c = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

⇒ In general a $\boxed{\text{a diff eq. is}}$ an eq. that contains an unknown fun. and its derivatives.

⇒ A fun. $y = f(x)$ is called a solution of a differential equation is satisfied when $y = f(x)$ and it's derevatives are substituted into the eq.

$$Ex: \frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 15x = 0$$ second ordenary diff. eq.

$$Ex_2: \frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial y^2} = w$$ PDE

$w(x,y)$

⇒ DOF = # varials − # eq.   عشان نقدر نحل المعادلات

$$Ex_3: \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$$ third ordenary diff eq.

dependent varial $\Rightarrow y'.y$; $\Rightarrow (y')^{a>1}$] إذا تواجدوا المعادلة تصبح non liner

* بغض النظر عن شكل independent وإذا مرفوعين لأس أو لا

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^4 u}{\partial x^4} = \frac{\partial^4 u}{\partial t}$$

*degree of diff eq. $= \left(\frac{dy}{dx}\right)^{\underline{\underline{2}}}$ → قيمته هي الدرجة ⇒ The power of the highest order derivative term

لازم أعلى مشتقة

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + 3 = 0 \Rightarrow \text{Degree} = \underline{\underline{3}}$$

*To find if it is linear:

1) dependent derrivatives and variables are of the power of (1).

2) no product between dependent derrivatives and dependant var

3) no transcendental fun of the dependent variable or it's derivative are accure

Ex: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 9y = \underline{\underline{0}}$ liner

→ $\cos(x)$ linear عادي

→ $\cos(y)$ non linear

Ex: $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$ → non linear

Ex: $\frac{dy}{dx} = \sin y$ → non linear

Ex: $x^2 \frac{d^2y}{dx^2} + y\frac{dy}{dx} = x^3$ → non linear

* some of examples in the slides:

⇒ $y' = x + 6y$ ⇒ $x = y^6 - 6y$ ⇒ ordenary

$y' = \frac{dy}{dx}$, $y(x)$; $x(y)$

⇒ linear

⇒ ordinary

⇒ first order

⇒ first degree

In depen → y

dep → x

⇒(3) $\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} - 2y = x^3$ → dep = y, in = x, deg = 3

$y(x)$

ordenary, nonlinear, second order

(2) $y'' = 4y + y^3$ → ordinary,

$\frac{d^2y}{dx^2}$

(4) $y'' + 2xy' + 4y = \cos(2x)$ ~~non~~ linear

(5) $\frac{dy}{dx} = \frac{x^2 - 1}{y - 4}$ nonlinear ← $\frac{dy}{dx}(y-4) = x^2 - 1$

(6) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

First order deff. eq. forms:

1. derevative form

$$\underbrace{a_1(x)}\frac{dy}{dx} + \underbrace{a_0(x)}y = g(x)$$

could be constant or a function of x

2. Defferential form

$$\underbrace{M(x,y)}dx + \underbrace{N(x,y)}dy = 0$$

fun

$\Rightarrow \frac{dy}{dx} = Ky \Rightarrow y' - Ky = 0$ derevative form

$a(x)\ y' + a_0(x)\ y = g(x)$

$dy = y'\,dx$

$\Rightarrow dy = y'\,dx$

$dy - (Ky)\,dx = 0$ Deff. form

$\frac{dy}{dx} = Ky = f(x,y)$ Explicit form المشتقة بطرف والباقي بطرف آخر

$\frac{dy}{dx} - Ky = 0$

$f(y, y', x) = 0$ emplicit form

Ex: $x^{-3}y' - 4y^2 = 0 \Rightarrow \frac{y'}{x^3} - 4y^2 = 0$

$y' = 4y^2x^3$ $\qquad y' = 4y^2x^3$ explicit ~~[illegible]~~

$f($

$y = h(x)$ solution, it's graph called solution curve

$$\frac{dy}{dx} = ky$$

$$\int \frac{dy}{y} = \int k\,dx$$

$$\ln y = kx + c$$

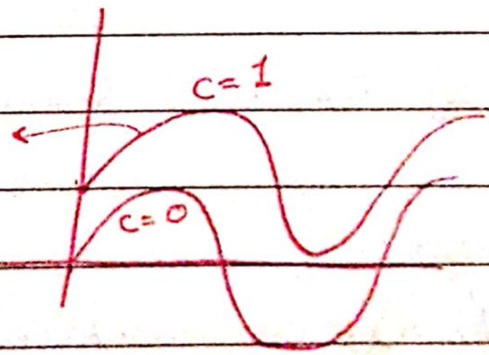
$$y = e^{kx+c}$$ explicit solution

تغير قيمة C يؤدي إلى ظهور curve جديدة

Ex: $\frac{dy}{dx} = \cos x \Rightarrow \int dy = \int \cos x \, dx$

$$y = \sin x + c$$

and so on



] family of solution

$$\frac{d^2y}{dx^2}=0$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=0$$

First order ... = 1 constant   second = 2 constant

Constant =

Ex: $\frac{dy}{dx}=xy \Rightarrow \int\frac{dy}{y}=\int x\,dx = \ln y = \frac{x^2}{2}+c$ implicit

$y=e^{\frac{x^2}{2}+c}=e^{\frac{x^2}{2}}\,e^{c}=ce^{\frac{x^2}{2}}=y$ explicit

*. Newton's law of cooling

$$\frac{dT}{dt}=K(T-T_s)$$

* Growth and Decay

$$\frac{dy}{dt}=Ky$$

$* \quad \frac{dy}{dt} = ky \Rightarrow \int \frac{dy}{y} = \int k\,dt \Rightarrow \ln_e y = kt + c$

$y = e^{kt}\, e^{c}$

in chemistry:

A $\quad C_A$

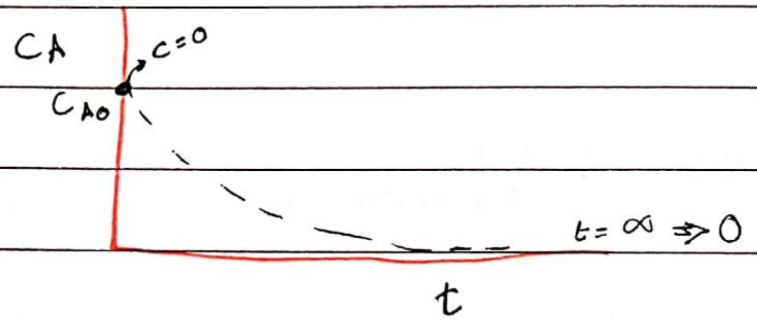
$A \longrightarrow$ product

$-\frac{dC_A}{dt} \propto C_A$

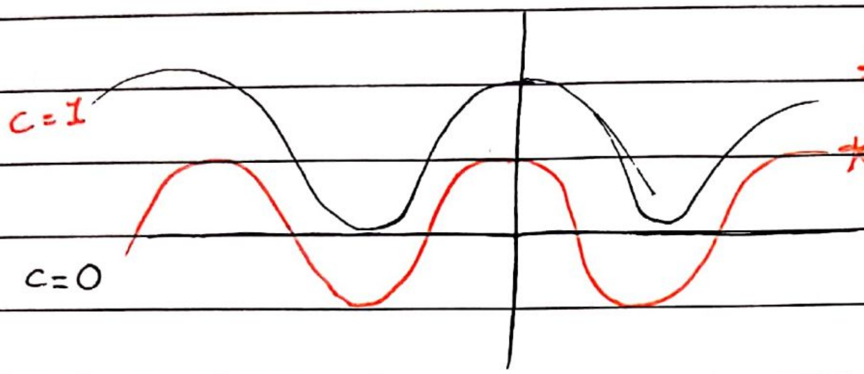
تناقص

$\frac{dC_A}{dt} = -k\,C_A$

$\int \frac{dC_A}{C_A} = \int -k\,dt \Rightarrow \ln C_A = -kt + c \Rightarrow C_A = c\,e^{-kt}$



* $\frac{dy}{dx} = \cos x \Rightarrow \int dy = \int \cos x \, dx \Rightarrow y = \sin x + C$

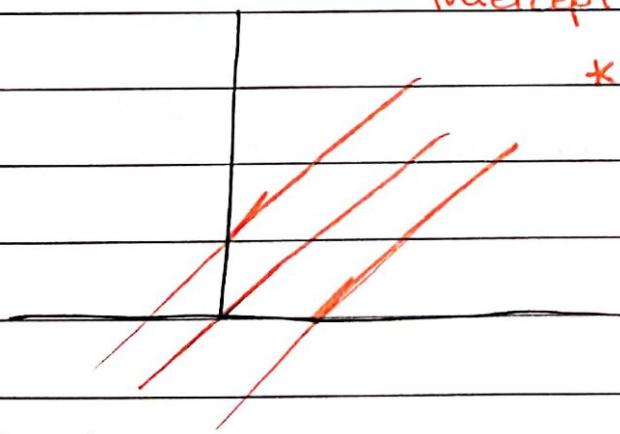


* family of curves
* Must be parallel

* $\frac{dy}{dx} = \underline{2} \Rightarrow$ rate of change = constant = slope

sol: $\int dy = \int 2\,dx + C \Rightarrow y = \underline{2}x + \underline{C}$ linear equation $(ax+b)$

intercept



* same slope but different intercept

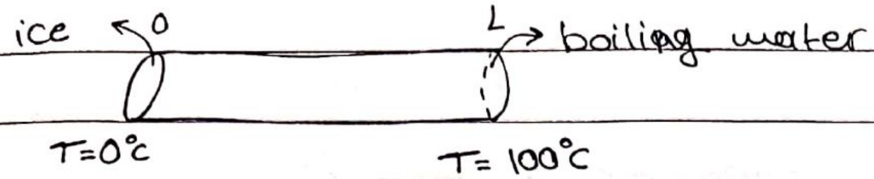
* To find $\underline{C}$ we must have $y(x_0)$ ⇒ certain point

* $y'' = 6x + e^x \Rightarrow y = x^3 + e^x + C_1 x + C_2$

To find C_1 and C_2 ⟶ $y(x_0) = \alpha$ inetial value

⟶ $y'(x_0) = \beta$ problem

كأنهم طرفين على نفس النقطة



$T(x=\underline{0}) = 0°c$

$T(x=\underline{L}) = 100°c$

معرفين على نقطتين مختلفتين إذاً

Boundary value problem

* $16yy' + 9x = 0$

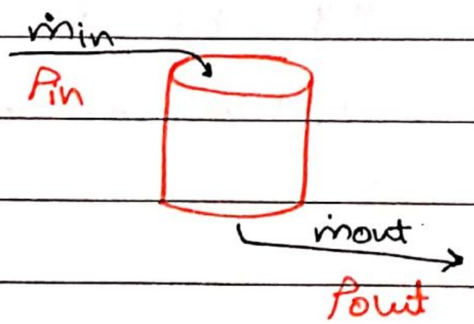
$\int (16yy' + 9x)\,dx = c_1 \Rightarrow \int 16\,y(x)\,y'(x)\,dx + \int 9x\,dx = c_1$

$\int 16y\,dy + \frac{9}{2}x^2 = c_1 \Rightarrow 8y^2 + \frac{9}{2}x^2 = c_1 \Rightarrow 8y^2 + 9x^2 = 2c_1$

ellips

Family of ellipses

* $\underline{Acc} = in - out + Gne - con$
* rate of change with time



$\dot m_{in} - \dot m_{out} = \frac{d}{dt}(m\ system)$

$\Rightarrow \dot m\ sys = \rho sys \cdot V sys$

$\rho in = \rho out = \rho sys$

$\Rightarrow \dot V in - \dot V out = \frac{d}{dt} V sys$

Geometric =...

* Methods of solution

* separation of variables (separate y from x)

1^{st} ODE

$y' = \frac{dy}{dx} = f(x,y)$ explicit form

$y' = f(x,y) = f(x)\, g(y)$

$\frac{dy}{dx} = f(x)\, g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)\, dx + c$

so we can sep. it

Ex: $\frac{dy}{dx} = -\frac{x^2+1}{y-2} = f(x) \quad g(y)$ where $f(x) = x^2+1$

$g(y) = \frac{1}{y-2}$

$\int dy\,(y-2) = \int -(x^2+1)\, dx + c$

$\frac{y^2}{2} - 2y = -\frac{x^3}{3} - x + C$ emplicit sol.

$\frac{y^2}{2} - 2y + \frac{x^3}{3} + x + C = 0$

⊙ Remark: $ax^2 + bx + c = 0$ here $ay^2 + by + c = 0$

explicit * الأفضل إنهاء الحل ك

$a = \frac{1}{2}, \quad b = -2 \quad c = \frac{x^3}{3} + x + C$

$y_{1,2} = \frac{2 \pm \sqrt{4 - 2\left(\frac{x^3}{3} + x + C\right)}}{1}$

Subject Date No.

Ex: $y' = e^{-y}(2x-4)$, $y(x=5)=0$

$$\frac{dy}{dx} = e^{-y}(2x-4) \Rightarrow \frac{dy}{e^{-y}} = (2x-4)\,dx \Rightarrow \int dy\, e^{y} = \int (2x-4)\,dx$$

$$\Rightarrow e^{y} = x^2 - 4x + c \quad \text{implicit}$$

ln ln

$$y = \ln \underbrace{x^2 - 4x + c} \quad \text{explicit} = \ln x^2 - 4x - 4$$

$\rightarrow x^2 - 4x + c > 0$

$\hookrightarrow$ we can find it

$$e^0 = 25 - 20 + c$$

$$1 = 5 + c$$

$$\boxed{c = -4}$$

$$x^2 - 4x - 4 > 0$$

$$x_{1,2} = \frac{4 \pm \sqrt{16+16}}{2}$$

+ x_1 − x_2 +

⇒ نتجه للمنطقة الموجبة داخل ln

Ex: $\frac{dy}{dx} = -Ky$

sol:

$\int \frac{dy}{y} = \int K\, dx + c \Rightarrow \ln_e^y = -Kx + c \Rightarrow y = e^{-Kx} e^{c} = c_1 e^{-Kx} = y$

$c_1 = y e^{Kx} = f(x, y(x))$

* $\left(0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x}\right) * dx = \left(\frac{\partial F}{\partial x}\right) dx + \left(\frac{\partial F}{\partial y}\right) dy$

Exact differential

$\frac{dy}{dx} = -Ky \Rightarrow dy = -Ky\,dx \Rightarrow dy + (Ky)dx = 0$

* $M(x,y)\,dx + N(x,y)\,dy = 0$

about the last examble *: $\frac{\partial F}{\partial x} = M(x,y)$, $\frac{\partial F}{\partial y} = N(x,y)$

⊙ Remark: $f(x,y) = 2x^2y + xy^2$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 4xy + y^2 = 4x + 2y$

same

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2x^2 + 2xy = 4x + 2y$

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Exact condition N&S for exactness

$\underline{F}$ بنقدر نستخدمها لإيجاد

Ex: $$\frac{dy}{dx} = -\frac{2x+y+1}{2y+x+1}$$

$$(2x+y+1)\,dx + (2y+x+1)\,dy = 0$$

$$M(x,y) = 2x+y+1$$

$$N(x,y) = 2y+x+1$$

$$\frac{\partial M}{\partial y} = \underline{1} \quad , \quad \frac{\partial N}{\partial x} = \underline{1} \rightarrow \text{Exact ODE}$$

$$\frac{\partial F}{\partial x} = M = 2x+y+1 \longrightarrow (1)$$

$$\frac{\partial F}{\partial y} = N = 2y+x+1 \longrightarrow (2)$$

integrate eq.1

$$F = \int (2x+y+1)\,dx + C \Rightarrow x^2 + xy + x + \underbrace{C_1}_{y^2+y} = F$$

integrate eq.2

$$F = \int (2y+x+1)\,dy \Rightarrow y^2 + xy + y + \underbrace{C_2}_{x^2+x} = F$$

→ صحيح يكونوا متساويات

$\Rightarrow F = y^2 + xy + y + x^2 + x = c_3$ emplicit

$ay^2 + by + c = 0$

$F = x^2 + xy + x + c_1(y)$

$\frac{\partial F}{\partial y} = x + c_1'(y)$

$\frac{\partial F}{\partial y} = N = 2y + x + 1$

$\Rightarrow \cancel{x} + c_1'(y) = 2y + \cancel{x} + 1$

$c_1'(y) = 2y + 1$

$\frac{dc_1}{dy} = 2y + 1 \Rightarrow \int dc_1 = \int (2y+1)\, dy$

$c_1 = y^2 + y$

$F = x^2 + xy + x + y^2 + y$

* Reduction of 1st ODE to a separable form

$f(x,y) = x^2 + xy$

$r \to x$, $r \to y$ $\Rightarrow rx, \; ry$

$f(rx, ry) = x^2 r^2 + r^2 xy \Rightarrow r^{2}(x^2+xy)$ degree = 2

* if M, N homogeneous functions of the same degree we can write it in a seprable form

$\tan\left(\frac{x}{y}\right) = \frac{\sin\left(\frac{x}{y}\right)}{\cos\left(\frac{x}{y}\right)} \Rightarrow \tan\frac{rx}{ry} = \frac{\sin\frac{rx}{ry}}{\cos\frac{rx}{ry}} \Rightarrow r^{0}\frac{\sin\left(\frac{x}{y}\right)}{\cos\left(\frac{x}{y}\right)}$

dgree = 0

* $f(rx, ry) = r^n f(x,y)$, homogeneous function with dgree of $\underline{n}$

* 1st ODE of M, N can be seprable if they have the same degree

* $\frac{y(x)}{x} = v(x)$

$y(x) = x\,v(x)$

$y'(x) = x\,v'(x) + v(x)$ Chain rule diff equation

Rem

$(x\,y\,z)' = x'yz + xyz' + xy'z$

$* \; y'(x) = \dfrac{dy}{dx} = f(x,y)$

Ex: $\left(xy\dfrac{dy}{dx} - x^2 - y^2\right)$ dx

$(x^2-y^2)\,dx - (xy)\,dy = 0$

$M = x^2 - y^2 \qquad N = -xy$

$\dfrac{\partial M}{\partial y} = -2y \qquad \dfrac{\partial N}{\partial x} = -y \qquad -2y \neq -y$ not exact

$M(rx, ry) = r^2x^2 - r^2y^2 \Rightarrow r^2(x^2-y^2) = r^{2}(M)$

dgree = 2

$N(rx, ry) = -r^2xy = r^{2}N$

dgree = 2

M's dgree = N's dgree

$v(x) = \dfrac{y(x)}{x}$

$y' = v + xv' \Rightarrow$ نعوضها في المعادلة الأصلية

simplification + نعوضها y'

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x^2-y^2}{xy} \Rightarrow \dfrac{\cancel{x^2}\left(1-\frac{y^2}{x^2}\right)}{\cancel{x}\,y} = \dfrac{1-\frac{y^2}{x^2}}{\left(\frac{y}{x}\right)} = \dfrac{1-v^2}{v} = v + xv' \Rightarrow$

$v = \dfrac{y}{x}$

$$v + v'x = \frac{1 - v^2}{v} \quad ----- = \frac{(dv)\,v}{1 - 2v^2} = \frac{1}{x}\,dx$$

separable function

$$\int \frac{v\,dv}{1 - 2v^2} = \ln x + c$$

$$\frac{1 - v^2}{v} - v = v'x$$

$$= \frac{1 - 2v^2}{v} = \frac{dv}{dx}x$$

$$\frac{dx}{x} = \frac{dv\,(v)}{1 - 2v^2}$$

$$\frac{-1}{4} \ln^{1-2v^2} = \ln x + c$$

$$4\ln x + \ln^{1-2v^2} = 4c = c_1$$

$$\ln^{x^4} + \ln^{1-2v^2} = c_1$$

$$\ln_{c}^{x^4(1-2v^2)} = \frac{c_1}{e}$$

$$x^4(1 - 2v^2) = e^{c_1} = c_2$$

$$\frac{c_2}{x^4} = 1 - 2v^2$$

نعوض قيمة v ونحل

Subject linear 1^{st} ODE Date No.

* standard form:

$$\frac{dy}{dx} + P(x)y = g(x) \rightarrow (1)$$

- if $g(x) = \rightarrow 0$ homogenous ODE

$\rightarrow \neq 0$ Nonhomogenous ODE

if eq_n 1 is not Exact we will make it exact!
we will use the following function $\mu(x)$ magic function
subject the following condition

$$\mu(x)\,P(x) = \mu'(x)$$

we multiply eq_n 1 by $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + \mu(x)\,P(x)\,y = \mu(x)\,g(x)$$

use: $\mu'(x) = \mu(x)\,P(x)$

$$\Rightarrow \underbrace{\mu(x)\frac{dy}{dx} + \mu'(x)y}_{LHS} = \underbrace{\mu(x)g}_{RHS}$$

LHS: $\mu(x)\frac{dy}{dx} + \mu'(x)y(x) = (\mu(x)\,y)'$

To prove that: $\mu(x)\,y' + y\,\mu'(x)$

Eq. N:

$$(\mu(x)y)' = \mu(x)\,g$$

$$\frac{d}{dx}(\mu(x)y) = \mu(x)\,g$$

$$y(x) = \frac{C_2 \int \left(e^{\int p(x)dx}\right) g(x)\,dx}{e^{\int p(x)dx}} + \frac{C}{C_2 \; e^{\int p(x)dx}}$$

$$y(x) = \frac{\int e^{\int p(x)\,dx} \cdot h(x)}{e^{\int p(x)dx}} + C_3 \Rightarrow$$ لا تفرق معها

$$\underbrace{\mu(x) = e^{\int p(x)dx}}_{\text{integrating factor}} \quad \{C_2 = 1\}$$

Ex: $\dfrac{dy}{dx} - y = e^{2x}$

standard form: $\dfrac{dy}{dx} + p(x)y = g(x)$ يجب كتابة المعادلة على هذه الصيغة دائماً

$p(x) = -1$

$g(x) = e^{2x}$

$\Rightarrow \mu(x) = e^{\int p(x)\,dx}$

$$\mu(x) = e^{\int -dx} = \underbrace{e^{-x} = \mu(x)}_{\text{The integrating factor}}$$

To make it exact:

$$\mu(x) * \left(\frac{dy}{dx} - y = e^{2x}\right) \Rightarrow e^{-x}\frac{dy}{dx} - e^{-x}y = e^{x}$$

$$(e^{-x}y)' = e^{x}$$

$$\int d(e^{-x}y) = \int e^{x}\,dx + C \Rightarrow e^{-x}y = e^{x} + C \rightarrow \text{Domain } (-\infty, \infty) \text{ so valid}$$

$$\underbrace{y = e^{2x} + Ce^{x}}_{\text{The solution}}$$

N O T E B O O K

$$\left(\frac{dy}{dx} - y = e^{2x}\right) * dx$$

$$dy - y\,dx - e^{2x}\,dx = 0$$

$$dy - dx\,(y + e^{2x}) = 0$$

$$\left.\begin{array}{l} N = 1 \\ M = y + e^{2x} \end{array}\right) \rightarrow \left.\begin{array}{l} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 1 \end{array}\right)$$ Not exact

Ex: Initial value problem: $\frac{dy}{dx} + y = x \qquad y(0) = 2$

Sol: $p(x) = 1$

$g(x) = x$

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int dx} = e^{x} = \mu(x) \rightarrow$$ integrating factor

$$\mu(x) * \left(\frac{dy}{dx} + y = x\right) = e^{x}\frac{dy}{dx} + e^{x}y = e^{x}x$$

$$(e^{x}y)' = e^{x}x$$

$$\Rightarrow \int d(e^{x}y) = \int e^{x}x\,dx + C$$

$$e^{x}y = \underbrace{\int e^{x}x\,dx + C}_{\text{by parts}}$$

$$e^{x}y = e^{x}x - \int e^{x}\,dx + C \Rightarrow \frac{e^{x}y}{e^{x}} = \frac{e^{x}x - e^{x} + C}{e^{x}}$$

$$y = x - 1 + \frac{C}{e^{x}}, \quad y(0) = 2$$

$$2 = -1 + C \Rightarrow \boxed{C = 3}$$

$$y = x - 1 + 3e^{-x}$$

Subject Non exact ODE Date 24. Feb. 2020 No.

$M(x,y)\,dx + N(x,y)\,dy = 0$ — ①

* if eq 1 is not exact we will force it to be.

Multiply eq 1 by an integrating factor $\mu(x,y)$

$\mu(x,y)\, M(x,y)\, dx + \mu(x,y)\, N(x,y)\, dy = 0$

we apply ~~its~~ condition of exactness

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

chain rule

$$M \frac{\partial \mu}{\partial y} + \mu \frac{\partial M}{\partial y} = N \frac{\partial \mu}{\partial x} + \mu \frac{\partial N}{\partial x}$$

let $\mu_y = \frac{\partial \mu}{\partial y}$ $\quad \mu_x = \frac{\partial \mu}{\partial x}$

$M_y = \frac{\partial M}{\partial y}$ $\quad N_x = \frac{\partial N}{\partial x}$

$$M \mu_y + \mu M_y = N \mu_x + \mu N_x$$

if $\mu(x,y) = \mu(x)$

$\Rightarrow \mu_y = 0$

$$\mu M_y = N \mu_x + \mu N_x$$

$$\frac{1}{N} \frac{\partial M}{\partial y} = \frac{1}{\mu} \frac{\partial \mu}{\partial x} + \frac{1}{N} \frac{\partial N}{\partial x}$$

$$\underbrace{\frac{1}{\mu} \frac{\partial \mu}{\partial x}}_{LHS} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = R(x,y)$$

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LHS:

$$\frac{1}{\mu}\frac{\partial \mu}{\partial x} = \frac{d}{dx}[\ln \mu(x)] = R(x,y)$$ separate & integrate:

$$\int d(\ln \mu(x)) = \int R(x,y)\,dx$$ نعتبر $c=0$

$$\ln \mu(x) = \int R(x,y)\,dx$$

$y(x)$ ⇐ separation بدي

if $R(x,y) = R(x)$ fun of x only

$$\ln \mu(x) = \int R(x)\,dx$$

$$\mu(x) = e^{\int R(x)\,dx}$$

$$M\mu_y + \mu M_y = N\mu_x + \mu N_x$$

if $\mu(x,y) = \mu(y) \Rightarrow \mu_x = 0$

$$M\mu_y + \mu M_y = \mu N_x$$

$$\underbrace{\frac{1}{\mu}\frac{d\mu}{dy}}_{[\ln \mu(y)]'} = \frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = R^*(x,y)$$

$$[\ln \mu(y)]' = R^*(x,y)$$

if $R^*(x,y) = R^*(y)$

$$\mu(y) = e^{\int R^*(y)\, dy}$$

$$R(x,y) = \frac{1}{N}(M_y - N_x)$$

$$R^*(x,y) = \frac{-1}{M}(M_y - N_x)$$

$$\mu(x) = e^{\int R(x)\, dx}$$

Rem:

$$M(x,y)\, dx + N(x,y)\, dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{Error}$$

Ex: $(2y^2+3x)\,dx + (2xy)\,dy = 0$ ---- (*)

$$\frac{dy}{dx} = \frac{-(2y^2+3x)}{2xy}$$ ← الأصل → Non linear

$$M = \frac{\partial F}{\partial x} = 2y^2+3x \Rightarrow \frac{\partial M}{\partial y} = 4y$$

$$N = \frac{\partial F}{\partial y} = 2xy \Rightarrow \frac{\partial N}{\partial x} = 2y$$

The condition:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

but here it isn't so Not Exa

* we must calculat $R(x,y)$ or $R^*(x,y)$ ⇒ إذا نضرب نستطيع نحولها Exact

$$R(x,y) = \frac{1}{2xy}(4y - 2y) = \frac{1}{\cancel{2}x\cancel{y}}\, \cancel{2y} \quad \text{function of } x$$

$$R(x) = \frac{1}{x}$$

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x = \mu$$

Multiply (*) by $x \to \mu$

$$[(2y^2 + 3x)\,dx + (2xy)\,dy = 0] \times$$

$$(2y^2x + 3x^2)\,dx + (2x^2y)\,dy = 0$$

Now.. it is exact or not?

$$M = 2y^2x + 3x^2 = \frac{\partial M}{\partial y} = 4yx$$
$$N = 2x^2y = \frac{\partial N}{\partial x} = 4xy$$
exact!

$$\frac{\partial F}{\partial x} = 2y^2x + 3x^2$$ integrate it:

$$F = \int (2y^2x + 3x^2)\,dx + c_1 \Rightarrow F = y^2x^2 + x^3 + \underbrace{C_1}_{\to 0}$$

$$\frac{\partial F}{\partial y} = 2x^2y$$ integrate it:

$$F = \int 2x^2y\,dy + c_2 = x^2y^2 + \underbrace{C_2}_{\to x^3}$$

$$F = x^2 + y^2 + x^3$$

to find C_2:

$$F(x) = x^2 y^2 + C_2(x)$$
$$\frac{\partial F}{\partial x} = 2xy^2 + C_2'(x)$$
$$\cancel{2y^2x} + 3x^2 = \cancel{2xy^2} + C_2'(x)$$
$$3x^2 = \frac{dC_2}{dx} \Rightarrow \boxed{C_2 = x^3}$$

Subject Bernoulli ODE   Date   No.

$$1\frac{dy}{dx} + p(x)\,y = g(x)\,y^n \qquad \text{standard form}$$

$, n \neq 0, 1 \rightarrow$ linear eqs

Devide it by $\underline{y^n}$

$$\Rightarrow \frac{dy}{dx}\,y^{-n}(x) + P(x)\,y^{1-n}(x) = g(x)$$

let $v(x) = y^{1-n}(x)$

$$v'(x) = (1-n)\,y^{-n}\,\frac{dy}{dx}$$

$$y^{-n}(x)\,\frac{dy}{dx} = \frac{v'(x)}{1-n}$$

$$\frac{v'(x)}{1-n} + P(x)\,v = g(x)$$

linear in $v(x)$, solve it by the integrating factor

Ex: $\frac{dy}{dx} + \frac{4}{x}y = x^3\,y^2$, $y(2) = -1$, $x > 0$

$$\frac{dy}{dx}\,y^{-2} + \frac{4}{xy} = x^3 \qquad n = 2$$

$$v(x) = y^{1-n} = y^{-1}$$

$$v'(x) = -y^{-2}\,\frac{dy}{dx}$$

$$-v' + \frac{4v}{x} = x^3$$

$$v' - \frac{4v}{x} = -x^3$$

Subject Date No.

$P(x) = -\frac{4}{x}$ $\qquad g(x) = -x^3$

$\mu = e^{\int -\frac{4}{x} dx} = e^{-4 \ln x} = e^{\ln x^{-4}} = x^{-4} = \mu(x)$

$\frac{1}{x^4}\left(\frac{dv}{dx} - \frac{4v}{x} = -x^3\right) = \frac{dv}{dx} x^{-4} - \frac{4v}{x^5} = -\frac{1}{x}$

$(v x^{-4})' = -\frac{1}{x}$

$\Rightarrow \int d(v x^{-4}) = \int -\frac{1}{x} dx + c$

$v x^{-4} = -\ln x + c$

$v = -\ln x \, x^4 + c x^4$

$\frac{1}{y} = -\ln x \, x^4 + c x^4 \Rightarrow y = \frac{1}{c x^4 - \ln x \, x^4}$

$\Rightarrow y = -1$ at $x = 2$

$-1 = -\ln 2 \; 16 + 16c$

$\frac{\ln 2 \; 16 - 1}{16} = c$

Subject: linear 2^{nd} ODE — Date: 26. Feb. 2019 No.

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = g(x)$$

if $g(x) = 0$ → homogeneous 2nd ODE

$\neq 0$ → Nonhomogeneous 2^{nd} ODE

coefficients → $p(x)$, $q(x)$, $r(x)$ → a, b & c → if they are constant

$$ay'' + by' + cy = g(x)$$

homogeneous $g(x) = 0$

$$ay'' + by' + cy = 0$$

Ex8 Determine some solution to the following:

$$y'' - 9y = 0$$

$y(x), y'(x), y''(x)$

$$y'' = 9y$$

* يعني لازم خذ اقتران لما نشتقه مرتين ينتج لدينا $9y$

* $y_1 = e^{3x}$

$y' = 3e^{3x}$

$y'' = 9\underbrace{e^{3x}}_{y} \Rightarrow y'' = 9y$

* $y_2 = e^{-3x}$

$y_2' = -3e^{-3x}$

$y_2'' = 9\underbrace{e^{-3x}}_{y} \Rightarrow y_2'' = 9y$

* $y_4 = e^{3x} - e^{-3x}$

it's also a solution

$y_1, y_2 \Rightarrow$ مع الثوابت

$y_3 = y_1 + y_2 \Rightarrow e^{3x} + e^{-3x}$

Any linear combination of y_1 & y_2 is a solution

$y_3' = 3e^{3x} - 3e^{-3x}$

$y_3'' = 9e^{3x} + 9e^{-3x} \Rightarrow 9\underbrace{(e^{3x} + e^{-3x})}_{y} = 9y$

$y'' - 9y = 0$

$y = e^{rx}$ r is a parameter

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$r^2 e^{rx} - 9e^{rx} = 0$

$e^{rx}(r^2 - 9) = 0$

$r^2 - 9 = 0$

$r_{1,2} = \pm 3$

$\to y_1 = e^{3x}$, $y_2 = e^{-3x}$

$\to$ Basis of solution

$y(x) = c_1 y_1 + c_2 y_2$

$y(x) = c_1 e^{3x} + c_2 e^{-3x}$

c_1 & c_2 are constant

* To find c_1 & c_2 we need to condition of $y(x)$

Rem:

$ay'' + by' + cy = 0$

$y(x_0) = \alpha$

$y'(x_0) = \beta$

x_0

$y(x_0)$

$y'(x_0)$

inetial value problem $\Rightarrow$ على نفس x_0

$y(x_0) = \alpha$ $y(x_1) = \beta$

x_0 x_1 $\Rightarrow$ Boundary value problem

$y'(x_0) \neq$ $y'(x_1)$ القيم مختلفة

نفس المثال السابق مع إضافات

Ex: $y'' - 9y = 0$

$y(x_0 = 0) = 2 \quad , \quad y'(x_0 = 0) = -1$

Sol:

$y = C_1 e^{3x} + C_2 e^{-3x} \rightarrow 1$

$y' = 3C_1 e^{3x} - 3C_2 e^{-3x} \rightarrow 2$

$\Rightarrow y(x_0 = 0) = 2 \quad , \quad y'(x_0 = 0) = -1$

$1 \rightarrow \; 2 = C_1 + C_2$

$2 \rightarrow \; -1 = 3C_1 - 3C_2$

Eliminate C_1 from eq. 1

$C_1 = 2 - C_2$

$3(2 - C_2) - 3C_2 = -1$

$6 - 3C_2 - 3C_2 = -1$

$-6C_2 = -7$

$C_2 = \frac{7}{6}$

or: $6 = 3C_1 + \cancel{3C_2}$

$-1 = 3C_1 - \cancel{3C_2}$

$5 = 6C_1$

$C_1 = 5/6$

$C_2 = 7/6$

$ay'' + by' + cy = 0$

$y = e^{rx}$

$y' = re^{rx}$ → بنعوض فيهم في المعادلة

$y'' = r^2 e^{rx}$

$\Rightarrow ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$

$\underbrace{e^{rx}}_{}(ar^2 + br + c) = 0 \quad \Rightarrow \quad ar^2 + br + c = 0$

لاكلك مجبان

لايساوي صفر

↳ quadratic eq. in r

(polynomial of dgree of 2)

Rem:

polynomial

$P_n(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_1 x + a_0$

$n \Rightarrow$ integer $n > 0$

Rem:

$ax^2 + bx + c = 0$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Delta = \sqrt{b^2 - 4ac}$

if → $\Delta > 0 \rightarrow$ 2 real solution

↳ $\Delta = 0 \rightarrow$ 1 real solution

→ $\Delta < 0 \rightarrow$ 2 complex conjogate solution

Subject Date No.

Rem 8

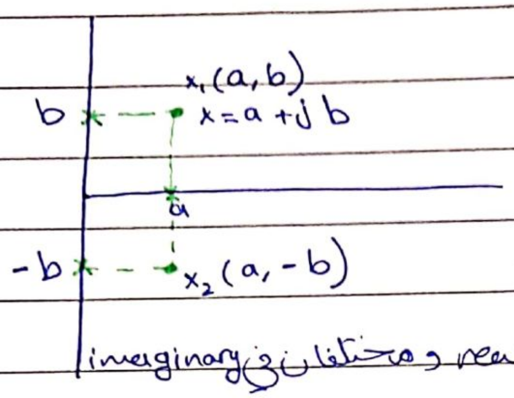
complex number

$* \; \alpha = -2 \pm \sqrt{-5}$

let $j^2 = -1$

$\alpha = -2 \pm \sqrt{(-1)5} = -2 \pm \sqrt{j^2 5} = \underbrace{-2}_{\text{real part}} \pm \underbrace{j\sqrt{5}}_{\text{Imaginary part}}$

$x = a + bj$



$\Rightarrow$ 2 complex conjugate Solution

حلان مشتركان في real part و مختلفان في imaginary

Ex: Solve the following initial value problem:

$y'' + 11y' + 24y = 0 \; , \; y(0) = 0 \; , \; y'(0) = -7$

Sol:

$y = e^{rx} \; , \; y' = re^{rx} \; , \; y'' = r^2 e^{rx}$

$\Rightarrow r^2 e^{rx} + 11 r e^{rx} + 24 e^{rx} = 0 \Rightarrow r^2 + 11r + 24 = 0$

$(r+8)(r+3) = 0$

$r_1 = -8 \qquad r_2 = -3$

$y_1 = e^{-8x} \qquad y_2 = e^{-3x}$

$y(x) = y_1 C_1 + y_2 C_2 \Rightarrow y(x) = e^{-8x} C_1 + e^{-3x} C_2 \rightarrow (1)$

$y'(x) = -8e^{-8x} C_1 - 3e^{-3x} C_2 \rightarrow (2)$

→ (1) $y=0$ at $x=0$

$0 = C_1 + C_2$

→ (2) $y' = -7$ at $x = 0$

$-7 = -8C_1 - 3C_2$

To find C_1, C_2 → Elimination or subsitution $\boxed{A}$

↳ Cramer's rule $\boxed{B}$

$\boxed{A}$ $C_1 = -C_2$

$-7 = 8C_2 - 3C_2$

$-7 = 5C_2 \Rightarrow C_2 = -7/5$

$C_1 = 7/5$

$\boxed{B}$ Rem:

Cramer's rule

$a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

Where: $a_{11}, a_{12}, a_{21}, a_{22}$ → Coefficients

b_1, b_2 → Constants

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$A \cdot x = b$$

$A \to \det(A)$

$\det(A) = a_{11} a_{22} - a_{21} a_{12}$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Back to our Q:

$C_1 + C_2 = 0$

$+3C_1 + 8C_2 = +7$

$a_{11} = 1 \quad a_{12} = 1 \quad a_{21} = 3 \quad a_{22} = 8 \quad b_1 = 0 \quad b_2 = 7$

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 8 \end{bmatrix}, \det(A) = 1(8) - 3 = 5$$

$$C_1 = \frac{\begin{vmatrix} 0 & 1 \\ -7 & -8 \end{vmatrix}}{5} = \frac{7}{5}$$

$$C_2 = \frac{\begin{vmatrix} 1 & 0 \\ -3 & -7 \end{vmatrix}}{5} = \frac{-7}{5}$$

$ay'' + by' + cy = 0$

characteristic eq:

$ar^2 + br + c = 0$

$r_{1,2} \Rightarrow$ roots

Real — complex conjugate

distinct — repeted

Distict

$y_1 = e^{r_1 x}$ $y_2 = e^{r_2 x}$ $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

if $r_{1,2}$ complex conjugate

$r_{1,2} = Re( ) \pm j\,Im(r)$

$r_{1,2} = \underbrace{\lambda}_{\text{Real part}} \pm \underbrace{\mu j}_{\text{Imaginary part}}$

$y_1 = e^{(\lambda + \mu j)x}$ $y_2 = e^{(\lambda - \mu j)x}$

Rem

Euler's formula

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$y_1 = e^{(r+\mu j)x} = e^{rx} \cdot e^{(\mu x) j}$, $\theta = \mu x$

$y_1(x) = e^{rx}\left(\cos(\mu x) + j \sin(\mu x)\right)$

$\mu_2 = e^{(r-\mu j)x} = e^{rx} \; e^{-(\mu x) j}$, $\theta = \mu x$

$y_2 = e^{rx}\left(\cos(\mu x) - j \sin(\mu x)\right)$

* Add y_1 to y_2

$y_1 + y_2 = 2 e^{rx} \cos(\mu x)$

$\frac{y_1}{2} + \frac{y_2}{2} = e^{rx} \cos(\mu x)$ ⇒ General solution

* Substract y_1 & y_2

$y_1 - y_2 = -2 j e^{rx} \sin(\mu x)$

$\underbrace{\frac{-1}{2j}}_{\text{constant } \alpha} y_1 + \underbrace{\frac{1}{2j}}_{\text{constant } b} y_2 = e^{rx} \sin(\mu x)$

$y(x) = c_1 y_1 + c_2 y_2$

$= e^{rx}\left[\underbrace{(c_1 + c_2)}_{A} \cos(\mu x) + \underbrace{j (c_1 - c_2)}_{B} \sin(\mu x)\right]$

$y(x) = e^{rx}\left[A \cos(\mu x) + B \sin(\mu x)\right]$ ⇒ General solution

$y(x) = e^{Re(r) \cdot x}\left[A \cos(Im(r) \cdot x) + B \sin(Im(r) \cdot x)\right]$

$y(x) = A e^{r_1 x} + B e^{r_2 x}$

Ex: Solve the following IVP: $y'' - 4y' + 9y = 0$, $y(0) = 0$, $y'(0) = -8$

Sol:

$ay'' + by' + c = 0$

$y = e^{rx}$

$y' = re^{rx} \Rightarrow r^2e^{rx} - 4re^{rx} + 9e^{rx} = 0 \Rightarrow \underbrace{r^2 - 4r + 9 = 0}_{\text{Characteristic eq.}}$

$y'' = r^2 e^{rx}$

$$r_{1,2} = \frac{4 \pm \sqrt{16-36}}{2} = \frac{2 \pm \sqrt{(-1)20}}{2} = \frac{2 \pm \sqrt{j^2 20}}{2}$$

$Re(r) = +2$

$\Rightarrow r_{1,2} = 2 \pm j\frac{\sqrt{20}}{2}$

$Im(r) = \sqrt{5}$

$\Rightarrow y_1 = e^{(\lambda + \mu j)x} = e^{\left(2 + j\frac{\sqrt{20}}{2}\right)x} = e^{(2 + j\sqrt{5})x}$

$\Rightarrow y_2 = e^{(\lambda - \mu j)x} = e^{\left(2 - j\frac{\sqrt{20}}{2}\right)x} = e^{(2 - j\sqrt{5})x}$

$y(x) = e^{2x}\left(A\cos(\sqrt{5}x) + B\sin(\sqrt{5}x)\right)$

$\odot$ $A = ?!$ $B = ?!$

$y(x) = Ae^{2x}\cos(\sqrt{5}x) + e^{2x}B\sin(\sqrt{5}x)$

$y'(x) = -Ae^{2x}\sqrt{5}\sin(\sqrt{5}x) + 2Ae^{2x}\cos(\sqrt{5}x) + B\sqrt{5}e^{2x}\cos(\sqrt{5}x) + 2Be^{2x}\sin(\sqrt{5}x)$

$*$ y at $x=0$ $= 0$

$0 = A + 0 \Rightarrow A = 0$

$y(x) = e^{2x}\left(\frac{-8}{\sqrt{5}}\sin(\sqrt{5}x)\right)$ → The solution

$*$ y' at $x = 0$ $= -8$

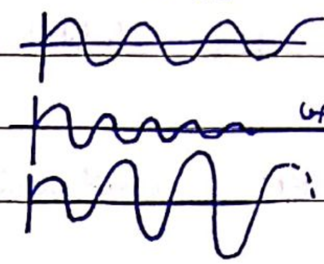
$-8 = 2A + \sqrt{5}B \Rightarrow B = \frac{-8}{\sqrt{5}}$

$y = e^{rx}\left[A\cos(\mathrm{Im}(r)x) + B\sin(\mathrm{Im}(r)x)\right]$

if the real part $\rightarrow = 0 \rightarrow$ ---> ∞ Stable

$\hookrightarrow$ -ive $\rightarrow$ تلاشى

$\hookrightarrow$ +ive $\rightarrow$: explode



* Real & repeated roots of the charastaristic eq:

$ar^2 + br + c = 0$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Delta = b^2 - 4ac = 0$

$r_{1,2} = -\frac{b}{2a}$ real & repeated

$y_1(x) = e^{-\left(\frac{b}{2a}\right)x}$ but it has a second solution

To find it: Multibly by α

$y_2 = \alpha y_1(x) = \alpha e^{-\left(\frac{b}{2a}\right)x} = \alpha e^{-rx}$

$y_2 = \alpha e^{-rx}$

$y_2' = -r\alpha e^{-rx}$ $\Rightarrow a r^2 \alpha e^{-rx} - r\alpha b e^{-rx} + \alpha e^{-rx} c = 0$

$y_2'' = r^2 \alpha e^{-rx}$ $\quad \alpha e^{-rx}(\underbrace{ar^2 - br + c}_{\text{Charastarictic}}) = 0$

if we know a solution, we could use it to find another solution by Multiblying 1^{st} solution by a fun. of x.

↳ Reduction of order method

$y_1 = e^{-\left(\frac{b}{2a}\right)x} = e^{rx}$

$y_2 = y_1(x)\ \underline{V(x)}$

↳ Another fun, we need to determain the form of $V(x)$.

$ay'' + by' + cy = 0$

$y_2' = y_1(x)\, V'(x) + V(x)\, y_1'(x)$

$y_2'' = y_1(x)\, V''(x) + V'(x) y_1'(x) + V(x)\, y_1''(x) + y_1'(x)\, V'(x)$

$= V'' e^{rx} + 2V'(x)\, r e^{rx} + V r^2 e^{rx}$

$\Rightarrow a\left(V'' e^{rx} + 2V'(x) r e^{rx} + V r^2 e^{rx}\right) + b\left(V' e^{rx} + V r e^{rx}\right) + c\left(V e^{rx}\right) = 0$

$e^{rx}\left(aV'' + \underbrace{(2ar+b)}_{\text{must}=0} V' + \underbrace{(ar^2+br+c)}_{\text{must}=0} V\right) = 0$

must $= 0$ ⇒ characteristic eq. $= 0$

$\Downarrow$

$r_{1,2} = r = -\frac{b}{2a} \Rightarrow 2ra + b = 0$

$\Rightarrow a e^{rx} V'' \neq 0$

$a, e^{rx} \neq 0$ so $V''(x) = 0$

$\frac{d}{dx} V'(x) = 0 = C_1$

$V(x) = C_1 x + C_2$

Subject Date No.

$y_2(x) = y_1(x)\ V(x)$

$y_2(x) = e^{rx}(C_1 x + C_2)$

$y_1 = e^{rx}$

$y(x) = A y_1(x) + B y_2(x)$

$= A e^{rx} + B e^{rx}(C_1 x + C_2)$

$= A e^{rx} + B e^{rx} C_1 + B e^{rx} C_2$

$= \underbrace{(A + C_2 B)}_{D} e^{rx} + \underbrace{(BC_1)}_{E} x e^{rx}$

$y(x) = D e^{rx} + E x e^{rx} = e^{rx}(D + E x) = y(x)$

General sol

Ex8

r_1, r_2 8

1) real & dis

$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

2) Conjogate & complex

$y(x) = e^{Re(r)} [C_1 \cos(Im(r) \cdot x) + C_2 \sin((Im(r)) \cdot x)]$

3) repeated

$y(x) = (C_1 + C_2 x) e^{rx}$

Ex8 IVP $y'' - 4y' + 4y = 0$, $y(x=0) = 12$, $y'(x=0) = -3$

Sol:

$ay'' + by' + cy = 0$

$r^2 e^{rx} - 4re^{rx} + 4e^{rx} = 0$

$r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$r_1 = 2 \Rightarrow y_1 = e^{2x}$

$y_2 = e^{2x}(c_1 x + c_2) \Rightarrow 12 = c_2$

$y_2' = e^{2x} c_1 + 2e^{2x}(c_1 x + c_2) \Rightarrow -3 = c_1 + 24 \Rightarrow c_1 = -27$

$$y(x) = e^{2x}(12 - 27x)$$

Subject Reduction of order method Date No.

$F(x, y, y', y'') = 0$ 2^{nd}, linear or nonlinear

* can be reduced to 1^{st} order diff eqn if:
* x (independent variable) doesn't appear in eqn
* y (dependent variable) doesn't appear in eqn
* if 2^{nd} order diff eqn is homogenous we can use it to fin another solution y_1 is known

Ex: $x y'' - 2y' = 0$ ⇒ y doesn't appear

Sol:

let $z = y'$

$z' = y''$

$x z' - 2z = 0$

$x \frac{dz}{dx} = 2z$

$\int \frac{dz}{z} = \int \frac{2}{x} dx + C = \ln^{z} = 2\ln^{x} + C$

$z = e^{\ln^{x^2} + C_1}$

$z = \frac{dy}{dx}$ ← $z = C_2 x^2$

or

$\ln^{z} - \ln^{x^2} = C_1$

$\ln \frac{z}{x^2} = C_1$

$z = x^2 C_2$

$\frac{dy}{dx} = C_2 x^2$ ⇒ $\int dy = \int C_2 x^2 dx + C_3$

$y = C_4 x^3 + C_3$

Subject Date No.

Ex: $y'' - 4y = 0$

$y = e^{rx}$ $y' = re^{rx}$ $y'' = r^2 e^{rx}$

$r^2 e^{rx} - 4e^{rx} = 0$

$r^2 - 4 = 0$

$r = \pm 2$

$y(x) = c_1 e^{2x} + c_2 e^{-2x}$

حسب الدرس السابقة

$z = y' = \frac{dy}{dx}$

$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$z' \neq y''$

$y'' = \frac{d}{dx} z = \frac{dz}{dx} \cdot \frac{dy}{dy} = \frac{dz}{dy} \underbrace{\frac{dy}{dx}}_{y'}$

$= y' \frac{dz}{dy} = y''$

$\underbrace{y'}_{z} \frac{dz}{dy} - 4y = 0$

$z \frac{dz}{dy} - 4y = 0$

$\int z\,dz = \int 4y\,dy + c_1 \Rightarrow \frac{z^2}{2} = 2y^2 + c_1$

$z = \pm\sqrt{4y^2 + 2c_1}$

let $c_1 = 0$

$z = \pm 2y$

$\frac{dy}{dx} = 2y$ $\frac{dy}{dx} = -2y$

Subject Date No.

$$\frac{dy}{dx} = 2y \Rightarrow \int \frac{dy}{y} = \int 2dx + C$$

$$\ln y = 2x + C$$

$$\underbrace{y_1 = e^{2x}}$$

$$\frac{dy}{dx} = -2y \Rightarrow \int \frac{dy}{y} = \int -2dx + C$$

$$\underbrace{y_2 = e^{-2x}}$$

Ex: $2x^2y'' + xy' - 3y = 0$

Given that $y_1 = \frac{1}{x}$ is a solution, find y_2?

$y_1 = \frac{1}{x}$ $\quad y_1' = -\frac{1}{x^2}$ $\quad y_2' = \frac{2}{x^3}$

$2\cancel{x^2}\frac{2}{x^{\cancel{3}}} + \cancel{x}\frac{1}{x^{\cancel{2}}} - \frac{3}{x} = \frac{4}{x} - \frac{1}{x} - \frac{3}{x} = 0$

it's a solution

$y_2 = v(x)\, y_1 = \frac{v}{x} = y_2$

$y_2' = \frac{-v}{x^2} + \frac{v'}{x}$

$y_2'' = \frac{+2v}{x^3} - \frac{v'}{x^2} - \frac{v'}{x^2} + \frac{v''}{x}$

$y_2'' = \frac{2v}{x^3} - \frac{2v'}{x^2} + \frac{v''}{x}$

$2x^2\left(\frac{2v}{x^3} - \frac{2v'}{x^2} + \frac{v''}{x}\right) + x\left(\frac{v'}{x} - \frac{v}{x^2}\right) - \frac{3v}{x} = 0$

$\frac{4v}{\cancel{x}} - 4v' + 2xv'' + v' - \frac{v}{\cancel{x}} - \frac{3v}{\cancel{x}} = 0$

$2xv'' - 3v' = 0$, let $z = v'$ $\quad z' = v''$

$2xz' - 3z = 0$

$2x\frac{dz}{dx} = 3z \Rightarrow \int\frac{dz}{z} = \int\frac{3}{2}\frac{dx}{x} + c$

Subject Date No.

$\ln^z = \frac{3}{2} \ln^x + c$

$\ln\left(\frac{z}{x^{\frac{3}{2}}}\right) = c$

$z = c_1 x^{\frac{3}{2}}$, $z = v'$

$\int dv = \int c_1 x^{\frac{3}{2}} dx + c_2$

$v = \frac{2c_1}{5} x^{\frac{5}{2}} + c_2$

let $c_1 = \frac{5}{2}$ & $c_2 = 0 \Rightarrow v \Rightarrow$ function of x

إذا لازم نتخلص من الثوابت

$v = x^{\frac{5}{2}}$

$y_2 = x^{\frac{5}{2}} x^{-1} = x^{\frac{3}{2}}$

$y(x) = c_1 x^{\frac{3}{2}} + c_2 x^{-1}$

Subject Euler - Cauchy Equation Date No.

it's form: $\underline{x^2 y''} + \underline{a x y'} + \underline{b y} = 0$

$$x^2 y'' + a x y' + b y = 0$$

$y(x) \rightarrow$ unknown function of x

let $y(x) = x^m$ a solution

$$y'(x) = m x^{m-1}$$

$$y''(x) = (m^2 - m) x^{m-2}$$

$$x^2 (m^2 - m) x^{m-2} + a x m x^{m-1} + b x^m = 0$$

$$(m^2 - m) x^m + a x^m m + b x^m = 0$$

$$x^m (m^2 - m + am + b) = 0$$

$$\underline{x^m} (m^2 + (a-1) m + b) = 0$$

Characteristic eqn

<u>if</u> $y = x^m$ a solution $\longleftrightarrow$ $m^2 + (a-1) m + b = 0$

m is a root

$m_{1,2}$ → Real & distinct

$m_{1,2}$ → Real & repeated

$m_{1,2}$ → complex conjugate

$$m_{1,2} = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}$$

Note: $\left.\begin{array}{l} (a-1)^2 = a^2 - 2a + 1 \\ (1-a)^2 = 1 - 2a + a^2 \end{array}\right)$ same

$$m_{1,2} = \frac{(1-a)}{2} \pm \frac{\sqrt{(1-a)^2 - 4b}}{2}$$

$$m_{1,2} = \frac{1}{2}(1-a) \pm \sqrt{\frac{1}{4}(1-a)^2 - b}$$

$$\Delta = \frac{1}{4}(1-a)^2 - b$$

if Δ $\rightarrow > 0$ Real and distinct

$\rightarrow < 0$ complex conjugate

$\rightarrow = 0$ Real & repeated

Case 1

$m_{1,2}$ real & distinct

$$y_1 = x^{m_1}$$

$$y_2 = x^{m_2}$$

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

Ex: $x^2 y'' + \frac{3}{2} x y' - 0.5\, y = 0$

Sol: $x^2 y'' + a x y' + b y = 0$

$a = \frac{3}{2}$ $\quad$ $b = \frac{-1}{2}$

$$m^2 + \frac{m}{2} - \frac{1}{2} = 0$$

A^2 $\quad 1 \quad \frac{-1}{2}$

$$\frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{8}{4}}}{2} = -\frac{1}{4} \pm \frac{3}{4}$$

$$m_1 = \frac{-1}{4} + \frac{3}{4} = \frac{1}{2} \Rightarrow y_1 = x^{\frac{1}{2}}$$

$$m_2 = \frac{-1}{4} - \frac{3}{4} = -1 \Rightarrow y_2 = x^{-1}$$

$$y(x) = c_1 x^{\frac{1}{2}} + c_2 x^{-1}$$

Ex: $x^2 y'' + 0.6 x y' + 16.04 y = 0$

$\left(\frac{4}{10}\right)^2 = \frac{16}{100} -$

$a = 0.6 \qquad b = 16.04$

$$m^2 - 0.4m + 16.04 = 0$$

$(4\, 10^{-1})^2 - (4$

$$m_{1,2} = \frac{0.4 \pm \sqrt{(0.4)^2 - (4 \;\, 16.04)}}{2}$$

$$= 0.2 \pm \sqrt{\frac{-400}{25}} = 0.2 \pm 4j$$

$Re(m) = 0.2 \qquad Im(m) = 4$

$$y = x^{0.2} \; x^{4j}$$

$$= x^{0.2} \, e^{\ln x^{4j}} = x^{0.2} \, e^{4j \ln x}$$

$$= x^{0.2} \, e^{(4 \ln x) j}$$

x

O T E B O O

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Euler's eq:

$$e^{\pm \theta j} = \cos\theta \pm j\sin\theta$$

$$x^{m_1} = y_1 = x^{0.2}\left[\cos(4\ln x) + j\sin(4\ln x)\right]$$

$$x^{m_2} = y_2 = x^{0.2}\left[\cos(4\ln x) - j\sin(4\ln x)\right]$$

$$x^{m_1} + x^{m_2} = 2x^{0.2}\cos(4\ln x)$$

$$x^{m_1} - x^{m_2} = 2j\,x^{0.2}\sin(4\ln x)$$

$$y = x^{0.2}\left[A\cos(4\ln x) + B\sin(4x)\right]$$

$$y = x^{Re(m)}\left[A\cos(Im(m)\ln x) + B\sin(Im(m)\ln x)\right]$$

double roots (Real & repeated)

$\Delta = \frac{1}{4}(1-a)^2 - b = 0$

$b = \frac{(1-a)^2}{4}$

$m_1, m_2 = \frac{1}{2}(1-a)$

$y_1 = y_2 = x^{\frac{1}{2}(1-a)}$

we need to find $y_2(x) = ?!$ by reduction of order method

$y_2 = y_1 u(x)$

$y_1 = x^{\frac{1-a}{2}}$

For $y'' + P(x)y' + q(x)y = 0$

$y_2' = y_1' u(x) + y_1 u'(x)$

$y_2'' = y_1' u'(x) + y_1'' u(x) + y_1 u''(x) + y_1' u'(x)$, with $y_1' u'(x) + y_1' u'(x) = 2y_1' u'(x)$

نعوضهم في المعادلة الأصلية

$\Rightarrow u'' + u'\left(\frac{2y_1' + Py_1}{y_1}\right) = 0$

let $v = u'$

$v' = u''$

$\Rightarrow v' + v\left(\frac{2y_1' + Py_1}{y_1}\right) = 0$ separate & integrate

$\ln v = \ln \frac{1}{y_1^2} - \int P(x)\,dx$

$$\Rightarrow V = e^{\ln \frac{1}{y_1^2}} \cdot e^{-\int P(x)\,dx} = \frac{1}{y_1^2} e^{-\int P(x)\,dx}$$

$$\Rightarrow u = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx} \, dx$$

$y_2 = y_1 u$ so we find y_2

Euler - Cauhy

$$\frac{x^2 y'' + a x y' + b y = 0}{x^2} \Rightarrow y'' + \underbrace{\frac{a}{x}}_{P(x)} y' + \underbrace{\frac{b}{x^2}}_{q_n(x)} y = 0$$

$$\Delta = \frac{1}{4}(1-a)^2 - b$$

$$b = \frac{1}{4}(1-a)^2 \quad , \quad q_n(x) = \frac{(1-a)^2}{4x^2}$$

$$y_1 = x^{\frac{1-a}{2}}$$

$$u = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx} \, dx$$

$$\int \frac{1}{x^{1-a}} \cdot e^{-\int \frac{a}{x} dx} \cdot dx = \int x^{a-1} e^{\ln x^{-a}} dx$$

$$= \int x^{a-1} x^{-a} = \int \frac{1}{x} dx = \underbrace{\ln x}_{u}$$

$$y_2 = y_1 u = x^{\frac{1-a}{2}} \cdot \ln x$$

Summary:

$$x^2 y'' + a x y' + b y = 0$$

$$y'' + \underbrace{\frac{a}{x}}_{p(x)} y' + \underbrace{\frac{b y}{x^2}}_{q(x)} = 0$$

$$y_2 = y_1 \underline{\underline{u}}$$

$$u = \int \frac{1}{y_1^2} e^{-\int p(x)\,dx} \cdot dx$$

* Non homogenerous 2^{nd} order linear diff eq

$$y'' + p(x)\, y' + q(x) \cdot y = r(x)$$

$\hookrightarrow = 0$ homogeneous

Thrm:

if $\bar{Y}_1(x), \bar{Y}_2(x)$ are a solutions to the non homogeneous and if $y_1(x)$ & $y_2(x)$ fundamental set of solutions (Basis) for the homogeneous eq then $\bar{Y}_1(x) - \bar{Y}_2(x)$ solution of the homogeneous

$$\bar{Y}_1(x) - \bar{Y}_2(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$\bar{Y}_1'' + P(x)\bar{Y}_1' + q(x)\bar{Y}_1 = r(x)$$

$\ominus$

$$\bar{Y}_2'' + P(x)Y_2' + q(x)\bar{Y}_2 = r(x)$$

$$\bar{Y}_1'' - Y_2'' + P(x)(\bar{Y}_1' - \bar{Y}_2) + q(x)(\bar{Y}_1 - \bar{Y}_2) = 0$$

$$\Big(\underbrace{\bar{Y}_1'' + P(x)\bar{Y}_1' + q(x)\bar{Y}_1}_{r(x)}\Big) - \Big(\underbrace{Y_2'' + P(x)\bar{Y}_2 + q(x)\bar{Y}_2}_{r(x)}\Big) = 0$$

so

$$\bar{Y}_1(x) - \bar{Y}_2(x) = c_1 y_1(x) + c_2 y_2(x)$$

let $y(t) = \bar{Y}_1(x)$ General solution of non

$y_p(t) = \bar{Y}_2(x)$ particular solution to the non

$$y(t) - y_p(t) = c_1 y_1(x) + c_2 y_2(x)$$

$$y(t) = y_p(t) + \underbrace{c_1 y_1(x) + c_2 y_2(x)}_{\text{Homo. sol}}$$ General sol

$$y(t) = y_p(\ ) + y_c(\ )$$

y_p → No constants

$(\cancel{c_1}, \cancel{c_2})$

Ex: $y'' - 4y' - 12y = \underbrace{3e^{5x}}_{r(x)}$

$y'' - 4y' - 12y = 0$ Homogeneous

$y = e^{rx}$ $y' = re^{rx}$ $y'' = r^2 e^{rx}$

characteristic eq. $\to r^2 - 4r - 12 = 0$

$(r-6)(r+2) = 0$

$r_1 = 6$ $r_2 = -2$

$y_1 = e^{6x}$ $y_2 = e^{-2x}$

$y_c(x) = c_1 e^{6x} + c_2 e^{-2x}$ complementary solution

Now $\Rightarrow$ particular solution

$r(x) = 3e^{5x}$

$y_p = Ae^{5x}$ our guess $\to$ $25Ae^{5x} - 20Ae^{5x} - 12Ae^{5x} = 3e^{5x}$

$\cancel{e^{5x}}(25A - 20A - 12A) = 3\cancel{e^{5x}}$

$y_p' = 5Ae^{5x}$

$A = \frac{-3}{7}$

$y_p'' = 25Ae^{5x}$

So

$y_p = \frac{-3}{7} e^{5x}$

$y(x) = c_1 e^{6x} + c_2 e^{-2x} - \frac{3}{7} e^{5x}$

$\to$ Method of undetermined coefficients if $y(x=0) = 18/7$ & $y'(x=0) = -\frac{1}{7}$ find c_1, c_2?

$$y(x) = c_1 e^{-2x} + c_2 e^{6x} - \frac{3}{7} e^{5x}$$

$$y'(x) = -2c_1 e^{-2x} + 6c_2 e^{6x} - \frac{15}{7} e^{5x}$$

$\boxed{1}$ at $x=0$ $\quad y = \frac{18}{7}$

$$\left(c_1 + c_2 - \frac{3}{7} = \frac{18}{7}\right) \times 7$$

$$7c_1 + 7c_2 - 3 = 18$$

$$c_1 + c_2 = 3 \quad - \boxed{1}$$

$\boxed{2}$ at $x=0$ $\quad y' = -\frac{1}{7}$

$$\left(-2c_1 + 6c_2 - \frac{15}{7} = \frac{-1}{7}\right)7 \Rightarrow -14c_1 + 42c_2 - 15 = -1$$

$$\frac{42c_2 - 14c_1 = 14}{14}$$

$$\Rightarrow 3c_2 - c_1 = 1 \quad - \boxed{2}$$

$\boxed{1} + \boxed{2}$

$$\cancel{c_1} + c_2 = 3$$

$$3c_2 \cancel{- c_1} = 1$$

$$c_2 = 1 \qquad c_1 = 2$$

Ex: $y'' - 4y' - 12y = \sin(2x)$

$y(x) = y_c + y_p$

$y_c = c_1 e^{-2x} + c_2 e^{6x}$

$\Rightarrow y_p = ?!$

RHS:

$r(x) = \sin(2x)$ let $y_p = A\sin(2x)$, $y_p' = 2A\cos(2x)$ & $y_p'' = -4A\sin(2x)$

$\Rightarrow -4A\sin(2x) - 8A\cos(2x) - 12A\sin(2x) = \sin(2x)$

$-16A = 1 \Rightarrow A = \frac{-1}{16}$

$-8A = 0 \Rightarrow A = 0$

يجب أن تكون لـ A قيمة واحدة إذا y_p خاطئة

let $y_p = A\cos(2x) + B\sin(2x)$

$y_p' = -2A\sin(2x) + 2B\cos(2x)$

$y_p'' = -4A\cos(2x) - 4B\sin(2x)$

$\Rightarrow -4A\cos(2x) - 4B\sin(2x) + 8A\sin(2x) - 8B\cos(2x) - 12A\cos(2x) - 12B\sin(2x) = \sin(2x)$

$\sin(2x)(-4B + 8A - 12B) = \sin(2x)$

$8A - 16B = 1$

$-4A\cos(2x) - 8B\cos(2x) - 12A\cos(2x) = 0$

$-4A - 8B - 12A = 0$

$-16A - 8B = 0 \Rightarrow -2A = B$

$\Rightarrow 8A + 32A = 1 \Rightarrow A = \frac{1}{40}$ / $B = \frac{-1}{20}$

$$y_p = \frac{\cos(2x)}{40} - \frac{\sin(2x)}{20}$$

$$y(x) = c_1 e^{-2x} + c_2 e^{6x} + \frac{\cos(2x)}{40} - \frac{\sin(2x)}{20}$$

Ex: $y'' - 4y' - 12y = 2x^3 - x + 3$

$y(x) = y_c + y_p$

$y_c = c_1 e^{-2x} + c_2 e^{6x}$

$y_p = ?!$

let $y_p = ax^3 + bx^2 + cx + d$

$y_p' = 3ax^2 + 2bx + c$

$y_p'' = 6ax + 2b$

$y_p = \frac{-1}{6}x^3 + \frac{1}{6}x^2 + \frac{1}{9}x - \frac{7}{27}$

$y(x) = c_1 e^{-2x} + c_2 e^{6x} - \frac{1}{6}x^3 + \frac{1}{6}x^2 + \frac{1}{9}x - \frac{7}{27}$

$$\boxed{6ax} + 2b - 12ax^2 \boxed{-8bx} - 4c - 12ax^3 - 12bx^2 \boxed{-12cx} - 12d = 2x^3 - x + 3$$

$$-12ax^3 - 12x^2(a+b) - 2x(-3a+4b+6c) + (2b-4c-12d) = 2x^3 - x + 3$$

1) $-12a = 2 \Rightarrow a = \frac{-1}{6}$

2) $-12\cancel{x}(a+b) = 0$

$-12b = -2 \Rightarrow b = \frac{1}{6}$

3) $6a - 8b - 12c = -1 \Rightarrow \cancel{-1} - \frac{4}{3} - 12c = \cancel{-1}$

$\cancel{-12}c = +\frac{\cancel{4}}{3}\ \frac{1}{3\cancel{12}} \Rightarrow c = \frac{1}{9}$

4) $2b - 4c - 12d = 3 \Rightarrow \frac{1}{\cancel{3}} - \frac{4}{9} - 12d = 3 - \frac{1}{3} + \frac{4}{9} \Rightarrow -12d = \frac{28}{9} \Rightarrow d = \frac{-7}{27}$