



MATHEMATICAL

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$$y = f(x)$$

Dependent $\rightarrow$ indep

$$y = f(x, z)$$

$$\frac{dy}{dx} \text{ or } \frac{dy}{dz}$$

* equation / Algebraic
 \ Differential

$$y' = \frac{2x + y}{x}$$

Dependent $\rightarrow$ indep

* Unknown function of the x

* Order degree

Differential Eqn

ordinary	partial	} مرتبة لاكثر من واحد جزئية
$y(x)$ $y'(x)$ $y''(x)$	$y(x, z)$ y_x y_z y_{xz}	

* linear $\Rightarrow$ ما يكونوا مرتبين بـ (independent var) أو مرفوع لقوة
 * non linear $\Rightarrow$ يكونوا مرتبين أو مرفوع لقوة (dependent var)

} only for independent variable.

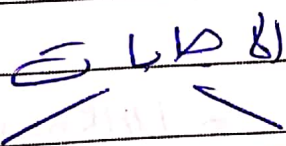
$$\frac{dy}{dx} = 2x + 3 \Rightarrow \text{linear}$$

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^5 + 3 = 0 \Rightarrow \text{non linear}$$

* Taylor - series

$f(x)$, Given x_0

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$$



general

partical

$$y = \sin x + C$$

$$y = \sin x + \frac{1}{x}$$

لا تأت غير معرف

معرف القيمة

المعاد

expect

implicit

$$y = \cos x$$

$$y = \cos x + \cos y$$

	ordinary/partical/linear	order	degree	# of dep	# of nondep
$y'' = 4y + y^3$	ordinam	linear	2	1	
$y' = x + 6y$	"	"	1	1	
$\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = 2y - x^3$	"	non	2	3	
$y'' + 2xy' + 4y = \cos 2x$	"	linear	2	1	1
$\frac{dy}{dx} = \frac{x^2 - 1}{y^2 - y}$	"	"	1	1	
$\frac{du}{dx} + \frac{du}{dy} + \frac{dw}{dz} = 0$	partical	"	1	1	3
$\frac{d^2u}{dx^2} = C^2 \frac{\partial^2 y}{\partial t^2}$	"	"	2	1	2

Degree: then power should be integer

$$\text{Ex: } y'' - 2y' - 15y = 0 \quad \text{is } y = e^{5x}$$
$$25e^{5x} - 10e^{5x} - 15e^{5x} \stackrel{?}{=} 0$$
$$0 = 0 \quad \checkmark$$
$$y' = 5e^{5x}$$
$$y'' = 25e^{5x}$$

and $y = e^{-3x}$ is a solution?

$$\begin{array}{l} y' = -3e^{-3x} \\ y'' = 9e^{-3x} \end{array} \quad \left| \quad \begin{array}{l} 9e^{-3x} + 6e^{-3x} - 15e^{-3x} \stackrel{?}{=} 0 \\ 0 = 0 \end{array} \right.$$

$y_1 = e^{5x} \rightarrow$ a solution
+ $y_2 = e^{-3x} \rightarrow$ another solution

$y_3 = e^{5x} + e^{-3x}$ another solution

$$y'_3 = 5e^{5x} - 3e^{-3x}$$

$$y''_3 = 25e^{5x} + 9e^{-3x}$$

$$\# \quad 25e^{5x} + 9e^{-3x} - 10e^{5x} + 6e^{-3x} - 15e^{5x} - 15e^{-3x} \stackrel{?}{=} 0$$
$$0 = 0 \quad \checkmark$$

$$\text{Ex: } \frac{\partial^2 \omega}{\partial x^2} + 2 \frac{\partial^2 \omega}{\partial x \partial y} = \omega$$

1st. $\omega(x,y) = e^{3x} \cdot \sin 2y$ a solution

$$\frac{\partial \omega}{\partial x} = 3e^{3x} \sin 2y$$

$$\frac{\partial^2 \omega}{\partial x^2} = 9e^{3x} \sin 2y$$

$$\frac{\partial \omega}{\partial y} = 2e^{3x} \cos 2y$$

$$\frac{\partial^2 \omega}{\partial y^2} = 4e^{3x} \cdot -\sin 2y$$

$$9e^{3x} \sin y + 8e^{3x} - 2 \sin 2y$$

$$\text{Ex: } y' = \frac{-x}{y} \quad \text{is } y = (a - x^2)^{1/2}$$
$$y' = \frac{1}{2}(a - x^2)^{-1/2} = \frac{-x}{y} \quad \left\{ \begin{array}{l} (-3 < x < 3) \end{array} \right.$$

y is amount of radio activity material

$$-\frac{dy(t)}{dt} \propto y$$

$$\Rightarrow \frac{dy(t)}{dt} \propto -y(t)$$

$$\frac{dy(t)}{dt} = -ky(t)$$

subject to $y(0) = 0.5$
initial value problem

$$\frac{dy(t)}{dt} = -ky(t)$$

k : constant

$$[y] = [g]$$

$$\left[\frac{dy}{dt}\right] = \left[\frac{g}{s}\right]$$

$$[k] = ? \left[\frac{1}{s}\right] = [s^{-1}]$$

$$y(t=0) = y(0) = 0.5g \quad , \quad k = 1.4 \times 10^{-11} [s^{-1}]$$

$$\int \frac{dy}{y} = \int -k dt$$

$$\ln y = -kt + c$$

{ implicit solution form }

General.

$$y = e^{-kt+c}$$
$$= e^{-kt} \cdot e^c$$

$$y = C_1 e^{-kt}$$

$$\text{when } t=0 : y(0) = 0.5 = C_1(e^0)$$

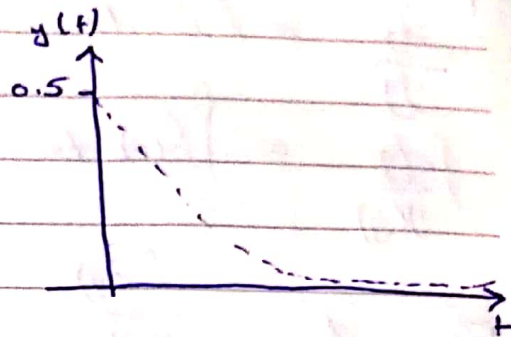
$$C_1 = 0.5$$

$$y(t) = 0.5 e^{-kt}$$

$$y(t) = \begin{cases} 0.5 & t=0 \\ 0 & t \rightarrow \infty \end{cases}$$

$$\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} 0.5 e^{-kt}$$

$$e^{-kt} = \frac{1}{e^{kt}} = \frac{1}{\infty} = 0$$



* 1st order Ordinary differential eqn. (ODE)

$$y' = \frac{dy}{dx} = ky$$

$$\frac{dy}{dx} = f(x,y) = ky : \text{Explicit form}$$

$$\frac{dy}{dx} - ky = 0$$

$$F(x,y,y') = 0 \text{ implicit form}$$

$$k dx = \frac{dy}{y} \rightarrow k dx - \left(\frac{1}{y}\right) dy = 0 \quad \left\{ \text{Differential form} \right\} \text{ (Derivative)}$$

$$M(x,y) dx + N(x,y) dy = 0$$

homogeneous

$$g(x) = 0$$

$$g(x) \neq 0$$

Non Homogeneous

$$* y' = f(x,y)$$

$$f(x,y,y') \neq 0$$

linear

$$a_1(x) \frac{dy}{dx} + a_0(x) y' = g(x)$$

separate Diff eqn

$$\frac{dy}{dx} = f(x,y) = f(x) \cdot g(y) = \frac{f(x)}{g(y)}$$

sep

$$\frac{dy}{g(y)} = f(x) dx$$

$$\int \frac{dy}{g(y)} = \int f(x) dx + C$$

Ex: $\frac{dy}{dx} = \underbrace{f(x)}_{f(x)} \cdot \underbrace{g(y)}_{g(y)} = f(x) \cdot g(y)$

$$\int x dx = \int \frac{1}{y} dy \rightarrow \frac{x^2}{2} = \ln y + C$$
$$\ln y = \frac{x^2}{2} + C_1$$

$$y = e^{x^2/2} \cdot e^{C_1}$$

Ex2: $y' = y + 1$

$$\frac{dy}{dx} = \frac{1}{f(x)} \cdot \frac{g(y)}{g(y)}$$

$$\int \frac{dy}{y+1} = \int dx + C$$

Change of variables

let $z = y + 1$

$$dz = dy$$

$$\int \frac{dy}{y+1} = \frac{dz}{z} = \ln z$$
$$= \ln(y+1)$$

$$\ln(y+1) = x + C$$

$$y+1 = e^{x+C} = e^x \cdot e^C = C_1 e^x$$

$$y = C_1 e^x - 1$$

$$-\infty < x < \infty$$

interval for valid solution

Ex: IVP

$$\frac{dy}{dx} = 6xy^2, \quad y(1) = \frac{1}{25}$$

$$\int \frac{dy}{y^2} = \int 6x dx$$

$$-1 y^{-1} = 6 \cdot \frac{x^2}{2} + C = -\frac{1}{y}$$

$$-\frac{1}{y} = 3x^2 + C$$

$$y = \frac{1}{3x^2 + C}$$

to estimate a $\square$

$$y(1) = \frac{1}{3+C} = \frac{1}{25} \rightarrow C = -28$$

particular soln $y = \frac{1}{3x^2 - 28}$

$$-(3x^2 + 28) \neq 0$$

$$28 - 3x^2 \neq 0$$

$$x^2 = \frac{28}{3}$$

$$x = \pm \sqrt{\frac{28}{3}} = \pm 3.1$$

$$-\infty < x < -3.1$$

$$3.1 < x < \infty$$

$$-\infty \quad -3.1 \quad 0 \quad +3.1 \quad +\infty$$

$$-3.1 < x < 3.1$$

Ex: $\frac{dy}{dx} = \frac{y^2}{x}, \quad y(1) = 2$

$$\int \frac{dy}{y^2} = \int \frac{1}{x} dx + C$$

$$-\frac{1}{y} = \ln x + C$$

$$y = \frac{1}{\ln x + C} = \frac{1}{C - \ln x}$$

to find C_1

$$y(1) = 2 = \frac{1}{C_1 - \ln(1)}$$

$$C_1 = \frac{1}{2}$$

$$y = \frac{1}{0.5 - \ln x}$$

$$\ln x = 2.303 \log x$$

$$\ln(10) = 2.303$$

$$\ln(0.1) = -2.303$$

$$x > 0$$

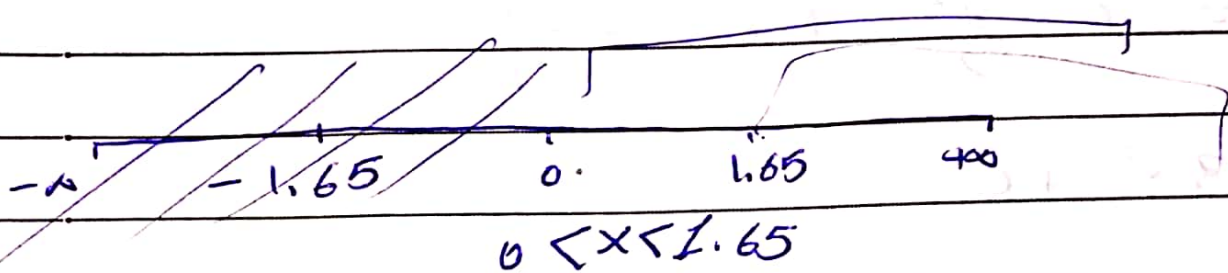
$$0.5 - \ln x \neq 0$$

$$\ln x = 0.5$$

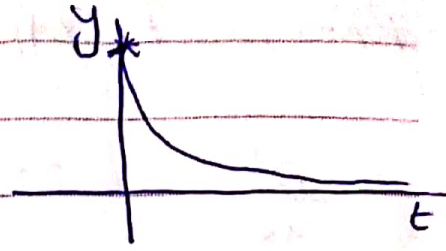
$$x = e^{0.5}$$

$$x^2 = e$$

$$x = \pm \sqrt{e} = \pm \sqrt{2.718} = \pm 1.65$$



Exact Differential eqn :



$$\frac{dy}{dx} = -kx$$

$\Rightarrow y = c e^{-kx}$ general solution, c : constant
solve * for c

$$c = \underbrace{y \cdot e^{kx}}_{f(x,y)}$$

$$\Rightarrow f(x,y) = c$$

$$\left(\frac{df}{dx}\right) + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial x} = 0$$

How to find !!

$$f(x,y) = M(x,y)$$

$$\boxed{\frac{\partial F}{\partial x} dx + \left(\frac{\partial F}{\partial y}\right) dy = 0} \#$$

Diff

$$M(x,y) dx + N(x,y) dy = 0$$

$$\rightarrow -kx dx = dy$$

$$kx dx + dy = 0$$

$$\frac{\partial F}{\partial x} = M(x,y)$$

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y}$$

$$\frac{\partial F}{\partial y} = N(x,y)$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

Rem. $f(x,y) = x^2 + y^3$

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 3y^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0 \quad , \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

* $f(x,y) = yx^2 + y^3x^2$

$$\frac{\partial f}{\partial x} = 2yx + 2xy^3 \quad , \quad \frac{\partial f}{\partial y} = x^2 + 3y^2x^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x + 6xy^2 \quad , \quad \frac{\partial^2 f}{\partial x \partial y} = 2x^2 + 6xy^2$$

which is =

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad : \quad \text{Necessary to sufficient condition for exactness}$$

$\frac{\partial F}{\partial x} = M(x,y)$	$\Rightarrow$ Integrate w.r.t x	we get F $F(x,y) = C$
$\frac{\partial F}{\partial y} = N(x,y)$	$\Rightarrow$ Integrate w.r.t y	

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Ex: in Diff form

N

$$M(2x + y + 1) dx + (2y + x + 1) dy = 0$$

$$\frac{dy}{dx} = - \frac{2x + y + 1}{2y + x + 1}$$

$$M(x, y) = 2x + y + 1 = \frac{\partial F}{\partial x}$$

$$N(x, y) = 2y + x + 1 = \frac{\partial F}{\partial y}$$

Test for exactness

$$\text{is } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial N}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = 1$$

exact Diff Eqn
 $f(x, y) = C$

$$\star \frac{\partial F}{\partial x} = 2x + y + 1 \rightarrow \text{sep int}$$

$$\int dF = \int (2x + y + 1) dx + C_1$$

$$F = x^2 + yx + x + C_1 \quad \text{--- (1)}$$

$$\star \frac{dF}{dy} = 2y + x + 1$$

$$\int dF = \int (2y + x + 1) dy + C_2$$

$$f = y^2 + xy + y + C_2 \quad \text{--- (2)}$$

Compare (1) with (2)

$$F = x^2 + xy + x + C_1$$

$$F = y^2 + xy + y + C_2$$

$$F = x^2 + xy + x + y^2 + y$$

$$F_2 = y^2 + xy + y + x^2 + x$$

$$F(x, y) = x^2 + xy + x + y^2 + y = C$$

implicit general Solution

مقسوم علیه مساوی الی

$$C_1 = y^2 + y \Rightarrow 1$$

$$C_2 = x^2 + x \Rightarrow 2$$

$$C_1 = f(y)$$

$$C_2 = f(x)$$

Solution that we must do

$$y^2 + y(1+x) + x^2 + x - c = 0 \quad \rightarrow \quad 2^{\text{nd}} \text{ degree polynomial in } y;$$

Rem: $ax^2 + bx + c = 0$ a, b, c constant

$$x_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x^2 + x + 1 = 0$$

$$x_{1/2} = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2} = 1$$

$$(x-1)(x-1) = 0$$

if $b^2 - 4ac \geq 0$ repeated
real / distinct
roots
if $b^2 - 4ac < 0$ complex

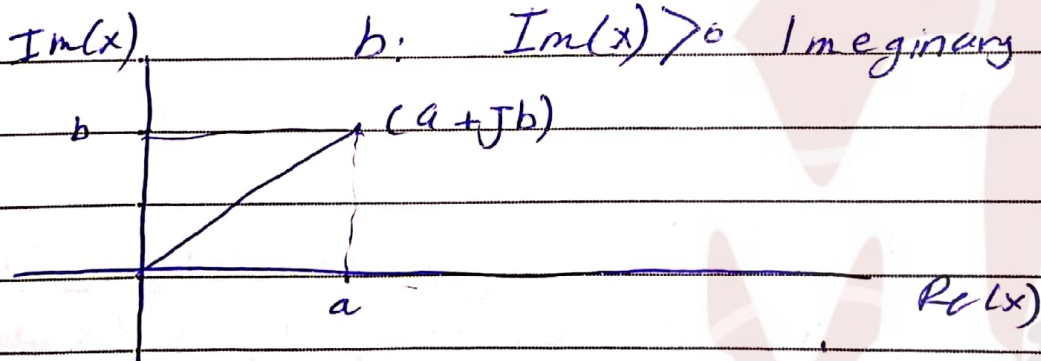
Rem: $x = a$

In general any number x :

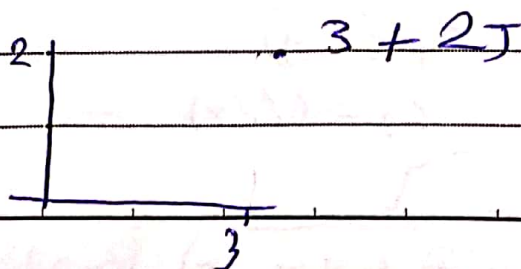
$$x = a + jb$$

a : $\text{Re}(x) > 0$ Real part

b : $\text{Im}(x) > 0$ Imaginary part



$$x = 3 + 2j$$



$$\sqrt{-9} \quad \text{let } j^2 = -1$$

$$\sqrt{9 \cdot j^2} = \pm 3j$$

إذا الرقم الحقة الجبر ← - Complex
 حقة الحمين ← + real

$$\frac{\partial F}{\partial y} = N(x, y) = 2y + x + 1$$

$$\Rightarrow F = x^2 + xy + x + C_1(y)$$

$$\frac{\partial F}{\partial y} = x + C_1'(y) = x + \frac{dC_1}{dy}$$

$$x + \frac{dC_1}{dy} = 2y + x + 1$$

$$\frac{dC_1}{dy} = 2y + 1$$

$$\int dC_1 = \int (2y + 1) dy$$

$$C_1 = y^2 + y$$

$$\Rightarrow F = x^2 + xy + x + y^2 + y$$

Homogenous function: A function of x and y is homogenous of degree n if there exist a constant n such that for every parameter λ : $f(\lambda x, \lambda y) = \lambda^n f(x, y)$

$$\begin{matrix} f(x, y) \\ x \rightarrow \lambda x \\ y \rightarrow \lambda y \end{matrix} \quad f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

Ex:

$$f(x,y) = x^2 + xy$$

$\lambda x, \lambda y$

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^2 + (\lambda x)(\lambda y) \\ &= \lambda^2 x^2 + \lambda^2 xy = \lambda^{(2)} (x^2 + xy) \\ &= \lambda^{(2)} f(x,y) \end{aligned}$$

$\Rightarrow$ degree ②

$$\text{Ex: } f(x,y) = \tan\left(\frac{x}{y}\right) = \frac{\sin(x/y)}{\cos(x/y)}$$

$$\begin{aligned} f(\lambda x, \lambda y) &= \frac{\sin(\lambda x / \lambda y)}{\cos(\lambda x / \lambda y)} = \lambda^0 \tan(x/y) \\ &\quad \{\text{degree zero}\} \end{aligned}$$

A 1st ODE (ordinary differential equation)

$M(x,y)dx + N(x,y)dy = 0$
is homogeneous function if $\left. \begin{matrix} M(x,y) \\ N(x,y) \end{matrix} \right\}$ are homogeneous functions of the same degree

$\Rightarrow$ 1st ODE is separable by changing variable

let $V = \frac{y}{x}$

$\Rightarrow y = xV$

$y(x) = V(x)$

$V(x) = \frac{y(x)}{x}$

$y(x) = xV(x)$

$y'(x) = \frac{dy}{dx} = V(x) + xV'(x)$

$\frac{dy}{dx} = V + x \frac{dV}{dx}$

* $\frac{dy}{dx} = f(x,y) = f(x,V)$
explicit form

$V + x \frac{dV}{dx} = f(x,V) = f(V)$

$V dx + x dV = f(V) dx$

$x dV = [f(V) - V] dx$

separate

$\frac{dV}{f(V)-V} = \frac{dx}{x}$

separable eqn

ex: $xy \frac{dy}{dx} = x^2 - y^2$ Differential form

$$\Rightarrow (x^2 - y^2) dx - (xy) dy = 0$$

$$M(x, y) = x^2 - y^2$$

$$N(x, y) = -xy$$

if M and N homogeneous of the same degree

$$\Rightarrow v = \frac{y}{x} \Rightarrow y = xv$$

$$* M(x, y) = x^2 - y^2 = x^2(1 - v^2) = x^2 M(x, y)$$

$$* N(x, y) = -xy = -x^2 v = x^2 N(x, y) \quad \text{Same degree}$$

$$\frac{dy}{dx} = v + \frac{dv}{dx}$$

$$x(xv) \left[v + \frac{dv}{dx} \right] = x^2 - x^2 v^2$$

$$x \frac{dv}{dx} + v = \frac{x^2(1 - v^2)}{x^2 v} = \frac{1 - v^2}{v}$$

$$v + x \frac{dv}{dx} = \frac{1 - v^2}{v} \rightarrow \text{separable}$$

$$x \frac{dv}{dx} = \frac{1 - v^2}{v} - v$$

sep

(LHS)

$$\frac{1 - v^2}{v} - v = \frac{1 - v^2 - v^2}{v} = \frac{1 - 2v^2}{v}$$

$$LHS = RHS$$

sep

$$x \frac{dv}{dx} = \frac{1 - 2v^2}{v} \Rightarrow \frac{v}{1 - 2v^2} dv = \frac{dx}{x}$$

$$C + \int \frac{v}{1-2v^2} dv = \int \frac{1}{x} dx$$

$$C + \int \frac{-4v}{4(1-2v^2)} dv = \ln|x|$$

$$C + \frac{1}{4} \ln|1-2v^2| = \ln(x)$$

$$-4 \ln x + C_1 = \ln|1-2v^2|$$

$$C_1 = \ln|1-2v^2| + 4 \ln x$$

$$C_1 = \ln(1-2v^2) + 4 \ln x$$

$$x^4(1-2v^2) = e^{C_1} = C_2$$

$$(1-2v^2) = x^{-4} C_2$$

$$1-2\left(\frac{y}{x}\right)^2 = C_2 x^{-4}$$

$$2x^2 y^2 + x^4 + C_2 = 0$$

$$a = 2x^2, \quad b = 0, \quad C = -x^4 + C_2$$

check

quadratic in y

$$\text{Ex: } \frac{dy}{dx} = \frac{x-y}{x+y}$$

$$(x-y) dx - (x+y) dy = 0$$

$$M(x,y) = x-y$$

$$N(x,y) = -(x+y)$$

$$\begin{aligned} M(\lambda x, \lambda y) &= \lambda x - \lambda y = \lambda(x-y) = \lambda' M(x,y) \\ N(\lambda x, \lambda y) &= -(\lambda x + \lambda y) = -\lambda(x+y) = \lambda' N(x,y) \end{aligned} \quad \left. \begin{array}{l} \text{same} \\ \text{degree 1} \end{array} \right\}$$

So that separable

$$u = \frac{y}{x}$$

$$\frac{dy}{dx} = v + \frac{dv}{dx}$$

$$v + x \frac{dv}{dx} = \frac{x - xv}{x + vx} = \frac{x(1-v)}{x(1+v)}$$

$$x \frac{dv}{dx} = \frac{1-2v-v^2}{1+v}$$

$$\text{sep} \int \left[\frac{1+v}{1-2v-v^2} \right] dv = \int \frac{1}{x} dx$$

$$\frac{-1}{2} \ln |1-2v-v^2| = \ln(x) + C$$

$$\ln |1-2v-v^2| + 2 \ln x = -2C = C_1$$

$$\ln |(1-2v-v^2) \cdot x^2| = C_1$$

$$(1-2v-v^2) \cdot x^2 = e^{C_1} = C_2$$

$$\left(1 - 2 \frac{y}{x} - \frac{y^2}{x^2}\right) \cdot x^2 = C_2$$

$$1 - 2yx - y^2 = C_2$$

~~$$\frac{x^2}{y} - 2xy - \frac{x^4}{y^2} = C_2$$~~

$$y^2 + 2yx - 1 + C_2$$

$$a=1, b=2x$$

~~$$\frac{2x^3}{y} + \frac{x^4}{y^2} = x^2 = C_2$$~~

$$C = -1 + C_2$$

$$\log 10 = 1$$

$$\log 100 = 2$$

$$\ln x = 2.303 \log x$$

$$\ln 10 = 2.303$$

$$\ln(0.1) = -2.303$$

$$\# \frac{dy}{dx} = f(ax + by)$$

$$\text{let } u = ax + by$$

$$u' = a + by'$$

$$y' = \frac{u' - a}{b}$$

$$f(ax + by) = f(u)$$

$$\frac{1}{b}(u' - a) = f(u) \quad \text{separable}$$

$$\frac{1}{b} \left(\frac{du}{dx} - a \right) = f(u)$$

$$\frac{1}{b} du = f(u) dx + (a/b) dx$$

$$\frac{1}{b} du = (f(u) + a/b) dx$$

$$\frac{du}{a + b f(u)} = dx$$

Ex: IVP

$$y' - (4x - y + 1)^2 = 0 \quad - y(0) = 2$$

$$ax + by \rightarrow a = 4 \rightarrow b = -1$$

$$u = 4x - y \rightarrow u' = 4 - y' \rightarrow (y' = 4 - u') \text{ in LHS}$$

$$\text{تعويض} \rightarrow 4 - u' - (u + 1)^2 = 0$$

$$\frac{du}{dx} = 4 - (u + 1)^2$$

المعادلة غير خطية

فلا نرمحلها

$$\frac{du}{4 - (u + 1)^2} = dx$$

$$4 - (u + 1)^2$$

$$\rightarrow 4 - (u^2 + 2u + 1) = -u^2 - 2u + 3 = -(u^2 + 2u - 3)$$

$$= -(u - 1)(u + 3)$$

$$\frac{du}{4 - (u + 1)^2} = -\frac{du}{(u - 1)(u + 3)}$$

$$\int \frac{dw}{(w-1)(w+3)} = \int -dx + C$$

ask & receive

④ $\rightarrow 4 - v - (v+1)^2 = 0$

$$\frac{dv}{dx} = 4 - (v+1)^2$$

$$\frac{dv}{4 - (v+1)^2} = dx$$

$$4 \cdot (v+1)^2 = 4 \cdot (v^2 + 2v + 1)$$

$$= -v^2 - 2v - 1 + 4$$

$$= -v^2 - 2v + 3$$

$$= -(v^2 + 2v - 3)$$

$$= -(v-1)(v+3)$$

$$\int \frac{dv}{4 - (v+1)^2} = \int \frac{dv}{-(v-1)(v+3)} = dx$$

$$\int \frac{dv}{(v-1)(v+3)} = \int -dx + C$$

* Integrating factor method for 1st (ODE) linear

$$\left[\frac{dy}{dx} + p(x)y(x) = g(x) \right]_{**} \text{Linear}$$

We multiply Eq(*) by an integrating factor $\mu(x)$ ($f(x)$) that will make (*) an exact diff.

$$\frac{dy}{dx} = y'(x)$$

$$\mu(x) \cdot y'(x) + \underbrace{\mu(x)p(x)}_{\mu'(x)} y(x) = \mu(x) g(x)$$

- The magic of $\mu(x)$ is that satisfied the following.

$$\mu(x) p(x) = \mu'(x).$$

* Summary:-

$$y' + p(x)y = q(x)$$

$$\underbrace{\mu(x)y'}_{\text{LHS}} + \underbrace{\mu(x)p(x)y}_{\text{LHS}} = \underbrace{\mu(x)q(x)}_{\text{RHS}} \quad *$$

$$\mu'(x) = \mu(x)p(x)$$

$$\{(x, y(x), z(x))'\}$$

$$= xz' + xy'z' + yz.$$

LHS:-

$$= \mu(x)y'(x) + \mu'(x)y(x)$$

$$= (\mu(x)y(x))' \text{ chain Rule. } **$$

- Combine * and **

$$(\mu(x)y(x))' = \mu(x)q(x)$$

$$\frac{d}{dx} (\mu(x)y(x)) = \mu(x)q(x)$$

Separate and integrate

$$\int d(\mu(x)y(x)) = \int \mu(x)q(x)dx + C$$

$$\mu(x)y(x) = \int \mu(x)q(x)dx + C.$$

$$y(x) = \frac{\int \mu(x)q(x)dx + C}{\mu(x)}$$

* Condition ($\mu(x)$)

$$\mu(x) p(x) = \mu'(x)$$

$$p(x) = \frac{\mu'(x)}{\mu(x)}$$

$$\frac{\mu'(x)}{\mu(x)} = [\ln \mu(x)]'$$

$$p(x) = [\ln \mu(x)]'$$

$$\frac{d}{dx} [\ln \mu(x)]' = p(x)$$

$$\int d [\ln \mu(x)] = \int p(x) dx \Rightarrow \text{Linear.}$$

$$\ln \mu(x) = \int p(x) dx.$$

$$\Rightarrow \mu(x) = e^{\int p(x) dx} \quad \text{integrating factor.}$$

* Examples:-

$$\frac{dy}{dx} - y = e^{2x} \quad (\text{Linear } y).$$

$$dy - y dx - e^{2x} dx = 0.$$

$$(y + e^{2x}) dx - y dy = 0.$$

$$M(x,y) = y + e^{2x}$$

$$\rightarrow \frac{M(x,y)}{dy} = 1$$

$$N(x,y) = -1 \rightarrow \frac{N(x,y)}{dx} = 0$$

ask is not exact. & recieve

$$p(x) = -1$$

$$\mu(x) = e^{\int p(x) dx} = e^{\int -1 dx} = e^{-x}$$

Multiply oDE $\mu(x)$:-

$$e^{-x} y' - e^{-x} y = e^{-x} \cdot e^{2x}$$

$$e^{-x} y' - e^{-x} y = e^x$$

$$(e^{-x} y)' = e^x$$

$$(e^{-x} y' - e^{-x} y) = (e^{-x} y)'$$

$$\rightarrow \frac{d}{dx} (e^{-x} y)' = e^x$$

Separate and integrate ~~both~~

$$\int d(e^{-x} y)' = \int e^x dx + c$$

$$e^{-x} y = e^x + c$$

$$y = \frac{e^x}{e^{-x}} + \frac{c}{e^{-x}}$$

$$y = e^{2x} + e^x c, \quad -\infty < x < \infty$$

* Example: Irf

$$\frac{dy}{dx} + y = x \quad y(0) = 2$$

$$p(x) = 1 \\ \mu(x) = e^{\int p(x) dx} = e^{\int 1 dx} = e^x$$

$$\underline{e^x y' + e^x y = x e^x}$$

$$(e^x y)' = x e^x$$

$$\frac{d}{dx} (e^x y) = x e^x$$

Separate and integrate

$$\int d(e^x y) = \int x e^x dx + C$$

$$e^x y = \int \underbrace{x e^x}_{\text{ابزار}}$$

$$\int u dv = u \cdot v - \int v du$$

$$u = x \rightarrow du = dx$$

$$dv = e^x dx \rightarrow v = e^x$$

$$\int x e^x dx = x \cdot e^x - \int e^x dx \\ = x e^x - e^x$$

$$e^x y = x e^x - e^x + C$$

$$y = (x - 1) + C e^{-x}$$

$$y(0) = 2$$

$$y(0) = 2$$

$$2 = -1 + C$$

$$C = 3$$

particular solution

$$y = (x - 1) + 3e^{-x}$$

$$-\infty < x < \infty$$

* Bernoulli's Differential equation. (non-linear) Linear

1st ODE.

$$y' + p(x)y(x) = Q(x)y^n \quad \dots *$$

$$n \neq 1$$

$$\text{if } n = \begin{cases} 0 & y' + p(x)y(x) = Q(x) \text{ Linear (non homogeneous)} \\ 1 & y' + (p(x) - Q(x)) = 0 \text{ Linear (homogeneous)} \end{cases}$$

Divide Eq * by y^n :-

$$\frac{y'}{y^n} + p(x) \frac{y(x)}{y^n} = Q(x).$$

$$y' y^{-n} + y^{-n} p(x) y = Q(x).$$

$$y' y^{-n} + y^{-n} p(x) = Q(x).$$

$$\begin{aligned} (y^{1-n})' &= (1-n) \cdot y^{-n} \cdot y' \\ &= (1-n) y^{-n} \cdot y' \end{aligned}$$

$$y^{-n} \cdot y' = \frac{1}{1-n} (y^{1-n})' \rightarrow \text{substitution}$$

$$\frac{1}{1-n} (y^{1-n})' + p(x) y^{1-n} = Q(x).$$

$$\text{Let } V = y^{1-n}$$

$$\frac{1}{1-n} V' + p(x) V = Q(x) \quad \text{Linear in } V$$

* Bernoulli's equation

$$y' + P(x)y = Q(x)y^n$$

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$

$$(y^{1-n})' = (1-n)y^{-n} \cdot y'$$

$$\Rightarrow y^{-n}y' = \left(\frac{1}{1-n}\right)(y^{1-n})'$$

$$\Rightarrow \left(\frac{1}{1-n}\right) \frac{d}{dx} y^{1-n} + P(x)y^{1-n} = Q(x)$$

$$\text{Let } v = y^{1-n}$$

$$\frac{1}{1-n} \frac{dv}{dx} + P(x)v = Q(x) \text{ Linear in } v.$$

* Example :- solve the following IVP and find the interval of validity for the solution.

$$y' + \frac{4}{x}y = x^3 \quad y(2) = 1 \quad x > 0.$$

divide by y^2 :

$$\boxed{y^{-2}y'} + \frac{4}{x} \frac{y}{y^2} = x^3$$

$$* \boxed{y^{-2}y'} + \frac{4}{x} (y^{-1}) = x^3$$

$$\frac{d}{dx}(y^{-1}) = -1 y^{-2} \cdot y'$$
$$\Rightarrow y^{-2}y' = -\frac{dy^{-1}}{dx} \rightarrow * \text{تعويض}$$

$$- \frac{dy^{-1}}{dx} + \frac{4}{x} y^{-1} = x^3$$

$$\text{Let } v = y^{-1}$$

$$- \frac{dv}{dx} + \frac{4}{x} v = x^3 \text{ Linear in } v$$

integrating factor method $\mu(x)$.

$$\frac{dv}{dx} - \frac{4}{x} v = -x^3$$

* Remark

$$y = e^{\ln x}$$

$$\Rightarrow \ln y = \ln x$$

$$y = x$$

$$\frac{dy}{dx} - \frac{4}{x} y = Q(x).$$

$$P(x) = -\frac{4}{x}$$

$$\mu(x) = e^{\int P(x) dx} = e^{\int -\frac{4}{x} dx} = e^{-4 \ln x} = x^{-4}$$

$$\boxed{\mu(x) = x^{-4}}$$

* multiply by $\mu(x)$

$$x^{-4} \frac{dv}{dx} - \frac{4}{x} v \cdot x^{-4} = -x^3 \cdot x^{-4}$$

$$x^{-4} \frac{dv}{dx} - 4v \cdot x^{-5} = -x^{-1}$$

$$\text{Note } \frac{d}{dx} (x^{-4} \cdot v) = x^{-4} \frac{dv}{dx} - 4x^{-5} v$$

~~Separate and integrate~~

$$\frac{d}{dx} (x^{-4} \cdot v) = -x^{-1} = -\frac{1}{x}$$

Separate and integrate

$$\int d(x^{-4} \cdot v) = - \int \frac{1}{x} dx + C$$

$$x^{-4} \cdot v = \ln x + C$$

$$v = x^4 c - x^4 \ln x$$

$$x > 0$$

$$v = x^4 (c - \ln x)$$

$$0 < x^4 (c - \ln x) \neq 0$$

$$v = y^{-1} = x^4 (c - \ln x)$$

$$c - \ln x \neq 0$$

$$\Rightarrow y = \frac{1}{x^4 (c - \ln x)}$$

* Particular Solution. $y(2) = -1$

$$-1 = \frac{1}{16 (c - \ln 2)}$$

$$c = \ln 2 - \frac{1}{16}$$

~~$$c = \ln 2 - \frac{1}{16}$$~~

* Find the interval:

$$y(x) = \frac{-16}{x^4 (16 \ln \frac{x}{2} + 1)}$$

$$16 \ln \frac{x}{2} + 1 = 0$$

$$\ln \frac{x}{2} = -\frac{1}{16}$$

$$\frac{x}{2} = e^{-1/16} = 0.9394$$

$$x = 2 \times 0.9394 = 1.8788$$

$$\begin{array}{c} | \quad \quad \quad * \quad \quad \quad | \\ 0 \quad \quad \quad 1.8788 \quad \quad \quad \infty \end{array}$$

$$1.8788 < x < +\infty \checkmark$$

$$0 < x < 1.8788 \Rightarrow 2 \text{ fields}$$

* Radiation reaction

$$m(t)$$

$$\frac{dm(t)}{dt} \propto -m(t)$$

$$\frac{dm}{dt} = -km$$

* Bacteria

$$\frac{dm(t)}{dt} \propto m$$

$$\frac{dm}{dt} = km$$

* Reaction

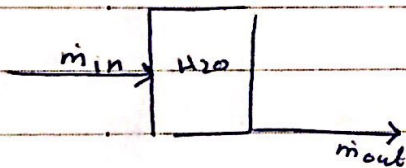
$$C_{A_0}$$

Rate of reaction

$$dC_A \propto C_A$$

$$\frac{dC_A}{dt} = -kC_A$$

*



conservation ← mass
energy
momentum

$$m_{in} - m_{out} = \frac{dm_{system}}{dt} = A \frac{d(PV)}{dt} = \frac{d(PAh)}{dt} = PA \frac{dh}{dt} \Rightarrow \text{...}$$

$$m = \rho V$$

$$m = \rho Ah \cdot \rho$$

* Geometry - cylindrical

$$V = Ah$$

$$m_{in} - m_{out} = \rho A \frac{dh}{dt} \quad \text{--- (1)}$$

Variable: $m_{in}(t)$

$$m_{out}(t)$$

$$h$$

equation → 1

$$\text{DoF} = 3 - 1 = 2$$

$h(t)$ represent state of the system

* we should specify either m_{in} or m_{out} . (Dof = 2)

$$m_{out} \propto h^p$$

$$m_{out} = \beta h^p \quad \text{--- (2)}$$

① & ② degrees

$$m_{in} = \beta h^p = \rho A \frac{dh}{dt} \quad \text{Dof} = 1$$

$\left[\frac{m^3}{s} \right] [m]$

$$\frac{\rho A dh}{\rho \beta} + h = \frac{1}{\rho \beta} m_{in}$$

$$\left[\frac{A}{\beta} \right] = \left[\frac{m^2}{m^2/s} \right] = [s] \text{ time.}$$

$$\frac{A}{\beta} \frac{dh}{dt} + h = \frac{1}{\rho \beta} m_{in}$$

* Hw#1

Set 1.1: 2, 4, 5, 8, 10, 12, 16.

Set 1.3: 2, 4, 6, 8, 10, 12, 14, 40.

Due on: Feb. 27, 2019.

* Non exact oDEs.

$$\begin{aligned} & P(x, y) dx + Q(x, y) dy = 0 \quad \text{text book} \\ & \rightarrow M(x, y) dx + N(x, y) dy = 0 \quad \text{--- **} \end{aligned}$$

$\mu(x)$ integrating factor

eq ** $\mu(x) M(x, y) dx + \mu(x) N(x, y) dy = 0$

exactness condition

$$\frac{\partial (\mu M)}{\partial y} = \frac{\partial (\mu N)}{\partial x}$$

$$M \frac{\partial M}{\partial y} + N \frac{\partial M}{\partial x} = N \frac{\partial N}{\partial y} + M \frac{\partial N}{\partial x}$$

$$\text{Let: } \frac{\partial M}{\partial y} = M_y \quad \left| \quad \frac{\partial M}{\partial x} = M_x$$

$$\frac{\partial M}{\partial y} = M_y \quad \left| \quad \frac{\partial N}{\partial x} = N_x$$

$$M M_y + N M_x = N N_x + M N_y$$

$$\text{if } M = M(x) \Rightarrow M_y = 0$$

$$M M_y = N M_x + M N_x \rightarrow \text{divided by } M \text{ and } N.$$

$$\Rightarrow \frac{1}{M} \frac{\partial M}{\partial x} = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

$$= R(x, y).$$

$$\frac{1}{M} \frac{\partial M}{\partial x} = R(x, y) \quad \text{if } R = R(x).$$

$$\frac{1}{M} \frac{\partial M}{\partial x} = R(x) \quad \text{separate and integrate.}$$

$$\int \frac{1}{M} dM = \int R(x) dx$$

$$\ln M = \int R(x) dx.$$

$$M = e^{\int R(x) dx}.$$

* In Text book $M = F$.

$$\frac{1}{F} \frac{dF}{dx} = R = \frac{1}{Q} \left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right]$$

$$F(x) = \int e^{\int R(x) dx}.$$

$$\text{if } M = M(y) \Rightarrow M_x = 0$$

$$\frac{1}{M} \frac{\partial M}{\partial y} = \frac{1}{N} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = R^*(x, y)$$

$$\text{if } R^*(x, y) = R^*(y).$$

$$M = e^{\int R^*(y) dy}.$$

Example:

$$(2y^2 + 3x)dx + (2xy)dy = 0$$

$$M(x,y) = 2y^2 + 3x \rightarrow \frac{\partial M}{\partial y} = 4y$$

$$N(x,y) = 2xy \rightarrow \frac{\partial N}{\partial x} = 2y \neq$$

$$\text{check: } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{it's not exact}$$

$$R(x,y) = \frac{1}{2xy} [4y - 2y] \Rightarrow \text{مماثل إذا ما لم تكن صفرية}$$

في x بمربى y (مماثلة)

$$R(x,y) = \frac{1}{x} = R(x)$$

$$\int R(x) dx = \int \frac{1}{x} dx = \ln x$$
$$M(x) = e = e = e = x$$

$$M(x) = x$$

$$x(2y^2 + 3x)dx + x(2xy)dy = 0$$

$$3x^2 dx + 2y^2 x dx + 2x^2 y dy = 0$$

$$\text{Note: } d(x^2 y^2) = (2xy^2)dx + (2x^2 y)dy$$

$$3x^2 dx + d(x^2 y^2) = 0$$

$$d(x^2 y^2) = -3x^2 dx$$

$$\int d(x^2 y^2) = \int -3x^2 dx$$

$$x^2 y^2 = -x^3 + y$$

$$y^2 = -x + x^{-2} C$$

* Linear 2nd ODE :-

$$p(x)y'' + q(x)y' + r(x)y = g(x)$$

$p(x), q(x), r(x)$ variable coefficients

$$\text{if } \begin{cases} g(x) = 0 & \text{homogeneous} \\ g(x) \neq 0 & \text{nonhomogeneous} \end{cases} \quad \left. \vphantom{\begin{matrix} g(x) = 0 \\ g(x) \neq 0 \end{matrix}} \right\} \text{2nd ODE}$$

Homogeneous Linear with constant coefficient :-

$$ay'' + by' + cy = 0$$

a, b, c constant

Example:- Determine the solution.

$$\text{e.g. : } y'' - 9y = 0$$

$$\Rightarrow y'' = 9y$$

$$y_1 = e^{3x}$$

$$y_1' = 3e^{3x}$$

$$y_1'' = 9e^{3x} \Rightarrow 9y_1 \neq 0$$

$$y_2 = e^{-3x}$$

$$y_2' = -3e^{-3x}$$

$$y_2'' = 9e^{-3x} = 9y_2 \neq 0$$

$$y_3 = y_1 + y_2$$

$$= e^{3x} + e^{-3x}$$

$$y_3' = 3e^{3x} - 3e^{-3x}$$

$$y_3'' = 9e^{3x} + 9e^{-3x}$$

$$= 9(e^{3x} + e^{-3x})$$

$$= 9(y_1 + y_2)$$

$$y(x) = C_1 y_1 + C_2 y_2$$

$$y(x) = C_1 e^{3x} + C_2 e^{-3x}$$

General Solution / superposition principle.

particular solution $\begin{matrix} C_1 \\ C_2 \end{matrix}$

* conclusion: Look for $y = e^{rx}$

Condition on $y(x)$

$$x_0 \rightarrow \begin{cases} y(x_0) \\ y'(x_0) \end{cases} \text{ initial condition.}$$

initial value problem (Ivp).

$$\begin{cases} x_0 \rightarrow y(x_0) \\ x_1 \rightarrow y(x_1) \end{cases} \begin{cases} \text{Boundary value problem} \\ \text{BVP} \end{cases}$$

$$\begin{array}{c|c} x_0 & x_1 \\ \hline y(x_0) & y(x_1) \end{array}$$

$$\text{Example 2: } y'' - 9y = 0 \quad \begin{cases} y(0) = 2 \\ y'(0) = -1 \end{cases} \quad x_0 = 0$$

$$y = C_1 e^{3x} + C_2 e^{-3x}$$

$$y' = 3C_1 e^{3x} - 3C_2 e^{-3x}$$

$$x_0 = 0 \rightarrow y(0) = 2 = C_1 + C_2$$

$$y'(0) = -1 = 3C_1 - 3C_2$$

$$C_1 + C_2 = 2$$

$$3C_1 - 3C_2 = -1$$

} Linear algebraic system

$$-6C_2 = -7$$

in C_1 and C_2

$$C_2 = \frac{7}{6}$$

$$C_1 = \frac{2 \cdot 6}{6} - \frac{7}{6} = \frac{5}{6}$$

$$y = \frac{5}{6} e^{3x} + \frac{7}{6} e^{-3x}$$

20/2

x All solution of

$$ay'' + by' + cy = 0 \quad \text{Linear}$$

have the following form

$$y = e^{rx} \quad r \text{ parameter}$$

if it's a solution, it should satisfy ODE:

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

substituted in ODE:

$$ar^2 e^{rx} + br e^{rx} + ce^{rx} = 0$$

$$e^{rx}(ar^2 + br + c) = 0$$

For nontrivial solution

$$e^{rx} \neq 0 \quad \text{Called characteristic equation}$$

it means that:

$$ar^2 + br + c = 0$$

quadratic equation

roots

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow \begin{matrix} > 0 \\ < 0 \end{matrix} \rightarrow \text{real roots}$$

r_1, r_2 real and distinct

called eigenvalues

Pole

$$b^2 - 4ac > 0 \rightarrow r_1, r_2 \text{ real and distinct}$$

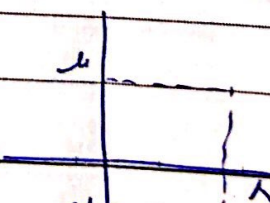
$$b^2 - 4ac = 0 \rightarrow r_1 = r_2 = r \text{ real and repeated}$$

$$b^2 - 4ac < 0 \rightarrow r_1, r_2 \text{ complex conjugate of } r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \lambda \pm \mu$$

λ real (r)

μ : imaginary (r)



Example: Solve IVP.

$$\textcircled{y}'' + 11y' + 24y = 0 \quad y(0) = 0, \quad y'(0) = -7.$$

Solution:

write characteristic equation:-

$$r^2 + 11r + 24 = 0.$$

$$\text{roots: } (r+3)(r+8) = 0.$$

$$r_1 = -3 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Stable}$$

$$r_2 = -8$$

$$\text{Solutions: } y_1 = e^{-3x}$$

$$y_2 = e^{-8x}$$

General solutions:-

$$y(x) = C_1 y_1 + C_2 y_2$$

$$y(x) = C_1 e^{-3x} + C_2 e^{-8x}$$

$$y'(x) = -3C_1 e^{-3x} - 8C_2 e^{-8x} \quad \text{particular solution}$$

ICs:-

$$x_0 = 0 \rightarrow y(0) = 0 = C_1 + C_2$$

$$y'(0) = -7 = -3C_1 - 8C_2$$

$$C_1 + C_2 = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Linear system}$$

$$-3C_1 - 8C_2 = -7 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{in } C_1 \leftrightarrow C_2$$

$$C_1 = -C_2$$

$$-5C_2 = -7$$

$$C_2 = \frac{7}{5}$$

$$C_1 = -\frac{7}{5}$$

$\Rightarrow$ particular solution

$$y(x) = \frac{7}{5} (e^{-8x} - e^{-3x})$$

$$\lim_{x \rightarrow \infty} y(x) = 0$$

$$* e^{-8x} = \frac{1}{e^8} = \frac{1}{\infty} = 0$$

* Case i:-

root r_1, r_2 complex conjugate

$$r_{1,2} = \lambda \pm \mu j$$

2 solutions:-

$$y_1 = e^{r_1 x} = e^{(\lambda - \mu j)x} = e^{\lambda x} \cdot e^{-\mu x j} = e^{\lambda x} (\cos(\mu x) - j \sin(\mu x)) \quad *$$

$$y_2 = e^{r_2 x} = e^{(\lambda + \mu j)x} = e^{\lambda x} \cdot e^{\mu x j} = e^{\lambda x} (\cos(\mu x) + j \sin(\mu x)) \quad **$$

use Euler's Formula:-

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

our case: $\theta = \mu x$

* Add * and **:-

$$y_1 + y_2 = 2 e^{\lambda x} \cos(\mu x)$$

$$\frac{1}{2} y_1 + \frac{1}{2} y_2 = e^{\lambda x} \cos(\mu x) \rightarrow C_1 = \frac{1}{2}, C_2 = \frac{1}{2}$$

* Subtract * and **:-

$$y_1 - y_2 = -2j e^{\lambda x} \sin(\mu x)$$

$$-\frac{1}{2j} y_1 + \frac{1}{2j} y_2 = \sin(\mu x) \cdot e^{\lambda x}$$

$C_1 = \frac{1}{2j}, C_2 = \frac{1}{2j}$

$$y(x) = C_1 e^{\lambda x} \cos(\mu x) + C_2 e^{\lambda x} \sin(\mu x)$$

$$\lambda = \operatorname{Re}(r)$$

$$y(x) = e^{\lambda x} [C_1 \cos(\mu x) + C_2 \sin(\mu x)]$$

$$\mu = \operatorname{Im}(r)$$

$$y(x) = \boxed{e^{\operatorname{Re}(r)x}} [C_1 \cos(\operatorname{Im}(r)x) + C_2 \sin(\operatorname{Im}(r)x)]$$

تبدیل به
 $\operatorname{Re}(r)$

اذا كانت Zero تكون البنية

اذا كانت Zero تكون البنية

اذا كانت Zero تكون البنية

Example:- solve Ivp

$$y'' - 4y' + 9y = 0$$

$$y(0) = 0, y'(0) = -8$$

* Solution:-

$$r^2 - 4r + 9 = 0$$

$$r_{1,2} = \frac{4 \pm \sqrt{16 - 4 \cdot 9}}{2} = \frac{4 \pm \sqrt{-20}}{2}$$

$$r_{1,2} = 2 \pm (\sqrt{5})i$$

$$\operatorname{Re}(r) = 2$$

$$\operatorname{Im}(r) = \sqrt{5}$$

* General Solution:-

$$y(x) = e^{2x} [C_1 \cos(\sqrt{5}x) + C_2 \sin(\sqrt{5}x)]$$

$$\text{I.C.s:- } y(0) = e^0 [C_1 \cos(0) + C_2 \sin(0)] = 0$$

$$C_1 = 0$$

$$\Rightarrow y(x) = e^{2x} \cdot C_2 \sin(\sqrt{5}x)$$

$$y'(x) = 2C_2 e^{2x} \sin(\sqrt{5}x) + C_2 e^{2x} \cos(\sqrt{5}x) \cdot \sqrt{5}$$

$$y'(0) = -8 = 2C_2 \cdot \sqrt{5}$$

$$C_2 = \frac{-8}{2\sqrt{5}}$$

$$y(x) = \frac{-8}{2\sqrt{5}} e^{2x} \sin(\sqrt{5}x)$$

25/2/2019.

* Real and Repeated (double roots).

$$ay'' + by' + cy = 0. \quad \text{---} \quad (*)$$

characteristic equation.

$$ar^2 + br + c = 0.$$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2a.

$$b^2 - 4ac = 0$$

$$r_{1,2} = -\frac{b}{2a} \Rightarrow r_1 = r_2 = r$$

1st solution ~~is~~ $y_1(x) = e^{-(b/2a)x}$ ①

To find 2nd solution:

we use Reduction of order method.

- if we know a solution, we can use it to find other solution.

Assume: 2nd solution.

$$y_2(x) = v(x) y_1(x)$$

↓
unknown.

To determine $v(x)$.

$$y_2'(x) = v'(x) y_1(x) + v(x) y_1'(x).$$

$$y_2''(x) = v'' y_1 + v' y_1' + v' y_1' + v y_1''$$

$$y_1(x) = e^{-rx} = e^{-(b/2a)x}$$

① $y_1 = e^{rx}$ ($-b/2a$) x

$$y_2(x) = v(x) e^{rx} \rightarrow y_1 = e^{rx}$$

$$y_2'(x) = v' e^{rx} + v r e^{rx}$$

$$y_2''(x) = v'' e^{rx} + 2v' r e^{rx} + v r^2 e^{rx}$$

* e^{rx} is a solution

$$a(v' e^{rx} + 2v' r e^{rx} + v r^2 e^{rx}) + b(v e^{rx} + v r e^{rx}) + v e^{rx} = 0.$$

$$e^{rx} [a v'' + (2ar + b) v' + (r^2 a + br + c) v] = 0.$$

* Note $\rightarrow r = \frac{-b}{2a} \Rightarrow \underline{2ar + b = 0}$

r root.

$$ar^2 + br + c = 0.$$

$$\Rightarrow e^{rx} (a v'') = 0$$

Since e^{rx} is a solution: $e^{rx} \neq 0$.

$$\Rightarrow v'' = 0$$

$$\frac{d}{dx}(v') = 0 \Rightarrow v' = c_1$$

$$\frac{dv}{dx} = c_1 \Rightarrow v(x) = c_1 x + c_2$$

$$y_2 = v y_1 = (c_1 x + c_2) e^{rx}$$

* general solution:-

$$y(x) = \alpha y_1 + \beta y_2$$

$$= \alpha e^{rx} + \beta (c_1 x + c_2) e^{rx}$$

$$y(x) = \underbrace{(\alpha + \beta c_2)}_{C_1} e^{rx} + \underbrace{(\beta c_1)}_{C_2} x e^{rx}$$

$$y(x) = C_1 e^{rx} + C_2 x e^{rx} = (C_1 + C_2 x) e^{rx}$$

Example 3: IVP.

$$y'' - 4y' + 4y = 0 \quad y(0) = 12$$

$$y'(0) = -3$$

characteristic equation:-

$$r^2 - 4r + 4 = 0$$

$$r_{1,2} = 2$$

$$y_1 = e^{2x}$$

$$y_2 = (c_1 + c_2 x) e^{2x}$$

$$y(x) = (c_1 + c_2 x) e^{2x} = c_1 e^{2x} + c_2 x e^{2x}$$

$$y' = 2c_1 e^{2x} + c_2 e^{2x} + 2x c_2 e^{2x}$$

ICs:-

$$y(0) = 12 = c_1$$

$$y'(0) = -3 = 2c_1 + c_2 = 24 + c_2$$

$$c_2 = -27$$

particular solution

$$y(x) = (12 - 27x) e^{2x}$$

* Reduction of order method:-

$$F(x, y, y', y'') = 0 \text{ Linear or non linear.}$$

can be reduced to 1st ODE if:-

* y doesn't appear

* x doesn't appear

* $y_1(x)$ solution, use it to find another solution

$$y_2 = v(x) y_1(x)$$

Example:-

$$xy'' = 2y' \quad y \text{ doesn't appear.}$$

$$\text{Let } z = 2y'$$

$$z' = y''$$

$$xz' = 2z$$

Separate and integral

$$\int \frac{dz}{z} = \int \frac{2}{x} dx + c$$

$$\ln z = 2\ln x + c$$

$$\Rightarrow \ln z - \ln x^2 = c$$

$$\ln \left(\frac{z}{x^2} \right) = c$$

$$\frac{z}{x^2} = e^c = C_1$$

$$z = x^2 C_1$$

$$\text{Since } z = y'$$

$$\frac{dy}{dx} = x^2 C_1$$

$$y = C_3 x^3 + C_2$$

Example: Find general solution.

not constant.

$$(2x^2)y'' + (x)y' - 3y = 0.$$

given that 1st solution

$$y_1(x) = \frac{1}{x}$$

Solution:

$\Rightarrow$ check if $y_1 = \frac{1}{x}$ a solution.

$$y_1 = x^{-2}$$

$$y_1' = -2x^{-3}$$

$$2x^2(-2x^{-3}) + x(-x^{-2}) - 3x^{-1} = 0.$$

$$-4x^{-1} + (-x^{-1} - 3x^{-1}) = 0.$$

$$-4x^{-1} - 4x^{-1} = -8x^{-1} = 0. \quad \checkmark \text{ It's a solution.}$$

Let 2nd

$$y_2 = v(x)y_1 = v \frac{1}{x} = vx^{-1}$$

$$y_2' = v'x^{-1} - vx^{-2}$$

$$y_2'' = v'x^{-2} - 2v'x^{-3} + v^{-1}v''$$

$$\Rightarrow (2x)y'' - 3v' = 0. \Rightarrow v \text{ doesn't appear.}$$

Solve for v.

$$\text{Let } z = v', \quad z' = v''$$

$$(2x)z' - 3z = 0.$$

$$\int \frac{dz}{z} = \int \frac{3}{2} \frac{dx}{x} + C.$$

$$\ln z = \frac{3}{2} \ln x + C.$$

$$\frac{z}{x^{3/2}} = C_1$$

$$z = x^{3/2} C_1$$

$$v' = C_1 x^{3/2}$$

$$\int dv = \int C_1 x^{3/2} dx + C_2$$

$$v = C_1 \frac{2}{5} x^{5/2} + C_2$$

$$\text{Let } C_1 = \frac{5}{2}, C_2 = 0.$$

$$\Rightarrow v = x^{5/2}$$

$$y_2(x) = v y_1$$

$$y_2(x) = x^{5/2} \cdot \frac{1}{x}$$

$$y_2(x) = x^{3/2}$$

$$y(x) = C_1 \frac{1}{x} + C_2 x^{3/2}$$

* Euler - Cauchy equation.

$$x^2 y'' + a x y' + b y = 0 \quad (*)$$

a, b constab.

$$x^2 + a x + b = 0$$

polynomial equation degree 2.

Let Solution $y(x) = x^m$

$$y'(x) = m x^{m-1}$$

$$y''(x) = m(m-1) x^{m-2}$$

? $\rightarrow$ $\neq$ p' de g' or

$$x^2 (m(m-1) x^{m-2}) + (a x) (m x^{m-1}) + b x^m = 0$$

$$m(m-1) x^m + a m x^m + b x^m = 0$$

$$x^m (m(m-1) + am + b) = 0.$$

$$y = x^m \neq 0 \text{ solution.}$$

$$\Rightarrow m(m-1) + am + b = 0.$$

$$m^2 + (a-1)m + b = 0. \quad \#$$

quadratic equation in m .

HW. 2#

Due on March 11.

Set. 1.43: 2, 4, 6, 8, 10, 12, 14

Set. 1.51: 4, 6, 15.

Review question 1, 12, 14, 22, 32, 38.

class $\rightarrow$ Hence, $y = x^m$ a solution

$\Leftrightarrow m$ root of equation # (quadratic in m).

$$m_{1,2} = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}$$

Note: $(a-1)^2 = a^2 - 2a + 1$

$$(1-a)^2 = 1 - 2a + a^2.$$

$$m_{1,2} = \frac{1}{2} (1-a) \pm \sqrt{\frac{1}{4} (1-a)^2 - b}.$$

Roots $\rightarrow$ Real and distinct.

$\rightarrow$ Real and repeated.

$\rightarrow$ Complex.

* Real and distinct.

$$(m_1, m_2)$$

$$y_1 = x^{m_1}, y_2 = x^{m_2}$$

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

Example:- $x^2 y'' + 1.5xy' - 0.5y = 0$.

$$a = 1.5, b = -0.5$$

$$m^2 + (a-1)m + b = 0$$

$$m^2 + (1.5-1)m - 0.5 = 0$$

$$m^2 + 0.5m - 0.5 = 0$$

$$m^2 + m_1 = \frac{1}{2}$$

$$m_2 = -1$$

$$y_1 = x^{1/2} = \sqrt{x}$$

$$y_2 = x^{-1} = \frac{1}{x}$$

$$y(x) = C_1 \sqrt{x} + C_2 \frac{1}{x}$$

- Reduction of order method

$$y'' + p(x)y' + q(x)y = 0$$

y_1 known.

$$y_2 = u(x) y_1$$

$$y_2' = u' y_1 + u y_1'$$

$$y_2'' = u'' y_1 + 2u' y_1' + u y_1''$$

sub in eq

is solution.

$$u'' y_1 + u' (2y_1' + p y_1) + u (\underbrace{y_1'' + p y_1' + q y_1}_{=0}) = 0$$

$$u'' + \left(\frac{2y_1' + p y_1}{y_1} \right) u' = 0 \quad u \text{ doesn't appear.}$$

Let $v = u'$

$$\frac{dv}{dx} + \left(\frac{2y_1'}{y_1} + P y_1 \right) v = 0$$

Note:-

$$\int \frac{2y_1' + P y_1}{y_1} = \left(\int \frac{2y_1'}{y_1} + \int P \right) \times dx$$

$$\ln 2 \ln y + \int P(x) dx$$

$$\int \frac{dv}{v} = - \left(\frac{2y_1' + y_1}{y_1} \right) dx$$

Final $\Rightarrow \ln v = \ln \frac{1}{y_1} - \int P(x) dx$

$$\ln(1/y_1) - \int P(x) dx$$

$$v = e$$

$$= e^{\ln(1/y_1) - \int P(x) dx}$$

$$V = \frac{1}{y_1^2} e^{-\int P(x) dx}$$

Let $v = u'$

$$\frac{du}{dx} = u' = \frac{1}{y_1^2} e^{-\int P(x) dx}$$

$$\int du = \int \frac{1}{y_1^2} e^{-\int P(x) dx} \cdot dx$$

$$u = \int \frac{1}{y_1^2} e^{-\int P(x) dx} \cdot dx$$

$$y_2 = u y = y \left(\int \frac{1}{y_1^2} e^{-\int P(x) dx} \cdot dx \right)$$

$$x^2 y'' + ax y' + by = 0.$$

roots: repeated

$$m_{1,2} = \frac{1}{2} (1-a).$$

~~4~~

$$y_1 = x^{\frac{1}{2}(1-a)}.$$

$$\text{Note: } \frac{1}{4} (1-a)^2 - b = 0.$$

$$\Rightarrow b = \frac{1}{4} (1-a)^2.$$

$$\rightarrow y'' + \left(\frac{a}{x}\right)y' + \left(-\frac{(1-a)^2}{4x^2}\right)y = 0.$$

$$P(x) = \frac{a}{x}$$

$$Q(x) = \frac{(1-a)^2}{4x^2}.$$

$$\begin{aligned} \frac{-\int P(x) dx}{e} &= \frac{-\int \frac{a}{x} dx}{e} = \frac{-a \ln x}{e} = \frac{\ln x^{-a}}{e} = x^{-a} = \frac{1}{x^a} \end{aligned}$$

$$y_2 = u y_1$$

$$\frac{1}{y_1^2} \frac{-\int P(x) dx}{e} = \frac{1}{\left[x^{\frac{(1-a)}{2}}\right]^2} \cdot \frac{1}{x^a}$$

$$\left[x^{\frac{(1-a)}{2}}\right]^2 = x^{\frac{(1-a)}{2}} \cdot x^{\frac{(1-a)}{2}} = x^{(1-a)}.$$

$$L = \frac{1}{x^{(1-a)}} \cdot \frac{1}{x^a} = \frac{1}{x}$$

$$y_2 = x^{(1-a/2)} \cdot \int \frac{1}{x} dx = x^{\frac{(1-a)}{2}} \cdot \ln x.$$

* general solution.

$$y(x) = c_1 x^{\frac{1}{2}(1-a)} + c_2 x^{\frac{1}{2}(1-a)} \ln(x).$$

$$y(x) = (c_1 + c_2 \ln x) x^{\frac{1}{2}(1-a)}$$

Example:- $x^2 y'' - 5xy' + 9y = 0.$

$$a = -5, b = 9.$$

$$m = m_{1,2} = \frac{1}{2}(1-a) = \frac{1}{2}(1+5) = \frac{6}{2} = 3.$$

$$y_1 = x^m = x^3.$$

$$\Rightarrow y_2 = x^3 \ln x.$$

$$y = (c_1 + c_2 \ln x) x^3.$$

$$m^2 - 6m + 9 = 0$$

$$(m-3)(m-3) = 0$$

$$m_1 = m_2 = m = 3,$$

double.

ask ; believe & recieve

* Case 3:- Complex roots.

Example:-

$$\underline{x^2 y'' + 0.6 x y' + 16.04 y = 0}$$

$$a = 0.6$$

$$b = 16.04$$

$$m^2 + (a-1)m + b = 0$$

$$m^2 + (0.6-1)m + 16.04 = 0.$$

$$m_{1,2} = 0.2 \pm 4j$$

$$y_1 = x^{m_1} = x^{0.2+4j} = x^{0.2} \cdot x^{4j} = x^{0.2} e^{4j \ln x} = x^{0.2} \cdot e^{(4 \ln x)j}$$

$$y_2 = x^{m_2} = x^{(0.2-4j)} = x^{0.2} \cdot x^{-4j} = x^{0.2} e^{-4j \ln x} = x^{0.2} \cdot \frac{1}{e^{4j \ln x}} = x^{0.2} \cdot e^{-j(4 \ln x)}$$

* Euler's Formula:-

$$e^{\pm i\theta} = \cos\theta \pm j \sin\theta.$$

$$\Rightarrow \theta = 4 \ln x.$$

$$\Rightarrow y_1 = x^{0.2} \cos(4 \ln x)$$

$$y_2 = x^{0.2} [\cos(4 \ln x) - j \sin(4 \ln x)]$$

$$y_1 + y_2 = 2x^{0.2} \cos(4 \ln x)$$

$$\frac{1}{2} y_1 + \frac{1}{2} y_2 = x^{0.2} \cos(4 \ln x)$$

$$y_1 - y_2 = 2j x^{0.2} \sin(4 \ln x)$$

$$\frac{1}{2} j y_1 - \frac{1}{2} j y_2 = x^{0.2} \sin(4 \ln x)$$

$$y = C_1 y_1 + C_2 y_2 = x^{0.2} [C_1 \cos(4 \ln x) + C_2 \sin(4 \ln x)]$$

* Fundamental (Basis) set of solution.

$$p(x)y'' + q(x)y' + r(x)y = 0$$

$$y(x_0) = y_0$$

$$y'(x_0) = y'_0$$

if y_1 and y_2 are solution.

$$\Rightarrow y(x) = C_1 y_1 + C_2 y_2 \text{ general solution.}$$

if $y(x)$ satisfies initial condition.

$$y(x_0) = y_0$$

$$y'(x_0) = y'_0$$

* Remark:-

$$y'' + by' + cy = 0.$$

$$r_{1,2} = \operatorname{Re}(r) \pm j \operatorname{Im}(r).$$

$$y = e^{\operatorname{Re}(r)x} [C_1 \cos(\operatorname{Im}(r)x) + C_2 \sin(\operatorname{Im}(r)x)]$$

ask & recieve

Question:- Is there any constant c_1 and c_2 that satisfy ICD?

$$y_0 = \begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = y_0 \\ y'(x_0) = c_1 y'_1(x_0) + c_2 y'_2(x_0) = y'_0 \end{cases}$$

Note:-

$$c_1 \begin{vmatrix} y_1(x_0) \\ y'_1(x_0) \end{vmatrix} + c_2 \begin{vmatrix} y_2(x_0) \\ y'_2(x_0) \end{vmatrix} = \begin{vmatrix} y_0 \\ y'_0 \end{vmatrix} \quad \text{Linear system.}$$

$$c_1 \begin{vmatrix} y_1(x_0) \\ y'_1(x_0) \end{vmatrix} + c_2 \begin{vmatrix} y_2(x_0) \\ y'_2(x_0) \end{vmatrix} = \begin{vmatrix} y_0 \\ y'_0 \end{vmatrix} \quad \text{in } c_1 \text{ and } c_2$$

$$AX = b$$

Cramer's Rule.

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y'_0 \end{bmatrix}$$

$$c_1 \begin{bmatrix} y_1(x_0) \\ y'_1(x_0) \end{bmatrix} + c_2 \begin{bmatrix} y_2(x_0) \\ y'_2(x_0) \end{bmatrix} = \begin{bmatrix} y_0 \\ y'_0 \end{bmatrix}$$

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(x_0) \\ y'_0 & y'_2(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} y_1(x_0) & y_0 \\ y'_1(x_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix}}$$

$$\det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{bmatrix} \neq 0.$$

$$= y_1(x_0) y'_2(x_0) - y_2(x_0) y'_1(x_0)$$

wronskian

$$W(y_1, y_2)(x_0)$$

c_1 and c_2 unique for y

Wronskian.

$f(x)$ and $g(x)$.

$$W(f, g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

* Abel's Theorem :-

if y_1 and y_2 two solution to $y'' + p(x)y' + q(x)y = 0$. then

wronskian of the solution is $\int_{x_0}^x -p(x) dx = -\int_{x_0}^x p(x) dx$.

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0) = e^{\int_{x_0}^x -p(x) dx} = C e^{\int_{x_0}^x -p(x) dx}$$

Example:-

$$2x^2 y'' + xy' - 2y = 0 \quad \left| \quad y'' + \frac{1}{2x} y' - \frac{1}{x^2} y = 0 \right.$$

$$\begin{matrix} y_1(x) = \frac{1}{x} \\ y_2(x) = x^{3/2} \end{matrix} \quad \left. \vphantom{\begin{matrix} y_1(x) = \frac{1}{x} \\ y_2(x) = x^{3/2} \end{matrix}} \right\} \text{ Do they form a basis (fundamental) set of solution.}$$

$$W(y_1, y_2)(x) = \begin{vmatrix} \frac{1}{x} & x^{3/2} \\ -x^{-2} & \frac{3}{2} x^{1/2} \end{vmatrix} = \frac{1}{2} \frac{1}{\sqrt{x}} \neq 0 \quad x \neq 0$$

$$y = c_1 \frac{1}{x} + c_2 x^{3/2} \text{ general.}$$

$$\text{Example:- } x^4 y'' - 2x^3 y' - x^8 y = 0$$

$$y'' - \left(\frac{2}{x}\right) y' - x^4 y = 0$$

$$p(x) = -\frac{2}{x} \quad x \neq 0$$

$$-\int \frac{2}{x} dx$$

$$W(y_1, y_2)(x) = C e$$

$$= C e^{+2 \ln x} = C e^{\ln x^2} = C x^2$$

$$W(y_2, y_1) = C x^2 \neq 0 \quad \text{if } x \neq 0$$

* Non homogeneous 2nd order and linear.

$$y'' + p(x)y' + q(x)y = r(x).$$

$r(x) \neq 0$. Non homogeneous.

$$y'' + p(x)y' + q(x)y = 0 \text{ homogeneous}$$

If $\bar{y}_1(x)$ and $\bar{y}_2(x)$ solution to Non homogeneous (eq 1)

and $y_1(x)$ and $y_2(x)$ fundamental to homogeneous (eq 2).

Then

$\bar{y}_1(x) - \bar{y}_2(x)$ is a solution to homogeneous (eq 2).

$$\bar{y}_1(x) - \bar{y}_2(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$\bar{y}_1'' + p(x)\bar{y}_1' + q(x)\bar{y}_1 = r(x).$$

$$\bar{y}_2'' + p(x)\bar{y}_2' + q(x)\bar{y}_2 = r(x).$$

$$(\bar{y}_1'' - \bar{y}_2'') + p(x)(\bar{y}_1' - \bar{y}_2') + q(x)(\bar{y}_1 - \bar{y}_2) = 0.$$

$\Rightarrow \bar{y}_1 - \bar{y}_2$ solution to homogeneous.

$$* \bar{y}_1 - \bar{y}_2 = c_1 y_1(x) + c_2 y_2(x)$$

$y_1(x)$ } fundamental set for homogeneous.
 $y_2(x)$ }

$\bar{y}_1(x)$ } for non homogeneous
 $\bar{y}_2(x)$ }

Let $y(x)$ General Solution to the Nonhomogeneous.

and $y_p(x)$ particular Solution to the Nonhomogeneous.

$$\rightarrow y(x) - y_p(x) = C_1 y_1(x) + C_2 y_2(x).$$

$$\Rightarrow y(x) = C_1 y_1 + C_2 y_2$$

$$y_c(x) = C_1 y_1 + C_2 y_2.$$

$$y(x) = y_c^{(x)} + y_p^{(x)}$$

y_c : complementary solution to homogeneous ~~equation~~ eq.

y_p : particular solution to Non homogeneous eq.

* Method of undetermined coefficient.

$$\text{Example 1: } y'' - 4y' - 12y = (3e^{5x}) g(x).$$

$$y(x) = y_c(x) + y_p(x)$$

$y_c(x)$:

$$y'' - 4y' - 12y = 0.$$

$$r^2 - 4r - 12 = 0 \quad \leftarrow \text{Characteristic equation.}$$

$$(r+2)(r-6) = 0.$$

$$r_1 = -2, r_2 = +6.$$

$$y_1 = e^{-2x}, y_2 = e^{6x}$$

$$y_c(x) = C_1 e^{-2x} + C_2 e^{6x}.$$

Root $\begin{cases} \text{eigenval} \\ \text{poles.} \end{cases}$

ask, believe & receive

* Guess y_p :-

~~Form~~ Form of y_p as $g(x)$.

$$y_p = A e^{5x}$$

$$y_p' = 5A e^{5x}, \quad y_p'' = 25A e^{5x}$$

$$25A e^{5x} - 4(5A e^{5x}) - 12(A e^{5x}) = 3e^{5x}$$

$$e^{5x} (25A - 20A - 12A) = 3e^{5x}$$

$$(-7A) e^{5x} = 3e^{5x}$$

Set coefficients equal:-

undetermined coefficient.

$$e^{5x} : -7A = 3 \Rightarrow A = -\frac{3}{7}$$

$$y(x) = C_1 e^{-2x} + C_2 e^{6x} - \frac{3}{7} e^{5x} \quad \leftarrow \text{General solution for non homogene.}$$

Example i: $y'' - 4y' - 12y = \sin(2x)$

different LH and RH

Find particular solution

$$g(x) = \sin 2x$$

$$y_p(x) = A \sin 2x$$

$$y_p'(x) = 2A \cos 2x, \quad y_p'' = -4A \sin 2x$$

$$-4A \sin 2x - 8A \cos 2x - 12A \sin 2x = \sin 2x$$

$$-16A \sin 2x - 8A \cos 2x = \sin 2x$$

Set coefficient equal:-

$$-16A = 1 \Rightarrow A = -\frac{1}{16}$$

$$-8A = 0 \Rightarrow A = 0$$

بمعنى y_p zero يعني complementary solution

← complementary solution

$$y_p = B \sin 2x + A \cos 2x.$$

$$y_p' = 2B \cos 2x - 2A \sin 2x.$$

$$y_p'' = -4B \sin 2x - 4A \cos 2x.$$

$$[-4A \cos(2x) - 4B \sin(2x)] - 4[-2A \sin(2x) + 2B \cos(2x)] - 12[A \cos 2x + B \sin 2x] = \sin 2x.$$

$$(-16A - 8B) \cos 2x + (8A - 16B) \sin 2x = \sin 2x.$$

Set coefficients equal:-

$$\cos(2x) = -16A - 8B = 0 \Rightarrow A = -\frac{1}{2}B.$$

$$\sin(2x) = 8A - 16B = 1 \Rightarrow B = -\frac{1}{20}$$

$$y_p(x) = \left(-\frac{1}{20}\right) \cos 2x - \frac{1}{20} \sin 2x.$$

Example: $y'' - 4y' - 12y = 2x^3 - x + 3$ $P_3(x) = 9_3x^3 + 9_2x^2 + 9_1x + 9_0$ (coefficient. أربع)

Find particular solution.

$$y_p(x) = Ax^3 + Bx^2 + Cx + D.$$

$$y_p'(x) = 3Ax^2 + 2Bx + C.$$

$$y_p''(x) = 6Ax + 2B.$$

$$(6Ax + 2B) - 4(3Ax^2 + 2Bx + C) - 12(Ax^3 + Bx^2 + Cx + D) = 2x^3 - x + 3.$$

$$-12Ax^3 + (-12A - 12B)x^2 + (6A - 8B - 12C)x + (2B - 4C - 12D) = 2x^3 - x + 3.$$

$$x^3: -12A = 2 \Rightarrow A = -\frac{1}{6}$$

$$x^2: -12A - 12B = 0$$

$$-12A - 12B = 0 \Rightarrow B = \frac{1}{6}$$

$$x: 6A - 8B - 12C = -1 \Rightarrow C = \frac{1}{9}$$

$$x^0: 2B - 4C - 12D = 3 \Rightarrow D = -\frac{5}{27}.$$

$$\begin{matrix} g(x) \\ \beta x \\ a e \end{matrix}$$

$$\begin{matrix} y_p(x) & \text{guess} \\ A \beta x & e \end{matrix}$$

$$a \cos(\beta x)$$

$$A \cos(\beta x) + B \sin(\beta x)$$

$$a \cos(\beta x) + b \sin(\beta x)$$

$$A \cos(\beta x) + B \sin(\beta x)$$

n^{th} degree polynomial

$$A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0$$

$$P_n(x)$$

Example:- Find particular solution.

$$y'' - 4y' - 12y = xe^{4x}$$

$$y_p(x): g(x) = x e^{4x} \rightarrow \text{exp.}$$

1st degree polynomial

$$y_p(x) = (Ax + B) \cdot e^{4x}$$

$$= [(Ac)x + Bc] e^{4x}$$

$$= (Ax + B) e^{4x}$$

$$y_p' = 4e^{4x}(Ax + B) + Ae^{4x}$$

$$y_p'' = 16Ax e^{4x} + 16e^{4x}B + BAe^{4x}$$

$$\cancel{16Ax e^{4x}} + \cancel{16e^{4x}B} + (4A - 12B)e^{4x} = xe^{4x}$$

Set coefficient equal:-

$$xe^{4x}: -12A = 1 \Rightarrow A = -\frac{1}{12}$$

$$e^{4x}: 4A - 12B = 0 \Rightarrow B = -\frac{1}{36}$$

$$y_p(x) = \left(-\frac{1}{12}x - \frac{1}{36}\right) e^{4x}$$

Example:-

$$y'' - 4y' - 12y = e^{6x}$$

$$y_p(x) = Ae^{6x}$$

$$y_p'(x) = 6Ae^{6x}$$

$$y_p''(x) = 36Ae^{6x}$$

$$36Ae^{6x} - 24Ae^{6x} - 12Ae^{6x} = e^{6x}$$

$$Zero = e^{6x} \Rightarrow \text{problem}$$

y_c

$$y'' - 4y' - 12y = 0$$

$$r^2 - 4r - 12 = 0$$

$$(r+2)(r-6) = 0$$

$$r_1 = -2$$

$$r_2 = 6$$

$$y_1 = e^{-2x}, y_2 = e^{6x}$$

$$y_c = c_1 e^{-2x} + c_2 e^{6x}$$

$$y_p = Ae^{6x} \Rightarrow x \text{ نفي}$$

$$(y_p)_{\text{new}} = Axe^{6x}$$

$$\xi y_p = Axe^{6x}$$

$$\begin{cases} y_p' = e^{6x}(A + 6Ax) \\ y_p'' = e^{6x}(12A + 36Ax) \end{cases}$$

$$(36A - 24A - 12A)x e^{6x} + 8Ae^{6x} = e^{6x}$$

$\underbrace{36A - 24A - 12A}_{\text{Zero}}$

$$8Ae^{6x} = e^{6x}$$

$$y_p = \frac{1}{8} x e^{6x}$$

$$A = \frac{1}{8}$$

Example:-

$$y'' + 3y' - 28y = 7x + e^{-7x} - 1$$

$$g(x) = 7x + e^{-7x} - 1$$

$$= \underbrace{(7x - 1)}_{\text{Polynomial}} + \underbrace{e^{-7x}}_{\text{exp}}$$

$$y_p = (Ax + B) + \underbrace{ce^{-7x}}$$

check if guess is correct.

$$\cancel{y'' + 3y' - 28y} = 0$$

$$(r - 4)(r + 7) = 0$$

$$r_1 = 4, r_2 = -7$$

$$y_1 = e^{-4x} \quad y_2 = \underbrace{e^{-7x}}$$

$$(y_p)_{\text{new}} = (Ax + B) + ce^{-7x}$$

ask ; believe & recieve

Example:-

$$y'' - 100y = 9x^2 e^{10x} + \cos x - x \sin x.$$

$$y'' - 100y = 0.$$

$$r^2 - 100 = 0.$$

$$r = \pm 10.$$

$$r_1 = e^{10x}$$

$$r_2 = e^{-10x}$$

$$9x^2 e^{10x} = (Ax^2 + Bx + C)e^{10x}$$

$$\cos x = D \cos x + E \sin x$$

$$x \sin x = (Fx + H)(\cos x + J \sin x)$$

$$= x(F \cos x + H \sin x)$$

$$y_p = (Ax^2 + Bx + C)e^{10x} + (Fx + D)\cos x + (Hx + E)\sin x$$

if y_{p1} particular solution $y'' + p(x)y' + q(x)y = g_1(x)$.

if y_{p2} " " " $y'' + p(x)y' + q(x)y = g_2(x)$.

then

$$y_p = y_{p1} + y_{p2} \text{ solution} = g_1(x) + g_2(x)$$

$$y'' - 4y' - 12y = \begin{cases} 3e^{5x} \rightarrow y_{p1} = Ae^{5x} \\ \sin 12x \rightarrow y_{p2} = C \sin 12x + D \cos 12x \\ x e^{4x} \rightarrow y_{p3} = (Ex + F)e^{4x} \end{cases}$$

First Al o l o

* Variation of parameters method: ~~3-1-1~~

$$P(x)y'' + Q(x)y' + r(x)y = g(x)$$

$$y_c = C_1 y_1 + C_2 y_2$$

$$P(x)y'' + Q(x)y' + r(x)y = 0$$

y_1, y_2 } Fundamental set of solution
for the homogeneous eqn.

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$u_1(x), u_2(x)$ unknown function.

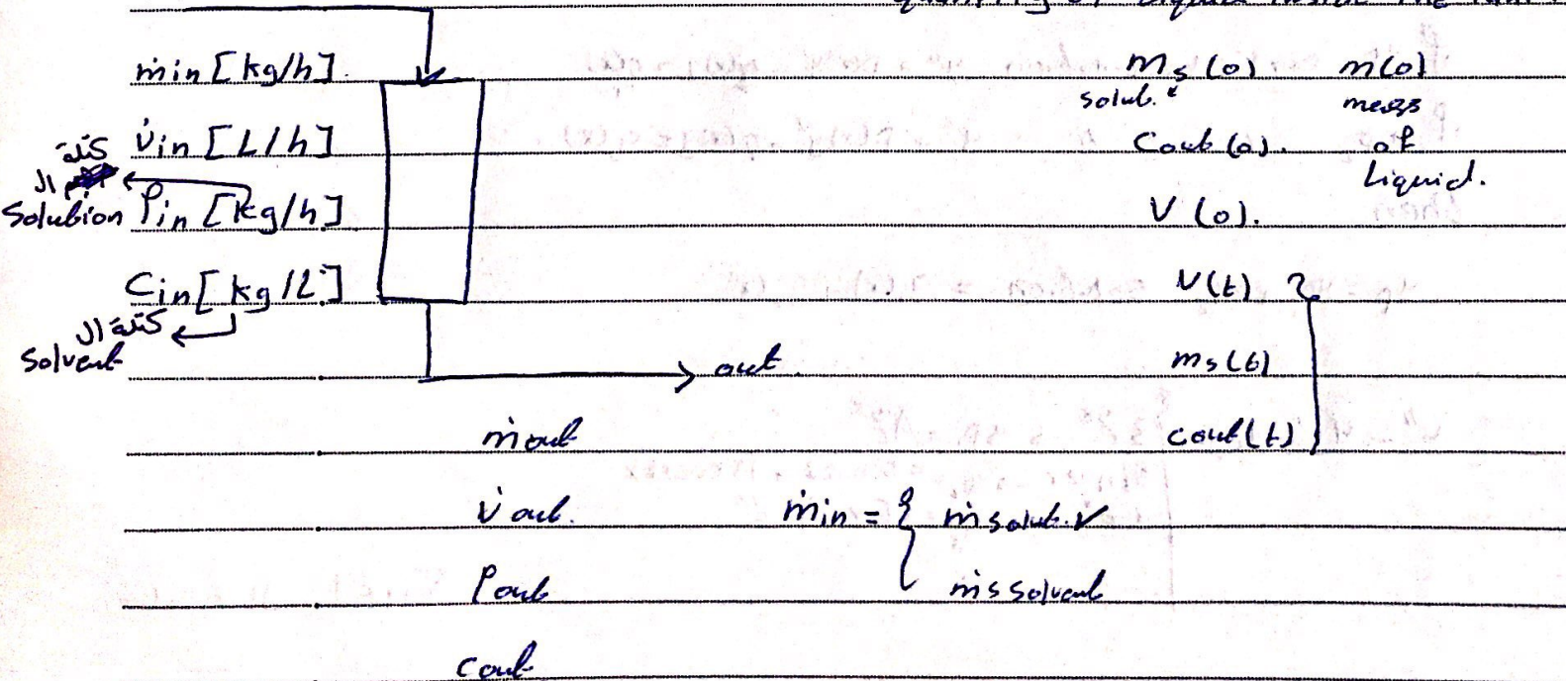
Mathematical.

* Modeling in chemical engineering. ~~دانیال~~

- mixing problem:

in

quantity of liquid inside the tank.



overall material balance

$$m_{in}(t) - m_{out}(t) = \frac{d(m(t))}{dt}$$

$$m = \rho \dot{V}$$

$$m = \rho_{out} \dot{V}$$

$$\rho_{in} \dot{V}_{in} - \rho_{out} \dot{V}_{out} = \frac{d(\rho_{out} V)}{dt}$$

Assume ρ constant

$$\rho (\dot{V}_{in} - \dot{V}_{out}) = \rho \frac{dV}{dt}$$

$$\boxed{\dot{V}_{in} - \dot{V}_{out} = \frac{dV}{dt}} \quad (1)$$

$$V(t) = B$$

component MB. for dissolved substances

$$C \left[\frac{\text{kg of } s}{L} \right] \dot{V} \left[\frac{L}{h} \right] = C \dot{V} \left[\frac{\text{kg of } s}{h} \right]$$

$$C_{in} \dot{V}_{in} - C_{out} \dot{V}_{out} = \frac{d(m_s)}{dt}$$

$$m_s = C_{out} V$$

$$[\text{kg of } s] = \left[\frac{\text{kg of } s}{L} \right] [L]$$

$$\Rightarrow C_{out} = \frac{m_s}{V}$$

$$C_{in} \dot{V}_{in} - \left(\frac{\dot{V}_{out}}{V} \right) m_s(t) = \frac{dm_s}{dt}$$

$$\boxed{\frac{dm_s}{dt} + \left(\frac{\dot{V}_{out}}{V} \right) m_s = C_{in} \dot{V}_{in}}$$

I.C: $m_s(0) = 0 \rightarrow \text{constant}$

$$C_{in} \dot{V}_{in} - C_{out} \dot{V}_{out} = \frac{d(V C_{out})}{dt}$$

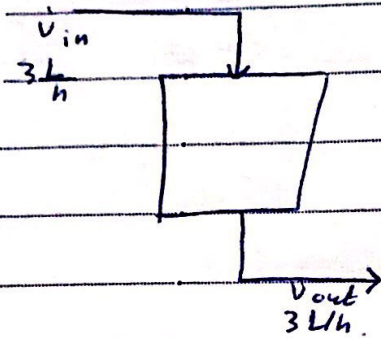
$$\boxed{\frac{d(V C_{out})}{dt} + \dot{V}_{out} C_{out} = C_{in} \dot{V}_{in}} \quad (3)$$

$$C_{out}(t_0) = r$$

$$\frac{dm_s}{dt} + (\dot{V}_{out}) m_s = C_{in} V_{in} \quad (2)$$

Comb: Eq 1 and 3

m_s : Eq 1 and 2



$$\dot{V}_{in} - \dot{V}_{out} = \frac{dV}{dt}$$

$$\dot{V}_{in} = \dot{V}_{out}$$

$$\frac{dV}{dt} = 0 \Rightarrow V(t) \text{ constant.}$$

Equation 2

$$\dot{V}_{in} = \dot{V}_{out} = \dot{V}$$

$$\frac{dm_s}{dt} + \left(\frac{\dot{V}}{V}\right) m_s = C_{in} V_{in}$$

$$\left(\frac{\dot{V}}{V}\right) = \left[\frac{\frac{1}{h}}{\frac{L}{h}}\right] = \left[\frac{1}{L}\right]$$

$$\left(\frac{V}{\dot{V}}\right) = \tau \text{ time constant.}$$

Ans.

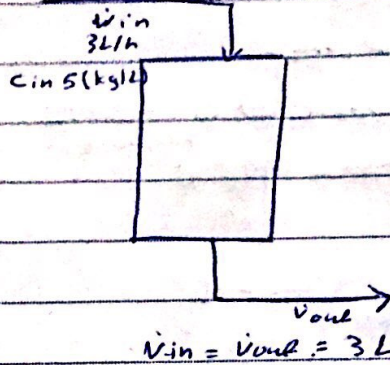
$$\frac{dm_s}{dt} + \frac{1}{\tau} m_s = C_{in} V_{in}$$

$$\xrightarrow{\text{Zero.}} \tau \frac{dm_s}{dt} + m_s = \tau C_{in} V_{in}$$

Example:-

$$m_s(0) = 2 \text{ kg} \quad \left. \begin{array}{l} \text{initial} \\ \text{condition} \end{array} \right\}$$

$$V = 800 \text{ L}$$



$$\frac{dm_s}{dt} + \left(\frac{v}{V}\right) m_s = c_{in} v$$

$$\frac{dm_s}{dt} + \left(\frac{3}{800}\right) m_s = 15$$

$$v_{in} = v_{out} = 3 \text{ L/h}$$

$$m_s(0) = 2$$

integrating factor.

$$y' + p(x)y = q(x)$$

$$M = e^{\int p(x) dx}$$

$$p(x) = \frac{3}{800}$$

$$\int \frac{3}{800} dt = \frac{3}{800} t$$

$$M = e^{\frac{3}{800} t} \rightarrow \text{multiplying both sides}$$

$$\frac{3}{800} t e^{\left(\frac{3}{800}\right)t} m_s' + e^{\left(\frac{3}{800}\right)t} \left(\frac{3}{800}\right) m_s = e^{\left(\frac{3}{800}\right)t} \cdot 15$$

$$\left(m_s \cdot e^{\left(\frac{3}{800}\right)t}\right)' = m_s' \cdot e^{\left(\frac{3}{800}\right)t} + m_s \cdot \frac{3}{800} e^{\left(\frac{3}{800}\right)t}$$

$$\frac{d}{dt} \left[m_s e^{\left(\frac{3}{800}\right)t} \right] = \left[m_s e^{\left(\frac{3}{800}\right)t} \right]' = 15 e^{\left(\frac{3}{800}\right)t}$$

$$\int d(m_s e^{\frac{3}{800}t}) = \int 15 e^{\frac{3}{800}t} dt + C$$

$$m_s e^{\left(\frac{3}{800}\right)t} = 15 \cdot \frac{1}{\frac{3}{800}} e^{\frac{3}{800}t} + C$$

$$\begin{aligned} m_s e^{\left(\frac{3}{800}\right)t} &= 5 \times 800 e^{\frac{3}{800}t} + C \\ &= 4000 e^{\frac{3}{800}t} + C \end{aligned}$$

$$m_s = 4000 + e^{-\frac{3}{800}t} C \Rightarrow$$

* Find C_3 .

at $t=0$ $m_s=2$

$$2 = 4000 + C \Rightarrow C = -3998$$

$(-3/800)t$

$$m_s = 4000 - 3998 e$$

author
solution
→ $m'_s + \left(\frac{3}{800}\right) m_s = 15$

$$y = y_c + y_p$$

$$m'_s + \left(\frac{3}{800}\right) m_s = 0$$

$$r + \frac{3}{800} = 0$$

$$r = -\frac{3}{800}$$

$$m_s = e^{(-3/800)t}$$

Gross m_p , $g(x) = 15$

$$m_p = A$$

$$m'_p = 0$$

$$\left(\frac{3}{800}\right) A = 15$$

$$A = 5 \times 800$$

$$A = 4000$$

$$(-3/800)t$$

$$y = e^{\dots} + C + 4000$$

$$2 = C + 4000$$

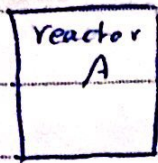
$$C = -3998$$

$$m_s(t) = 4000 - 3998 e^{(-3/800)t}$$

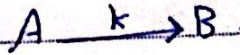
$$Cost = \frac{m_s(t)}{V} = \frac{4000 - 3998 e^{(-3/800)t}}{8000}$$

$$= 5 - \frac{3998}{8000} e^{(-3/800)t}$$

*



Batch



1st order.

$$[r_A] \propto -C_A(t)$$

$$\frac{dC_A}{dt} = r_A = -kC_A(t)$$

$$\frac{dC_A}{dt} = -kC_A(t)$$

$$C_A(0) = C_{A0}$$

$$\frac{dC_A}{dt} = -k dt$$

$$\ln C_A = -kt + C$$

$$C_A = e^{-kt+C} = e^{-kt} \cdot e^C$$

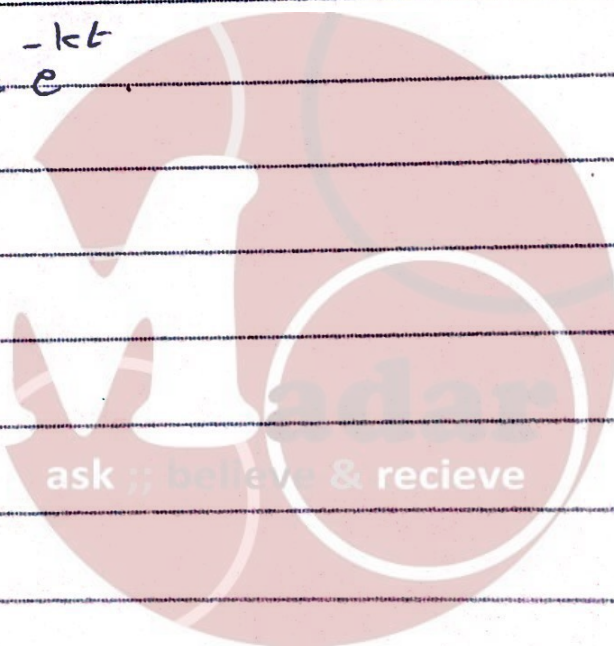
$$C_A = C_1 e^{-kt}$$

$$\boxed{C_1 = C_{A0}} \Rightarrow \frac{C_A}{C_{A0}} = e^{-kt}$$

if 2nd order:-

$$r_A \propto C_A^2$$

$$\frac{dC_A}{dt} = -kC_A^2$$



* Variation parameter method:

$$p(x)y'' + Q(x)y' + r(x)y = g(x) \quad \text{--- } \textcircled{*}$$

$$y_c = C_1 y_1 + C_2 y_2 \quad \text{complementary solution}$$

$$\text{Let } y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

y_1, y_2 set of fundamental solution

$u_1(x), u_2(x)$ unknown function?

To determine $u_1(x)$ and $u_2(x)$ we need 2 equation.

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

$$\text{Assume: } u_1'y_1 + u_2'y_2 = 0$$

$$\Rightarrow y_p'(x) = u_1y_1' + u_2y_2'$$

$$y_p''(x) = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

$\Rightarrow$ Substitute in equation *:

$$p(x)(u_1'y_1' + u_2'y_2') + u_1(x)(\overset{=0}{p(x)y_1'' + Q(x)y_1' + r(x)y_1}) + u_2(x)(\overset{=0}{p(x)y_2'' + Q(x)y_2' + r(x)y_2}) = g(x)$$

$$\Rightarrow p(x)(u_1'y_1' + u_2'y_2') = g(x)$$

$$\text{2 eqs: } u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1 + u_2'y_2 = \frac{g(x)}{p(x)}$$

if $p(x) = 1$

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(x) \end{cases} \quad \text{Linear system in } u_1' \text{ and } u_2'$$

$$u_1' \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} + u_2' \begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(x) \end{bmatrix}$$

$$u_1' = -\left(\frac{y_2}{y_1}\right) u_2'$$

$$u_1' = \frac{-y_2 g(x)}{y_1 y_2' - y_2 y_1'}$$

$$u_2' = \frac{y_1 g(x)}{y_1 y_2' - y_2 y_1'}$$

Note $\rightarrow y_1 y_2' - y_2 y_1' \neq 0$

y_1 and y_2 fundamental set

$$\begin{aligned} W(y_1, y_2)(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ \text{Wronskian} &= y_1 y_2' - y_2 y_1' \end{aligned}$$

$$u_1(x) = - \int \frac{y_2 g(x)}{y_1 y_2' - y_2 y_1'} dx$$

$$u_2(x) = \int \frac{y_1 g(x)}{y_1 y_2' - y_2 y_1'} dx$$

$$y_p = y_c + y_p, \quad y_p = u_1 y_1 + u_2 y_2$$

$$y_p(x) = \underbrace{-y_1 \int \left(\frac{y_2 g(x)}{W(y_1, y_2)} \right) dx}_{u_1} + \underbrace{y_2 \int \left(\frac{y_1 g(x)}{W(y_1, y_2)} \right) dx}_{u_2}$$

Example: Find the general solution.

$$2y'' + 18y = 6 \tan(3x).$$

$$P(x) = 1$$

$$y'' + 9y = 3 \tan(3x).$$

$$y'' + 9y = 0.$$

$$r^2 + 9 = 0.$$

$$r_{1,2} = \pm 3i = 0 \mp 3i$$

$$y_c = e^{\frac{P}{2}x}$$

$$y_c = e^{\frac{P}{2}x} (C_1 \cos(Im \cdot x) + C_2 \sin(Im \cdot x))$$

$$y_c = C_1 \cos(3x) + C_2 \sin(3x).$$

$$y_1 = \cos(3x) \rightarrow y_1 = -3 \sin(3x).$$

$$y_2 = \sin(3x) \rightarrow y_2 = 3 \cos(3x).$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix}$$

$$= -3 \cos^2(3x) + 3 \sin^2(3x).$$

$$= 3 (\cos^2(3x) + \sin^2(3x))$$

$$W(y_1, y_2) = 3.$$

$$u_1 = \int \frac{\sin(3x) \cdot 3 \tan(3x)}{3} dx.$$

$$= \int \frac{\sin^2(3x)}{\cos(3x)} dx.$$

$$u_2 = \int \frac{\cos(3x) \cdot 3 \tan(3x)}{3} dx = \int \frac{\cos(3x) \cdot \sin(3x)}{\cos(3x)} dx.$$

$$u_2 = \int \sin(3x) dx.$$

* Higher order linear ODE :-

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = g(x) \quad \text{non homogeneous}$$

$$= 0 \quad \text{homogeneous}$$

$$y = e^{rx}$$

$$y' = r e^{rx}$$

$$y'' = r^2 e^{rx}$$

⋮

$$y^{(n)} = r^n e^{rx}$$

$$e^{rx} (a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0) = 0$$

$P_n(x)$ characteristic

polynomial degree n ~~equation~~
(equation)

$$e^{rx} \neq 0$$

$$P_n(x) = a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$$

Solve for r

n^{th} order, Linear ODE:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = g(x)$$

$$\text{ICs: } y(x_0) = \alpha_1, y'(x_0) = \alpha_2, \dots, y^{(n-1)}(x_0) = \alpha_n$$

Variation of parameters in ICs

$$y_p = \sum_{k=1}^n y_k(x) U_k(x) = \sum_{k=1}^n y_k(x) \int \frac{w_k(x)}{W(x)} g(x) dx$$

$$y_p = y_1(x) \underbrace{\int \frac{w_1(x)}{W(x)} g(x) dx}_{u_1} + \dots + y_n(x) \underbrace{\int \frac{w_n(x)}{W(x)} g(x) dx}_{u_n}$$

y_k $k=1, \dots, n$ complementary solution

y_p particular solution of Non homogeneous

W_k is obtained by replacing the k^{th} column of W by $[0 \dots 0 1]^T$

$$\begin{array}{c} W = \begin{vmatrix} y_1 & y_2 & \dots & y_k & \dots & y_n \\ y_1' & y_2' & \dots & y_k' & \dots & y_n' \\ \vdots & \vdots & & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_k^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \\ \text{---} \\ \begin{vmatrix} y_1 & y_2 & \dots & y_k & \dots & y_n \\ y_1' & y_2' & \dots & y_k' & \dots & y_n' \\ \vdots & \vdots & & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_k^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} = W_k \end{array}$$

$$n=2$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2'' \\ 1 & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix}$$

Example: $y^{(3)} - 2y' - 21y' - 18y = 3 + 4e^x$

$$r^3 - 2r^2 - 21r - 18 = 0$$

$$r_1 = -3, r_2 = -1, r_3 = 6$$

$$y_1 = e^{-3x}, y_1' = -3e^{-3x}, y_1'' = 9e^{-3x}$$

$$y_c = c_1 e^{-3x} + c_2 e^{-x} + c_3 e^{6x}$$

$$y_2 = e^{-x}, y_2' = -e^{-x}, y_2'' = e^{-x}$$

$$y_3 = e^{6x}, y_3' = 6e^{6x}, y_3'' = 36e^{6x}$$

$$W = \begin{vmatrix} e^{-3x} & e^{-x} & e^{6x} \\ -3e^{-3x} & -e^{-x} & 6e^{6x} \\ 9e^{-3x} & e^{-x} & 36e^{6x} \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} & e^{6x} \\ 0 & -e^{-x} & 6e^{6x} \\ 1 & e^{-x} & 36e^{6x} \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^{-3x} & 0 & e^{6x} \\ -3e^{-3x} & 0 & 6e^{6x} \\ 9e^{-3x} & 1 & 36e^{6x} \end{vmatrix}$$

$$W_3 = \begin{vmatrix} e^{-3x} & e^{-x} & 0 \\ -3e^{-3x} & -e^{-x} & 0 \\ 9e^{-3x} & e^{-x} & 1 \end{vmatrix}$$

$$W = 126 e^{2x}$$

$$W_1 = 7 e^{5x}$$

$$W_2 = -9 e^{3x}$$

$$W_3 = 2 e^{-4x}$$

$$u_1 = \int \frac{W_1}{W} g(x) dx =$$

$$= \int \frac{7 e^{5x}}{126 e^{2x}} (3 + 4 e^{4x}) dx = \frac{1}{18} \int (3 e^{3x} + 4 e^{2x}) dx$$

$$\Rightarrow y = e^{-3x} \cdot u_1 = \frac{1}{18} (e^{3x} + 2 e^{2x})$$

$$u_2 = \int \frac{W_2}{W} g(x) dx$$

$$= \int \frac{-9 e^{3x}}{126 e^{2x}} (3 + 4 e^{4x}) dx = -\frac{1}{14} (3 e^x + 4x)$$

$$u_3 = \int \frac{W_3}{W} g(x) dx$$

$$= \int \frac{2 e^{-4x}}{126 e^{2x}} (3 + 4 e^{4x}) dx = \frac{1}{63} \left(-\frac{1}{2} e^{-6x} - \frac{4}{7} e^{-7x} \right)$$

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

$$= -\frac{1}{6} + \frac{5}{49} e^{-x} - \frac{2}{7} x e^{-x}$$

$$\Rightarrow y = c_1 e^{-3x} + c_2 e^{-x} + c_3 e^{2x} - \frac{1}{6} + \frac{5}{49} e^{-x} - \frac{2}{7} x e^{-x}$$

*Euler - Cauchy equation.

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x.$$

$$\Rightarrow y''' - \frac{3}{x} y'' + \frac{6}{x^2} y' - \frac{6}{x^3} y = x \ln x.$$

$$\text{Solution: } y = x^m, y' = m x^{m-1}, y'' = m(m-1) x^{m-2}, y''' = m(m-1)(m-2) x^{m-3}$$

$\Rightarrow$ Sub in Eq i-

characteristic equation $m^3 - 6m^2 + 11m - 6 = 0$.

Solve form

$$m_1 = 1 \quad y_1 = x$$

$$m_2 = 2 \quad y_2 = x^2$$

$$m_3 = 3 \quad y_3 = x^3$$

$$y_c = C_1 x + C_2 x^2 + C_3 x^3$$

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

$$U_1 = \int \frac{W_1(x)}{W(x)} g(x) dx$$

$$* y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

(by part)

$$= \int \frac{x^4}{2x^3} x \ln x dx = \frac{1}{2} \left[\frac{x^3}{3} \ln x - \frac{1}{9} x^3 \right]$$

$$U_2 = \int \frac{-2x^3}{2x^3} x \ln x dx = - \left(\frac{x^2}{2} \ln x - \frac{x^2}{4} \right)$$

$$U_3 = \int \frac{x^2}{2x^3} x \ln x dx = x \ln x - x$$

$$\begin{aligned} * y_1' &= f_1(y_1, y_2) \\ y_2' &= f_2(y_1, y_2) \end{aligned} \quad \left. \begin{array}{l} \text{if } y_1 \text{ and } y_2 \text{ are functions of } x \\ \text{coupled} \end{array} \right\}$$

إذا كانا متغيرين

* System of Linear 1st order ODE:-

* Homogeneous

2 1st ODE, Linear in

$$y_1' = a_{11} y_1 + a_{12} y_2 \quad \text{--- (1)}$$

$$y_2' = a_{21} y_1 + a_{22} y_2 \quad \text{--- (2)}$$

$[a_{ij}]_{i,j=1,2}$ coefficient.

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \begin{array}{l} * y_1 = y_1(t) \\ * y_2 = y_2(t) \end{array}$$

$$y' = Ay$$

* method of substitution:-

* substitution method:- convert system to a single ODE

1. Direct method

Guess solution & verify the method

- From eq ①

$$y_2 = \frac{1}{a_{12}} [y_1' - a_{11}y_1] \quad \text{--- (3)}$$

- Diff eq ①

$$y_1'' = a_{11}y_1' + a_{12}y_2' \quad \text{--- (4)}$$

- Sub ③ in ④

$$y_1'' - (a_{11} + a_{22})y_1' + (a_{22}a_{11} - a_{12}a_{21}) = 0$$

2nd ODE Homogeneous.

$$y_1'' + ay_1' + by_1 = 0$$

$$\left\{ \begin{array}{l} y_1 = C_1 e^{r_1 t} + C_2 e^{r_2 t} \rightarrow r_{1,2} \text{ distinct and real} \end{array} \right.$$

$$y_1 = (C_1 + C_2 t) e^{rt} \rightarrow r \text{ real and repeated}$$

$$y_1 = e^{P \pm iQ} [C_1 \cos(\text{Im}(v)t) + C_2 \sin(\text{Im}(v)t)] \rightarrow \text{Complex}$$

Solution in eq ③ to get y_2 .

Substitution

$$\text{Ex: } X' = 4X + 2y.$$

$$y' = x + y.$$

$$y = \frac{1}{2} [4x - x'].$$

$$y = 2x - \frac{x'}{2} \quad \text{--- (3)}$$

$$x'' = 4x' + 2(x+y) \quad \text{--- (4)}$$

$$x'' - 5x' + 6x = 0.$$

$$r^2 - 5r + 6 = 0.$$

$$(r-3)(r+2) = 0.$$

$$\begin{aligned} x = y_1 &= c_2 e^{3t} + c_1 e^{2t} \\ x' &= 3c_2 e^{3t} + 2c_1 e^{2t} \quad \text{sub in (3)} \end{aligned}$$

$$y = y_2 = 2(c_1 e^{2t} + c_2 e^{3t}) - \frac{1}{2}(2c_1 e^{2t} + 3c_2 e^{3t})$$

$$\begin{aligned} y_2 = y_1 &= c_1 e^{2t} + \frac{1}{2} c_2 e^{3t} \\ y_1 = x &= c_1 e^{2t} + c_2 e^{3t} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{ solutions.}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 & \frac{1}{2} c_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{3t} \end{bmatrix}$$

OR.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 \\ c_1 \end{bmatrix}}_{\text{eigenvector}} e^{2t} + \underbrace{\begin{bmatrix} \frac{1}{2} c_2 \\ c_2 \end{bmatrix}}_{\text{eigenvector}} e^{3t}$$

eigen values

* Direct method for system of ODE:-

in dependent variable $\swarrow$ $\searrow$ dependent variable

$$\begin{cases} y_1' = f_1(t, y_1, y_2, \dots, y_n) \\ y_2' = f_2(t, y_1, y_2, \dots, y_n) \\ \vdots \\ y_n' = f_n(t, y_1, y_2, \dots, y_n) \end{cases} \quad (1)$$

Solution of (1):-

$$y_1 = h_1(t)$$

$$y_2 = h_2(t)$$

$$\vdots$$

$$y_n = h_n(t)$$

- Linear system is given by:-

$$\begin{cases} y_1' = a_{11}(t)y_1(t) + a_{12}(t)y_2(t) + \dots + a_{1n}(t)y_n(t) + g_1(t) \\ y_2' = a_{21}(t)y_1(t) + a_{22}(t)y_2(t) + \dots + a_{2n}(t)y_n(t) + g_2(t) \\ \vdots \\ y_n' = a_{n1}(t)y_1(t) + a_{n2}(t)y_2(t) + \dots + a_{nn}(t)y_n(t) + g_n(t) \end{cases}$$

in vector for n:-

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}_{n \times 1} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n \times n} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n \times 1} + \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}_{n \times 1}$$

$\bar{y}' = A \bar{y} + g$

if $q = 0 \Rightarrow$ homogeneous

$\bar{y}' = A\bar{y} \Rightarrow$ homogeneous linear system

- if $\bar{y}^{(1)}$ and $\bar{y}^{(2)}, \dots, \bar{y}^{(n)}$ solution.

then $\bar{y} = c_1 \bar{y}^{(1)} + c_2 \bar{y}^{(2)} + \dots + c_n \bar{y}^{(n)}$

$\bar{y}^{(1)} ? , \bar{y}^{(2)} ?$

Eigenvalue problem.

* 1st ODE Linear.

$y' = -ky$ Linear.

$$y' + ky = 0$$

$$r + k = 0$$

$$r = -k$$

$$y = e^{-kt}$$

$$y = c e^{+kt}$$

$$y_1 = c_1 e^{+kt} \Rightarrow x_1 e^{+kt}$$

$$y_2 = c_2 e^{+kt} = x_2 e^{+kt}$$

$$y'' + ky' + y = 0$$

$$y_n = x_n e^{+kt}$$

$$\bar{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} e^{\lambda t} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$y_1 = x_1 e^{\lambda t}$$

$$\vdots$$

$$y_n = x_n e^{\lambda t}$$

} λ ?

$$\rightarrow y_1' = \lambda x_1 e^{\lambda t}$$

$$y_2' = \lambda x_2 e^{\lambda t}$$

$\vdots$

$$y_n' = \lambda x_n e^{\lambda t}$$

$$\bar{Y}' = A\bar{Y}$$

$$\begin{bmatrix} \lambda x_1 e^{\lambda t} \\ \lambda x_2 e^{\lambda t} \\ \vdots \\ \lambda x_n e^{\lambda t} \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_n e^{\lambda t} \end{bmatrix}$$

$$\lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} e^{\lambda t} = \boxed{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} e^{\lambda t}$$

$$e^{\lambda t} \neq 0$$

$$\lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \boxed{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\bar{X} \quad A \quad \bar{X}$$

$$\Rightarrow \lambda \bar{X} = A \bar{X}$$

$$\Rightarrow A \bar{X} = \lambda \bar{X} \text{ is eigen value problem.}$$



Determine

eigen value $\leftarrow \lambda$

eigen vector $\leftarrow \bar{X}$

$$A \bar{X} = \lambda \bar{X}$$



Linear system

Solution is eigen value

$$\lambda_1, \lambda_2, \dots, \lambda_n$$

n eigen vector

$$\bar{X}^{(1)}, \bar{X}^{(2)}, \dots, \bar{X}^{(n)}$$

General solution.

~~$\forall \lambda \in \mathbb{C}, \lambda \neq 0$~~

$$\bar{Y} = C_1 \bar{X}^{(1)} e^{\lambda_1 t} + C_2 \bar{X}^{(2)} e^{\lambda_2 t} + \dots + C_n \bar{X}^{(n)} e^{\lambda_n t}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = C_1 \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} e^{\lambda_1 t} + C_2 \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} e^{\lambda_2 t} + \dots + C_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} e^{\lambda_n t}$$

$\bar{Y} \qquad \bar{X}^{(i)}$

$$-A\bar{X} - \lambda\bar{X} = 0$$

$$(A - \lambda I)\bar{X} = 0$$

$$\bar{X} \neq 0.$$

$$\Rightarrow (A - \lambda I) \Rightarrow \text{Singular.}$$

$$\det(A - \lambda I) = 0$$

This is given: polynomial of degree in λ

$P_n(\lambda) \rightarrow$ characteristic polynomial

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$y_1 = x_1 e^{\lambda t} \rightarrow y_1' = x_1 \lambda e^{\lambda t}$$

$$y_2 = x_2 e^{\lambda t} \rightarrow y_2' = x_2 \lambda e^{\lambda t}$$

$$y_n = x_n e^{\lambda t} \rightarrow y_n' = x_n \lambda e^{\lambda t}$$

$x_1, x_2, \dots, x_n$ constants

$$\bar{y}' = A \bar{y}$$

$$y_1' = a_{11} y_1 + a_{12} y_2 + \dots + a_{1n} y_n$$

$$y_n' = a_{n1} y_1 + a_{n2} y_2 + \dots + a_{nn} y_n$$

$$\begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$\bar{y}' \quad \quad \quad A \quad \quad \quad \bar{y}$
 $n \times 1 \quad \quad \quad n \times n \quad \quad \quad n \times 1$

$$\bar{y}' = A\bar{y} \text{ homogeneous.}$$

$$\lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} e^{\lambda t} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} e^{\lambda t}$$

$e^{\lambda t} \neq 0$

Vector form.

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \underline{A\bar{x} = \lambda \bar{x}}$$

$A \quad \quad \quad \bar{x} \quad \quad \quad \bar{x}$

Eigen value problem.

$$A\bar{x} = \lambda \bar{x}$$

↗ scalar.

$$A\bar{x} - \lambda \bar{x} = 0$$

$$\bar{x} (A - \lambda I) = 0$$

Singular.
 $\bar{x} \neq 0$

$$P_n(\lambda) = \det(A - \lambda I) = 0.$$

find λ_i 's

for each λ_i find corresponding eigen vector $\bar{x}^i$

$$\bar{y} = c_1 \bar{y}_1 + c_2 \bar{y}_2 + \dots + c_n \bar{y}_n$$

$$\bar{y} = c_1 \bar{x}_1 e^{\lambda_1 t} + c_2 \bar{x}_2 e^{\lambda_2 t} + \dots + c_n \bar{x}_n e^{\lambda_n t}$$

different roots.

Example:- $y_1' = -3y_1 + y_2$

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$y_2' = y_1 - 3y_2$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\bar{y}' = A \bar{y}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t} \rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \lambda e^{\lambda t}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \lambda e^{\lambda t}$$

$$= \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \lambda e^{\lambda t}$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$A\bar{x} = \lambda\bar{x}$ eigen value problem

$$(A - \lambda I)\bar{x} = 0$$

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix}$$

$$\Rightarrow \det(A - \lambda I) = 0$$

$$(-3-\lambda)(-3-\lambda) = 0$$

$$\lambda^2 + 6\lambda + 8 = 0$$

$$(\lambda + 2)(\lambda + 4) = 0$$

$$\lambda_1 = -2$$

$$\lambda_2 = -4$$

* Remark:-

$$A = [a_{ij}] A^{-1} = \frac{A}{a_{ii}} \Rightarrow 1$$

$$A^{-1}A = I$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I\bar{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & a_3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{a_1} & & 0 \\ & \frac{1}{a_2} & \\ 0 & & \frac{1}{a_3} \end{bmatrix}$$

ask :: believe & recieve

for $\lambda_1 = -2$ find $\bar{x}^1$

$$(A - \lambda I) \bar{x} = 0$$

$$A - \lambda_1 I = \begin{bmatrix} -3 - \lambda_1 & 1 \\ 1 & -3 - \lambda_1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + x_2 = 0$$

$$x_1 = x_2$$

$$\bar{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\bar{x}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = -4$ find $\bar{x}^2$

$$A - \lambda_2 I = \begin{bmatrix} -3 - \lambda_2 & 1 \\ 1 & -3 - \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + x_2 = 0 \\ x_1 = -x_2$$

$$\bar{x}^2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$y_1 = c_1 e^{-2t} + c_2 e^{-4t}$$

$$y_2 = c_1 e^{-2t} - c_2 e^{-4t}$$

* repeated:-

$$y_1 = e^{\lambda t}$$

$$y_2 = t e^{\lambda t}$$

$$y = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

$$\bar{y}' = A \bar{y}$$

$$x \rightarrow t$$

$$c_1 \rightarrow \bar{x}$$

$$\bar{y}_1 = \bar{x} e^{\lambda t}$$

$$\bar{y}_2 = t \bar{x} e^{\lambda t} \rightarrow \bar{y}_2' = \bar{x} e^{\lambda t} + t \lambda \bar{x} e^{\lambda t}$$

$$\underbrace{\bar{x} e^{\lambda t} + t \lambda \bar{x} e^{\lambda t}}_{\bar{y}_2'} = \underbrace{A t \bar{x} e^{\lambda t}}_{\bar{y}_2}$$

$$e^{\lambda t} \neq 0$$

$$\bar{x} + t \lambda \bar{x} = A t \bar{x}$$

Set coeff.

$$t^1: \lambda \bar{x} = A \bar{x}$$

$$t^0: 1 = 0$$

$\bar{y}_2$ modification

$$\bar{y}_2 = t \bar{x} e^{\lambda t} + \vec{\eta} e^{\lambda t}$$

?
new guess.

$$\bar{y}_2' = e^{\lambda t} \bar{x} + t \lambda e^{\lambda t} \bar{x} + \lambda e^{\lambda t} \vec{\eta}$$

$$\text{Sub. in } \bar{y}_2' = A \bar{y}_2$$

$$\Rightarrow A(t \bar{x} e^{\lambda t} + \vec{\eta} e^{\lambda t} + \bar{x} e^{\lambda t} + t \bar{x} e^{\lambda t}) = A t \bar{x} e^{\lambda t}$$

$$(\bar{x} + \lambda \vec{\eta}) + t(\lambda \bar{x}) = t(A \bar{x}) + A \vec{\eta}$$

set coeff. equal

$$L: \lambda \bar{x} = A \bar{x}$$

$$L^0: (\bar{x} + \lambda \bar{z}) = A \bar{z}$$

$$\begin{cases} A \bar{x} = \lambda \bar{x} \\ \bar{x} + \lambda \bar{z} = A \bar{z} \end{cases}$$

~~NOTE~~ $A \bar{z} - \lambda \bar{z} = \bar{x}$

$$(A - \lambda I) \bar{z} = \bar{x}$$

$$A \bar{x} = \lambda \bar{x} \quad \text{--- ①} \Rightarrow \text{Solve eigenvalue problem.}$$

$$(A - \lambda I) \bar{z} = \bar{x} \quad \text{--- ②}$$

$$A \bar{x} = \lambda \bar{x} \text{ find } \bar{x}$$

for unknown $\bar{x}$ find $\bar{z}$

General solution:-

$$\bar{y} = c_1 \bar{x} e^{\lambda t} + c_2 (t \bar{x} e^{\lambda t} + \bar{z} e^{\lambda t})$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t} + c_2 \left(t e^{\lambda t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + e^{\lambda t} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right)$$

Example:- $y_1' = 4y_1 - y_2$ $\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$y_2' = y_1 + 2y_2$$

$$A \bar{x} = \lambda \bar{x}$$

$$(A - \lambda I) \bar{x} = 0$$

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 4 - \lambda & -1 \\ 1 & 2 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 6\lambda + 9 = 0.$$

$$(\lambda_1 - 3)(\lambda_2 - 3) = 0.$$

$$\lambda_1 = \lambda_2 = 3$$

$$(A - \lambda I)\vec{x} = 0.$$

$$\begin{bmatrix} 4-3 & -1 \\ 1 & 2-3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$x_1 - x_2 = 0 \Rightarrow x_1 = x_2.$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda I)\vec{v} = \vec{x}.$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$v_1 - v_2 = 1$$

$$v_1 = 1 + v_2$$

$$\vec{v} = \begin{bmatrix} 1 + v_2 \\ v_2 \end{bmatrix}$$

$$\therefore v_2 = 1 \rightarrow v_1 = 2.$$

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \left(t e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right). \quad \#$$

$$y_1 = c_1 e^{3t} + c_2 t e^{3t} + c_2 \cdot 2 e^{3t}$$

$$y_2 = c_1 e^{3t} + c_2 t e^{3t} + c_2 e^{3t}$$

* Non homogeneous:-

- undetermined method coefficient.

$$\bar{Y}' = A\bar{Y} + g \quad \bar{Y} = \bar{Y}_c + \bar{Y}_p$$

Example:- $y_1' = -3y_1 + y_2 - 6e^{-2t}$

$y_2' = y_1 - 3y_2 + 2e^{-2t}$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix}$$

$$\bar{Y}' = A \bar{Y} + \underline{g}$$

* $\bar{Y}_c$: $\bar{Y}' = A\bar{Y} \rightarrow$ solve $A\bar{X} = \lambda I$.

$\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} \quad \left| \quad \det(A - \lambda I) \right.$$

$= \lambda^2 + 6\lambda + 8 = 0.$

$\lambda_1 = -2$

$\lambda_2 = -4.$

$\bar{Y}_c = \begin{bmatrix} y_{1c} \\ y_{2c} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} \quad / \quad \bar{X}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \bar{X}^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$g_1 = -6e^{-2t}, g_2 = 2e^{-2t}$

$\bar{Y}_p = \begin{bmatrix} y_{p1} \\ y_{p2} \end{bmatrix} = t e^{\lambda t} \bar{X} + e^{\lambda t} \bar{Z}$

$= (\bar{Z} + t\bar{X}) e^{\lambda t}$

$\begin{bmatrix} (\bar{Z}_1 + t\bar{X}_1) e^{\lambda t} \\ (\bar{Z}_2 + t\bar{X}_2) e^{\lambda t} \end{bmatrix} = \begin{bmatrix} (v_1 + u_1 t) e^{-2t} \\ (v_2 + u_2 t) e^{-2t} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{-2t}$

ask receive

$\underline{v}$ $\underline{u}$

$$A\bar{x} = \lambda\bar{x}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$u\vec{v} = \vec{0}$$

$$v = \vec{v}$$

$$\vec{y}_p = \begin{bmatrix} (v_1 + u_1 t)e^{-2t} \\ (v_2 + u_2 t)e^{-2t} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{-2t} + t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{-2t}$$

$$\vec{y}_p = \vec{v} e^{-2t} + t \vec{u} e^{-2t}$$

$$\begin{bmatrix} y_{p1}' \\ y_{p2}' \end{bmatrix} = -2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{-2t} - 2t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{-2t}$$

$$= \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 + u_1 t \\ v_2 + u_2 t \end{bmatrix} e^{-2t} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$\vec{y}' = A\vec{y} + \vec{g}$$

$$\begin{bmatrix} y_{p1}' \\ y_{p2}' \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \vec{v} \end{bmatrix} + \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \vec{u} \end{bmatrix} t + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

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$$\begin{bmatrix} \vec{u} \end{bmatrix} - 2 \begin{bmatrix} \vec{u} \end{bmatrix} t - 2 \begin{bmatrix} \vec{v} \end{bmatrix}$$

Set coefficient equal:-

$$t: A\vec{u} = -2\vec{u} \rightarrow \text{eigen value problem } \lambda = -2$$

$$t^0: \vec{u} - 2\vec{v} = A\vec{v} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$\vec{u} = \vec{x}$$

$$\vec{v} = \vec{z}$$

$$\Rightarrow A\bar{x} = -2\bar{x}$$

$$\bar{x} - 2\bar{x} = A\bar{x} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$A\bar{x} + 2\bar{x} = \bar{x} - \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$(A+2I)\bar{x} = \bar{x} - \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$-x_1 + x_2 = x_1 + 6$$

$$-x_1 + x_2 = -x_2 + 2$$

$$x_1 + 6 = x_2 + 2$$

$$\begin{array}{l|l} -x_1 + x_2 = x_1 + 6 & x_1 + 6 = -x_2 + 2 \quad \times [!]
-x_1 + x_2 = -x_2 + 2 & -x_2 + 2 \quad | \quad x_1 + x_2 = -4 \quad x_1 = x_2 \end{array}$$

$$2x_1 = -4$$

$$-x_1 + x_2 = -2 + 6$$

$$x_1 = -2$$

$$= 4$$

$$x_2 = 4 + x_1 \quad \text{if } x_1 = k \quad \text{if } x_1 = 0$$

$$\vec{x} = \begin{bmatrix} x_1 \\ 4 + x_1 \end{bmatrix} = \begin{bmatrix} k \\ 4 + k \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} y_{p1} \\ y_{p2} \end{bmatrix} &= \begin{bmatrix} (x_1'' + t x_1') e^{-2t} \\ (x_2'' + t x_2') e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} (0 - 2t) e^{-2t} \\ (4 - 2t) e^{-2t} \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} e^{-2t} \end{aligned}$$

Example 1- $y' = 2y_1 - 4y_2 + 2t^2 + 10t$

$y_2' = y_1 - 3y_2 + t^2 + 9t + 3$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$$

$\underline{Y}' \qquad \qquad \underline{A} \qquad \qquad \underline{Y} \qquad \qquad \underline{g}$

$A\underline{x} = \lambda \underline{x}$

$\lambda_1 = -2 \rightarrow \underline{x}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1 \rightarrow \underline{x}^2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\underline{Y}_p = \begin{bmatrix} y_{p1} \\ y_{p2} \end{bmatrix} = \begin{bmatrix} A_1 + B_1 t + C_1 t^2 \\ A_2 + B_2 t + C_2 t^2 \end{bmatrix}$

$\underline{Y}'_p = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + 2 \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} t$

$\underline{B} \qquad \qquad \underline{C}$

$= \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} A_1 + B_1 t + C_1 t^2 \\ A_2 + B_2 t + C_2 t^2 \end{bmatrix} + \begin{bmatrix} 2t^2 + 10t \\ t^2 + 9t + 3 \end{bmatrix}$

$\underline{A}$

* Remark:-

$\begin{bmatrix} A \\ A \end{bmatrix} + \begin{bmatrix} A \\ A \end{bmatrix} [B] t + \begin{bmatrix} A \\ A \end{bmatrix} [C] t^2 = \begin{bmatrix} 2 & -4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} A_1 + B_1 \\ A_2 + B_2 \end{bmatrix}$

$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + 2t \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} A \\ A \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} A \\ A \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} t + \begin{bmatrix} A \\ A \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} t^2 = \begin{bmatrix} 2(A_1 + B_1) - 4(A_2 + B_2) \\ 2(A_1 + B_1) - 3(A_2 + B_2) \end{bmatrix}$

$+ t^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 10 \\ 9 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Set coeff. equal:-

$t^2: \begin{bmatrix} A \\ A \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \text{Solve for } \underline{C}$

$t^1: 2 \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} A \\ A \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 9 \end{bmatrix} \text{ solve for } \underline{B}$

$t^0: \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A \\ A \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \text{ solve for } \underline{A}$

or

$\Rightarrow \begin{bmatrix} 2 & -4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A \\ A \end{bmatrix}$

$\therefore \underline{A} \underline{A} + \underline{AB}$

$$C = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y_p = A_1 + B_1 t + C_1 t^2$$

$$= 0 + 0 + 1t^2$$

$$y_{p2} = 0 + 3t$$

* Variation of parameters:-

$$\bar{Y}' = A(t)\bar{Y} + g(t) *$$

Example:- $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$

homogeneous & complementary solution

$$\bar{Y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}, \bar{Y}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} \Rightarrow \text{linearly independent}$$

$$\bar{Y}_c = C_1 \bar{Y}_1 + C_2 \bar{Y}_2 = \begin{bmatrix} t e^{-2t} & -4t \\ e^{-2t} & -e^{-2t} \end{bmatrix} + \begin{bmatrix} C_2 e^{-4t} \\ -C_2 e^{-4t} \end{bmatrix}$$

$$\bar{Y}_c = \begin{bmatrix} \bar{Y}_1 & \bar{Y}_2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$\bar{Y}_c = \bar{Y}_* C$$

$$\bar{Y}_c = \bar{Y}_* C$$

$\bar{Y}$ fundamental set of solution knowing $\bar{Y}(t)$

$$\bar{Y}_p = \bar{Y}(t) \cdot U(t)$$

Particular solution.

$$\bar{Y}_p' = \bar{Y}' \cdot U + \bar{Y} \cdot U' = A(t) \bar{Y} U + g *$$

$$\bar{Y}' = A \bar{Y} \text{ homogenous}$$

$$A \bar{Y} u + \bar{Y} u' = A \bar{Y} u + g$$

$$\Rightarrow \bar{Y} u' = g \Rightarrow u' = \bar{Y}^{-1} g$$

$$\det \bar{Y} \neq 0$$

$$\bar{Y}^{-1} = \frac{1}{2} \begin{bmatrix} 2e^{2t} & 2e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}$$

$$\Rightarrow u' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2e^{2t} & 2e^{2t} \\ 2e^{4t} & -e^{4t} \end{bmatrix} \begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix}$$

$$u' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix}$$

$$u(t) = \int_0^t \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix} dt$$

$$u = \begin{bmatrix} -2t |_0^t \\ -2e^{2t} |_0^t \end{bmatrix} = \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

$$\Rightarrow \bar{Y} p = \bar{Y} u = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

$$\bar{Y} p = \begin{bmatrix} -2t \\ -2t \end{bmatrix} e^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t} + e^{-4t} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\bar{Y}_{\text{General}} = c_1 \bar{Y}_1 + c_2 \bar{Y}_2 + \bar{Y} p$$

* procedure :-

• Find $\bar{Y}_c \Rightarrow \bar{Y} [\bar{Y}_1 \bar{Y}_2 \dots \bar{Y}_n]$

• Find $\bar{Y}^{-1}$

• Find $Y' = \bar{Y}^{-1} q$

• Find $Y = \int_0^t Y^{-1}(u) q(u) du$

• Find $\bar{Y}_p$

$\bar{Y}_p = \bar{Y} Y$

• Find general solution

$\bar{Y}_G = \bar{Y}_c + \bar{Y}_p$

* Remarks :-

A^{-1} ; $\det(A) \neq 0$.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots & \\ & & & d_n \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{1}{d_1} & & 0 \\ & \frac{1}{d_2} & \\ 0 & & \ddots & \\ & & & \frac{1}{d_n} \end{bmatrix}$

$A^{-1} \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$* A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{24-14} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 6/10 & -7/10 \\ -2/10 & 4/10 \end{bmatrix}$$

* Laplace Transforms:-

$$\begin{array}{ccc} \text{P(t)} & \xrightarrow{\mathcal{L}} & \text{F(s)} \\ \text{t-domain} & & \text{s-domain} \end{array}$$

$$\mathcal{L} [P(t)] = F(s)$$

$$\mathcal{L}^{-1} [F(s)] = P(t)$$

$$P(t) \quad t \geq 0$$

Definition:-

$$\mathcal{L} [P(t)] = \int_0^{\infty} P(t) e^{-st} dt = F(s)$$

Example:-

$$P(t) = a \text{ constant}$$

$$\mathcal{L} [a] = \int_0^{\infty} a e^{-st} dt = a \int_0^{\infty} e^{-st} dt$$

$$= a \left(-\frac{1}{s} e^{-st} \right) \Big|_0^{\infty} = a \left(0 + \frac{1}{s} \right) = \frac{a}{s} = F(s)$$

$$\text{if } a=1 \Rightarrow \mathcal{L} [1] = \frac{1}{s}$$

$$* f(t) = e^{-bt} \quad b > 0.$$

$$\begin{aligned} \mathcal{L}[e^{-bt}] &= \int_0^{\infty} e^{-bt} \cdot e^{-st} dt \\ &= \int_0^{\infty} e^{-(b+s)t} dt \end{aligned}$$

$$= \frac{-1}{b+s} e^{-(b+s)t} \Big|_0^{\infty}$$

$$= \frac{1}{s+b} \Rightarrow \mathcal{L}[e^{-bt}] = \frac{1}{s+b}$$

$$\mathcal{L}[e^{bt}] = \frac{1}{s-b}$$

$$* f(t) = t.$$

$$\mathcal{L}[t] = \int_0^{\infty} t e^{-st} dt \quad \text{by part.}$$

$$dv = e^{-st} dt$$

$$u = t$$

Linear equation. ~~2nd part~~ ~~find~~ *

$$* f(t) = \sin at$$

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2}$$

$$= \frac{f(t)}{a} + \frac{g(t)}{b}$$

$$\begin{aligned} \mathcal{L}[af(t) + bg(t)] &= \mathcal{L}[af(t)] + \mathcal{L}[bg(t)] \\ &= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] \\ &= aF(s) + bG(s) \end{aligned}$$

$$\begin{aligned} * \mathcal{L}^{-1}[aF(s) + bG(s)] &= a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)] \\ &= af(t) + bg(t) \end{aligned}$$

Example:- ~~Let~~ $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

$$\begin{aligned} &\mathcal{L}[6e^{-5t}] + \mathcal{L}[e^{3t}] + \mathcal{L}[5t^3] - \mathcal{L}[9] \\ &= 6\mathcal{L}[e^{-5t}] + \mathcal{L}[e^{3t}] + 5\mathcal{L}[t^3] - \mathcal{L}[9] \\ &= 6 \times \frac{1}{s+5} + \frac{1}{s-3} + 5\mathcal{L}[t^3] - \frac{9}{s} \end{aligned}$$

$t^n, n=1, 2, 3, \dots$

$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

$\downarrow 5 \times \frac{3!}{s^{3+1}}$

$$F(s) = \frac{6}{s} - \frac{1}{s+8} + \frac{4}{s-3}$$

$$\begin{aligned} \mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}\left[\frac{6}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+8}\right] + \mathcal{L}^{-1}\left[\frac{4}{s-3}\right] \\ &= 6 - e^{-8t} + 4e^{3t} \end{aligned}$$

Example:- $F(s) = \frac{19}{s+2} + \frac{1}{3s-5} + \frac{7}{s^5}$

$f(t) = 19e^{-2t} + \frac{1}{3}e^{\frac{5}{3}t} + \frac{7}{4!}t^4$

$L^{-1}\left[\frac{7}{s^5}\right] = 7L^{-1}\left[\frac{1}{s^5}\right]$

$= 7\left[\frac{1}{s^5} \frac{4!}{4!}\right]$

$= \frac{7}{4!} \left[\frac{4!}{s^5}\right] = \frac{7}{4!} t^4$

Example:- $F(s) = \frac{8}{3s^2+12} + \frac{3}{s^2-49}$

$= \frac{8}{3} \cdot \frac{1}{s^2+4} + \frac{3}{s^2-49}$

$= \frac{8}{3} \cdot \frac{1}{s^2+2^2} + \frac{3}{s^2-7^2}$

$\rightarrow \sin(2t) = \frac{2}{s^2+2^2}$

$= \frac{4}{3} \sin(2t) + \frac{3}{7} \sinh(7t)$

$\sin(at) = \frac{a}{s^2+a^2}$

$\sinh(7t) = \frac{7}{s^2-49}$

$\cos(at) = \frac{s}{s^2+a^2}$

$\cosh(at) = \frac{s}{s^2-a^2}$

Example:- $F(s) = \frac{1-3s}{s^2+8s+21}$

$s^2+8s+21 \rightarrow$ complex root.

we have to complete square

use $s^2 + d_1s + d_0$

$\frac{d_1}{2} \rightarrow \frac{d_1^2}{4}$

$= (s + \frac{d_1}{2})^2 + (d_0 - \frac{d_1^2}{4})$

$d_1 = 8$

$\frac{d_1}{2} = 4 \quad (s+4) + (d_0 - \frac{64}{4}) = (s+4)^2 + 5$

$$\Rightarrow F(s) = \frac{1-3s}{(s+4)^2+5} = \frac{13-3(s+4)}{(s+4)^2+5}$$

$$= \frac{13\sqrt{5}}{\sqrt{5}((s+4)^2+5)} - \frac{3(s+4)}{(s+4)^2+5}$$

$$\Rightarrow = \frac{13}{\sqrt{5}} e^{-4t} \sin(\sqrt{5}t) - 3e^{-4t} \cos(\sqrt{5}t)$$

* Note:-

$$e^{qt} \cos wt = \frac{s-a}{(s-a)^2+w^2}$$

$$e^{qt} \sin wt = \frac{w}{(s-a)^2+w^2}$$

$$G(s) = \frac{s+7}{(s+2)(s-5)} \quad \text{partial fraction}$$

$$G(s) = \frac{A}{s+2} + \frac{B}{s-5}$$

$$s+7 = A(s-5) + B(s+2)$$

$$s=5 \Rightarrow 12 = 7B \Rightarrow B = 12/7$$

$$s=-2 \Rightarrow A = -\frac{5}{7}$$

$$g(t) = A e^{-2t} + B e^{5t}$$

$$= -\frac{5}{7} e^{-2t} + \frac{12}{7} e^{5t}$$

* Factor denominator:-

$$(ax+b)$$

$$(ax+b)^k$$

$$ax^2+bx+c$$

$$(ax^2+bx+c)^k$$

* Term in partial fraction decomposition

$$\frac{A}{ax+b} + \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_k}{(ax+b)^k}$$

$$\frac{Ax+B}{ax^2+bx+c}$$

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

Example:- $G(s) = \frac{2-5s}{(s-6)(s^2+11)} = \frac{A}{s-6} + \frac{Bs+C}{s^2+11}$

$$\frac{Bs}{s^2+11} + \frac{C}{s^2+11}$$

$$\Rightarrow 2-5s = A(s^2+11) + (Bs+C)(s-6)$$

$$2-5s = (A+B)s^2 + (-6B+C)s + (11A-6C)$$

Set coefficient equal

$$s^2: A+B=0$$

$$s: -6B+C=5$$

$$s^0: 11A-6C=2$$

$$\Rightarrow A = -\frac{28}{47}, B = \frac{28}{47}, C = -\frac{67}{47}$$

$$G(s) = \frac{1}{47} \left(\frac{-28}{s-6} + \frac{28s}{s^2+11} - \frac{67}{s^2+11} \right)$$

$$g(t) = \frac{1}{47} \left(-28e^{6t} + 28\cos(\sqrt{11}t) - \frac{67}{\sqrt{11}}\sin(\sqrt{11}t) \right)$$

* Solving Diff Equ. with Laplace transform:-

2 of derivatives:-

$$f(t), f'(t), f''(t), \dots, f^{(n)}(t).$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

$$= s \mathcal{L}[f(t) - f(0)]$$

$$\mathcal{L}[f''(t)] = s^2 F(s) - sf(0) - f'(0)$$

$$\mathcal{L}[f'''(t)] = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots$$

n terms

$$\times \int_0^t f(t) dt$$

$$\mathcal{L}\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s}$$

Example:- Solve Ivp

$$\mathcal{L}[y''(t) - 10y'(t) + 9y] = \mathcal{L}[5t], \quad y(0) = -1$$

$$y'(0) = -2$$

$$\mathcal{L}[y''(t)] - 10\mathcal{L}[y'(t)] + 9\mathcal{L}[y(t)] = 5\mathcal{L}[t]$$

$$(s^2 Y(s) - sy(0) - y'(0)) - 10(sY(s) + y'(0)) + 9Y(s) = 5 \frac{1}{s^2}$$

$$Y(s)(s^2 - 10s + 9) + (s - 12) = \frac{5}{s^2}$$

Solve for $Y(s)$ (solution in s -domain)

$$Y(s) = \frac{5}{s^2 + 10s + 9} - \frac{s - 12}{s^2 + 10s + 9}$$

$$\checkmark \checkmark Y(s) = \frac{5}{(s-1)(s-9)} - \frac{s-12}{(s-1)(s-9)}$$

$$\checkmark \checkmark Y(s) = \frac{5+12s^2-s^3}{s^2(s-1)(s-9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

Take $\mathcal{L}^{-1}$.

$$\mathcal{L}^{-1}[Y(s)] = y(t) = \mathcal{L}^{-1}\left[\frac{A}{s}\right] + \mathcal{L}^{-1}\left[\frac{B}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{C}{s-9}\right] + \mathcal{L}^{-1}\left[\frac{D}{s-1}\right]$$

$$y(t) = A + Bt + Ce^{9t} + De^t$$

$$5 + 12s^2 - s^3 = As(s-1)(s-9) + B(s-1)(s-9) + Cs^2(s-1) + Ds^2(s-9)$$

or set coefficient

$$\text{if } s=0 \Rightarrow B = \frac{5}{9}$$

$$s=9 \Rightarrow C = \frac{5+12(9)^2-(9)^3}{(9)^2(9-1)} = \frac{31}{81}$$

$$s=1 \Rightarrow 5+12-1 = D(1)^2(1-9)$$

$$D = -2$$

To find A select s that different from 0, 9, 1

$$\text{if } s=2 \Rightarrow A = \frac{50}{81}$$

Example: $2y'' + 3y' - 2y = t^{-2}e^{-t}$ $y(0)=0, y'(0)=-2$

$$* t^n e^{at} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}[2y''] + \mathcal{L}[3y'] - \mathcal{L}[2y] = \mathcal{L}[t^{-2}e^{-t}]$$

From table

$$2(s^2 Y(s) - s y'(0) - y'(0)) + 3(s Y(s) - y(0)) - 2Y(s) = \frac{1!}{(s+2)^2}$$

$$(2s^2 + 3s - 2)Y(s) + 4 = \frac{1}{(s+2)^2}$$

Solve for Y(s)

$$Y(s) = \frac{1}{(2s-1)(s+2)} - \frac{4}{(2s-1)(s+2)}$$

$$Y(s) = \frac{-4s^2 - 16s - 15}{(2s-1)(s+2)^3} = \frac{A}{2s-1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

$$2^{-1} \Rightarrow y(t) = \frac{A}{2} e^{112t} + B e^{-2t} + C t e^{-2t} + \frac{D}{2!} t^2 e^{-2t}$$

$$-4s^2 - 16s - 15 = (A+2B)s^3 + (6A+7B+2C)s^2 + (12A+4B+3C+2D)s + (8A-4B-2C-D)$$

* $\rightarrow$ set coefficients equal.

$$s^3: A + 2B = 0$$

$$s^2: 6A + 7B + 2C = -4$$

$$s: 12A + 4B + 3C + 2D = -16$$

$$s^0: 8A - 4B - 2C - D = -15$$

$$A = \frac{-192}{125}, B = \frac{962}{125}, C = \frac{-2}{25}, D = \frac{-1}{5}$$

$$y(t) = \left(-96 e^{112t} + 96 e^{-2t} + 10t e^{-2t} - \frac{25}{2} t^2 e^{-2t} \right) \times \frac{1}{125}$$

Example: $y'' + 4y' = \cos(t-3) + 4t$, $y(3) = 0$, $y'(3) = -7$.

$y(0)?$, $y'(0)?$

Let $x = t - 3 \Rightarrow t = x + 3$

$$y''(t) + 4y'(t) = \cos(t-3) + 4t$$

$$y''(x+3) + 4y'(x+3) = \cos(x) + 4(x+3) = \cos x + 4x + 12$$

Let $u(x) = y(x+3)$.

$$u'(x) = \frac{du}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = y'(x+3)$$

$$u''(x) = y''(x+3)$$

$$u(0) = y(0+3) = y(3) = 0$$

$$u'(0) = y'(0+3) = y'(3) = -7$$

$u''(x) + 4u'(x) = \cos x + 4x + 12$ $u(0) = 0$, $u'(0) = -7$

$$2 \Rightarrow (s^2 \bar{U}(s) - s u(0) - u'(0)) + 4(s \bar{U}(s) - u(0)) = \frac{s}{s^2+1} + \frac{4+12s}{s^2}$$

Solve For $\bar{U}(s)$.

$$s(s+4) \bar{U}(s) = \frac{s}{s^2+1} + \frac{4+12s}{s^2} + 7s^2$$

$$\bar{U}(s) = \frac{1}{(s+4)(s^2+1)} + \frac{4+12s+7s^2}{s^3(s+4)}$$

partial Fraction.

$$\frac{1}{(s+4)(s^2+1)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+1} \Rightarrow A = \frac{1}{17}, B = -\frac{1}{17}, C = \frac{4}{17}$$

$$\frac{4+12s+7s^2}{s^3(s+4)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+4}$$

$$\mathcal{L}^{-1} \bar{U}(s) = \frac{1}{s^3} \times \frac{x^2}{2!} + \frac{\frac{11}{4}}{s^2} + \frac{\frac{17}{16}}{s} - \frac{\frac{273}{272}}{s+4} + \frac{1}{17} \left[\frac{4}{s^2+1} - \frac{s}{s^2+1} \right]$$

$$u(x) = \frac{1}{2!} x^2 + \frac{11}{4} x + \frac{17}{16} - \frac{273}{272} e^{-4x} + \frac{1}{17} [4 \sin x - \cos x]$$

$$y(x) = u(x) = y(x+3).$$

$$x = t - 3$$

$$y(x) = u(x) = u(t-3).$$

$$y(t) = \frac{1}{2} (t-3)^2 + \frac{11}{4} (t-3) + \frac{17}{16} - \frac{273}{2} e^{-4(t-3)} + \frac{1}{17} [4 \sin(t-3) + \cos(t-3)]$$