

Mathematical

* Concepts of D.E

$$y(x) = f(x) = \dots$$

$\swarrow$ dependant $\searrow$ independent

variables
 $\swarrow$ $\searrow$
 dependant independent
 (unknown) (known)
 only one at least one 1, 2, 3, ...

Equations

Algebraic differential

ordinary D.E
 (one independent
 variable)

$$\rightarrow y = 2x^2 + 3$$

$$\frac{dy}{dx} = 4x$$

partial D.E
 at least 2
 independent variable

$$\rightarrow z(x, y) = 2x^2 + y^3 + 3xy$$

$$\frac{\partial z}{\partial x} = 4x + 3y$$

$$\frac{\partial z}{\partial y} = 3y^2 + 3x$$

$$\rightarrow \frac{\partial^2 z}{\partial x \partial y} = 3$$

Ex: $\frac{d^2 y}{dx^2} = 2x + y^2$ dependant 1 term is *

2nd deg. is $\rightarrow (\frac{dy}{dx})^2 = 2x + y^2$ degree 1, 2 terms

تريع الطرف $\rightarrow y = (\frac{dy}{dx})^4 + \dots$ 4th deg. term

$$\rightarrow \frac{d^2 y}{dx^2} = 2 \frac{dy}{dx} - 15y = 0$$

$y = e^{5x}$ Is solution?

$$\frac{dy}{dx} = 5e^{5x} \quad \frac{d^2 y}{dx^2} = 25e^{5x} \rightarrow 25e^{5x} - 10e^{5x} - 15e^{5x} \stackrel{?}{=} 0$$

General sol.

$\checkmark \quad 0 = 0$



* $y_2 = e^{-3x}$ is a solution? يحققة جزء من المعادلة ليس الحل

$$\frac{dy}{dx} = -3e^{-3x} \quad \frac{dy^2}{dx^2} = 9e^{-3x}$$

$$\rightarrow 9e^{-3x} + 2x \cdot 3e^{-3x} - 15e^{-3x} \stackrel{?}{=} 0 \Rightarrow 15e^{-3x} - 15e^{-3x} = 0 \quad \checkmark \quad \text{general solution.}$$

* $y_3 = e^{4x}$ is a solution?

$$\frac{dy}{dx} = 4e^{4x} \quad \frac{dy^2}{dx^2} = 16e^{4x}$$

$$\Rightarrow 16e^{4x} - 2x \cdot 4e^{4x} - 15e^{4x} \stackrel{?}{=} 0$$

$$-23e^{4x} \neq 0$$

not a solution

الشرطيات (جزء من المعادلة) يكون

$$\rightarrow y'' - 2y' - 15y = 0$$

$$y_1 = e^{5x}$$

$$y_2 = e^{-3x}$$

عشوائي

$C_1, C_2 \rightarrow$ arbitrary constants

try

$$\begin{aligned} y_{\text{general}} &= C_1 y_1 + C_2 y_2 \\ y_g &= C_1 e^{5x} + C_2 e^{-3x} \\ y'_g &= 5C_1 e^{5x} - 3C_2 e^{-3x} \\ y''_g &= 25C_1 e^{5x} + 9C_2 e^{-3x} \end{aligned}$$

$$25C_1 e^{5x} + 9C_2 e^{-3x} - 10C_1 e^{5x} + 6C_2 e^{-3x} - 15C_1 e^{5x} - 15C_2 e^{-3x} = 0$$

$$0 = 0 \quad \checkmark$$

✓ zero ← C_2, C_1 فقرة عشوائية

$y(0)=1$ $y'(0)=2$ particular solution (constant), initial condition estimation جزء *

$$y_g = C_1 e^{5x} + C_2 e^{-3x} \rightarrow y'_g = 5C_1 e^{5x} - 3C_2 e^{-3x}$$

$$1 = C_1 + C_2$$

$$2 = 5C_1 - 3C_2$$

$$C_1 = 5/8$$

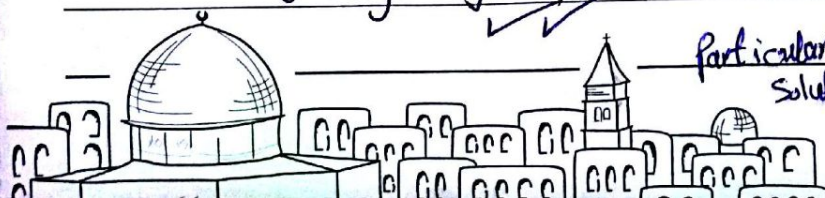
$$C_2 = 3/8$$

$$y_{\text{particular}} = \frac{5}{8} e^{5x} + \frac{3}{8} e^{-3x}, \quad y'_p = \frac{25}{8} e^{5x} - \frac{9}{8} e^{-3x}$$

$$y'' - 2y' - 15y = 0 \quad \checkmark \quad \checkmark$$

$$y''_p = \frac{125}{8} e^{5x} + \frac{27}{8} e^{-3x}$$

particular solution + initial condition



ask & receive

طيفة هذه الحارة بكون السحاب

تحقق، معاودة مع الشروط

↳ unique solution

* منہ ال dependent-
var. جرد اذا implied او

* انا بقدر اقول من implicit و explicit جز

$y = (C_1 e^{5x} + C_2 e^{-3x})^2$ ← explicit & up
implicit

$2y^3 + 3\cos y + 2x^2 = 0$: مثال 2
explicit y in terms of x
for x

explicit & implicit $\uparrow$
for x

$$a_0(x) \left(\frac{d^n y}{dx^n} \right)' + a_1 \left(\frac{d^{n-1} y}{dx^{n-1}} \right)' + a_{n-1} \left(\frac{d^1 y}{dx^1} \right)' + \dots = f(x)$$

~~dependent var.~~ ~~is~~ ~~independent var.~~ ~~to~~ ~~be~~ ~~used~~ ~~as~~ ~~a~~ ~~test~~

(dependent) 1) exponential $\sin y \rightarrow y$ is π

$$y'' + 2y \sin y = f(x) \quad \text{IC}$$

(متراب ۲) مترم ثانوی ۶ ال ۶ متر ثانیا (ایک ۲ sin) X لائن linear

$$y'' - 2y' - 15y = 0$$

$$y(0) = 1$$

$$y'(1) = -4$$

boundary



* initial or boundary. linear second order no *

$$y_g = C_1 e^{5x} + C_2 e^{-3x}$$

$$y_p = \frac{5}{8} e^{5x} + \frac{3}{8} e^{-3x} \rightarrow \text{non singular}$$

$$y_p = -2e^{5x} \rightarrow \text{non singular}$$

$$y = 2\sqrt{x}$$

$\rightarrow$ singular

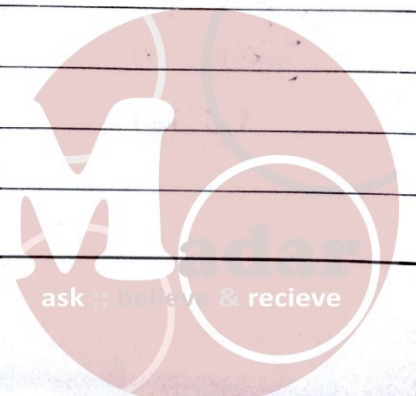
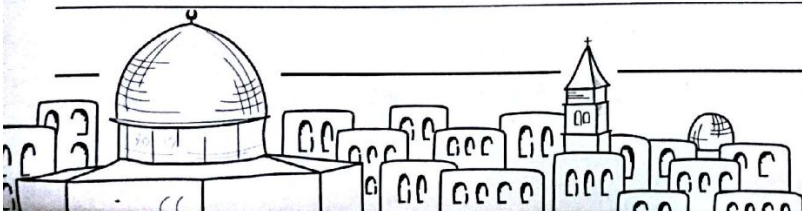
↑ general form استجابة في شكل

$$(y')^2 - xy' + y = 0$$

$$y_g = C_1 x - C_2^2$$

$$y'_1 = C \rightarrow \text{Try}$$

$$C^2 - xC + Cx \stackrel{?}{=} 0 \quad \checkmark \text{ solution}$$



* Single 1st order ordinary differential equation

$$\frac{dy}{dx} = F(x, y)$$

- | | | |
|-------------------------------------|---------------------------|-------------|
| ① separable D.E | 2) Reduction to separable |] for which |
| ③ exact D.E | 4) Integrating factor | |
| 5) Linear 1 st order D.E | 6) Applications | |

① Separable 1st order D.E:

$$\frac{dy}{dx} = f(x, y) = F(x) \cdot G(y)$$

$$\frac{dy}{G(y)} = F(x) dx \quad \text{separable}$$

$\downarrow$ $\downarrow$
 $G(y)$ $F(x)$
 only only

$$\int \frac{dy}{G(y)} = \int F(x) dx + C$$

← arbitrary constant
 for which $y(0) = 0$
 $y(x) = 0$ is a solution

* Ex: $x \frac{dy}{dx} - y = 2x^2 y$

$$x \frac{dy}{dx} = 2x^2 y + y$$

$$\frac{x}{y} \frac{dy}{dx} = \frac{y(2x^2 + 1)}{x}$$

$$\frac{dy}{dx} = y \left(2x + \frac{1}{x} \right)$$

$$\int \frac{dy}{y} = \int \left(2x + \frac{1}{x} \right) dx + C$$

$$\ln y = x^2 + \ln x + C$$

$$e^{\ln y} = e^{x^2 + \ln x + C}$$

$$y = e^{x^2} \cdot x \cdot e^C$$

$$y = e^{x^2} \cdot x \cdot C_1$$



ex: $x \sin y \, dx + (x^2+1) \cos y = 0$

$$\frac{dy}{dx} = \frac{-x \sin y}{(x^2+1) \cos y}$$

$\downarrow$ $\downarrow$
 $F(x)$ $G(y)$

$$\int \frac{\cos y}{\sin y} \, dy = \int \frac{-x}{x^2+1} \, dx$$

$$e(\ln \sin y = -\frac{1}{2} \ln(x^2+1) + C)$$

$$\sin^{-1} \left(\sin y = \frac{C_1}{\sqrt{x^2+1}} \right)$$

$$y = \sin^{-1} \left(\frac{C_1}{\sqrt{x^2+1}} \right)$$

② Reduction to separable

if $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, let $u = \frac{y}{x}$

$$xu = y$$

$$u + x \frac{du}{dx} = \frac{dy}{dx}$$

$$\frac{u}{-u} + x \frac{du}{dx} = \frac{f(u) - u}{x}$$

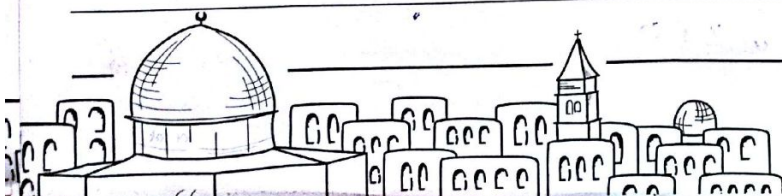
$$\frac{du}{dx} = \frac{f(u) - u}{x}$$

$\uparrow$ $\uparrow$
 $G(u)$ $F(x)$

$$\int (f(u) - u) \, du = \int \frac{dx}{x} + C$$

$$\int (f(u) - u) \, du = \ln x + C$$

$\frac{y}{x} \leftarrow u$ is in $G(u)$



$$\textcircled{Q} \frac{dy}{dx} = \frac{x-y}{x+y} = \frac{(1-\frac{y}{x})}{(1+\frac{y}{x})} \quad \text{Let } \frac{y}{x} = u \quad \text{Ratio of } x, y$$

$$u + x \frac{du}{dx} = \frac{1-u}{1+u}$$

$$\star \text{ (let) } \frac{y}{x} = u \quad xu = y$$

$$\left(u + x \frac{du}{dx} = \frac{dy}{dx} \right)$$

$$\frac{du}{dx} = \frac{1}{x} \left(\frac{(1-u)}{(1+u)} - u \right)$$

$$\frac{du}{dx} = \frac{1}{x} \left(\frac{1-2u-u^2}{1+u} \right)$$

$$\int \left(\frac{1+u}{1-2u-u^2} \right) du = \int \frac{dx}{x} + C$$

$$\left(-\frac{1}{2} \ln(1-2u-u^2) = \ln x + C \right) \times e$$

$$\frac{1}{\sqrt{1-2u-u^2}} = C_1 x \quad \text{u is a function of x}$$

$$\frac{1}{\sqrt{1-2\frac{y}{x} - \left(\frac{y}{x}\right)^2}} = C_1 x$$

1st O.O.D.E

$$y' = f(x, y)$$

① separable D.E

② Reduction to separable

① if $y' = f\left(\frac{y}{x}\right) \quad u = \frac{y}{x}$

② if $y' = \sqrt{f(x+y)} = [f(x+y)]^{\frac{1}{2}} \quad \text{fraction}$

$$\star x+y = u^2$$

$$x+y = u^2$$

$$x + \frac{dy}{dx} = 2u \frac{du}{dx} - 1$$

$$2u \frac{du}{dx} - 1 = \sqrt{u^2}$$

$$2u \frac{du}{dx} = u + 1$$

$$\int \frac{2u}{u+1} du = \int 1 dx \quad \checkmark$$



ex $\frac{dy}{dx} = \sqrt{2x+3y}$

let $2x+3y = u^2$

$2 + 3\frac{dy}{dx} = 2u \frac{du}{dx}$

$2u \frac{du}{dx} = \sqrt{u^2} = 3u+2$

$\frac{dy}{dx} = \frac{1}{3} \left[2u \frac{du}{dx} - 2 \right]$

$\int \frac{2u}{3u+2} du = \int 1 dx + C$

$$\begin{array}{r} \frac{2/3}{3u+2} \overline{) 2u} \\ \underline{-2u + \frac{4}{3}} \\ -\frac{4}{3} \end{array} \quad \frac{2}{3} - \frac{4}{3} \cdot \frac{1}{3u+2}$$

$\left(\frac{2}{3} - \frac{4}{3(3u+2)} \right) du = \int 1 dx + C$

$\frac{2}{3}u - \frac{4}{9} \ln(3u+2) = x + C$

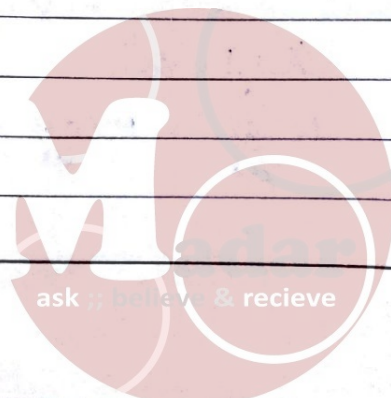
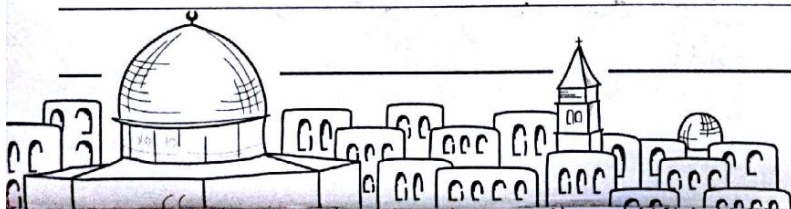
Ex 11.1
 (c) if $\frac{dy}{dx} = [f(x+y)]^n$ where n is an integer

let $u = x+y$

$\frac{du}{dx} = 1 + \frac{dy}{dx} \quad \therefore \frac{dy}{dx} = \frac{du}{dx} - 1$

$\frac{du}{dx} - 1 = [f(x+y)]^n = u^n$
 $\frac{du}{dx} = u^n + 1$

$\int \frac{1}{u^{n+1}} du = \int dx + C$



$$\textcircled{ex} \frac{dy}{dx} = (x+y)^2 \quad n=2$$

$$(x+y) = u$$

$$1 + \frac{du}{dx} = \frac{du}{dx} \quad \therefore \frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{du}{dx} - 1 = u^2 + 1$$

$$\int \frac{1}{u^2+1} du = \int (1) dx + C$$

$$\arctan(u) = x + C$$

$$\textcircled{d} \text{ if } \frac{dy}{dx} = \sin(f(x,y)) \text{ or cos or tan } \dots$$

$$\text{let } f(x,y) = u$$

$$\textcircled{ex} \frac{dy}{dx} = \tan(x+y) - 1$$

$$\text{let } u = x+y$$

$$\frac{du}{dx} = 1 + \frac{dy}{dx} \quad \therefore \frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{du}{dx} - 1 = \tan(u) - 1$$

$$\int \frac{\cos(u)}{\sin(u)} du = \int (1) dx$$

$$\ln \sin(u) = x + C$$



(e) if $\frac{dy}{dx} = f(ax+by+c)$ where a, b, c are constants.

$$\text{let } (ax+by+c) = u \quad a+b\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{b} \left[\frac{du}{dx} - a \right] = f(u) + a \quad \therefore \frac{dy}{dx} = \frac{1}{b} \left[\frac{du}{dx} - a \right]$$

$$\int \frac{du}{bf(u)+a} = \int dx + C$$

(ex) $\frac{f_y}{f_x} = (x+y-7)^2$

$$\text{let } x+y-7 = u \quad 1+\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{du}{dx} - 1 = u^2$$
$$\frac{du}{dx} = u^2 + 1$$

$$\therefore \frac{du}{dx} - 1 = \frac{dy}{dx}$$

$$\int \frac{du}{u^2+1} = \int dx + C$$

$$\arctan(u) = x + C$$

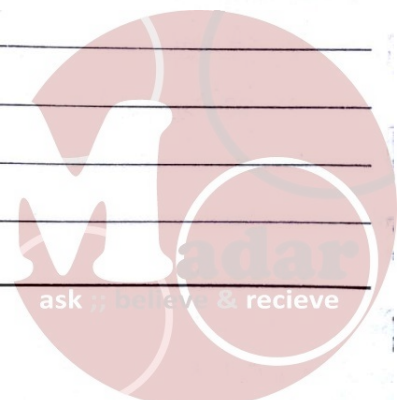
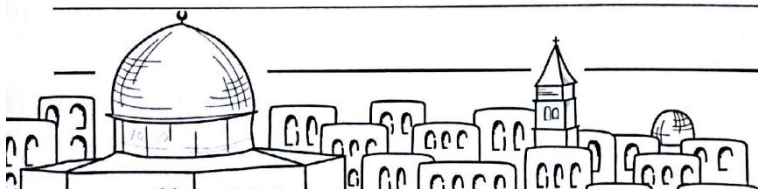
(f) if $(a_1x+b_1y+c_1)dx + (a_2x+b_2y+c_2)dy = 0$

$a_1, a_2, b_1, b_2, c_1, c_2$ are all constants

case 1 $\frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow \text{let } u = a_1x + b_1y$
or $u = a_2x + b_2y$

is C.I. in simple form

• $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$



ex 1

$$(x + 2y + 3) dx + (2x + 4y - 1) dy = 0$$

$$\frac{a_1 = 1}{a_2 = 2} = \frac{b_1 = 2}{b_2 = 4}$$

ratio is similar

$$\text{let } x + 2y = u$$

$$1 + 2 \frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore \frac{1}{2} \left[\frac{du}{dx} - 1 \right] = \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{(x + 2y + 3)}{(2x + 4y - 1)}$$

$$\frac{du}{dx} - 1 = -\frac{(u + 3)}{(2u - 1)} \Rightarrow \frac{du}{dx} = \frac{-7}{(2u - 1)}$$

$$\int (2u - 1) du = \int -7 dx + C$$

$$u^2 - u = -7x + C$$

* Case 2

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

assume

$$\text{let } x = u + h$$

$$y = v + k \rightarrow \text{constants}$$

variables

to find h and k solve the following equations

$$a_1 h + b_1 k + C_1 = 0$$

$$a_2 h + b_2 k + C_2 = 0$$

ask

& remove

Ex $(5x + 2y - 1)dx + (2x + y + 1)dy = 0$

$a_1 = 5 \neq a_2 = 2$
 $b_1 = 2 \neq b_2 = 1$

let $x = u + h$
 $y = v + k$

$5h + 2k - 1 = 0$
 $2h + 1k + 1 = 0$

$h = 3$

$k = -7$

$x = u + 3 \Rightarrow dx = du$
 $y = v - 7 \Rightarrow dy = dv$

$[5(u+3) + 2(v-7) - 1]dv + [2(u+3) + (v-7) + 1]du = 0$

$[5u + 15 + 2v - 14 - 1]dv + [2u + 6 + v - 7 + 1]du = 0$

$[5u + 2v]dv + [2u + v]du = 0$

$\frac{5u + 2v}{2u + v} = -\frac{dv}{du}$

$\frac{-(5 + 2\frac{v}{u})}{2 + \frac{v}{u}} = \frac{dv}{du}$

assume
 $\frac{v}{u} = M$

$\frac{-5 - 2M}{2 + M} = M + u \frac{dM}{du}$
 $\frac{dv}{du} = M + u \frac{dM}{du}$

$u \frac{dM}{du} = \frac{-(5 + 4M + M^2)}{2 + M}$

$\int \frac{2 + M}{-(5 + 4M + M^2)} dM = \int \frac{-1}{u} du + C$

$-\frac{1}{2} \ln(5 + 4M + M^2) = -\ln u + C$



③ exact D.E

$$M(x,y) dx + N(x,y) dy = 0$$

other var. $\rightarrow$ $\frac{d}{dx}$ or $\frac{d}{dy}$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{if we prove this} \rightarrow \text{it's exact}$$

ex $(x^3 + 3xy^2) dx + (3x^2y + y^3) dy = 0$

$M(x,y)$ $N(x,y)$

$$\frac{\partial M}{\partial y} = 6xy \quad \frac{\partial N}{\partial x} = 6xy \rightarrow \text{exact D.E}$$

The solution of exact D.E is

step 1 find $u(x,y) = \int M(x,y) dx + k(y)$ constant as a function of y

step 2 solve the following

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left[\int M(x,y) dx \right] + \frac{dk}{dy} = N(x,y)$$

$$\int dk = \int \left[N(x,y) - \frac{\partial}{\partial y} \left[\int M(x,y) dx \right] \right] dy + C$$

$$k = \int \left[N(x,y) - \frac{\partial}{\partial y} \left[\int M(x,y) dx \right] \right] dy + C$$

step 3

$$u(x,y) = \int M(x,y) dx + \int k dy = 0$$

$$\Rightarrow u(x,y) = \int (x^3 + 3xy^2) dx + k(y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + k(y)$$

$$\Rightarrow \frac{du}{dy} = 3x^2y + \frac{dk}{dy} = 3x^2y + y^3$$

$$\int dk = \int y^3 dy + C$$

$$k = \frac{1}{4}y^4 + C$$

$$\therefore u(x,y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + C = 0$$

$$\frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 = C_1 \checkmark \checkmark \checkmark$$

homework
 $\int N(x,y) dy$
Subtract with



ex $(2xy) dx + (x^2 + \cos y) dy = 0$

$\frac{\partial M}{\partial y} = 2x$ $\frac{\partial N}{\partial x} = 2x \rightarrow \text{exact D.E}$

$u(x,y) = \int (x^2 + \cos y) dy + h(x)$

$u(x,y) = x^2 y + \sin y + h(x)$

$\frac{du}{dx} = \cancel{2xy} + \frac{dh}{dx} = \cancel{2xy}$

$\int dh = \int 0 dx + C$

$h = C_1 + C = C_2$

$u(x,y) = x^2 y + \sin y + C_2 = 0$

$x^2 y + \sin y = C_3$

ex $(2x \cos y + 3x^2 y) dx + (x^3 - x^2 \sin y - y) dy = 0$

$\frac{\partial M}{\partial y} = -2x \sin y + 3x^2$, $\frac{\partial N}{\partial x} = 3x^2 - 2x \sin y \rightarrow \text{exact D.E}$

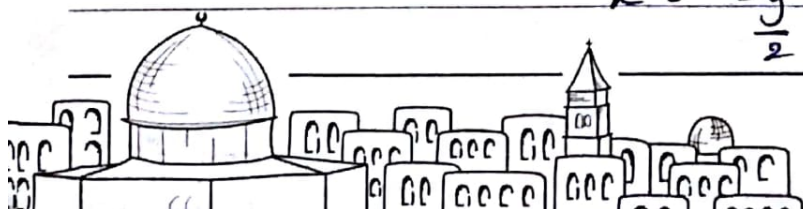
$u(x,y) = \int 2x \cos y + 3x^2 y dx + k(y)$

$u(x,y) = x^2 \cos y + x^3 y + k(y)$

$\frac{du}{dy} = -x^2 \sin y + x^3 + \frac{dk}{dy} = x^3 - x^2 \sin y - y$

$\int dk = \int -y dy$

$k = -\frac{y^2}{2} + C$



$$\therefore u(x, y) = x^2 \cos y + x^2 y - \frac{1}{2} y^2 = C_1$$

$$* \quad y \frac{dx}{x} - x \frac{dy}{y} = 0$$

$$\frac{du}{dy} = 1 \quad \frac{dv}{dx} = -1 \quad \therefore \frac{du}{dy} \neq \frac{dv}{dx} \quad \text{Not exact}$$

if we multiply by $\boxed{\frac{1}{x^2}}$ $\rightarrow$ integrating factor

$$\frac{y}{x^2} dx - \frac{1}{x} dy = 0$$

$$\frac{du}{dy} = \frac{1}{x^2} \quad \frac{dv}{dx} = \frac{1}{x^2} \Rightarrow \text{exact}$$

* How do we obtain the integrating factor?

① Find $R_1 = \frac{1}{v} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right)$; if R_1 is function of x only

then an integrating factor can be obtained from R_1

as $\underbrace{I.F.}_{\text{integrating factor}} = e^{\int R_1 dx}$

② Find $R_2 = \frac{1}{u} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)$, if R_2 is function of y only

then an I.F can be obtained from R_2 as $I.F. = e^{\int R_2 dy}$

$$R_1 = \frac{1}{-x} (1 + 1) = -\frac{2}{x} \quad I.F. = e^{\int -\frac{2}{x} dx} = e^{-2 \ln x} = e^{\ln x^{-2}} = x^{-2}$$

(2) $\frac{1}{x^2}$ is $\frac{1}{x^2}$ only $\frac{1}{x^2}$

$$\rightarrow \boxed{I.F. = \frac{1}{x^2}}$$



$$\underbrace{2xy}_{M} dx + \underbrace{(4y + 3x^2)}_{N} dy = 0 \rightarrow \text{not exact D.E}$$

$$\frac{\partial M}{\partial y} = 2x \quad \frac{\partial N}{\partial x} = 6x$$

$$\rightarrow R_1 = \frac{1}{4y + 3x^2} (2x - 6x) = \frac{-4x}{4y + 3x^2} \quad \text{not function of } x \text{ only}$$

$$\rightarrow R_2 = \frac{1}{2xy} (6x - 2x) = \frac{2}{y}, \quad \text{function of } y \text{ only}$$

$$\text{I.F.} = e^{\int \frac{2}{y} dy} = e^{2 \ln y} = y^2 \rightarrow \text{I.F. is function of } y \text{ only}$$

$$(2xy^3) dx + (4y^3 + 3x^2y^2) dy = 0$$

$$\frac{\partial M'}{\partial y} = 6xy^2 \quad \frac{\partial N'}{\partial x} = 6xy^2 \rightarrow \text{now it's exact D.E}$$

$$u(x, y) = \int 2xy^3 dx + k(y)$$

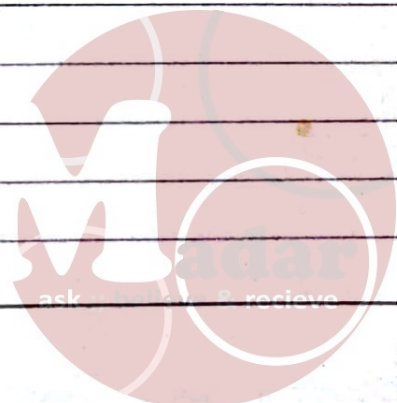
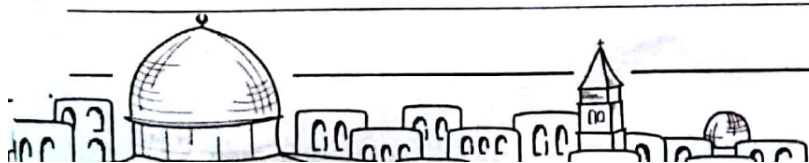
$$u(x, y) = x^2y^3 + k(y)$$

$$\frac{\partial u}{\partial y} = 3x^2y^2 + \frac{dk}{dy} = 4y^3 + 3x^2y^2$$

$$\int dk = \int 4y^3 dy$$

$$k = y^4 + C$$

$$u(x, y) = x^2y^3 + y^4 + C \Rightarrow \boxed{x^2y^3 + y^4 = C_1}$$



$$* (1 + 2x^2 + 4xy) dx + 2y dy = 0$$

$$\frac{\partial M}{\partial y} = 4x \quad \frac{\partial N}{\partial x} = 0 \rightarrow \text{not exact}$$

$$\Rightarrow R_1 = \frac{1}{2} (-4x) = -2x \quad \text{function of } x \text{ only,}$$

$$\text{I.F.} = e^{\int -2x dx} = e^{-x^2}$$

$$(e^{-x^2} + 2x^2 e^{-x^2} + 4xy e^{-x^2}) dx + 2y e^{-x^2} dy = 0$$

$$\frac{\partial M'}{\partial y} = 4x e^{-x^2} \quad \frac{\partial N'}{\partial x} = 4x e^{-x^2} \rightarrow \text{exact DE}$$

now

$$d(x,y) = \int 2y e^{-x^2} dy + l(x)$$

$$d(x,y) = y^2 e^{-x^2} + l(x)$$

$$\frac{\partial d}{\partial x} = 4xy e^{-x^2} + \frac{dl}{dx} = e^{-x^2} + 2x^2 e^{-x^2} + 4xy e^{-x^2}$$

$$\int dl = \int (e^{-x^2} + 2x^2 e^{-x^2}) dx + C$$

$$l = \int (e^{-x^2} + 2x^2 e^{-x^2}) dx$$

$$u = x^2$$

$$l = \int (e^{-u} + 2u e^{-u}) \frac{du}{2x}$$

$$du = 2x dx$$

$$l = \int e^{-u} (1 + 2u) du$$

$$l = x e^{-x^2} + C$$

$$\therefore d(x,y) = 4xy e^{-x^2} + x e^{-x^2} = 0$$



* 1st order linear D.E

$$\frac{dy}{dx} + P(x)y = r(x)$$

• if $r(x) = 0$, homogeneous 1st O.D.E

$$\frac{dy}{dx} + P(x)y = 0 \Rightarrow \frac{dy}{y} = -P(x)dx \quad \leftarrow \text{P(x)}$$

$$\int \frac{dy}{y} = \int -P(x)dx + C$$

$$\ln y = \int -P(x)dx + C$$

$$y = e^{-\int P(x)dx} \cdot e^C = C_1 e^{-\int P(x)dx}$$

• if $r(x) \neq 0$, nonhomogeneous 1st O.D.E

$$\frac{dy}{dx} = r(x) - P(x)y$$

$$\left[\frac{P(x)y - r(x)}{M(x,y)} \right] dx + \left[\frac{1}{N(x,y)} \right] dy = 0$$

$$\frac{\partial M}{\partial y} = P(x)$$

$$\frac{\partial N}{\partial x} = 0 \quad \therefore \text{not exact}$$

$$R_1 = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{1} (P(x) - 0) = P(x)$$

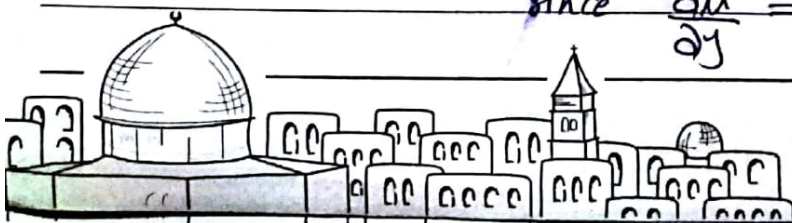
$$\text{I.F.} = e^{\int R_1 dx} = e^{\int P(x)dx}$$

$$\left[P(x)e^{\int P(x)dx} \cdot y - r(x)e^{\int P(x)dx} \right] dx + \left[e^{\int P(x)dx} \right] dy$$

$$\frac{\partial M'}{\partial y} = P(x) e^{\int P(x)dx}$$

$$\frac{\partial N'}{\partial x} = P(x) e^{\int P(x)dx}$$

$$\text{since } \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \Rightarrow \text{exact}$$



$$u(x,y) = \int e^{\int p(x) dx} dy + l(x)$$

$$u(x,y) = ye^{\int p(x) dx} + l(x)$$

$$\frac{du}{dx} = p(x) \cdot y \cdot e^{\int p(x) dx} + \frac{dl}{dx} = u' = p(x) e^{\int p(x) dx} y - r(x) e^{\int p(x) dx}$$

$$\frac{dl}{dx} = -r(x) e^{\int p(x) dx}$$

$$\int dl = \int -r(x) e^{\int p(x) dx} dx + C$$

$$l = \int e^{\int p(x) dx} \cdot r(x) dx + C$$

$$u(x,y) = ye^{\int p(x) dx} - \int e^{\int p(x) dx} \cdot r(x) dx + C = 0$$

$$ye^{\int p(x) dx} = \int \left(\frac{1}{e^{\int p(x) dx}} \cdot r(x) \right) dx + C_1$$

$$y = \frac{1}{e^{\int p(x) dx}} \left[\int \left(e^{\int p(x) dx} \cdot r(x) \right) dx + C_1 \right]$$

$$* y = e^{-\int p(x) dx} \left[\int \left(e^{\int p(x) dx} \cdot r(x) \right) dx + C_1 \right]$$

1st O.D.E



ex $y' - y = e^{2x}$

$p(x) = -1$, $r(x) = e^{2x}$

$y = e^{-\int -1 dx} \left[\int e^{\int -1 dx} \cdot e^{2x} dx + C_1 \right]$

$e^x \left[\int [e^{-x} \cdot e^{2x}] dx + C_1 \right]$

$e^x \left[\int e^x dx + C_1 \right] = \boxed{e^{2x} + C_1 e^x}$

Time work

الوقت

ex $(x^2 + 1) \frac{dy}{dx} + 4xy = x$

$\frac{dy}{dx} + \left(\frac{4x}{x^2 + 1} \right) y = \frac{x}{x^2 + 1}$

$p(x)$

$r(x)$

$\int p(x) dx = \int \frac{4x}{x^2 + 1} dx =$

$\boxed{2 \ln(x^2 + 1)}$

$y = e^{-2 \ln(x^2 + 1)} \left(\int e^{2 \ln(x^2 + 1)} \cdot \frac{x}{x^2 + 1} dx + C \right)$

$\frac{1}{(x^2 + 1)^2} \left(\int (x^2 + x) dx + C \right)$

$\boxed{\frac{1}{(x^2 + 1)^2} \left(\frac{x^3}{3} + \frac{x^2}{2} + C \right)}$



* linear 1st O.D.E

$$y' + p(x)y = r(x)$$

$$y = e^{-\int p(x) dx} \left[\int \left(e^{\int p(x) dx} r(x) \right) dx + C \right]$$

(ex) $\frac{dy}{dx} = \frac{y^2}{1+3xy}$

$$(1+3xy) dy - (y^2) dx = 0 \rightarrow \text{not linear of } y \rightarrow \text{linear of } x$$

$$\frac{dx}{dy} = \frac{(1+3xy)}{y^2}$$

$$\frac{dx}{dy} - \left(\frac{(1+3xy)}{y^2} \right) = 0$$

$$\frac{dx}{dy} - \frac{1}{y^2} - \frac{3x}{y} = 0$$

$$\underbrace{\frac{dx}{dy}}_{x'} + \underbrace{\left(\frac{-3x}{y} \right)}_{P(y)} = \underbrace{\frac{1}{y^2}}_{r(y)}$$

$$x = e^{-\int P(y) dy} \left[\int \left(e^{\int P(y) dy} \cdot r(y) \right) dy + C \right]$$

$$x = e^{-\int \frac{-3}{y} dy} \left[\int e^{\int \frac{-3}{y} dy} \cdot \frac{1}{y^2} dy + C \right]$$

$$x = y^3 \left[\int \frac{1}{y^5} dy + C \right]$$

$$x = y^3 \cdot \left[\frac{y^{-5+1}}{-5+1} + C \right]$$



* Non linear 1st . O.D.E

[بیرنگی کو بیان]

$$y' + P(x)y = g(x)y^a$$

→ if $a=0$, $\therefore y' + P(x)y = g(x)$ → this is linear $\frac{dy}{dx} + P(x)y = g(x)$

→ if $a=1$, $\therefore y' + P(x)y = g(x)y$

$$\therefore y' + (P(x) - g(x))y = 0 \quad \downarrow r(x)=0 \quad y = e_1 e^{-\int P(x) dx}$$

→ if $a \neq 0, a \neq 1$

$$y' + P(x)y = g(x)y^a$$

non linear D.E

$$\text{let } u(x) = y^{1-a}$$

$$\frac{du}{dx} = (1-a)y^{-a} \frac{dy}{dx}$$

$$\frac{du}{dx} = (1-a)y^{-a} \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^a}{(1-a)} \frac{du}{dx}$$

$$\left(\frac{y^a}{(1-a)} \frac{du}{dx} + P(x)y = G(x)y^a \right) / y^a$$

$$\frac{1}{(1-a)} \frac{du}{dx} + P(x)u = G(x)$$

$$\frac{1}{(1-a)} \frac{du}{dx} + P(x)u = G(x) \quad \text{linear in } u$$

$$\frac{du}{dx} + \underbrace{[(1-a)P(x)]}_{P(x)} u = \underbrace{(1-a)G(x)}_{R(x)}$$



ask : believe & recieve

$$u = e^{-\int p(x) dx} \left[\int e^{\int p(x) dx} \cdot r(x) dx + C \right]$$

ex $y' + (1)y = y^2$ $\therefore$ not linear

الابتداء من المعادلة، افتراض.

$a=2, \therefore$ let $u = y^{1-2} = y^{-1}$

$$\frac{du}{dx} = -y^{-2} \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = -y^2 \cdot \frac{du}{dx}}$$

$$\left(-y^2 \frac{du}{dx} + (1)y = (1)y^2 \right) / -y^2$$

$$\frac{du}{dx} - \frac{1}{y^2} y = -1$$

$$\frac{du}{dx} - \underbrace{\frac{y^{-1}}{y}}_{u} = -1$$

$$\frac{du}{dx} - \underbrace{u}_{-1=p(x)} = \underbrace{-1}_{r(x)}$$

$$u = e^{-\int -1 dx} \left[\int e^{\int -1 dx} \cdot -1 dx + C \right]$$

$$u = e^x \left[\int e^{-x} dx + C \right]$$

$$u = e^x (\bar{e}^{-x} + C)$$

$$u = 1 + e^x C$$

$$\frac{1}{y} = 1 + ce^x \rightarrow y = \frac{1}{1 + ce^x}$$



$$* (3y' + y = (1-2x)y^4) / 3 \rightarrow y' + \frac{1}{3}y = \frac{1}{3}(1-2x)y^4 \quad \text{H.W}$$

$$\text{let } u = y^{-4} = y^{-3} \rightarrow \frac{du}{dx} = -3y^4 \frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{-1}{3}y^4 \frac{du}{dx}$$

$$\left(\left(\frac{-1}{3}y^4 \frac{du}{dx} \right) + \frac{1}{3}y = \frac{1}{3}(1-2x)y^4 \right) / \frac{-1}{3}y^4$$

$$\frac{du}{dx} - y^3 = -(1-2x)$$

$$\frac{du}{dx} - u = (2x-1) \quad P(x) = -1 \quad r(x) = (2x-1)$$

$$u = e^{-\int -1 dx} \left[\int e^{\int -1 dx} \cdot (2x-1) dx + C \right]$$

$$u = e^x \left[\int \underbrace{e^{-x} \cdot (2x-1) dx}_{\text{by parts}} + C \right]$$

$$\begin{aligned} u &= 2x-1 \rightarrow dV = e^x dx \\ dU &= 2 dx \leftarrow V = -e^{-x} \\ &= -(2x-1)e^{-x} - \int -e^{-x} \cdot 2 dx \end{aligned}$$

$$u = e^x \left[\int 2e^{-x} dx - (2x-1)e^{-x} + C \right]$$

$$u = -2 - (2x-1) + e^x C$$

$$y^{-3} = -1 - 2x + e^x C$$

$$* \quad a=4 \rightarrow (1-a) = -3 \quad P(x) = -1 \quad G(x) = (2x+1) \quad \leftarrow \text{على القانون مباشرة}$$

$$\frac{du}{dx} + (-3) \cdot \frac{1}{3} u = (-3)(1-2x) \left(\frac{1}{3} \right)$$

$$\frac{du}{dx} - 3u = (2x-1)$$

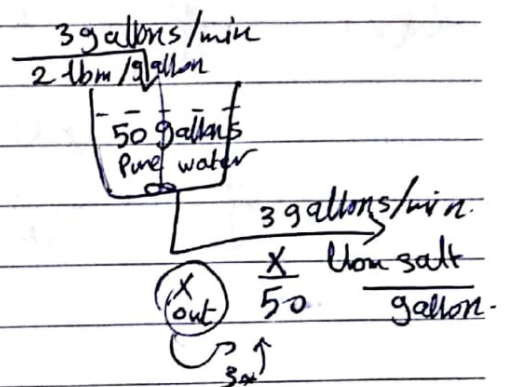
$$u = e^{-\int -3 dx} \left(\int e^{\int -3 dx} (2x-1) dx + C \right) \rightarrow$$



1st O.D.E Application

Q1: Let x be the amount of salt

$$\frac{\text{lbm salt}}{\text{min}} \frac{dx}{dt} = \underbrace{6}_{\substack{2 \text{ lbm salt} \\ \text{gallon}} \star \substack{3 \text{ gallons} \\ \text{min}}} - X_{\text{out}}$$



$$\frac{dx}{dt} = 6 - \frac{3}{50}x \quad x(t=0) = 0$$

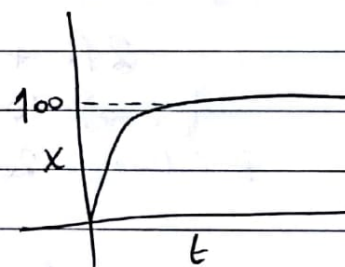
$$\frac{dx}{dt} + \frac{3}{50}x = 6$$

$$x = e^{\int \frac{3}{50} dx} \left(\int e^{\int \frac{3}{50} dx} \cdot 6 dx + C \right)$$

$$x = 100 + Ce^{-3/50 t}$$

$$0 = 100 + C(1) \rightarrow C = -100$$

$$* \quad x = 100 - 100 e^{-3/50 t}$$



Q2:

$$\frac{dk}{dt} = 10 - \frac{3x}{50+2t}$$

$$x(t=0) = 10$$

$$\frac{dk}{dt} = + \left(\frac{3}{50+2t} \right) x = 10$$

$\downarrow$ $\downarrow$
 $p(x)$ $r(x)$

5 gallons/min
2 lbm salt / gallon

50 gallon
10 lbm salt

3 gallons/min

Tank) $\xrightarrow{x \text{ lbm salt}}$ $(50+2t)$ gallon
 $V(t) = 50 + 2t$

Q3:



$$\frac{dC}{dt} = k (\text{amount}_A) (\text{Amount}_B)$$

$$2A + 1B = 3C \quad / \text{min.}$$

$$A(t=0) = 10 \rightarrow 10 - \frac{2}{3} C$$

$$B(t=0) = 20 \rightarrow 20 - \frac{1}{3} C$$

$$\frac{dC}{dt} = K \left(10 - \frac{2}{3} C \right) \left(20 - \frac{1}{3} C \right)$$

$$C(t=0) = 0$$

$$C(t=20) = 6$$

Q5:

$$\frac{dT}{dt} = K(T_a - T)$$

$\downarrow$
 Heating ambient Temp.

$T = 18$
 $T(t=0) = 10^\circ C$

$10^\circ C$

$$T(t=1) = 14^\circ C$$

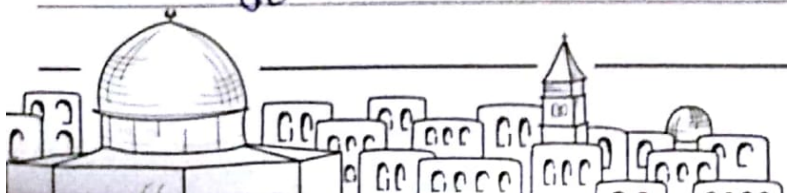
$$\frac{dT}{dt} =$$

k.w

$$\left[\frac{dT}{dt} = K(18 - T) \right]$$

ask & recieve

$$= K(T - 18)$$



* 2nd O.D.E

* $p(x)y'' + q(x)y' + l(x)y = r(x)$

Case 1 if $r(x) = 0$ and $p(x), l(x), q(x)$ are all constants.
homogenous D.E $y'' + ay' + by = 0$

Case 2 if $r(x) = 0$ and $p(x), q(x), l(x)$ are variables

$x^2y'' + axy' + by = 0$ Euler equation

Case 3 if $r(x) \neq 0$ and $p(x), q(x), l(x)$ are constants
nonhomogenous D.E $y'' + ay' + by = r(x)$

Case 4 if $r(x) \neq 0$ and $p(x), q(x), l(x)$ are not constants

$x^2y'' + axy' + by = r(x)$

* Theorem * (1) * For a homogenous linear D.E any linear combination of 2 solutions for $y'' + p(x)y' + q(x)y = 0$... (1)

Let y_1 and y_2 be the 2 sol. for eqn (1), then the linear combination $y = C_1y_1 + C_2y_2$

This linear combination is a solution for the equation

Example: $y'' + y' = 1$

$y_1 = 1 + \cos(x)$
 $y_1' = -\sin(x)$
 $y_1'' = -\cos(x)$

$y_2 = 1 + \sin(x)$
 $y_2' = \cos(x)$
 $y_2'' = -\sin(x)$

algebraically $\rightarrow$

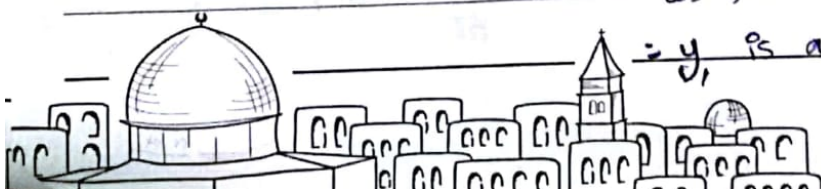
$-\cos(x) + 1 + \cos(x) = 1$

$\therefore y_1$ is a sol.

$-\sin(x) + 1 + \sin(x) = 1$

$\therefore y_2$ is solution

ask & solve & receive



linear comb. of y_2 and y_1

$$* y_3 = C_1 y_1 + C_2 y_2 = C_1 (1 + \cos(x)) + C_2 (1 + \sin(x))$$

$$\begin{cases} y_3 = C_1 + C_1 \cos(x) + C_2 + C_2 \sin(x) \\ y_3' = -C_1 \sin(x) + C_2 \cos(x) \\ y_3'' = -C_1 \cos(x) - C_2 \sin(x) \end{cases}$$

original DE $y'' + y = 1 \rightarrow -C_1 \cos(x) - C_2 \sin(x) + C_1 + C_1 \cos(x) + C_2 + C_2 \sin(x) = 1$
 $C_1 + C_2 = 1$

if we assume $C_1 = 2$ $C_2 = 4$

but
homogeneous
linear

$$\begin{cases} y_3 = 2 + 2\cos(x) + 4 + 4\sin(x) \\ y_3' = -2\sin(x) + 4\cos(x) \\ y_3'' = -2\cos(x) - 4\sin(x) \end{cases}$$

$y'' + y = 1 \rightarrow -2\cos(x) - 4\sin(x) + 2 + 2\cos(x) + 4 + 4\sin(x) = 1$
 $\rightarrow$ nonhomogeneous and $2 = 1 \therefore$ violation

$$y''y - xy' = 0$$



verification $\rightarrow$

$$y_1 = x^2$$

$$y_1' = 2x$$

$$y_1'' = 2$$

$$2x^2 - 2x^2 \stackrel{!}{=} 0$$

$\therefore y_1$ is sol.

$$y_2 = 1$$

$$y_2' = 0$$

$$y_2'' = 0$$

$$0(1) - x(0) \stackrel{!}{=} 0$$

$\therefore y_2$ is a sol.



linear comb. of y_1 and y_2

$$\begin{cases} y_3 = C_1 y_1 + C_2 y_2 = C_1 x^2 + C_2 \\ y_3' = 2C_1 x + 0 \\ y_3'' = 2C_1 \end{cases}$$

by
[homogeneous
linear]

جواب $\rightarrow (2C_1)(C_1 x^2 + C_2) - x(2C_1 x + 0) \stackrel{??}{=} 0$

$$2C_1^2 x^2 + 2C_1 C_2 - 2C_1 x^2 \stackrel{??}{=} 0$$

if $C_1 = 0 \checkmark$ * for any values of C_1 and C_2
the eqn. will hold

* The linear combination can only be used if y_1 and y_2 are basis of solution.

They are independent (y_1, y_2)

y_1, y_2 are independent only if $\frac{y_1}{y_2}$ or $\frac{y_2}{y_1}$ is

* not constant

OR the value of $w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq \text{zero}$

if $w \neq 0$ then y_1, y_2 are dependent solution

independent $\rightarrow$ if $w \neq 0$ then y_1, y_2 are independent solution



$$y'' - 2y' - 15y = 0$$

$$y_1 = e^{5x}$$

$$y_1' = 5e^{5x}$$

$$y_1'' = 25e^{5x}$$

$$y_2 = e^{-3x}$$

$$y_2' = -3e^{-3x}$$

$$y_2'' = 9e^{-3x}$$

$$\text{check} \rightarrow 25e^{5x} - 10e^{5x} - 15e^{5x} \stackrel{?}{=} 0$$

y_1 is soln.

$$9e^{-3x} + 6e^{-3x} - 15e^{-3x} \stackrel{?}{=} 0$$

y_2 is soln.

$$\star W = \begin{vmatrix} e^{5x} & e^{-3x} \\ 5e^{5x} & -3e^{-3x} \end{vmatrix} = -e^{5x} \cdot 3e^{-3x} - 5e^{5x} \cdot e^{-3x} \\ = -3e^{2x} - 5e^{2x} = -8e^{2x}$$

$W \neq 0 \rightarrow$ independent basis ✓

$$y_3 = C_1 y_1 + C_2 y_2 = C_1 e^{5x} + C_2 e^{-3x}$$

$$9C_2 e^{-3x} + 25C_1 e^{5x} - 10C_1 e^{5x} + 6C_2 e^{-3x} - 15C_1 e^{5x} - 15C_2 e^{-3x} \stackrel{?}{=} 0 \checkmark$$

* 2nd O.D.E

(I) 2nd O.D.E linear + homogeneous with constant coefficients

$$y'' + ay' + by = 0 \quad (I) \rightarrow a, b \text{ are constants}$$

Let's solve this

• Let $y = e^{\lambda x}$, where λ could be any number.

$$\begin{cases} y' = \lambda e^{\lambda x} \\ y'' = \lambda^2 e^{\lambda x} \end{cases}$$

substitute in (I)

$$\lambda^2 e^{\lambda x} + a \lambda e^{\lambda x} + b e^{\lambda x} = 0$$

$$e^{\lambda x} (\lambda^2 + a\lambda + b) = 0$$

must equal zero

$$\therefore \lambda^2 + a\lambda + b = 0 \quad \text{Quadratic}$$

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

according to $a^2 - 4b$, there are 3 cases:

* Case 1: If $(a^2 - 4b) > 0$, then there are 2 real distinct roots

$$\lambda_1 = \frac{-a}{2} + \frac{\sqrt{a^2 - 4b}}{2}$$

$$\lambda_2 = \frac{-a}{2} - \frac{\sqrt{a^2 - 4b}}{2}$$

* Case 2:

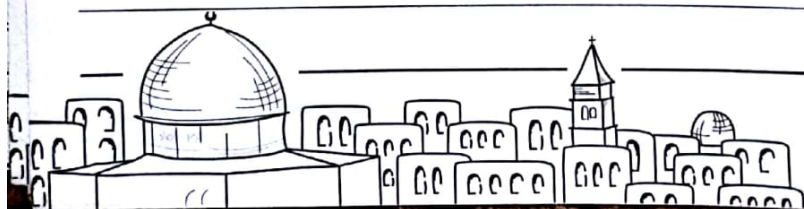
if $(a^2 - 4b) = 0$, there are two identical roots

$$\lambda_1 = \lambda_2 = \frac{-a}{2}$$

* Case 3: If $(a^2 - 4b) < 0$, then we have complex roots

$$\lambda_1 = \frac{-a}{2} + wi$$

$$\lambda_2 = \frac{-a}{2} - wi$$



→ case 1 if $\lambda_1 = \frac{-a}{2} + \frac{\sqrt{a^2-4b}}{2}$, substituted in the sol.

$$y = e^{\left(\frac{-a}{2} + \frac{\sqrt{a^2-4b}}{2}\right)x}$$

$$\lambda_2 = \frac{-a}{2} - \frac{\sqrt{a^2-4b}}{2}, \quad y = e^{\left(\frac{-a}{2} - \frac{\sqrt{a^2-4b}}{2}\right)x}$$

* $y_1 = e^{\lambda_1 x}$

$y_2 = e^{\lambda_2 x}$

→ to check the dependency

$$\frac{y_1}{y_2} = \frac{e^{\lambda_1 x}}{e^{\lambda_2 x}} = e^{(\lambda_1 - \lambda_2)x}, \text{ since } \lambda_1 \neq \lambda_2$$

$\therefore e^{(\lambda_1 - \lambda_2)x}$ is not constant $\Rightarrow y_1$ & y_2 are linearly independent

* Then a linear combination can be built up to right a general solution.

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

general sol. $\left[y = C_1 e^{\left(\frac{-a}{2} + \frac{\sqrt{a^2-4b}}{2}\right)x} + C_2 e^{\left(\frac{-a}{2} - \frac{\sqrt{a^2-4b}}{2}\right)x} \right]$

particular sol. $\left[y_p = \boxed{} e^{\left(\frac{-a}{2} + \frac{\sqrt{a^2-4b}}{2}\right)x} + \boxed{} e^{\left(\frac{-a}{2} - \frac{\sqrt{a^2-4b}}{2}\right)x} \right]$ تحقق النتيجة



ex) $y'' - 2y' - 15y = 0$

let $y = e^{\lambda x}$ $y' = \lambda e^{\lambda x}$ $y'' = \lambda^2 e^{\lambda x}$

$$(\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} - 15 e^{\lambda x}) = 0$$

$$e^{\lambda x} (\lambda^2 - 2\lambda - 15) = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 + 960}}{2} \Rightarrow \lambda_1 = 5 \quad \lambda_2 = -3$$

$$y_1 = e^{\lambda_1 x} = e^{5x} \quad y_2 = e^{\lambda_2 x} = e^{-3x}$$

$y_1' = 5e^{5x}, y_1'' = 25e^{5x}$

$\rightarrow 25e^{5x} - 10e^{5x} - 15e^{5x} = 0$

$\frac{y_1}{y_2} = \frac{e^{5x}}{e^{-3x}} = e^{8x} \neq \text{constan} \therefore$ independent $\rightarrow y_1, y_2$ are linearly independent

$$W = \begin{vmatrix} e^{5x} & e^{-3x} \\ 5e^{5x} & -3e^{-3x} \end{vmatrix} = 3e^{2x} - 5e^{2x} = -8e^{2x} \neq 0$$

$\therefore$ independent solutions

* now the general solution using the linear combination

$$y = C_1 e^{5x} + C_2 e^{-3x}$$

ex) $y'' + 2.2y' + .4y = 0$ $y(0) = -1.2$ $y(0) = 3.3$

let $y = e^{\lambda x}$ $y' = \lambda e^{\lambda x}$ $y'' = \lambda^2 e^{\lambda x}$

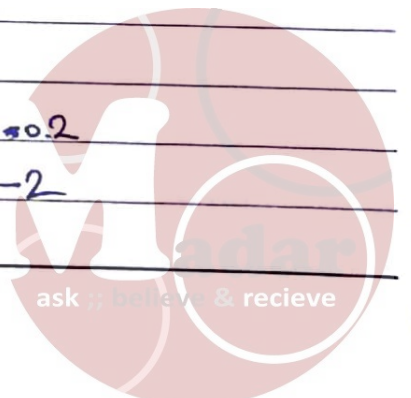
$$\lambda^2 e^{\lambda x} + 2.2\lambda e^{\lambda x} + .4e^{\lambda x} = 0$$

$$e^{\lambda x} (\lambda^2 + 2.2\lambda + 0.4) = 0$$

$$\lambda = \frac{-2.2 \pm \sqrt{2.2^2 - 4(.4)}}{2}$$

$$\lambda_1 = -1.2$$

$$\lambda_2 = -2$$



$$\text{for } \lambda_1 = -0.2 \rightarrow y_1 = e^{-0.2x}$$

$$\text{for } \lambda_2 = -2 \rightarrow y_2 = e^{-2x}$$

$$\frac{y_1}{y_2} = \frac{e^{-0.2x}}{e^{-2x}} = e^{1.8x} \neq \text{constant}$$

$\therefore y_1$ and y_2 are linearly independent

$$y = C_1 e^{-0.2x} + C_2 e^{-2x}$$

$$(y(0) = 3.3) \rightarrow 3.3 = C_1 + C_2 \quad \text{--- (1)}$$

$$y'(0) = -1.2 \rightarrow -1.2 = -0.2C_1 - 2C_2 \quad \text{--- (2)}$$

} solve.

$$C_1 = 3$$

$$C_2 = -3$$

$$\therefore y_p = 3e^{-0.2x} - 3e^{-2x}$$



* Case 2 if $(a^2 - 4b) = \text{zero}$, repeated values of λ

$$\lambda_1 = \lambda_2 = -\frac{a}{2}$$

for $\lambda_1 = -\frac{a}{2}$, $y_1 = e^{-\frac{a}{2}x}$
 $\lambda_2 = -\frac{a}{2}$, $y_2 = e^{-\frac{a}{2}x}$

* $\frac{y_1}{y_2} = 1$, since the ratio is constant, then y_1 and y_2 are dependent.

* To obtain another linearly independent solution, we will let

$y_2 = u y_1$ dependent var. (function of x) only

$$y'' + ay' + by = 0 \quad \left\{ \begin{array}{l} y_2' = u y_1' + u' y_1 \\ y_2'' = u y_1'' + u' y_1' + u' y_1' + u'' y_1 \end{array} \right.$$

$$u y_1'' + 2u' y_1' + u'' y_1 + a u y_1' + a u' y_1 + b u y_1 = 0$$

$$y_1 u'' + (2y_1' + a y_1) u' + (y_1'' + a y_1' + b y_1) u = 0$$

also zero $\left[y_1 = e^{-\frac{a}{2}x} \quad y_1' = -\frac{a}{2} e^{-\frac{a}{2}x} \quad y_1'' = \frac{a^2}{4} e^{-\frac{a}{2}x} \right]$

$$\frac{a^2}{4} e^{-\frac{a}{2}x} + a \left(-\frac{a}{2} \right) e^{-\frac{a}{2}x} + b e^{-\frac{a}{2}x} = 0$$

$$e^{-\frac{a}{2}x} \left(-\frac{a}{2} + b \right) = 0$$

now

$$y_1 u'' = 0$$

$$u'' = 0$$

$$\int \frac{d^2 u}{dx^2} = 0$$

$$\frac{du}{dx} = c_1$$

$$\int du = \int c_1 dx$$

$$u = C_1 x + C_2$$

$$\therefore y_2 = (C_1 x + C_2) y_1$$

$$\frac{y_1}{y_2} = \frac{e^{-\frac{a}{2}x}}{(C_1 x + C_2) e^{-\frac{a}{2}x}} = \frac{1}{C_1 x + C_2}$$

✓ not const



⇒ since the obtained $y_2 = (x+c)y_1$ is a linearly independent solution then the linear combination theory can be used to construct the general solution

$$\begin{aligned} y_g &= Ay_1 + By_2 \\ &= Ae^{-\frac{a}{2}x} + B(x+c_3)e^{-\frac{a}{2}x} \\ &= \underbrace{A}_{L}e^{-\frac{a}{2}x} + Bxe^{-\frac{a}{2}x} + \underbrace{Ac_3}_{L}e^{-\frac{a}{2}x} \end{aligned}$$

$$* y_g = e^{-\frac{a}{2}x} \left[\frac{L}{A+Ac_3} + Bx \right]$$

$$* y_g = (L+Bx)e^{-\frac{a}{2}x}$$

$$y_g = [C_1 + C_2x]e^{-\frac{a}{2}x}$$

(ex) $y'' - 6y' + 9y = 0$

$$\text{let } y = e^{\lambda x} \Rightarrow e^{\lambda x}(\lambda^2 - 6\lambda + 9) = 0$$

$$\lambda_1 = \lambda_2 = +3$$

$$\lambda = \frac{6 \pm \sqrt{36-36}}{2} \text{ أو صافين}$$

$$\therefore y_g = [C_1 + C_2x]e^{3x}$$

(ex) $y'' - 4y' + 4y = 0$ let $y = e^{\lambda x} \Rightarrow e^{\lambda x}(\lambda^2 - 4\lambda + 4) = 0$

$$\lambda_2 = \lambda_1 = +2$$

$$\therefore y_g = [C_1 + C_2x]e^{2x}$$



* Case 3

if $(a^2 - 4b) < \text{zero}$

$$i^2 = -1$$

$$\lambda = \frac{-a}{2} \pm \sqrt{a^2 - 4b} = \frac{-a}{2} \pm \sqrt{\frac{a^2 - 4b}{4}}$$

$$\lambda = \frac{-a}{2} \pm \sqrt{(-1) \left(\frac{4b - a^2}{4} \right)}$$

$$= \frac{-a}{2} \pm \sqrt{-1} \cdot \sqrt{\frac{4b - a^2}{4}} \omega$$

$$= \frac{-a}{2} \pm \omega i$$

$$\begin{aligned} \lambda_1 &= -a/2 + \omega i \\ \lambda_2 &= -a/2 - \omega i \end{aligned} \quad \left. \vphantom{\begin{aligned} \lambda_1 &= -a/2 + \omega i \\ \lambda_2 &= -a/2 - \omega i \end{aligned}} \right] \text{not repeated}$$

$$\left[\begin{aligned} \text{for } \lambda_1 &= \frac{-a}{2} + \omega i, y_1 = e^{\left(\frac{-a}{2} + \omega i\right)x} = e^{(-a/2)x} \cdot e^{\omega i x} \\ \text{for } \lambda_2 &= \frac{-a}{2} - \omega i, y_2 = e^{\left(\frac{-a}{2} - \omega i\right)x} = e^{(-a/2)x} \cdot e^{-\omega i x} \end{aligned} \right.$$

$$\rightarrow \text{From mathematics with } \begin{aligned} e^{\theta i x} &= (\cos(\theta x) + i \sin(\theta x)) \\ e^{-\theta i x} &= (\cos(\theta x) - i \sin(\theta x)) \end{aligned}$$

$$y_1 = e^{-\frac{a}{2}x} [\cos(\omega x) + i \sin(\omega x)]$$

$$y_2 = e^{-\frac{a}{2}x} [\cos(\omega x) - i \sin(\omega x)]$$

$\rightarrow$ since y_1 and y_2 are linearly independent, then the linear combination will give us the general sol.

$$y_g = C_1 y_1 + C_2 y_2$$



ask & recieve

$$\begin{aligned}
 y &= e^{(q/2)x} \left(C_1 (\cos(wx) + i \sin(wx)) + e^{-\frac{q}{2}x} C_2 (\cos(wx) - i \sin(wx)) \right) \\
 &= e^{(q/2)x} \left(C_1 \cos(wx) + C_1 i \sin(wx) + C_2 \cos(wx) - C_2 i \sin(wx) \right) \\
 &= e^{(q/2)x} \left[\cos(wx) \underbrace{[C_1 + C_2]}_{A \text{ Real const.}} + \sin(wx) \underbrace{[C_1 - C_2]i}_{B = C_3 i \text{ imaginary const.}} \right]
 \end{aligned}$$

$$\therefore y = e^{(q/2)x} [A \cos(wx) + B \sin(wx)]$$

ex) $y'' + 9y = 0$ let $y = e^{\lambda x}$

$$e^{\lambda x} [\lambda^2 + 9] = 0$$

$$\lambda^2 = -9$$

$$\lambda = \underbrace{0}_{\frac{-9}{2}} \pm \underbrace{3i}_w$$

$$y = e^0 [A \cos(3x) + B \sin(3x)]$$

ex) $y'' + 2y' + 5y = 0$ $y(0) = 1$ $y'(0) = 5$

let $y = e^{\lambda x}$

$$e^{\lambda x} (\lambda^2 + 2\lambda + 5) = 0 \quad \frac{-2 \pm \sqrt{-16}}{2} \quad \text{then } -1 \pm 2i$$

$$y = e^{-x} (A \cos(2x) + B \sin(2x))$$

$$1 = e^0 (A \cos(0) + B \sin(0)) \rightarrow A = 1$$

$$y' = -e^{-x} (A \cos(2x) + B \sin(2x)) + e^{-x} (-2A \sin(2x) + 2B \cos(2x))$$

$$5 = -(1) + 0 + 2B \rightarrow B = 3$$

$$* y_p = e^{-x} (\cos(2x) + 3 \sin(2x)) *$$



* 2nd O.D.E [homogenous, variable coefficients]

$$x^2 y'' + ax y' + by = 0$$

let $y = x^\mu$ constant

$$y' = \mu x^{\mu-1} \quad y'' = \mu(\mu-1) x^{\mu-2}$$

$$x^2 (\mu(\mu-1) x^{\mu-2}) + ax \mu x^{\mu-1} + b x^\mu = 0$$

$$x^\mu (\mu^2 - \mu + a\mu + b) = 0$$

$$x^\mu (\underbrace{\mu^2 + (a-1)\mu + b}_{=0}) = 0$$

* Depends on $[\mu^2 + (a-1)\mu + b = 0]$ there will be three cases

$$\mu = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}$$

case 1 if $[(a-1)^2 - 4b] > \text{zero}$, μ_1 and μ_2 are real distinct

case 2 if $[(a-1)^2 - 4b] = \text{zero}$, $\mu_1 = \mu_2 = \frac{1-a}{2}$

case 3 if $[(a-1)^2 - 4b] < \text{zero}$, μ_1 and μ_2 are complex.

* case 1: $\mu_1 \neq \mu_2$

for μ_1 , $y_1 = x^{\mu_1}$

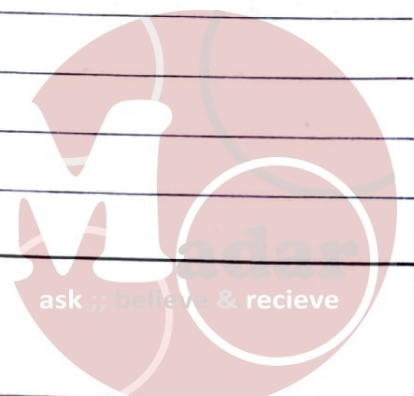
for μ_2 , $y_2 = x^{\mu_2}$

$$\frac{y_1}{y_2} = \frac{x^{\mu_1}}{x^{\mu_2}} = x^{\mu_1 - \mu_2} \neq 0$$

$\therefore y_1, y_2$ are linearly independent

→ Using the linear combination theorem

$$y = C_1 y_1 + C_2 y_2 = C_1 x^{\mu_1} + C_2 x^{\mu_2}$$



(ex) $x^2 y'' - 2.5 xy' - 2y = 0$

let $y = x^u$ $a = -2.5$
 $b = -2$

$$x^u (u^2 - 3.5u - 2) = 0$$

$$u = \frac{3.5 \pm \sqrt{(3.5)^2 + 8}}{2}$$

$\left[\begin{array}{l} u_1 = 4, y_1 = x^4 \\ u_2 = -0.5, y_2 = x^{-0.5} \end{array} \right]$

$$y_g = C_1 x^4 + C_2 x^{-0.5}$$

$$y_g = C_1 x^4 + \frac{C_2}{\sqrt{x}}$$

dependency = $\frac{y_1}{y_2} = x^{4.5} \neq \text{const}$

$y_1 = x^4, y_1' = 4x^3, y_1'' = 12x^2$

$$12x^2 \cdot x^2 - 2.5x \cdot 4x^3 - 2x^4 \stackrel{?}{=} 0$$

$$12x^2 - 10x^4 - 2x^4 = 0$$

$$0 = 0 \quad \checkmark$$

(ex) $x^2 y'' + xy' - 0.1y = 0$

let $y = x^u$ $a = 1$ $b = -0.1$

$$x^u (u^2 + u - 0.1) = 0$$

$$u = \frac{-0 \pm \sqrt{0^2 + 4 \cdot 0.1}}{2} = \frac{\pm 0.2}{2} = \pm 0.1$$

$$y_g = C_1 x^{0.1} + C_2 x^{-0.1}$$

$\begin{array}{l} u_1 = 0.1, y_1 = x^{0.1} \\ u_2 = -0.1, y_2 = x^{-0.1} \end{array}$



* Case 2 if $[(a-1)^2 - 4b] = 0$, $\mu_1 = \mu_2 = \frac{1-a}{2}$

To obtain y_2 we let $y_2 = u y_1$,

$$x^2 y'' + a x y' + b y = 0$$

$$\begin{cases} y_2' = u y_1' + u' y_1 \\ y_2'' = u y_1'' + u' y_1' + u' y_1' + u'' y_1 \\ y_2'' = u y_1'' + 2u' y_1' + u'' y_1 \end{cases}$$

$$x^2 (u y_1'' + 2u' y_1' + u'' y_1) + a x (u y_1' + u' y_1) + b u y_1 = 0$$

$$x^2 y_1'' u + 2x^2 u' y_1' + x^2 u'' y_1 + a x u y_1' + a x u' y_1 + b u y_1 = 0$$

$$x^2 y_1'' u + (2x^2 y_1' + a x y_1) u' + (x^2 y_1'' + a x y_1' + b y_1) u = 0$$

$(y_1 = x^{\frac{1-a}{2}} \quad y_1' = \frac{(1-a)}{2} x^{\frac{1-a}{2}-1})$ zero because y_1 is solution

$$2x^2 \frac{(1-a)}{2} \frac{\frac{1-a}{2}-1}{x^2} x + a x' x^{\frac{(1-a)}{2}} = ?$$

$$(x^{\frac{1-a}{2}} \cdot x - a x^{\frac{1-a}{2}} x + a x^{\frac{1-a}{2}} \cdot x)$$

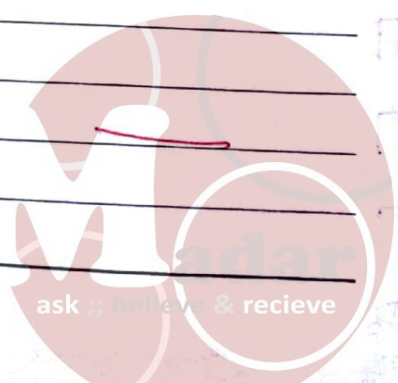
$x y_1^2$

$$\rightarrow (x^2 y_1 u'' + x y_1 u' = \text{zero}) / y_1 x$$

$$x u'' + u' = 0$$

$\rightarrow$ since u doesn't appear $\rightarrow$ [1st special case]
 $z = u' \quad z' = u''$

$$x z' + z = 0 \quad z' = -\frac{z}{x}$$



$$\int \frac{dz}{z} = \int -\frac{1}{x} dx + C$$

$$e(\ln z = -\ln x + C)$$

$$z = x^{-1} \cdot e^C$$

$$z = \frac{1}{x} \cdot C_2$$

$$z = u \quad \swarrow \quad z = \frac{C_2}{x}$$

$$\frac{du}{dx} = \frac{C_2}{x}$$

$$\int du = \int \frac{C_2}{x} dx + C_3$$

$$u = C_2 \ln x + C_3$$

$$\therefore y_2 = u y_1$$

$$y_2 = (C_2 \ln x + C_3) y_1$$

$\frac{y_1}{y_2} \neq \text{const.} \rightarrow \text{independent}$

$$y_g = A y_1 + B y_2$$

$$= A (x^{\frac{1-a}{2}}) + B (C_2 \ln x + C_3) x^{\frac{1-a}{2}}$$

$$y = x^{\frac{1-a}{2}} (A + B C_2 \ln x + B C_3)$$

$$* \boxed{y_g = x^{\frac{1-a}{2}} (L + u \ln x)} *$$



* special cases for 2nd O.D.E

① if in a 2nd O.D.E the independent variable y doesn't appear explicitly into the equation

D.E $f(y'', y', x) = 0$ / [lets y diso]
let $y' = z$ this assumption will 1st & 2nd in 1st diso
reduce the 2nd O.D.E to a 1st O.D.E

$$xy'' = 2y' \rightarrow f(x, y', y'') = 0$$

let $\begin{bmatrix} y' = z \\ y'' = z' \end{bmatrix} \uparrow$

$$(xz' - 2z = 0) / x$$

$$z' - \frac{2}{x}z = 0 \quad \leftarrow \text{linear}$$

$$z = e^{-\int \frac{-2}{x} dx} \cdot C_1$$

$$z = x^2 C_1$$

$$y' = x^2 C_1$$

$$\frac{dy}{dx} = x^2 C_1$$

$$\int dy = \int C_1 x^2 dx$$

$$y = \frac{1}{3} x^3 C_1 + C_2$$

② if in the 2nd O.D.E the independent variable x doesn't appear explicitly into the equation

$$f(y'', y', y) = 0$$

$$\text{let } y'' = \frac{dz}{dy} z \quad \text{and} \quad z = \frac{dy}{dx}$$



$$y'' + e^{2y} \cdot (y')^3 = 0$$

$$y'' = \frac{dz}{dy} z \quad \text{and} \quad z = \frac{dy}{dx}$$

$$\frac{dz}{dy} \cdot z + e^{2y} \cdot z^3 = 0$$

$$\frac{dz}{dy} + e^{2y} z^2 = 0$$

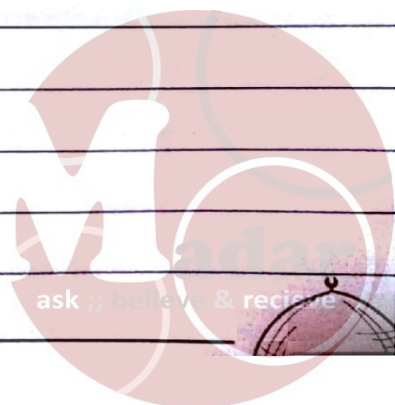
$$\int \frac{dz}{z^2} = \int -\frac{dy}{e^{2y}} + C$$

$$\frac{-1}{z} = \frac{e^{-2y}}{-2} + C$$

$$\frac{1}{z} = \frac{e^{-2y}}{2} + C_1$$

$$\int \frac{dx}{dy} = \int \left(\frac{1}{2} e^{2y} + C_1 \right) dy$$

$$C_2 + X = \frac{1}{4} e^{2y} + C_1 y$$



(ex) $x^2 y'' + 9xy' + 16y = 0$
 $a=9$ $b=16$

let $y = x^u$ $y' = u x^{u-1}$ $y'' = u(u-1)x^{u-2}$

$$x^u (u^2 + (a-1)u + b) = 0$$

$$u = \frac{-8 \pm \sqrt{64-64}}{2} = -4$$

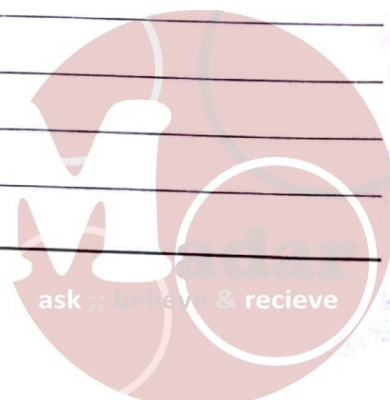
$$u_1 = u_2 = -4 \frac{-(a-1)}{2}$$

$y_g = x^{-4} (L + M \ln x)$

1) $x^2 y'' + 3xy' + y = 0$

hw

2) $4x^2 y'' + 4xy' - y = 0$ ($y(4) = 2$ $y'(4) = -.25$)



Case 3 $x^2 y'' + axy' + by = 0$

$$x^M (M^2 + (a-1)M + b) = 0$$

$$M = \frac{(1-a) \pm \sqrt{(a-1)^2 - 4b}}{2} \quad \sqrt{(a-1)^2 - 4b} \leq 0 \quad \text{complex roots}$$

$$= \left(\frac{1-a}{2} \right) \pm \sqrt{\frac{(a-1)^2 - 4b}{4}} \quad \text{complex}$$

$$= \frac{(1-a)}{2} \pm \sqrt{-1} \cdot \sqrt{\frac{4b - (a-1)^2}{4}} \quad \gamma$$

$$= \frac{(1-a)}{2} \pm \gamma i$$

$$M_1 = \frac{(1-a)}{2} \pm \gamma i \rightarrow y_1 = x^{\frac{(1-a)}{2} \pm \gamma i} = x^{\frac{1-a}{2}} \cdot x^{\pm \gamma i}$$

$$M_2 = \frac{(1-a)}{2} - \gamma i \rightarrow y_2 = x^{\frac{(1-a)}{2} - \gamma i} = x^{\frac{1-a}{2}} \cdot x^{-\gamma i}$$

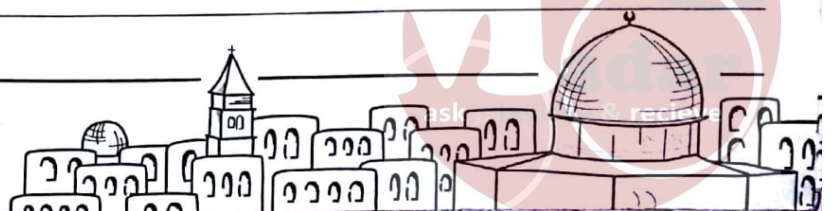
$$\rightarrow \begin{cases} e^{+i\theta x} = \cos(\theta x) + i \sin(\theta x) \\ e^{-i\theta x} = \cos(\theta x) - i \sin(\theta x) \end{cases}$$

$$\rightarrow \begin{cases} x^{\pm \gamma i} = \cos(\gamma \ln x) \pm i \sin(\gamma \ln x) \\ x^{\mp \gamma i} = \cos(\gamma \ln x) - i \sin(\gamma \ln x) \end{cases}$$

$$y_1 = x^{\frac{1-a}{2}} \cdot (\cos(\gamma \ln x) + i \sin(\gamma \ln x))$$

$$y_2 = x^{\frac{1-a}{2}} \cdot (\cos(\gamma \ln x) - i \sin(\gamma \ln x))$$

since y_1 and y_2 are linearly independent then
 $y_g = C_1 y_1 + C_2 y_2$



$$y = C_1 \left(X^{\frac{(-a+1)}{2}} (\cos(\sqrt{b} \ln x) + i \sin(\sqrt{b} \ln x)) \right) + C_2 \left(X^{\frac{1-a}{2}} (\cos(\sqrt{b} \ln x) - i \sin(\sqrt{b} \ln x)) \right)$$

$$y = X^{\frac{1-a}{2}} (C_1 \cos(\sqrt{b} \ln x) + C_1 i \sin(\sqrt{b} \ln x) + C_2 \cos(\sqrt{b} \ln x) - C_2 i \sin(\sqrt{b} \ln x))$$

$$y = X^{\frac{1-a}{2}} \left(\cos(\sqrt{b} \ln x) (C_1 + C_2) + i \sin(\sqrt{b} \ln x) (C_1 - C_2) \right)$$

$$\therefore y = X^{\frac{1-a}{2}} (A \cos(\sqrt{b} \ln x) + B \sin(\sqrt{b} \ln x))$$

ex $x^2 y'' + 7x y' + 13y = 0$ $a = 7$ $b = 13$

let $y = x^u$ $(a-1)$
 $x^u (u^2 + 6u + 13) = 0$

$$u = \frac{-6 \pm \sqrt{36 - 4 \times 13}}{2} = \frac{-6 \pm \sqrt{-16}}{2}$$

$\frac{1-a}{2} \rightarrow -3 \pm \sqrt{-1} \cdot 2$

$$y = x^{-3} (A \cos(2 \ln x) + B \sin(2 \ln x))$$

ex $x^2 y'' - x y' + 2y = 0$, $y(1) = -1$ $y'(1) = -1$

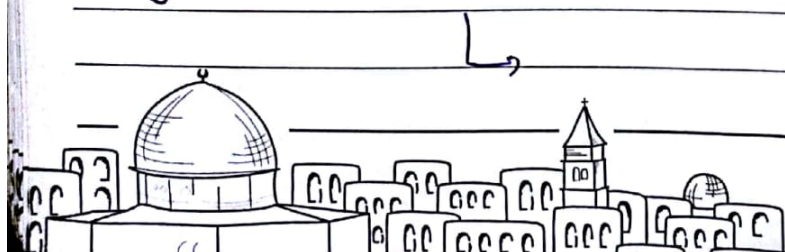
let $y = x^u$ $a = -1$
 $b = 2$

$$x^u (u^2 - 2u + 2) = 0$$

$$u = \frac{2 \pm \sqrt{4 - 4 \times 2}}{2}$$

$$y = x' (A \cos(\ln x) + B \sin(\ln x))$$

$$\frac{1-a}{2} \rightarrow 1 \pm \sqrt{-1} \cdot 1$$



$$y(1) = -1 \rightarrow -1 = (A \cos(0) + B \sin(0))$$

$$-1 = A$$

$$y'(1) = -1 \rightarrow y'_g = (A \cos(\ln x) + B \sin(\ln x)) + x \left(-\frac{A}{x} \sin(\ln x) + \frac{B}{x} \cos(\ln x) \right)$$

$$-1 = (A \cos(0) + B \sin(0)) + (-A \sin(0) + B \cos(0))$$

$$-1 = A + B$$

$$-1 = -1 + B$$

$$0 = B$$

$$\therefore y_p = x (-\cos(\ln x)) \leftarrow$$



* 2nd . o . D . E nonhomogeneous with constant coefficients.

$$y'' + ay' + by = r(x) \quad a, b \text{ are constants}$$

$$r(x) \neq 0$$

$$y_g = y_{\text{homogeneous}} + y_{\text{nonhomogeneous}}$$

→ where y_h is the solution of the 2nd . o . D . E but without $r(x)$

→ where y_N is the nonhomogeneous part of the solution
"free of any arbitrary constant" related to $r(x)$.

* To find y_N we use one of the following.

- ① Undetermined coefficients (specific) ② Variation of parameters (general)

1 →

$r(x)$	assumption y_N
① e^{kx}	$A e^{kx}$
② $X^n + X^{n-1} + X^{n-2} + \dots + X^0$	$AX^n + BX^{n-1} + CX^{n-2} + \dots + DX^0$
③ $\sin(nx), \cos(nx)$	$A \cos(nx) + B \sin(nx)$
$\tan(nx), \cot(nx)$	
$\sec(nx), \csc(nx)$	

$n \rightarrow$ is positive integer / notation

$\sin(2x) \rightarrow A \cos(2x) + B \sin(2x)$

$r(x) = \sin(2x) + \cos(3x)$

$\rightarrow A \sin(2x) + B \cos(2x) + C \sin(3x) + D \cos(3x)$

$$\rightarrow r(x) = e^{3x} + 2 \sin(3x)$$

$$y_N = Ae^{3x} + B \cos(3x) + C \sin(3x)$$

$$\rightarrow r(x) = e^{3x} \cdot \sin(3x)$$

$$= Ae^{3x} \cdot (B \cos(3x) + C \sin(3x))$$



* nonhomogeneous 2nd. o. D.E

$$y'' + ay' + by = r(x)$$

$$x^2 y'' + axy' + by = r(x)$$

$$y = y_h + y_n$$

$y_h \rightarrow$ homogenous solution

$y_n \rightarrow$ nonhomogeneous solution.
arbitrary const. $C_1, C_2, \dots$

* Undetermined coefficient, $r(x)$ must be simple $e^{ax}, \sin(ax), x^2 + ax + b, x^0, \dots$

(EX) $y'' - 3y' + 2y = e^x$

$$y = y_h + y_n$$

$y_h \Rightarrow y'' - 3y' + 2y = 0$
let $y = e^{\lambda x} \rightarrow$

$$e^{\lambda x} (\lambda^2 - 3\lambda + 2) = 0$$

$$\lambda = \frac{+3 \pm \sqrt{9 - 4 \times 2}}{2} \Rightarrow \lambda = 2, 1$$

$$\therefore y_{\text{homo}} = C_1 e^x + C_2 e^{2x}$$

let $y_n = A e^x$
 $y_n' = A e^x$
 $y_n'' = A e^x$

$$A e^x - 3A e^x + 2A e^x = e^x$$

هذا انك في بي الالف
dependent غير Lips
 $\frac{y_n}{y_h} = \text{const} !!$

since $y = A e^x$ is already exist in y_h then we have to
modify y_n to $y_n = \lambda x e^x \rightarrow$ independent



$$y'_N = ae^x + axe^x$$

$$y''_N = ax + axe^x + ae^x = axe^x + 2ae^x$$

$$2Ae^x + Axe^x - 3Ae^x - 3Axe^x + 2Axe^x = e^x$$

$$-Ae^x = e^x$$

$$\therefore A = -1$$

$$y_N = -Xe^x$$

$$\Rightarrow y_g = C_1 e^x + C_2 e^{2x} - Xe^x$$

$$y'_g = C_1 e^x + 2C_2 e^{2x} - e^x - Xe^x$$

$$y''_g = C_1 e^x + 4C_2 e^{2x} - e^x - e^x - Xe^x$$

$$C_1 e^x + 4C_2 e^{2x} - 2e^x - Xe^x$$

(تقريباً)

$$y'' - 3y' + 2y = e^x$$

$$e^x = C_1 e^x + 4C_2 e^{2x} - 2e^x - Xe^x - 3C_1 e^x - 6C_2 e^{2x} + 3e^x + 3Xe^x + 2e^x$$

$$+ 2C_2 e^{2x} - 2Xe^x$$

; simplification

$$e^x = e^x$$

ex $y'' - 2y' + y = e^x + x$

$$y_g = y_h + y_N$$

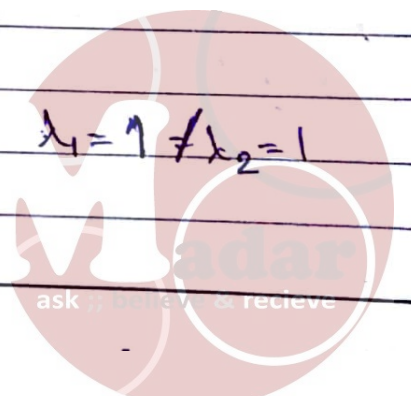
$$y_h \rightarrow y'' - 2y' + y = 0$$

$$y = e^{\lambda x}$$

$$e^{\lambda x} (\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda_1 = 1 \neq \lambda_2 = 1$$

$$y_h = C_1 e^x + C_2 x e^x$$



for $y_N \rightarrow$ let $y_N = \underbrace{Ae^x}_{\text{modify der. eq.}} + Bx + C$

modify $y_N = Ax^2e^x + Bx + C$

$$y'_N = 2Axe^x + Ax^2e^x + B$$

$$y''_N = 2Ae^x + \underbrace{2Axe^x + 2Axe^x}_{4Axe^x} + Ax^2e^x$$

$$2Ae^x + 4Axe^x + Ax^2e^x - 4Axe^x - 2Ax^2e^x - 2B + Ax^2e^x + Bx + C = e^x + x$$

$$2Ae^x = e^x \rightarrow A = \frac{1}{2}$$

$$Bx = x \rightarrow B = 1$$

$$-2B + C = 0 \rightarrow C = 2$$

$$y_N = \frac{1}{2}x^2e^x + x + 2$$

الحدود لا يكونه عندي
Conditions
يكونه عندي ولا مش y_h

$$\Rightarrow y_g = C_1e^x + C_2xe^x + \frac{1}{2}x^2e^x + x + 2$$

Ex $y'' + 2y' + 5y = 16e^x + \sin(2x)$
 $[Ae^x + B\cos(2x) + C\sin(2x)]$

$$y_h \Rightarrow y'' + 2y' + 5y = 0$$

$$\text{let } y = e^{\lambda x} \rightarrow e^{\lambda x}(\lambda^2 + 2\lambda + 5) = 0$$

$$\lambda_1 = -1 + 2i$$

$$\lambda_2 = -1 - 2i$$

$$y_h = e^{-x} [C_1 \cos(2x) + C_2 \sin(2x)]$$



$$y_N = Ae^x + B\cos(2x) + C\sin(2x)$$

$$y'_N = Ae^x - 2B\sin(2x) + 2C\cos(2x)$$

$$y''_N = Ae^x - 4B\cos(2x) - 4C\sin(2x)$$

$$Ae^x - 4B\cos(2x) - 4C\sin(2x) + 2Ae^x - 4B\sin(2x) + 4C\cos(2x) + 5Ae^x + 5B(\cos(2x)) + 5C\sin(2x) = 16e^x + \sin(2x)$$

$$8Ae^x = 16e^x \rightarrow A = 2$$

$$(C - 4B)\sin(2x) = \sin(2x)$$

$$C - 4B = 1 \quad \text{--- (1)}$$

$$-\frac{4}{17}$$

$$(B + 4C)\cos(2x) = 0$$

$$B + 4C = 0 \quad \text{--- (2)}$$

$$B = \frac{4}{17}$$

$$C = \frac{1}{17}$$

$$y_N = 2e^x + \frac{-4}{17}\cos(2x) + \frac{1}{17}\sin(2x)$$

$$\Rightarrow y = e^x [C_2 \sin(2x) + C_1 \cos(2x)] + 2e^x - \frac{4}{17}\cos(2x) + \frac{1}{17}\sin(2x)$$

ANSWER



* Variation of parameters

$$y'' + ay' + by = \boxed{r(x)} \quad \ln x, \sqrt{x}, x^{-1} \rightarrow \text{am} \rightarrow *$$

$$x^2 y'' + axy' + by = \boxed{R(x)} \quad (r(x) = \frac{R(x)}{x^2}) \rightarrow \text{تغير الطريقة} \quad \underline{\underline{r(x)}}$$

* $y_g = y_h + y_N$

$y_h \Rightarrow$ homogeneous solution

$$C_1 \boxed{y_1} + C_2 \boxed{y_2}$$

$$y_N = \boxed{\frac{y_1}{y_2}} \int \frac{W_1}{W} r(x) dx + \boxed{\frac{y_2}{y_1}} \int \frac{W_2}{W} r(x) dx$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = 0 - y_2$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1 - 0$$

$$\boxed{y_N = -y_1 \int \frac{y_2}{W} r(x) dx + y_2 \int \frac{y_1}{W} r(x) dx}$$



ex $y'' - 2y' + y = x^{3/2} \cdot e^x$
 $r(x)$

$y_h \Rightarrow y'' - 2y' + y = 0$
 $e^{\lambda x} (\lambda^2 - 2\lambda + 1) = 0 \quad \lambda_1 = 1 \quad \lambda_2 = 1$

$\Rightarrow y_h = c_1 \boxed{e^x} + c_2 \boxed{x e^x}$
 $y_1 \quad y_2$

$W = \begin{vmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{vmatrix} = x e^{2x} + e^x - x e^{2x} = e^{2x}$

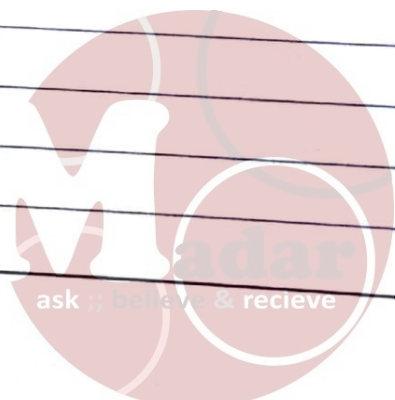
$W_1 = -y_2 = -x e^x \quad W_2 = y_1 = e^x$

$y_N = -e^x \int \frac{x e^x}{e^{2x}} \cdot x^{3/2} e^x dx + x e^x \int \frac{e^x}{e^{2x}} \cdot x^{3/2} e^x dx$
 $= -e^x \int x^{5/2} dx + x e^x \int x^{3/2} dx$
 $= -\frac{2}{7} x^{7/2} \cdot e^x + x e^x \cdot \frac{2}{5} x^{5/2}$
 $= -\frac{2}{7} x^{7/2} e^x + \frac{2}{5} x^{7/2} e^x$

$\boxed{y_N = \frac{4}{35} x^{7/2} e^x}$

$y_g = c_1 e^x + c_2 x e^x + \frac{4}{35} e^x x^{7/2}$

$\boxed{y = e^x \left(c_1 + x c_2 + \frac{4}{35} x^{7/2} \right)} *$



$$\textcircled{\text{ex}} \quad y'' + 4y' + 4y = \underbrace{x^{-2} \cdot e^{-2x}}_{r(x)}$$

$$y_h \Rightarrow y'' + 4y' + 4y = 0$$

$$e^{\lambda} (\lambda^2 + 4\lambda + 4) = 0$$

$$\lambda = \frac{-4 \pm \sqrt{(4)^2 - 16}}{2} = -2$$

$$\Rightarrow y_h = C_1 \underbrace{\frac{e^{-2x}}{y_1}}_{y_1} + C_2 \underbrace{\frac{x e^{-2x}}{y_2}}_{y_2}$$

$$W = \begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & e^{-2x} - 2x e^{-2x} \end{vmatrix} = \cancel{e^{-4x}} - 2x \cancel{e^{-4x}} + 2x \cancel{e^{-4x}} = e^{-4x}$$

$$W_1 = -x e^{-2x}$$

$$W_2 = e^{-2x}$$

$$y_N = -e^{-2x} \int \frac{x e^{-2x}}{e^{-4x}} \cdot x^2 \cdot e^{-2x} dx + x e^{-2x} \int \frac{e^{-2x}}{e^{-4x}} \cdot x^2 \cdot e^{-2x} dx$$

$$y_N = -e^{-2x} \int x^{-1} dx + x e^{-2x} \int x^{-2} dx$$

$$y_N = -e^{-2x} \ln x + \frac{-x e^{-2x}}{x}$$

$$y_N = -e^{-2x} \ln x - e^{-2x} = \boxed{-e^{-2x} (\ln x + 1)}$$

y_N
لا يوجد شرط ثوابت

$$\left[y_g = e^{-2x} C_1 + x e^{-2x} C_2 - e^{-2x} (\ln x + 1) \right] *$$



(ex) $(x^2 y'' - 4xy' + 6y = \frac{42}{x^4}) / x^2$

$$y'' - \frac{4}{x} y' + 6y = \boxed{\frac{42}{x^6} = r(x)}$$

$$x^2 y'' - 4xy' + 6y = 0$$

$$x^u (u^2 - 5u + 6) = 0$$

$$u = \frac{5 \pm \sqrt{25 - 4 \times 6}}{2}$$

$$\Rightarrow y_h = x^2 C_1 + x^3 C_2$$

$y_1 = w_2$ $y_2 = -w_1$

$$\begin{cases} u_1 = \frac{5+1}{2} = 3 \\ u_2 = \frac{5-1}{2} = 2 \end{cases}$$

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$$

$$y_N = -x^2 \int \frac{x^3}{x^4} \cdot \frac{42}{x^6} dx + x^3 \int \frac{x^2}{x^4} \cdot \frac{42}{x^6} dx$$

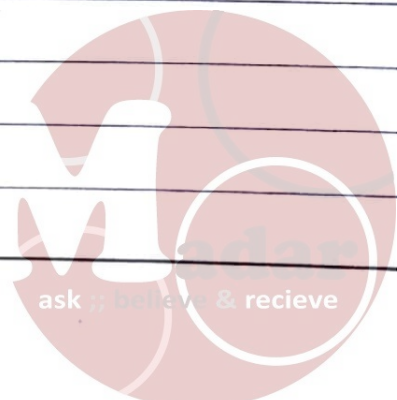
$$= -x^2 \int \frac{42}{x^7} dx + x^3 \int \frac{42}{x^8} dx$$

$$= -42x^2 \cdot \frac{x^{-6}}{-6} + 42x^3 \cdot \frac{x^{-7}}{-7}$$

$$= 7x^{-4} - 6x^{-4}$$

$$\boxed{y_N = x^{-4}}$$

$$\left[y_g = x^{-4} + c_1 x^2 + c_2 x^3 \right] *$$



ex $(x^2 y'' - 2xy' + 2y = 5x^3 \cdot \cos x) / x^2$

$5x \cdot \cos x = r(x)$

$x^2 y'' - 2xy' + 2y = 0$

$x^u (u^2 - 3u + 2) = 0$

$u = \frac{3 \pm \sqrt{9-8}}{2}$

* $y_h = x^{u_1} C_1 + x^{u_2} C_2$
 $u_1 = 2, u_2 = -1$

$u_1 = \frac{3+1}{2} = 2$

$u_2 = \frac{3-1}{2} = 1$

$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$

$y_N = -x \int \frac{x^2}{x^2} \cdot 5x \cos x \, dx + x^2 \int \frac{x}{x^2} 5x \cos x \, dx$

$= -5x \int x \cos x \, dx + 5x^2 \int \cos x \, dx$

$= -5x (x \sin x + \cos x) + 5x^2 \sin x$

$y_N = -5x \cos x$

$y_g = x C_1 + x^2 C_2 - 5x \cos x$ *

ex $x^2 y'' - 4xy' + 6y = x^{-4}$

$\frac{x^{-4}}{x^2} = r(x) = x^{-6}$

$y_h \Rightarrow x^2 y'' - 4xy' + 6y = 0$

$x^u (u^2 - 5u + 6) = 0$

$a = -4, b = 6$

$u = \frac{5 \pm \sqrt{25-42}}{2}$

$y_h = C_1 x^2 + C_2 x^3$
 $y_N = \frac{1}{42} x^{-4}$

$u_1 = 2$

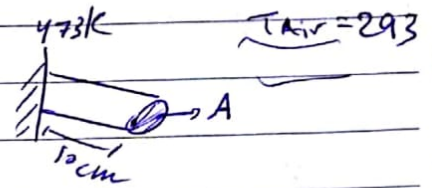
$u_2 = 3$

$y_g = C_1 x^2 + C_2 x^3 + \frac{1}{42} x^{-4}$



Applications.

[Q] $\frac{d^2 T}{dx^2} - \frac{h_c P}{k A_c} (T - T_s) = 0$

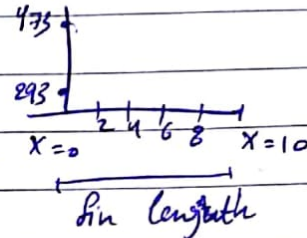


$h_c = 40$

$P = 0.15$

$k = 240$

$A_c = 1.6 \times 10^{-5}$



$\frac{d^2 T}{dx^2} - \frac{40 \times 0.15}{240 \times 1.6 \times 10^{-5}} (T - \overset{\text{ambient } T}{T_s}) = 0$ fin length

$\frac{d^2 T}{dx^2} - 156.25 (T - 293) = 0$

$\frac{d^2 T}{dx^2} - 156.25 T + 4.6 \times 10^4$
 $\left. \begin{aligned} \frac{d^2 T}{dx^2} - 156.25 T &= -4.6 \times 10^4 \end{aligned} \right\}$

$y \Rightarrow T'' - 156.25 T = 0$
 $e^{\lambda x} (\lambda^2 - 156.25) = 0 \quad \lambda = \sqrt{156.25}$

$y_h = C_1 e^{12.5x} + C_2 e^{-12.5x}$

$y \Rightarrow$ let $y_h = L$
 $y'_N = 0 \quad y''_N = 0$

$0 - 156.25 L = -4.6 \times 10^4$

$\therefore L = 293$

$\therefore y_g = C_1 e^{12.5x} + C_2 e^{-12.5x} + 293$

$\left. \begin{aligned} T(0) &= 473 \\ T(10) &= 293 \end{aligned} \right\}$
 10 cm



Higher O.D.E

* Higher O.D.E nonhomogeneous with constant coefficient

$$y''' + ay'' + by' + cy = r(x)$$

$$y'''' + ay'''' + by''' + cy'' + Dy' = r(x)$$

$$e^{\lambda x} (\underbrace{\lambda^3}_{\neq \text{zero}} + \underbrace{a\lambda + b\lambda + c}_{\text{zero}}) = 0$$

Case 1 $\lambda_1 \neq \lambda_2 \neq \lambda_3 \Rightarrow \left. \begin{array}{l} \frac{y_1}{y_2} \neq \text{const} \\ \frac{y_2}{y_3} \neq \text{const} \\ \frac{y_1}{y_3} \neq \text{const} \end{array} \right\} \text{ or } W = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & e^{\lambda_3 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \lambda_3 e^{\lambda_3 x} \\ \lambda_1^2 e^{\lambda_1 x} & \lambda_2^2 e^{\lambda_2 x} & \lambda_3^2 e^{\lambda_3 x} \end{vmatrix} \neq 0$

→ independent sol.

Case 2 $\lambda_1 = \lambda_2 = \lambda_3 \Rightarrow y_1 = e^{\lambda_1 x} \quad y_2 = x e^{\lambda_1 x} \quad y_3 = x^2 e^{\lambda_1 x}$

Case 3 $\lambda_1 = \lambda_2 \neq \lambda_3 \Rightarrow y_1 = e^{\lambda_1 x} \quad y_2 = x e^{\lambda_1 x} \quad y_3 = e^{\lambda_3 x}$

Case 4 λ_1 , 2 complex $\Rightarrow \lambda_2 = \mu + \nu i \quad y_1 = e^{\lambda_1 x} \quad y_2 = e^{\mu x} (\cos(\nu x) + i \sin(\nu x))$
 $\lambda_3 = \mu - \nu i \quad y_3 = e^{\mu x} (\cos(\nu x) - i \sin(\nu x))$



$$(ex) \quad y'''' - y = 4.5e^{-2x} \quad (x)$$

$$y_g = y_h + y_N$$

$$y_h \Rightarrow y'''' - y = 0$$

$$\text{let } y = e^{\lambda x}$$

$$e^{\lambda x} (\lambda^4 - 1) = 0$$

$$(\lambda^2 - 1)(\lambda^2 + 1) = 0$$

$$e^{\lambda x} (\lambda - 1)(\lambda + 1)(\lambda^2 + 1) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = -1$$

$$\lambda = 0 \pm \frac{\sqrt{-4}}{2}$$

$$\lambda_3 = 0 + i$$

$$\lambda_4 = -i$$

$$y_1 = e^x$$

$$y_2 = e^{-x}$$

$$y_3 = e^{ix} = e^i(A \cos(x) + B \sin(x))$$

$$y_4 = e^{-ix} = e^{-i}(A \cos(x) + B \sin(x))$$

$$[y_h = C_1 e^x + C_2 e^{-x} + A \cos(x) + B \sin(x)]$$

$$\text{for } y_N \Rightarrow \text{let } y_N = K e^{-2x}$$

$$16 K e^{-2x} - K e^{-2x} = 4.5 e^{-2x} \quad K \rightarrow \frac{4.5}{15}$$

$$y_N = \frac{4.5}{15} e^{-2x}$$

$$y'_N = -2 K e^{-2x}$$

$$y''_N = 4 K e^{-2x}$$

$$y'''_N = -8 K e^{-2x}$$

$$y''''_N = 16 K e^{-2x}$$

$$(y_g = C_1 e^x + C_2 e^{-x} + A \cos(x) + B \sin(x) + \frac{4.5}{15} e^{-2x})$$



$$\textcircled{\cdot e^x} y''' - 3y'' + 3y' - y = 30e^x \quad r(x)$$

$$y_h \rightarrow y''' - 3y'' + 3y' - y = 0$$

let $e^{\lambda x} = y$

$$e^{\lambda x} (\lambda^3 - 3\lambda^2 + 3\lambda - 1) = 0 \quad \lambda_1 = 1$$

$$e^{\lambda x} (\lambda - 1)(\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = \lambda_3 = +1$$

$$\lambda = \frac{2 \pm \sqrt{4-4}}{2}$$

$$\begin{array}{r} \lambda^2 - 2\lambda + 1 \\ \lambda - 1 \overline{) \lambda^3 - 3\lambda^2 + 3\lambda - 1} \\ \underline{-\lambda^3 + \lambda^2} \\ -2\lambda^2 + 3\lambda - 1 \\ \underline{+2\lambda^2 + 2\lambda} \\ \lambda - 1 \end{array}$$

$$y_1 = e^x$$

$$y_2 = x e^x$$

$$y_3 = x^2 e^x$$

$$y_h = C_1 e^x + C_2 x e^x + C_3 x^2 e^x$$

$$\text{let } y_N = k e^x \xrightarrow{\text{modify}} k x e^x \xrightarrow{\text{modify}} k x^2 e^x \xrightarrow{y_N} k x^3 e^x$$

$$y_N = k x^3 e^x \quad y'_N = 3k x^2 e^x + k x^3 e^x$$

$$y''_N = 6k x e^x + 3k x^2 e^x + 3x^2 k e^x + k x^3 e^x$$

$$= 6k x e^x + 6k x^2 e^x + k x^3 e^x$$

$$y'''_N = 6k e^x + 6k x e^x + 12x k e^x + 6k x^2 e^x + 3x^2 k e^x + k x^3 e^x$$

$$y'''_N = 18k x e^x + 6k e^x + 9k x^2 e^x + k x^3 e^x$$

$$18k x e^x + 6k e^x + 9k x^2 e^x + k x^3 e^x - 3(6k x e^x + 6k x^2 e^x + k x^3 e^x) + 3(3k x^2 e^x + k x^3 e^x) - (k x^3 e^x) = 30 e^x$$

$$k = 5$$

$$(y_N = 5 x^3 e^x)$$

$$y_g = C_1 e^x + C_2 x e^x + C_3 x^2 e^x + 5 x^3 e^x$$



ex $y''' + y'' - 2y = 2x^2 + 2x$

$$\begin{cases} y(0) = -1 \\ y'(0) = 0 \\ y''(0) = -4 \end{cases}$$

$y_h \Rightarrow y''' + y'' - 2y = 0$
let $y = e^{\lambda x}$

$$e^{\lambda x} (\lambda^3 + \lambda^2 - 2) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = -1 + i \quad \lambda_3 = -1 - i$$

$$y_1 = e^x$$

$$y_{2,3} = e^{-x} (C_2 \cos(x) + C_3 \sin(x))$$

$$y_h = C_1 e^x + e^{-x} (C_2 \cos(x) + C_3 \sin(x))$$

$y_N \Rightarrow$ let $y_N = Ax^2 + Bx + C$
 $y'_N = 2xA + B$
 $y''_N = 2A + 0$
 $y'''_N = 0$

$$0 + 2A - 2Ax^2 + -2Bx - 2C = 2x^2 + 2x$$

$$-2A = 2 \rightarrow A = -1$$

$$-2B = 2 \rightarrow B = -1$$

$$2A - 2C = 0 \rightarrow C = -1$$

$$y_N = -x^2 - x - 1$$

$$y_g = C_1 e^x + e^{-x} (C_2 \cos(x) + C_3 \sin(x)) - x^2 - x - 1$$

$$y(0) = -1 \rightarrow -1 = C_1 - C_2 - 1 \rightarrow 0 = C_1 - C_2 \quad \text{--- (1)}$$

بسته مرتبه و جواب
 y_p ال

H.W

H.W ① $y''' - 2y'' - y' + 2y = 2x^2$

② $y''' + 3y'' + 3y' + y = 16e^x + x + 3$



ask & recieve

* Higher O.D.E. cubic

$$x^2 y'' + ax y' + by = r(x)$$

let $y = x^u \rightarrow$

$$x^u [u^2 + (a-1)u + b] = 0$$

$$x^3 y''' + ax^2 y'' + bx y' + cy = r(x)$$

let $y = x^u$, $y' = u(x)^{u-1}$, $y'' = u(u-1)x^{u-2}$

$$y''' = u(u-2)(u-1)x^{u-3}$$

يعني

$$x^3 (u(u-2)(u-1)x^{u-3}) + ax^2 (u(u-1)x^{u-2}) + bx u x^{u-1} + cx^u = 0$$

$$x^u (u^3 + (a-3)u^2 + (2-a+b)u + c) = 0$$

$\downarrow$
= zero

ex $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$

$$a = -3$$

$$b = 6$$

$$c = -6$$

$$x^u (u^3 - 6u^2 + 11u - 6) = 0$$

$$u_1 = 1 \quad y_1 = x$$

$$u_2 = 2 \quad y_2 = x^2$$

$$u_3 = 3 \quad y_3 = x^3$$

$$y_h = C_1 x + C_2 x^2 + C_3 x^3$$



* Variation of Parameters

⇒ For a second order

$$y_N = y_1 \int \frac{W_1}{W} r(x) dx + y_2 \int \frac{W_2}{W} r(x) dx$$

⇒ For a third order

$$y_N = y_1 \int \frac{W_1}{W} r(x) dx + y_2 \int \frac{W_2}{W} r(x) dx + y_3 \int \frac{W_3}{W} r(x) dx$$

$$* W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$r(x) = \frac{R(x)}{x^3}$$

$$* W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix}$$

$$* W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix}$$

$$* W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix}$$

← مثال

ex $x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x$

$$y_h = C_1 x + C_2 x^2 + C_3 x^3$$

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$



ask & receive

$$r(x) = (x^4 \ln x) / x^3 = x \ln x$$

$$y_N = \frac{x}{2} \int \frac{x^4}{2x^3} \cdot x \ln x dx + x^2 \int \frac{-2x^3}{2x^3} \cdot x \ln x dx + x^3 \int \frac{x^2}{2x^2} x \ln x dx$$

$$y_N = \frac{x}{2} \int x^2 \ln x dx - x^2 \int x \ln x dx + \frac{x^3}{2} \int \ln x dx$$

$$= \frac{x}{2} \left(\frac{1}{3} x^3 \ln x - \frac{x^3}{9} \right) - x^2 \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) + \frac{x^3}{2} (x \ln x - x)$$

$$\boxed{y_N = \frac{x^4}{6} \left(\ln x - \frac{11}{6} \right)}$$

$$\boxed{y_g = C_1 x + C_2 x^2 + C_3 x^3 + \frac{x^4}{6} \left(\ln x - \frac{11}{6} \right)}$$

$$\textcircled{e} (x y''' + 3 y'' = e^x) x^2 \Rightarrow r(x) = \frac{e^x}{x}$$

$$(x^3 y''' + 3 x^2 y'') = e^x x^2 \quad a=3 \quad b=0 \quad c=0$$

$$y_h \rightarrow x^3 y''' + 3 x^2 y'' = 0$$

$$\int \text{let } y = x^\mu \quad y' = \mu x^{\mu-1} \quad y'' = \mu(\mu-1) x^{\mu-2} \quad y''' = \mu(\mu-1)(\mu-2) x^{\mu-3}$$

$$x^\mu [\mu^3 + (a-3)\mu^2 + (a+2b-a)\mu + c] = 0$$

$$x^\mu [\mu^3 + 0 - \mu^2 + -\mu + 0] = 0$$

$$x^\mu [\mu^3 - \mu] = 0$$

$$\mu_1 = 0, \mu_2 = 1, \mu_3 = -1$$

$$y_1 = 1 \quad y_2 = x \quad y_3 = x^{-1}$$

$$y_h = C_1 + C_2 x + \frac{C_3}{x}$$



-2-1
✓

$$y_N \rightarrow W = \begin{vmatrix} 1 & x & \frac{1}{x} \\ 0 & 1 & \frac{-1}{x^2} \\ 0 & 0 & \frac{2}{x^3} \end{vmatrix} = \frac{2}{x^3}$$

$$W_1 = \begin{vmatrix} 0 & x & \frac{1}{x} \\ 0 & 1 & \frac{-1}{x^2} \\ 1 & 0 & \frac{2}{x^3} \end{vmatrix} = -\frac{2}{x} \quad / y_1 = 1$$

$$W_2 = \begin{vmatrix} 1 & 0 & \frac{1}{x} \\ 0 & 0 & \frac{-1}{x^2} \\ 0 & 1 & \frac{2}{x^3} \end{vmatrix} = \frac{1}{x^2} \quad / y_2 = x$$

$$W_3 = \begin{vmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad / y_3 = x^{-1}$$

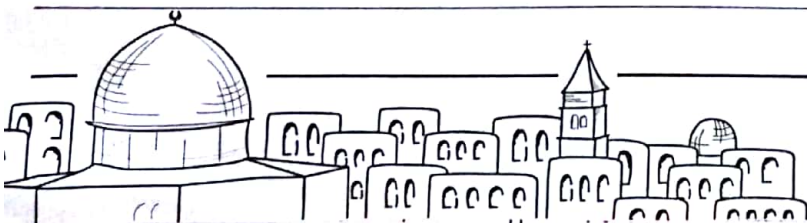
$$y_N = 1 \int \frac{-\frac{2}{x}}{\frac{2}{x^3}} \cdot \frac{e^x}{x} dx + x \int \frac{\frac{1}{x^2}}{\frac{2}{x^3}} \cdot \frac{e^x}{x} dx + \frac{1}{x} \int \frac{1}{\frac{2}{x^3}} \cdot \frac{e^x}{x} dx$$

$$y_N = -\int x e^x dx + \frac{1}{2} x \int e^x dx + \frac{1}{2x} \int x^2 e^x dx$$

$$y_N = \left[-(x e^x - e^x) + \frac{x}{2} (e^x) + \frac{1}{2x} (x^2 e^x - 2x e^x + 2e^x) \right]$$

$$y_N = \frac{e^x}{x}$$

$$\therefore y_g = C_1 + x C_2 + \frac{C_3}{x} + \frac{e^x}{x}$$



* system of 1st ODE

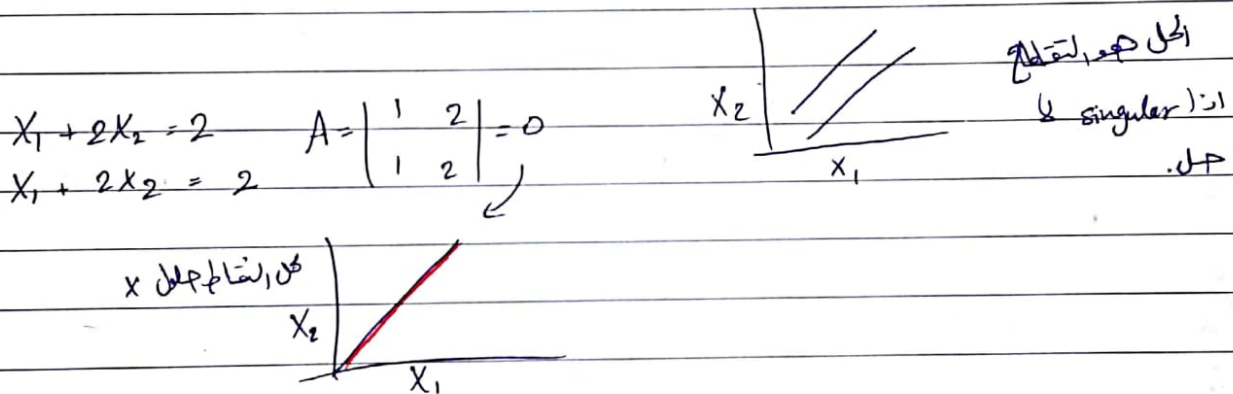
$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + r_1(x) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + r_2(x) \end{aligned} \quad \leftarrow \begin{matrix} \text{Xy, or } y_2 \\ y_g = y_h + y_p \end{matrix}$$

zero $\neq$ determinant $\leftarrow$ singular is fine -

$$[A] = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \neq \text{zero.}$$

فقط (فقط)

دو $\begin{cases} x_1 + 2x_2 = 2 \\ 2x_1 + 4x_2 = 4 \end{cases} \rightarrow [A] = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0 \rightarrow \text{singular matrix}$



* How to obtain eigen values for any system!?

$$|[A] - \lambda [I]| = \text{zero}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \text{zero}$$

matrix

find the determined

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = \text{zero}$$

$$a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{12}a_{21} = 0$$



$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\lambda = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4[a_{11}a_{22} - a_{12}a_{21}]}}{2}$$

$$* [A] = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \Rightarrow \text{Det}[A] = 10 - 4 = 6 \quad \therefore \text{non singular}$$

To find the eigen values.

$$\left| \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} (-5-\lambda) & 2 \\ 2 & (-2-\lambda) \end{vmatrix} = 0 \quad \begin{aligned} &(-5-\lambda)(-2-\lambda) - 4 = 0 \\ &= \lambda^2 + 7\lambda + 6 = 0 \end{aligned}$$

$$\lambda = \frac{-7 \pm \sqrt{49 - 24}}{2}$$

$$\lambda_1 = -6 \quad \lambda_2 = -1$$

$\Rightarrow$ for $\lambda_1 = -6$

$$\left| \begin{bmatrix} (-5+6) & 2 \\ 2 & (-2+6) \end{bmatrix} \right| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = \text{zero}$$

for $\lambda_2 = -1$

$$\left| \begin{bmatrix} -5+1 & 2 \\ 2 & (-2+1) \end{bmatrix} \right| = \begin{vmatrix} -4 & 2 \\ 2 & -1 \end{vmatrix} = \text{zero}$$



• How to find an eigen vector?

solve the following sys.

$$[A] - \lambda [I] \begin{Bmatrix} \text{vector} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

↓
eigen vector

if singular (فوق)

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

assumption (فرضية)

$$(a_{11} - \lambda) A + a_{12} B = 0 \quad \text{--- (1)}$$

$$a_{21} A + (a_{22} - \lambda) B = 0 \quad \text{--- (2)}$$

$$(1) A = (-2) B$$

$$\text{if } A = -2 \\ B = 1$$

بذلك نحصل على
نوعه كمال

نوعه كمال

$$A + 2B = 0 \\ A = -2B$$

eigen vectors.

$$\begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

$$\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$$

$$\begin{Bmatrix} 4 \\ -8 \end{Bmatrix}$$

✶ system of 1st O.D.E homogeneous with constant coefficients

$$y_1' = a_{11} y_1 + a_{12} y_2$$

$$y_2' = a_{21} y_1 + a_{22} y_2$$

$$y_1 = e^{\lambda x} \\ y_2 = e^{\lambda x}$$

$$\begin{Bmatrix} y_1' \\ y_2' \end{Bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{[A]} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$



Case 1 if the eigen values are different and real
 $\lambda_1 \neq \lambda_2$

• For λ_1 there will be an eigen vector

refer to y_1 $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$
 refer to y_2

• For λ_2 there will be another eigen vector

$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \leftarrow \begin{bmatrix} a_{11} - \lambda_2 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ for the first eigen vector we construct the first part of the solution as $y_1|_{\lambda_1} = A_1 e^{\lambda_1 x}$, $y_2|_{\lambda_1} = B_1 e^{\lambda_1 x}$

$\Rightarrow$ for the second eigen vector we construct the second part of the solution as $y_1|_{\lambda_2} = A_2 e^{\lambda_2 x}$, $y_2|_{\lambda_2} = B_2 e^{\lambda_2 x}$

* Using the linear combination theorem

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} e^{\lambda_1 x} + C_2 \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} e^{\lambda_2 x}$$

$$y_{1g} = C_1 A_1 e^{\lambda_1 x} + C_2 A_2 e^{\lambda_2 x}$$

$$y_{2g} = C_1 B_1 e^{\lambda_1 x} + C_2 B_2 e^{\lambda_2 x}$$

(ex) $\begin{cases} y_1' = 5y_1 + 2y_2 \\ y_2' = 2y_1 - 2y_2 \end{cases} \quad \begin{cases} y_1(0) = 1 \\ y_2(0) = -2 \end{cases}$

step 1 find the eigen values for the following matrix

$$\begin{bmatrix} -5 & 2 \\ 2 & -1 \end{bmatrix} \Rightarrow [A] = I \neq \text{zero}$$



$$| [A] - \lambda [I] | = 0$$

$$\left| \begin{vmatrix} -5 & 2 \\ 2 & -2 \end{vmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -5-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(-2-\lambda)-4=0$$

$$\lambda^2 + 7\lambda + 6 = 0$$

$$\lambda_1 = -6 \quad \lambda_2 = -1$$

step 2 find the eigen vectors related to each eigen value
for $\lambda_1 = -6$ $[[A] - \lambda [I]] \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$\left[\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{singulard}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{cases} A_1 + 2B_1 = 0 \\ 2A_1 + 4B_1 = 0 \end{cases} \rightarrow \text{choose one of them}$$

$$A_1 + 2B_1 = 0$$

option ①

$$A_1 = 2$$

$$-B_1 = 1$$

option ②

$$A_1 = 4$$

$$B_1 = -2$$

For the second eigen vector

$$\left[\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} + (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} A_2 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \rightarrow -4A_2 + 2B_2 = 0$$

$$4A_2 = 2B_2$$

option 1

$$\begin{matrix} A_2 = 1 \\ B_2 = 2 \end{matrix}$$

option 2

$$\begin{matrix} A_2 = 2 \\ B_2 = 4 \end{matrix}$$

ask & recieve



step 3 For the 1st λ , there will be

$$y_1|_{\lambda_1} = -2e^{6x} \text{ or } y_1|_{\lambda_1} = 4e^{-6x}$$

$$y_2|_{\lambda_1} = 1e^{-6x} \text{ or } y_2|_{\lambda_1} = -2e^{6x}$$

for the $\lambda_2 = -1$ there will be

$$y_1|_{\lambda_2} = 2e^{-x} \text{ or } y_1|_{\lambda_2} = e^{-x}$$

$$y_2|_{\lambda_2} = 4e^{-x} \text{ or } y_2|_{\lambda_2} = 2e^{-x}$$

using the linear combination theorem

$$y_1 g = C_1 (2)e^{6x} + C_2 (2)e^{-x}$$

$$y_2 g = C_1 (1)e^{6x} + C_2 (4)e^{-x}$$

or

$$y_1 g = C_1 (4)e^{6x} + C_2 (1)e^{-x}$$

$$y_2 g = C_1 (-2)e^{6x} + C_2 (2)e^{-x}$$

$$y_1 p = \frac{-8}{10} (2) e^{6x} + C_2 \left(\frac{-3}{10}\right) 2e^{-x}$$

$$y_2 p = \frac{-8}{10} (1) e^{6x} + C_2 \left(\frac{-3}{10}\right) 4e^{-x}$$

$$y_1 p = -\frac{16}{10} e^{6x} - \frac{6}{10} e^{-x}$$

$$y_2 p = -\frac{8}{10} e^{6x} - \frac{12}{10} e^{-x}$$



Case 2 system of 1st O.D.E

if $\lambda_2 = \lambda_1 \rightarrow$ repeated eigen values

$\Rightarrow$ if A_1, B_1, A_2 of the characteristic equation are real and equal, then $\lambda_1 = \lambda_2 = \lambda$

The system has two linearly independent solution of the form

$$\begin{aligned} y_1^{(1)} &= A_1 e^{\lambda x} \\ y_2^{(1)} &= B_1 e^{\lambda x} \end{aligned} \rightarrow \begin{cases} y_1^{(2)} = (A_2 x + A_3) e^{\lambda x} \\ y_2^{(2)} = (B_2 x + B_3) e^{\lambda x} \end{cases}$$

• using the linear combination

$$y = \begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \end{bmatrix} = C_1 \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} + C_2 \begin{bmatrix} A_2 x + A_3 \\ B_2 x + B_3 \end{bmatrix} e^{\lambda x}$$

(ex) $\begin{aligned} y_1' &= 4y_1 - y_2 \\ y_2' &= y_1 + 2y_2 \\ y_0 &= y_h \end{aligned}$

$$\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 - 6\lambda + 9 = 0$$

$$\lambda_2 = \lambda_1 = 3 \rightarrow \boxed{\lambda = 3}$$

for $\lambda = 3$

$$\begin{bmatrix} 4-3 & -1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad A_1 - B_1 = 0 \quad A_1 = B_1$$

$\angle$ $y_1^{(1)} = (1)e^{3x}$ / to find $y_2^{(2)}, y_1^{(2)}$
 $y_2^{(1)} = (1)e^{3x}$ / let $y_1^{(2)} = (A_2 x + A_3)e^{3x}$
 $y_2^{(2)} = (B_2 x + B_3)e^{3x}$

if $\begin{cases} A_1 = 1 \\ B_1 = 1 \end{cases}$



$$y_1^{(2)} = 3(A_2 x + A_3) e^{3x} + (A_2) e^{3x}$$

$$y_2^{(2)} = 3(B_2 x + B_3) e^{3x} + (B_2) e^{3x}$$

$$(3A_2 x + 3A_3) e^{3x} + A_2 e^{3x} = (4A_2 x + 4A_3) e^{3x} - B_2 x e^{3x} - B_3 e^{3x}$$

قانونياً
\$y_1 = 4y_2 - y_2\$

$$3A_2 = 4A_2 - B_2 \rightarrow B_2 = A_2$$

$$3A_3 + A_2 = 4A_3 - B_3 \rightarrow A_2 = A_3 - B_3$$

$$(2B_2 x + 2B_3 + B_2) e^{3x} = (A_2 x + A_3 + 2B_2 x + 2B_3) e^{3x}$$

تعويضاً
\$y_2 = y_1 - y_2\$

$$B_2 = A_2 \quad \text{في المعادلة}$$

$$x(B_3 + B_2) = A_3 \rightarrow B_2 = A_3 - B_3 \rightarrow \text{repeated eq.!!}$$

بستقيم واصل فقط من المعادلات

لها (eigen vector) فيكون في

$$\Rightarrow B_2 = A_2 \quad \text{if } A_2 = 1 \Rightarrow B_2 = 1$$

$$\Rightarrow A_2 = A_3 - B_3 \quad 1 = A_3 - B_3 \rightarrow A_3 = 1 + B_3$$

$$\text{if } A_3 = 1 \Rightarrow B_3 = 0$$

$$y_1^{(2)} = ((1)x + (1)) e^{3x}$$

$$y_2^{(2)} = (1(x) + (0)) e^{3x}$$

now using linear comp.

$$y_1(x) = C_1(e^{3x}) + C_2(x+1)e^{3x}$$

$$y_2(x) = C_1(e^{3x}) + C_2(x)e^{3x}$$

إذا لم تكن هذه المعادلات

$$y_g = \begin{bmatrix} y_1^{(2)} \\ y_2^{(2)} \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3x} + C_2 \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] e^{3x}$$



Case 3

The roots of the characteristic equation are complex $\lambda_1 = a + bi$ $\lambda_2 = a - bi$

The solution will be based on one of them and will be the same for λ_1 or λ_2

$$\begin{aligned} y_1 &= \begin{Bmatrix} A \\ B \end{Bmatrix} e^{(a+bi)x} = \begin{Bmatrix} A \\ B \end{Bmatrix} e^{ax} \cdot e^{bix} = \begin{Bmatrix} A \\ B \end{Bmatrix} e^{ax} [\cos(bx) + i \sin(bx)] \\ y_2 &= \begin{Bmatrix} A \\ B \end{Bmatrix} e^{(a-bi)x} \end{aligned}$$

eigen vector $\rightarrow$ complex eigen vector

$$\begin{aligned} y_1 &= y_1^{\text{Real}} + y_1^{\text{Imag}} \\ y_2 &= y_2^{\text{Real}} + y_2^{\text{Imag}} \end{aligned} = C_1 \begin{bmatrix} y_1^{\text{Real}} \\ y_2^{\text{Real}} \end{bmatrix} + C_2 \begin{bmatrix} y_1^{\text{Imag}} \\ y_2^{\text{Imag}} \end{bmatrix}$$

ex

$$\begin{aligned} y_1' &= 3y_1 + 2y_2 \\ y_2' &= -5y_1 + y_2 \end{aligned}$$

step ① $\begin{vmatrix} 3-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(1-\lambda) + 10 = 0$

$$\lambda^2 - 4\lambda + 13 = 0$$

$$\lambda_1 = 2 + 3i$$

$$\lambda_2 = 2 - 3i$$

$\downarrow$ $\downarrow$
 a b

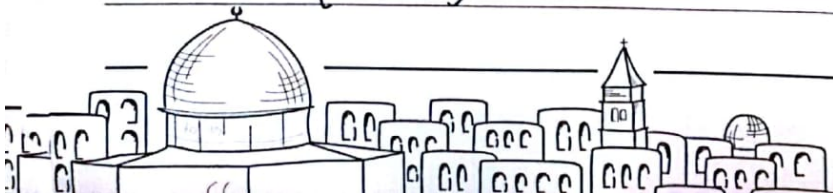
step ② for $\lambda_1 = 2 + 3i$

$$\begin{bmatrix} 3 - (2+3i) & 2 \\ -5 & 1 - (2+3i) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$(1-3i)A + 2B = 0$$

$$(3i-1)A = 2B \quad \text{if } A=2, B=3i-1$$

$$\begin{bmatrix} 2 \\ 3i-1 \end{bmatrix} \leftarrow \text{complex eigen vector}$$



ask : believe & recieve

$$y_1 = A e^{(2+3i)x} = 2e^{(2x)} \cdot e^{3ix} = 2e^{2x} [\cos(3x) + i \sin(3x)]$$

$$y_2 = B e^{(2+3i)x} = (3i-1)e^{2x} \cdot e^{3ix} = (3i-1)e^{2x} [\cos(3x) + i \sin(3x)]$$

$$y = e^{2x} \left[\underbrace{2 \cos(3x)}_{\text{real}} + \underbrace{2i \sin(3x)}_{\text{complex}} \right]$$

$$y_2 = e^{2x} \left(\underbrace{3i \cos(3x)}_{\text{comp.}} - \underbrace{3 \sin(3x)}_{\text{real}} - \underbrace{\cos(3x)}_{\text{real}} - \underbrace{i \sin(3x)}_{\text{comp.}} \right)$$

$$y_2 = e^{2x} \left(\underbrace{-3 \sin(3x) - \cos(3x)}_{\text{real}} + i \underbrace{(3 \cos(3x) - \sin(3x))}_{\text{imaginary complex}} \right)$$

$$y_1 = \left(C_1 \begin{bmatrix} 2 \cos(3x) \\ -3 \sin(3x) - \cos(3x) \end{bmatrix} + \underbrace{C_2}_{(C_2 i)} \begin{bmatrix} 2 \sin(3x) \\ 3 \cos(3x) - \sin(3x) \end{bmatrix} \right) e^{2x}$$

$$y = e^{2x} \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \cos(3x) + \begin{bmatrix} 0 \\ -3 \end{bmatrix} \sin(3x) \right\} C_1 + C_2 \left\{ \begin{bmatrix} 0 \\ 3 \end{bmatrix} \cos(3x) + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \sin(3x) \right\}$$

$$y_1^{(2)} = e^{2x} (2C_1 \cos(3x) + 2C_2 \sin(3x))$$

$$y_2^{(2)} = e^{2x} (-C_1 \cos(3x) - 3C_1 \sin(3x) + 3C_2 \cos(3x) - C_2 \sin(3x))$$

$$\lambda = 2 - 3i$$

up! ←



* non homogeneous system

$$y_1' = a_{11}y_1 + a_{12}y_2 + r_1(x)$$

$$y_2' = a_{21}y_1 + a_{22}y_2 + r_2(x)$$

↓ both or any of them

free of any arbitrary constants.

$$y_g = y_N + y_h$$

① undetermined coefficient

② variation of parameters

① undetermined coefficient

$$y_1' = 2y_1 - 4y_2 + \frac{2x^2 + 10x}{r_1(x)}$$

$$y_2' = -3y_1 + \frac{x^2 + 9x + 3}{r_2(x)}$$

$$y_h \Rightarrow y_1' = 2y_1 - 4y_2$$

$$y_2' = -3y_1 - 3y_2$$

for eigen values $\lambda_1 = 1, \lambda_2 = -2$

find eigen vector for $\lambda_1 \rightarrow \begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$ $\lambda_2 \rightarrow \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$\therefore y_h = c_1 \begin{Bmatrix} 4 \\ 1 \end{Bmatrix} e^x + c_2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{-2x}$$

$$\text{let } y_N = K + Lx + Mx^2$$

$$\therefore y_N^{(1)} = Mx^2 + L_1x + K_1$$

$$\therefore y_N^{(2)} = K_2 + L_2x + M_2x^2$$

$$L_1 + 2M_1x = 2K_1 + 2L_1x + 2M_1x^2 - 4K_2 - 4L_2x - 4M_2x^2 + 2x^2 + 10x$$

$$L_2 + 2M_2x = K_1 + L_1x + M_1x^2 - 3K_2 - 3L_2x - 3M_2x^2 + x^2 + 9x + 3$$

$$2M_1 - 4M_2 + 2 = 0 \quad \text{--- (1)}$$

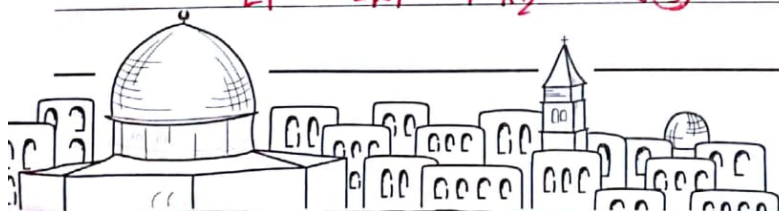
$$M_1 - 3M_2 + 1 = 0 \quad \text{--- (4)}$$

$$2M_1 = 2L_1 - 4L_2 + 10 \quad \text{--- (2)}$$

$$2M_1 = L_1 - 3L_2 + 9 \quad \text{--- (5)}$$

$$L_1 = 2K_1 - 4K_2 \quad \text{--- (3)}$$

$$L_2 = K_1 - 3K_2 + 3 \quad \text{--- (6)}$$



ask, believe & receive

system of nonhomogeneous D.E.

Solve eq:- $M_2=0$, $M_1=-1$, $L_2=3$, $L_1=0$, $k_2=k_1=0$

$$\therefore y_N^{(1)} = k_1 + L_1 x + M_1 x^2$$

$$= -x^2$$

$$y_N^{(2)} = k_2 + L_2 x + M_2 x^2$$

$$= 3x$$

$$y_g = C_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^x + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2x} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} x + \begin{bmatrix} -1 \\ 0 \end{bmatrix} x^2$$

$$y_g^{(1)} = 4C_1 e^x + C_2 e^{-2x} - x^2$$

$$y_g^{(2)} = C_1 e^x + C_2 e^{-2x} + 3x$$

$$\begin{aligned} y_1' &= y_2 + \underbrace{\begin{bmatrix} 6e^{2x} \\ -3e^{2x} \end{bmatrix}}_{r_1(x)} \\ y_2' &= y_1 - \underbrace{\begin{bmatrix} 6e^{2x} \\ -3e^{2x} \end{bmatrix}}_{r_2(x)} \end{aligned}$$

$$y_h \Rightarrow \begin{aligned} y_1' &= 0y_1 + (1)y_2 \\ y_2' &= (1)y_1 + (0)y_2 \end{aligned}$$

$$\begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 - 1 = 0 \quad \lambda_1 = +1, \lambda_2 = -1$$

$$\text{for } \lambda_1 = 1 \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ for } \lambda_2 = -1 \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y_h = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^x + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-x}$$

$$y_N \rightarrow \text{let } y_N = k e^{2x}$$

$$\left. \begin{aligned} y_N^{(1)} &= k_1 e^{2x} \\ y_N^{(2)} &= k_2 e^{2x} \end{aligned} \right\} \rightarrow \begin{aligned} y_N^{(1)} &= 2k_1 e^{2x} \\ y_N^{(2)} &= 2k_2 e^{2x} \end{aligned}$$



$$e^{2x} 2k_1 = k_2 e^{2x} + 6e^{2x} \rightarrow 2k_1 = k_2 + 6 \quad \text{--- (1)}$$

$$2k_2 = k_1 - 3 \quad \text{--- (2)}$$

$$\boxed{k_1 = 3} \quad \boxed{k_2 = 0}$$

$$y_N^{(1)} = 3e^{2x} \quad y_N^{(2)} = 0$$

$$y = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-x} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} e^{2x}$$

$$y^{(1)} = C_1 e^x + C_2 e^{-x} + 3e^{2x}$$

$$y^{(2)} = C_1 e^x - C_2 e^{-x}$$

$$y_1' = y_2 - 5 \sin(x)$$

$$y_2 = 4y_1 + 17 \cos(x)$$

$$\begin{vmatrix} 0-\lambda & 1 \\ 4 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 4 = 0$$

$$\lambda_1 = 0+2i \quad \lambda_2 = -2i$$

$$y = C_1 \begin{bmatrix} \cos(2x) \\ 2 \sin(2x) \end{bmatrix} + C_2 \begin{bmatrix} \sin(2x) \\ -2 \cos(2x) \end{bmatrix}$$

$$y_N = A \cos(x) + B \sin(x)$$

$$y_N^{(1)} = A_1 \cos(x) + B_1 \sin(x), \quad y_N^{(2)} = A_2 \sin(x) + B_2 \cos(x)$$

$$y_N^{(1)} = A_2 \cos(x) + B_2 \sin(x) \quad y_N^{(2)} = -A_2 \cos(x) - A_2 \sin(x)$$

$$-A_1 \sin(x) + B_1 \cos(x) = A_2 \cos(x) + B_2 \sin(x) - 5 \sin(x)$$

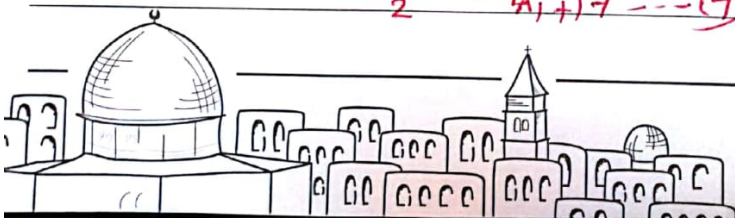
$$-A_1 = B_2 - 5 \quad \text{--- (1)}$$

$$B_1 = A_2 \quad \text{--- (2)}$$

$$-A_2 \sin(x) + B_2 \cos(x) = -4A_1 \cos(x) - 4B_1 \sin(x) + 17 \cos(x)$$

$$-A_2 = -4B_1 \quad \text{--- (3)}$$

$$B_2 = -4A_1 + 17 \quad \text{--- (4)}$$



$$A_1 = 4$$

$$B_1 = 0$$

$$A_2 = 0$$

$$B_2 = 1$$

$$y_g = C_1 \begin{bmatrix} \cos 2x \\ 2 \sin 2x \end{bmatrix} + C_2 \begin{bmatrix} \sin 2x \\ -2 \cos 2x \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} \cos x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin x$$

$$\begin{aligned} y_1' &= 2y_1 + 3y_2 + 2e^{2x} \\ y_2' &= y_1 + 4y_2 + 3e^{2x} \end{aligned} \quad \left. \begin{aligned} y_1(0) &= -\frac{2}{3} \\ y_2(0) &= \frac{1}{3} \end{aligned} \right\} \boxed{y_p}$$

$$y_h = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5x} + C_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^x$$

H.W

$$y_g = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5x} + C_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^x - \frac{1}{3} \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{2x}$$

* system of D.E

② Variation of Parameters.

$$y_1' = a_{11}y_1 + a_{12}y_2 + r_1(x)$$

$$y_2' = a_{21}y_1 + a_{22}y_2 + r_2(x)$$

$$y_g = \boxed{y_h} + y_N$$

$$y_N \Rightarrow \text{step 1} \quad \text{find } y_h = C_1 \begin{bmatrix} A_1 y_1 \\ B_1 y_2 \end{bmatrix} + C_2 \begin{bmatrix} A_2 y_1 \\ B_2 y_2 \end{bmatrix}$$

$$\begin{bmatrix} A_1 y_1 & A_2 y_1 \\ B_1 y_2 & B_2 y_2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$



step 2 find $Y^{-1} = \frac{1}{\text{Determinant}(Y)} \begin{bmatrix} B_2 y_2 & -A_2 y_1 \\ -B_1 y_2 & A_1 y_1 \end{bmatrix}$

step 3 find $Y^{-1} \cdot g(x) \rightarrow \begin{bmatrix} Y^{-1} \end{bmatrix}_{2 \times 2} \begin{bmatrix} g(x) \end{bmatrix}_{2 \times 1} = \begin{bmatrix} \end{bmatrix}_{2 \times 1}$

$$\begin{bmatrix} r_1(x) \\ r_2(x) \end{bmatrix}$$

step 4 find $u(x) = \int^x Y^{-1} \cdot g(x) dx + C$

(C) is condition (1) to (3) $\leftarrow x_0$
 x_0 is condition (3) $\leftarrow$

step 5 $\begin{bmatrix} Y \end{bmatrix}_{2 \times 2} \cdot u(x) \begin{bmatrix} \end{bmatrix}_{2 \times 1} = \begin{bmatrix} \end{bmatrix}_{2 \times 1} \leftarrow y_N$

$$\begin{aligned} y_1' &= 2y_1 + 3y_2 + 2e^{2x} \\ y_2' &= 4y_1 + 4y_2 + 3e^{2x} \end{aligned}$$

$\rightarrow y_h = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5x} + C_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^x$

$\rightarrow Y = \begin{bmatrix} (1) e^{5x} & (3) e^x \\ A_1 & A_2 \\ (1) e^{5x} & (-1) e^x \\ B_1 & B_2 \end{bmatrix} = \begin{bmatrix} -e^{5x+x} & -3e^{5x+x} \end{bmatrix} = -4e^{6x}$

$\rightarrow Y^{-1} \Rightarrow \frac{1}{-4e^{6x}} \cdot \begin{bmatrix} -1e^x & -3e^x \\ -e^{5x} & e^{5x} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} e^{-5x} & \frac{3}{4} e^{-5x} \\ \frac{1}{4} e^{-x} & -\frac{1}{4} e^{-x} \end{bmatrix}_{2 \times 2}$

$\rightarrow \begin{bmatrix} \frac{1}{4} e^{-5x} & \frac{3}{4} e^{-5x} \\ \frac{1}{4} e^{-x} & -\frac{1}{4} e^{-x} \end{bmatrix}_{2 \times 2} \cdot \begin{bmatrix} 2e^{2x} \\ 3e^{2x} \end{bmatrix}_{2 \times 1} = \begin{bmatrix} \frac{2}{4} e^{-5x} \cdot e^{2x} + \frac{9}{4} e^{-5x} \cdot e^{2x} \\ \frac{2}{4} e^{-x} \cdot e^{2x} - \frac{3}{4} e^{-x} \cdot e^{2x} \end{bmatrix}_{2 \times 1}$

$[Y^{-1}] \cdot g(x) = \begin{bmatrix} \frac{11}{4} e^{-3x} \\ -\frac{1}{4} e^x \end{bmatrix}$



$$\Rightarrow u(x) = \int_0^x \begin{bmatrix} \frac{11}{4} e^{-3x} \\ -\frac{1}{4} e^x \end{bmatrix} dx = \begin{bmatrix} -\frac{11}{12} e^{-3x} + \frac{11}{12} \\ -\frac{1}{4} e^x + \frac{1}{4} \end{bmatrix}_0^x$$

$$u(x) = \begin{bmatrix} -\frac{11}{12} e^{-3x} + \frac{11}{12} \\ -\frac{1}{4} e^x + \frac{1}{4} \end{bmatrix}$$

$$\Rightarrow Y * \{u\} = \begin{bmatrix} e^{5x} & 3e^x \\ e^{5x} & -e^x \end{bmatrix}_{2 \times 2} \begin{bmatrix} -\frac{11}{12} e^{-3x} + \frac{11}{12} \\ -\frac{1}{4} e^x + \frac{1}{4} \end{bmatrix}_{2 \times 1} =$$

$$\begin{bmatrix} e^{5x} \left(-\frac{11}{12} e^{-3x} + \frac{11}{12} \right) + 3e^x \left(-\frac{1}{4} e^x + \frac{1}{4} \right) \\ e^{5x} \left(-\frac{11}{12} e^{-3x} + \frac{11}{12} \right) - e^x \left(-\frac{1}{4} e^x + \frac{1}{4} \right) \end{bmatrix}$$

$$= \begin{bmatrix} \left(-\frac{11}{12} - \frac{9}{12} \right) e^{2x} + \left(\frac{11}{12} e^{5x} + \frac{3}{4} e^x \right) \\ \left(-\frac{11}{12} + \frac{3}{12} \right) e^{2x} + \left(\frac{11}{12} e^{5x} - \frac{1}{4} e^x \right) \end{bmatrix}$$

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$$y = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5x} + C_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^x + \begin{bmatrix} \frac{20}{12} \\ -\frac{8}{12} \end{bmatrix} e^{2x} + \begin{bmatrix} \frac{11}{12} \\ \frac{11}{12} \end{bmatrix} e^{5x} + \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix} e^x$$

$$= \begin{pmatrix} C_1 + \frac{11}{12} \\ C_2 + \frac{1}{4} \end{pmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5x} + \begin{pmatrix} C_2 + \frac{1}{4} \\ C_2 - \frac{1}{4} \end{pmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^x + \begin{bmatrix} \frac{20}{12} \\ -\frac{8}{12} \end{bmatrix} e^{2x} + \frac{11}{12} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5x} + \frac{1}{4} \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^x$$

$$y = C_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5x} + C_4 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^x + \begin{bmatrix} \frac{20}{12} \\ -\frac{8}{12} \end{bmatrix} e^{2x}$$

$$y = C_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5x} + C_4 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^x + \frac{1}{3} \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{2x}$$

$$y^{(1)} = C_3 e^{5x} + 3C_4 e^x - \frac{5}{3} e^{2x}$$

الشروط
الحدية

$$y^{(2)} = C_3 e^{5x} + -C_4 e^x - \frac{2}{3} e^{2x}$$



Laplace Transforms

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100

* Linear DE

* With constant coefficients.

* Initial value problem and all conditions must be given at $x_0 = 0$

$$f(t) \Rightarrow \mathcal{L}(f(t)) = \int_{t=0}^{t=\infty} f(t) \cdot e^{-st} dt$$

(ex) $\mathcal{L}(f(t)=1) = \int_0^{\infty} (1) e^{-st} dt = -\frac{e^{-st}}{s} \Big|_0^{\infty} = -\frac{1}{s} \left(\frac{e^{-\infty s}}{1} - \frac{e^{-0s}}{1} \right)$
 $\left[\frac{1}{e^{\infty s}} = \frac{1}{\infty} = 0 \right]$

$$= \frac{1}{s}$$

(ex) $\int_0^{\infty} t \cdot e^{-st} dt$

$$u = t$$

$$dv = e^{-st} dt$$

$$du = dt$$

$$v = -\frac{e^{-st}}{s}$$

$$-t \frac{e^{-ts}}{s} \Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-ts}}{s} dt$$

$$-t \frac{e^{-ts}}{s} \Big|_0^{\infty} + -\frac{e^{-ts}}{s^2} \Big|_0^{\infty}$$

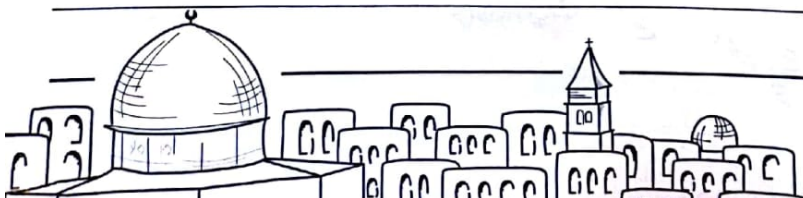
$$\left(-\frac{\infty e^{-\infty s}}{s} - 0 \right) - \left(\frac{e^{-\infty s}}{s^2} - \frac{e^{-0s}}{s^2} \right)$$

$$= 0 + \frac{1}{s^2} \checkmark$$

(ex) $\mathcal{L}(f(t)) = \mathcal{L}(e^{at}) = \int_0^{\infty} e^{at} \cdot e^{-st} dt = \int_0^{\infty} e^{(a-s)t} dt$

$$\frac{1}{(a-s)} e^{(a-s)t} \Big|_0^{\infty} = \frac{1}{(a-s)} \left[\frac{e^{(a-s)\infty}}{(a-s)} - \frac{e^{(a-s)0}}{(a-s)} \right]$$

$$= \frac{1}{s-a}$$



$$\Rightarrow \mathcal{L}(f(t)) = \mathcal{L}(\cos(\omega t)) = \frac{s}{\omega^2 + s^2}$$

$$\int_0^{\infty} \cos(\omega t) \cdot e^{-st} dt$$

$$\begin{cases} e^{i\theta x} = \cos(\theta x) + i\sin(\theta x) \\ e^{-i\theta x} = \cos(\theta x) - i\sin(\theta x) \end{cases}$$

$$\int_0^{\infty} \frac{e^{i\omega t} + e^{-i\omega t}}{2} \cdot e^{-st} dt$$

$$\cos(\theta x) = \frac{e^{i\theta x} + e^{-i\theta x}}{2}$$

$$\frac{1}{2} \int_0^{\infty} [e^{(i\omega - s)t} + e^{-(i\omega + s)t}] dt$$

$$\sin(\theta x) = \frac{e^{i\theta x} - e^{-i\theta x}}{2i}$$

$$\begin{cases} i = \sqrt{-1} \\ i^2 = -1 \end{cases}$$

$$\frac{1}{2} \left[\left(\frac{1}{i\omega - s} e^{(i\omega - s)t} \right) + \left(\frac{1}{-(s + i\omega)} e^{-(i\omega + s)t} \right) \right]_0^{\infty}$$

$$= \frac{1}{2} \left\{ \frac{1}{(i\omega - s)} (e^{(i\omega - s)\infty} - e^{(i\omega - s)(0)}) + \frac{1}{-(s + i\omega)} (e^{-(i\omega + s)\infty} - e^{-(i\omega + s)(0)}) \right\}$$

$$\frac{1}{2} \left\{ \frac{1}{-(s + i\omega)} + \frac{1}{(s + i\omega)} \right\} = \frac{s}{\omega^2 + i\omega s - i\omega s + s^2} = \frac{s}{\omega^2 + s^2}$$

$$* \mathcal{L}(f(t)) = \mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2 + \omega^2} \quad \text{H.W.}$$

$$\text{H.W. } \cos(\omega x), \sin(\omega x) \text{ H.W.}$$

$$y'' + 2y' + 3y = e^x$$

$$f(t) = \mathcal{L}(f'(t)) = \int_0^{\infty} f'(t) e^{-st} dt$$

$$u = e^{-st}$$

$$du = -s e^{-st} dt$$

$$dV = f'(t)$$

$$V = f(t)$$



$$\int_0^{\infty} f'(t) e^{-st} dt = e^{-st} \cdot f(t) \Big|_0^{\infty} + s \int_0^{\infty} f(t) \cdot e^{-st} dt$$

$$f(y') = e^{-s\infty} \cdot f(\infty) - f(0) \cdot e^{-s\infty} + s F(s)$$

$$L(y') = s F(s) - f(0)$$

* How to solve D.E using Laplace transforms

$$y''' + ay'' + cy' + dy = r(x)$$

$$y(0), y'(0), y''(0)$$

step 1 convert all terms in the ~~diff~~ terms in DE from (t) domain to (s) domain

$$L(y''') \quad L(y'') \quad L(y') \quad L(y) \quad L(r(x))$$

$$\text{ex) } y'' - 3y' + 2y = 12e^{4t} \quad y(t=0)=1 \quad y'(t=0)=0$$

$$\text{step 1 } L(y'') = s^2 F(s) - s y(0) - \dot{s} y(0) = s^2 F(s) - s$$

$$L(y') = s F(s) - s^0 y(0) = s F(s) - 1$$

$$L(y) = s^0 F(s) = F(s)$$

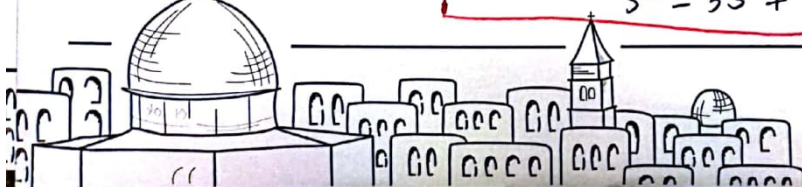
$$L(4t) = \frac{1}{s-4}$$

$$\underbrace{(s^2 F(s) - s)}_{y''} - 3 \underbrace{(s F(s) - 1)}_{y'} + 2 \underbrace{(F(s))}_y = \underbrace{\frac{12}{s-4}}_{e^{4t}}$$

step 2 Find the transfer function $\bar{F}(s) = F(s)$

$$\begin{aligned} s^2 F(s) - s - 3s F(s) + 3 + 2 F(s) &= \frac{12}{s-4} \\ s^2 F(s) - 3s F(s) + 2 F(s) &= \frac{12}{s-4} + s - 3 \\ F(s) (s^2 - 3s + 2) &= \frac{12}{s-4} + s - 3 \end{aligned}$$

$$\text{Transfer function } \bar{F}(s) = \frac{\left(\frac{12}{s-4} + s - 3 \right)}{s^2 - 3s + 2}$$



step 3 invert the transfer function to obtain the solution.

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{\frac{12}{s-4} + s^{-3}}{s^2 - 3s + 2}\right)$$

$$\rightarrow \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos(t)$$

$$\rightarrow \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t)$$

$$\rightarrow \mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{1}{2} \sin(t)$$

$$a) y'' + y = 16 \cos(t)$$

$$y(0) = 0, \quad y'(0) = 0$$

$$\mathcal{L}(y'') = s^2 F(s) - s y(0) - y'(0) = s^2 F(s)$$

$$\mathcal{L}(y) = s^0 (F(s)) = F(s)$$

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1}$$

$$s^2 F(s) + F(s) = \frac{16s}{s^2+1}$$

$$F(s)(s^2+1) = \frac{16s}{s^2+1}$$

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{16s}{(s^2+1)^2}\right)$$

$$y = 16 \mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right)$$

$$y = 16 \cdot \frac{1}{2} \cdot t \cdot \sin t = 8t \cdot \sin(t)$$



حل المسألة بطريقة المصفوفة

$$y_g = y_h + y_N$$

$$y'' + y = 0$$

$$e^{\lambda x} (\lambda^2 + 1) = 0$$

$$\lambda = 0 \pm i$$

$$y_h = A \cos(t) + B \sin(t)$$

$$y_N = C_1 \cos(t) + C_2 \sin(t) \leftarrow \text{repeated}$$

$$y_N = C_1 t \cos(t) + C_2 t \sin(t) \quad \begin{cases} C_1 = 0 \\ C_2 = 8 \end{cases}$$

$$y_g = A \cos(t) + B \sin(t) + C_1 t \cos(t) + C_2 t \sin(t)$$

$$A = B = 0$$

$$y_g = 8t \sin(t)$$

* Partial fraction method

$$y'' - 3y' + 2y = 12e^{4t} \quad y(0) = 1 \quad y'(0) = 0$$

$$\mathcal{L}(y'') = s^2 F(s) - s y(0) - y'(0) = s^2 F(s) - s$$

$$\mathcal{L}(y') = s F(s) - y(0) = s F(s) - 1$$

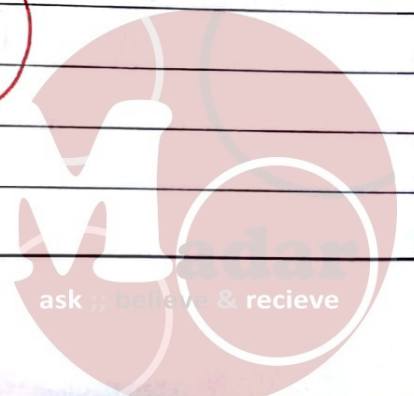
$$\mathcal{L}(y) = F(s)$$

$$\mathcal{L}(e^{4t}) = \frac{1}{s-4}$$

$$s^2 F(s) - s - 3s F(s) + 3 + 2F(s) = \frac{12}{s-4}$$

$$F(s) = \frac{\left(\frac{12}{s-4} + s - 3 \right)}{s^2 - 3s + 2}$$

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{\frac{12}{s-4} + s - 3}{s^2 - 3s + 2} \right)$$



$$y(t) = \mathcal{L}^{-1} \left(\frac{12 + (s-3)(s-4)}{(s^2 - 3s + 2)(s-4)} \right)$$

$$\begin{aligned} & \downarrow \\ & \frac{12 + s^2 - 4s - 5s + 12}{(s^2 - 3s + 2)(s-4)} \\ & \downarrow \\ & \frac{s^2 - 7s + 24}{(s-4)(s-1)(s-2)} \end{aligned}$$

$s=1$ $s=2$ $s=4$

$$y = \mathcal{L}^{-1} \left(\frac{A}{s-4} + \frac{B}{s-2} + \frac{C}{s-1} \right)$$

$$A = \lim_{s \rightarrow 4} \frac{(s-4)(s^2 - 7s + 24)}{(s-4)(s-2)(s-1)} = \lim_{s \rightarrow 4} \frac{16 - 28 + 24}{2 \times 3} = 2$$

$$B = \lim_{s \rightarrow 2} \frac{(s-2)(s^2 - 7s + 24)}{(s-4)(s-2)(s-1)} = \frac{4 - 14 + 24}{-2 \times 1} = -7$$

$$C = \lim_{s \rightarrow 1} \frac{(s-1)(s^2 - 7s + 24)}{(s-4)(s-2)(s-1)} = \frac{1 - 7 + 24}{-3 \times -1} = 6$$

$$\mathcal{L}^{-1} \left(\frac{2}{s-4} \right) + \mathcal{L}^{-1} \left(\frac{-7}{s-2} \right) + \mathcal{L}^{-1} \left(\frac{6}{s-1} \right)$$

$$y(t) = 2e^{4t} - 7e^{2t} + 6e^t$$



if $F(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{(s-q_1)(s-q_2)(s-q_3) \dots}$

where q_i are the roots of the polynomial $Q(s)$ which may be distinct, or repeated, then $F(s)$ can be represented as:

then $F(s) = \frac{A}{s-q_1} + \frac{B}{s-q_2} + \frac{C}{s-q_3} \dots$

to find the values of $A, B, C, \dots$

$A = \lim_{s \rightarrow q_1} \frac{(s-q_1) \cdot P(s)}{(s-q_1)(s-q_2)(s-q_3)} \checkmark$

$B = \lim_{s \rightarrow q_2} \frac{(s-q_2) P(s)}{(s-q_1)(s-q_2)(s-q_3)} \checkmark$

من تكرر
فمنه يفرق

For the case of repetition, if there is any repetition of the
 $F(s) = \frac{P(s)}{(s-q_1)^2(s-q_2)(s-q_3)} = \frac{A}{(s-q_1)^2} + \frac{B}{(s-q_1)} + \frac{C}{s-q_2} + \frac{D}{s-q_3}$ (roots)

$F(s) = \frac{P(s)}{(s-q_1)^3(s-q_2)(s-q_3)} = \frac{A}{(s-q_1)^3} + \frac{B}{(s-q_1)^2} + \frac{C}{s-q_1} + \frac{D}{s-q_2} + \frac{E}{s-q_3}$
 (من تكرر 3 مرات)

$A = \lim_{s \rightarrow q_1} \left(\frac{(s-q_1)^3 \cdot P(s)}{(s-q_1)^3(s-q_2)(s-q_3)} \right) \checkmark$

$B = \lim_{s \rightarrow q_1} \left(\frac{d}{ds} \left(\frac{P(s)}{(s-q_2)(s-q_3)} \right) \right) \cdot 1$

$C = \lim_{s \rightarrow q_1} \left(\frac{d^2}{ds^2} \left(\frac{P(s)}{(s-q_2)(s-q_3)} \right) \right) \cdot \frac{1}{2}$

$A_m = \lim_{s \rightarrow q_1} \left[\frac{d^{m-1}}{ds^{m-1}} \left(\frac{P(s)}{(s-q_2)(s-q_3)} \right) \right] \cdot \frac{1}{(m-1)!}$



ask : believe & recieve

ex $y'' - 2y' + y = e^t + t$
 $y(0) = 1$ $y'(0) = 0$

$$\mathcal{L}(y'') = s^2 F(s) - s$$

$$\mathcal{L}(y') = s F(s) - 1$$

$$\mathcal{L}(y) = F(s)$$

$$\mathcal{L}(e^t) = \frac{1}{s-1}$$

$$\mathcal{L}(t) = \frac{1}{s^2}$$

$$\frac{(s^2 - 2s + 1) F(s)}{(s-1)(s-1)} = \left(\frac{1}{s-1} + \frac{1}{s^2} + \frac{s-2}{1} \right)$$

$$F(s) = \frac{1}{(s-1)^3} + \frac{1}{s^2(s-1)^2} + \frac{s-2}{(s-1)^2}$$

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1} \left(\frac{s^2 + s - 1}{(s-1)^3 s^2} + \frac{s-2}{(s-1)^2} \right)$$

$$= \left(\frac{A}{(s-1)^3} + \frac{B}{(s-1)^2} + \frac{C}{s-1} + \frac{D}{s^2} + \frac{E}{s} \right) + \left(\frac{F}{(s-1)^2} + \frac{G}{(s-1)} \right)$$

$$y(t) = \mathcal{L}^{-1} \left(\frac{A}{(s-1)^3} \right) + \mathcal{L}^{-1} \left(\frac{B}{(s-1)^2} \right) + \mathcal{L}^{-1} \left(\frac{C}{s-1} \right) + \mathcal{L}^{-1} \left(\frac{D}{s^2} \right) + \mathcal{L}^{-1} \left(\frac{E}{s} \right) + \mathcal{L}^{-1} \left(\frac{F}{s-1} \right) + \mathcal{L}^{-1} \left(\frac{G}{s-1} \right)$$

$$D = \lim_{s \rightarrow 0} \left[s^2 \cdot \frac{(s^2 + s - 1)}{s^2(s-1)^3} \right] = +1 \quad \boxed{D=1}$$

$$E = \frac{d}{ds} \left(\frac{s^2 + s - 1}{(s-1)^3} \right) \cdot \frac{1}{1!} = \lim_{s \rightarrow 0} \frac{d}{ds} \left((s-1)^3 (2s+1) + (s^2 + s - 1)(-3)(s-1)^{-4} \right)$$

$$\boxed{E=2}$$

$$A = \lim_{s \rightarrow 1} \left[(s-1)^3 \cdot \frac{(s^2 + s - 1)}{(s-1)^3} \right] = 1 \quad \boxed{A=1}$$



$$B = \lim_{s \rightarrow 1} \frac{d}{ds} \frac{(s^2 + s - 1)s^2}{1!} =$$

$$\lim_{s \rightarrow 1} ((s^2 + s - 1)(-2s^{-3}) + s^{-2}(2s + 1)) = 1 \quad \boxed{B=1}$$

$$C = \lim_{s \rightarrow 1} \frac{d}{ds} \left(\frac{1}{2!} \right) = -2 \quad \boxed{C=-2}$$

$$F = \lim_{s \rightarrow 1} \left(\frac{s-2}{(s-1)^2} \cdot (s-1)^2 \right) = -1 \quad \boxed{F=-1}$$

$$G = \lim_{s \rightarrow 1} \frac{d}{ds} (s-2) = 1 \quad \boxed{G=1}$$

$$\begin{aligned} &= \cancel{L^{-1}\left(\frac{1}{s^2}\right)} + \cancel{L^{-1}\left(\frac{2}{s}\right)} + L^{-1}\left(\frac{1}{(s-1)^3}\right) + L^{-1}\left(\frac{1}{(s-1)^2}\right) + \cancel{L^{-1}\left(\frac{-2}{s-1}\right)} + \cancel{L^{-1}\left(\frac{-1}{(s-1)^2}\right)} + \\ &= t + 2 + \frac{1}{2}t^2e^t + te^t - 2e^t - te^t + e^t \\ &= \frac{t^2e^t}{2} - e^t + 2 + t \end{aligned}$$



ex $y''' + 4y'' + 5y' + 2y = 10 \cos t$
 $y(0) = y'(0) = 0 \quad y''(0) = 3$

$$(s^3 + 4s^2 + 5s + 2) Y(s) = \frac{10s}{s^2+1} + 3 \quad \text{or } \frac{10s + 3s^2 + 3}{s^2+1}$$

$$Y(s) = \frac{3s^2 + 10s + 3}{(s^2+1)(s^3+4s^2+5s+2)}$$

$$\frac{3s^2 + 10s + 3}{(s+1)^2(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{B}{(s+1)^2} + \frac{C}{s+1} + \frac{Ds+E}{s^2+1}$$

$$A = \lim_{s \rightarrow -2} \left((s+2) \cdot \frac{3s^2 + 10s + 3}{(s+1)^2(s^2+1)} \right) = \frac{12 + -20 + 3}{51 \times 5} = -1$$

$$B = \lim_{s \rightarrow -1} \left((s+1)^2 \cdot \frac{3s^2 + 10s + 3}{(s+1)^2(s+2)(s^2+1)} \right) = -2$$

$$C = \lim_{s \rightarrow -1} \frac{d}{ds} \left(\frac{3s^2 + 10s + 3}{(s^2+1)(s+2)} \right) = 2$$

لحد و هو 2، و هو 2، و هو 2 H.W

$$D = -1, E = 2$$

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1} \left(\frac{-1}{s+2} + \frac{-2}{(s+1)^2} + \frac{2}{s+1} + \frac{-s+2}{s^2+1} \right)$$

$$y(s) = -e^{-2t} + -2 \frac{t}{1!} e^{-t} + 2e^{-t} + \mathcal{L}^{-1} \left(\frac{-s}{s^2+1} \right) + \mathcal{L}^{-1} \left(\frac{2}{s^2+1} \right)$$

$$y(s) = -e^{-2t} + 2te^{-t} + 2e^{-t} - \cos t + 2 \sin t$$



Laplace transforms complex roots

$$Y(s) = \frac{F(s)}{(s+k_1+ik_2)(s+k_1-ik_2)}$$

roots = $-k_1 - ik_2$
 $-k_1 + ik_2$

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(F(s) + \frac{a_1+ib_2}{s+k_1+ik_2} + \frac{a_1-ib_2}{s+k_1-ik_2}\right)$$

$$y(t) = \mathcal{L}^{-1}(F(s)) + \mathcal{L}^{-1}\left(\frac{a_1+ib_2}{s+k_1+ik_2}\right) + \mathcal{L}^{-1}\left(\frac{a_1-ib_2}{s+k_1-ik_2}\right)$$

$$= \mathcal{L}^{-1}(F(s)) + (a_1+ib_2)\mathcal{L}^{-1}\left(\frac{1}{s+k_1+ik_2}\right) + (a_1-ib_2)\mathcal{L}^{-1}\left(\frac{1}{s+k_1-ik_2}\right)$$

$$y(t) = \mathcal{L}^{-1}(F(s)) + (a_1+ib_2)e^{-(k_1+ik_2)t} + (a_1-ib_2)e^{-(k_1-ik_2)t}$$

roots $-k_1-ik_2$ roots $-k_1+ik_2$

$$y(t) = \mathcal{L}^{-1}(F(s)) + e^{-k_1 t} \left[a_1 \cos(k_2 t) - a_1 \sin(k_2 t) + ib_1 \cos(k_2 t) + b_1 \sin(k_2 t) + a_2 \cos(k_2 t) + a_2 \sin(k_2 t) - ib_2 \cos(k_2 t) + b_2 \sin(k_2 t) \right]$$

$$= \mathcal{L}^{-1}(F(s)) + e^{-k_1 t} (2a_1 \cos(k_2 t) + 2b_2 \sin(k_2 t))$$



$$Y(s) = \frac{10s + 3s^2 + 3}{(s^2+1)(s+2)(s+1)^2} \Rightarrow \boxed{\frac{10s + 3s^2 + 3}{(s+2)(s+1)^2}} f(s)$$

$$\begin{array}{c} \downarrow \\ (s+1)(s-i) \\ \swarrow \quad \searrow \\ \text{root} = 0-i \quad \text{root} = 0+2 \end{array}$$

$k_1=0$
 $k_2=1$

$$\Rightarrow \frac{D_i}{s+i} + \frac{F_i}{s-1}$$

$$y(t) = \dots + e^{-0t} [2a_2 \cos(t) + 2b_1 \sin(t)]$$

where limit is $\rightarrow b_1, a_2$ etc.

$$D_i = \lim_{s \rightarrow -i} \left[(s+i) \times \frac{10s + 3s^2 + 3}{(s+2)(s+1)^2(s-1)(s+i)} \right]$$

$$= \frac{10(-i) + 3(-i)^2 + 3}{(2-i)(1+i)^2(-2i)} \rightarrow \frac{-10i}{-8+4i} \times \frac{-8-4i}{-8-4i}$$

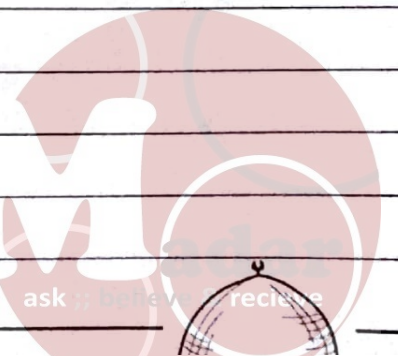
$$= \frac{+80i + -40}{64 - 32i + 32i + 16} = \frac{80i - 40}{80}$$

$$D_i = \cancel{10}i - \left(\frac{1}{2}\right)$$

$\swarrow \quad \nwarrow$
 $b_1 \quad a_1$

اذا دقتي لا،
مقابل في جواب i باليمين
بفكس لا،

$$y(t) = \dots + \left(-\frac{2}{2} \cos(t) + 2(1) \sin(t)\right)$$



Series

Taylor $y = \sum_{j=0}^{\infty} a_j (x-x_0)^j = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$

If $x_0=0 \Rightarrow y = \sum_{j=0}^{\infty} a_j x^j = a_0 + a_1 x + a_2 x^2 + \dots$
 ↳ Maclaurin

$y'' + 2x \sin(x) y' + y = 0$

$y(0) = ?$

$y'(0) = ?$

$\rightarrow y = \sum_{j=0}^{\infty} a_j x^j$

$y(1) = ?$

$y'(1) = ?$

$\rightarrow y = \sum_{j=0}^{\infty} a_j (x-x_0)^j$

Frobenius $y = \sum_{j=0}^{\infty} a_j (x-x_0)^{j+r}$

expansion pt. det. sing. point x_0
 regular singular point

- Points :

* ordinary, singular

$P(x) y'' + Q(x) y' + R(x) y = 0$

$y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0$

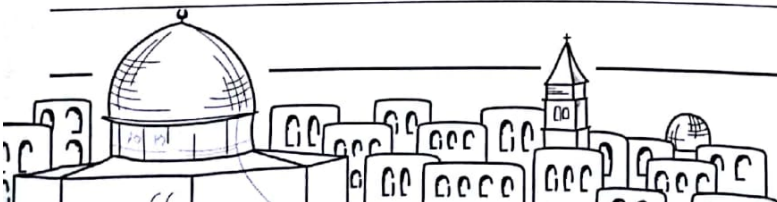
ex $(x-2)y'' + \sin(x) y' + y = 0$

$y'' + \frac{\sin(x)}{(x-2)} y' + \frac{1}{x-2} y = 0$

$x=2$ [singular]

we can't find solution
 about them directly

then the ordinary pts $(-\infty, \infty) - \{2\}$
 ordinary singular



$$A = \sum_{j=0}^{\infty} a_j (x-x_0)^j = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + \dots$$

$$B = \sum_{j=0}^{\infty} b_j (x-x_0)^j = b_0 + b_1 (x-x_0) + b_2 (x-x_0)^2 + \dots$$

$$\begin{aligned} A+B &= a_0 + b_0 + (a_1 + b_1)(x-x_0) + (a_2 + b_2)(x-x_0)^2 + \dots \\ &= \sum_{j=0}^{\infty} (a_j + b_j)(x-x_0)^j \end{aligned}$$

$$A = \sum_{j=0}^{\infty} a_j (x-x_0)^j = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + \dots$$

$$B = \sum_{j=0}^{\infty} b_j (x-x_0)^{j+1} = b_0 (x-x_0) + b_1 (x-x_0)^2 + \dots$$

$$A+B = a_0 + (a_1 + b_0)(x-x_0) + (a_2 + b_1)(x-x_0)^2 + \dots$$

$$A+B = \sum_{j=0}^{\infty} (a_j + b_{j-1})(x-x_0)^j$$

shifting Rule.

$$\sum_{j=0}^{\infty} a_j (x-x_0)^j + \sum_{j=0}^{\infty} b_j (x-x_0)^{j+1} \quad \text{--- (1)}$$

$$\sum_{j=0}^{\infty} a_j (x-x_0)^j + \sum_{j=1}^{\infty} b_{j-1} (x-x_0)^j = a_0 + \sum_{j=1}^{\infty} (a_j + b_{j-1})(x-x_0)^j$$

$$= \sum_{j=1}^{\infty} b_j (x-x_0)^j + a_0 + \sum_{j=1}^{\infty} (a_j + b_{j-1})(x-x_0)^j$$



$$A = \sum_{j=0}^2 a_j (x-x_0)^j = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2$$

$$B = \sum_{j=0}^2 b_j (x-x_1)^j = b_0 + b_1 (x-x_1) + b_2 (x-x_1)^2$$

$$A * B = a_0 b_0 + a_0 b_1 (x-x_1) + a_0 b_2 (x-x_1)^2 + b_0 a_1 (x-x_0) + \dots$$



$$y = \sum_{j=0}^{\infty} a_j (x-x_0)^j$$

$$y' = \sum_{j=1}^{\infty} j a_j (x-x_0)^{j-1}$$

1 من 14

$$y'' = \sum_{j=2}^{\infty} j(j-1) a_j (x-x_0)^{j-2}$$

في 1 machine
غير اول

example 3

$$y' - y = 0 \quad y(0) = 2$$

let

$$y = e^{kx}$$

$$y = C_1 e^{kx}$$

$$y = \sum_{j=0}^{\infty} a_j (x-x_0)^j$$

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$y' = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$\sum_{j=1}^{\infty} j a_j x^{j-1} - \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=1}^{\infty} (j+1) a_{j+1} x^j - \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=0}^{\infty} (j+1) a_{j+1} x^j - \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=0}^{\infty} \underbrace{(j+1) a_{j+1} - a_j}_{=0} x^j = 0$$

$$(j+1) a_{j+1} - a_j = 0$$

$$a_{j+1} = \frac{a_j}{j+1}$$

$$j=0, 1, 2, \dots$$

$$j=0 \rightarrow a_1 = \frac{a_0}{1} \rightarrow a_1 = a_0$$

$$j=1 \rightarrow a_2 = \frac{a_1}{2} \rightarrow a_2 = \frac{a_0}{2}$$

$$j=2 \rightarrow a_3 = \frac{a_2}{3} \rightarrow a_3 = \frac{a_0}{6}$$

$$j=3 \rightarrow a_4 = \frac{a_3}{4}$$

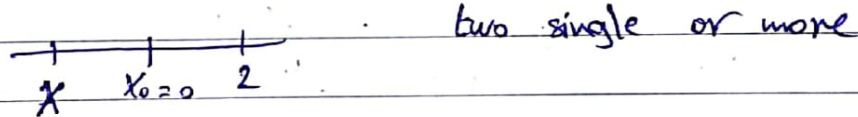
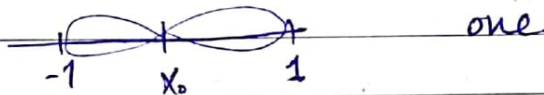
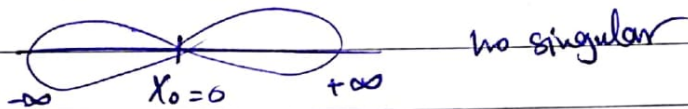


$$\therefore y = a_0 + a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{6} x^3 + \frac{a_0}{24} x^4 \dots$$

$$y = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots \right)$$

↓

$$y = a_0 e^x$$



$$y'' + xy' + (x^2 + 2)y = 0 \quad y(0) = 3, \quad y'(0) = 1$$

let $y = \sum_{j=0}^{\infty} a_j (x)^j$

$$y(0) = 3, \quad y'(0) = 1$$

$$\sum_{2+j=2}^{\infty} (j+2)(2+j-1)a_{j+2}x^{j-2+2} + \sum_{j=1}^{\infty} ja_jx^j + \sum_{2+j=0}^{\infty} a_jx^{j+2-2} + \sum_{j=0}^{\infty} 2a_jx^j$$

$$\sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2} x^j + \sum_{j=1}^{\infty} j a_j x^j + \sum_{j=2}^{\infty} a_{j-2} x^j + \sum_{j=0}^{\infty} 2a_j x^j = 0$$

$$(2) \cancel{1} a_2 x^0 + (3) (2) a_3 x^1 + \sum_{j=2}^{\infty} (j+2)(j+1) a_j x^j + (1) a_1 x^1 + \sum_{j=2}^{\infty} j a_j x^j +$$

$$\sum_{j=2}^{\infty} 2a_j x^j + \sum_{j=2}^{\infty} a_j x^j + 2a_0 x^0 + 2a_1 x^1 =$$

$$\cancel{\{2a_2 + 2a_3\}X^0} + \cancel{\{6a_3 + a_1 + 2a_2\}X^1} + \sum_{j=2}^{\infty} \underbrace{j(j+1)a_{j+2} + j a_j + a_{j+2} + 2a_j}_{\substack{\neq 0 \\ = 0}}$$

$$2a_2 + 2a_0 = 0 \rightarrow a_2 = -a_0$$

$$6a_3 + 3a_1 = 0 \rightarrow a_3 = -\frac{1}{2}a_1$$

$$(j+2)(j+1)a_{j+2} + j a_j + a_{j+2} + 2a_j$$

$$a_{j+2} = \frac{-(j+2)a_j + a_{j-2}}{(j+2)(j+1)} \quad \text{Recurrence formula}$$

$j=2, 3, 4, 5, \dots, \infty$

$$\text{for } \boxed{j=2} \quad a = \frac{-[4a_2 + a_0]}{(4)(3)} = \frac{a_0}{2}$$

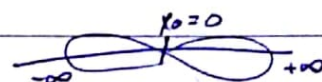
for $j=3$ $a_3 = \frac{-[5a_2 + a_1]}{(5)(4)} = \frac{3a_1}{40}$

$$y = \sum_{j=0}^{\infty} a_j (x)^j = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$= a_0 + a_1 x - a_2 x^2 - \frac{1}{2} a_1 x^3 + \frac{a_0}{4} x^4 + \frac{3}{40} a_1 x^5$$



$$y_0 = a_0 \left[1 - x^2 + \frac{1}{4}x^4 \right] + a_1 \left[x - \frac{x^3}{2} + \frac{3x^5}{40} \right]$$



$$y(0) = 3 \rightarrow a_0 + a_1(0) = 3 \rightarrow \boxed{a_0 = 3}$$

$$y'(0) = 1 \rightarrow y'_0 = a_0 \left[0 - 2x + \frac{1}{2}x^3 + \dots \right] + a_1 \left[1 - \frac{3}{2}x^2 + \dots \right]$$

$$1 = 0 + a_1 \rightarrow \boxed{a_1 = 1}$$

$$y_p = 3 \left[1 - x^2 + \frac{1}{4}x^4 \dots \right] + (1) \left[x - \frac{x^3}{2} + \frac{3x^5}{40} \right]$$

ex

$$y' = x + y + 1 \quad y(0) = 2$$

بعضی نقاط

independent

که مستقل است

$$y' - y = x + 1$$

$(-\infty, \infty) \rightarrow$ all ordinary pts.

$$\text{Let } y = \sum_{j=0}^{\infty} a_j (x)^j \quad y' = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$\sum_{j=1}^{\infty} j a_j x^{j-1} = \sum_{j=0}^{\infty} a_j x^j = 1x^0 + 1x^1 + 0x^2 + 0x^3 + 0x^4 \dots$$

$$\sum_{j+1=1}^{\infty} (j+1) a_{j+1} x^{j-1+1} = \sum_{j=0}^{\infty} a_j x^j = 1x^0 + 1x^1 + 0$$

$$\sum_{j=0}^{\infty} (j+1) a_{j+1} x^j = \sum_{j=0}^{\infty} a_j x^j = 1x^0 + 1x^1 + 0$$

$$\sum_{j=0}^{\infty} [(j+1) a_{j+1} - a_j] x^j = 1x^0 + 1x^1 + 0x^2$$

$$(1) a_1 - a_0 = 1 \quad (2) a_2 - a_1 = 1 \quad \sum_{j=2}^{\infty} [(j+1) a_{j+1} - a_j] x^j = x^0 + x^1 + 0$$

$$a_1 - a_0 = 1 \rightarrow \boxed{a_1 = 1 + a_0}$$

$$2a_2 - a_1 = 1 \rightarrow \boxed{a_2 = \frac{1+a_1}{2}} \rightarrow \boxed{a_2 = 1 + \frac{1}{2}a_0}$$

$$\frac{1 + a_0 + 1}{2} \quad \frac{a_0 + 2}{2}$$



ask : believe & recieve

$$a_{j+1} = \frac{a_j}{j+1}, \quad j=2, 3, 4, 5 \dots$$

$$\text{for } j=2 \quad a_3 = \frac{a_2}{3} = \frac{1 + \frac{1}{2}a_0}{3} = \frac{1}{3} + \frac{1}{6}a_0$$

$$\text{for } j=3 \quad a_4 = \frac{a_3}{4} = \frac{1}{12} + \frac{1}{24}a_0$$

$$\text{for } j=4 \quad a_5 = \frac{a_4}{5} = \frac{1}{60} + \frac{1}{120}a_0$$

$$y = \sum_{j=0}^{\infty} a_j x^j = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

$$a_0 x^0 + (1+a_0) + \left(1 + \frac{1}{2}a_0\right)x^2 + \left(\frac{1}{2} + \frac{a_0}{24}\right)x^4 + \left(\frac{1}{60} + \frac{a_0}{120}\right)x^5 + \left(\frac{1}{3} + \frac{a_0}{6}\right)x^3 + \dots$$

$$x \left(1 + x + \frac{1}{3}x^2 + \frac{1}{12}x^3 + \frac{1}{60}x^4 + \dots \right) + a_0 \left[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots \right]$$



$$xy' - y = x \rightarrow y' - \frac{1}{x}y = 1$$

$x=0$ is a singular pt

$$y(1) = 2$$

Let $y = \sum_{j=0}^{\infty} a_j (x-1)^j$, $y' = \sum_{j=1}^{\infty} j a_j (x-1)^{j-1}$

$$x \sum_{j=1}^{\infty} j a_j (x-1)^{j-1} - \sum_{j=0}^{\infty} a_j (x-1)^j = x$$

$$(x-1+1) \sum_{j=1}^{\infty} j a_j (x-1)^{j-1} - \sum_{j=0}^{\infty} a_j (x-1)^j = x$$

$$(x-1) \sum_{j=1}^{\infty} j a_j (x-1)^{j-1} + (1) \sum_{j=1}^{\infty} j a_j (x-1)^{j-1} - \sum_{j=0}^{\infty} a_j (x-1)^j = x$$

$$\sum_{j=1}^{\infty} j a_j (x-1)^j + \sum_{j=1}^{\infty} (j+1) a_{j+1} (x-1)^j + \sum_{j=0}^{\infty} a_j (x-1)^j = x+1-1$$

$$= \sum_{j=1}^{\infty} j a_j (x-1)^j + (1) a_1 (x-1)^0 + \sum_{j=1}^{\infty} (j+1) a_{j+1} (x-1)^j - a_0 (x-1)^0 -$$

$$\sum_{j=1}^{\infty} a_j (x-1)^j = (x-1) + 1$$

$$= (a_1 - a_0) + \sum_{j=1}^{\infty} [j a_j + (j+1) a_{j+1} - a_j] (x-1)^j = 1 + (x-1)$$

$a_1 - a_0 = 1$ $a_1 = 1 + a_0$
 $\begin{pmatrix} 1(x-1)^0 + 1(x-1)^1 \\ + 0(x-1)^2 + 0(x-1)^3 \end{pmatrix}$

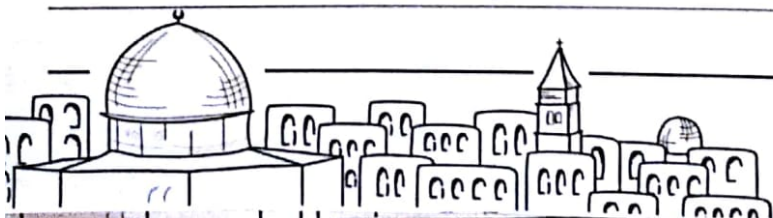
at $j=1$ $(1a_1 + 2a_2 - a_1)(x-1)^1 = 1(x-1)^1$

$2a_2 = 1$ $a_2 = \frac{1}{2}$

at $j=2$ now for all other j 's, $j=1, 2, 3, 4, \dots$

$$a_{j+1} = \frac{a_j - j a_j}{(j+1)} = \frac{(1-j) a_j}{(j+1)}$$

$j=2$ $a_3 = \frac{a_2 - 2a_2}{3} = -\frac{1}{6}$



for $j=3$, $a_4 = \frac{1-2a_3}{4} = \frac{1}{12}$

$$y = \sum_{j=0}^{\infty} a_j (x-1)^j = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4$$

$$= (x-1) + a_0 + a_0(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4$$

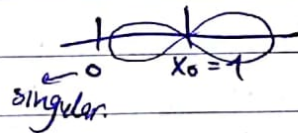
$$y = a_0(1+(x-1)) + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4$$

$$y_g = a_0(1+(x-1)) + (x-1) \left[1 + \frac{1}{2}(x-1) + \frac{1}{6}(x-1)^2 + \frac{1}{12}(x-1)^3 + \dots \right]$$

→ condition at x_0

$$2 = a_0(1) + 0 \longrightarrow a_0 = 2$$

$$y_p = 2x + (x-1) \left[1 + \frac{1}{2}(x-1) + \frac{1}{6}(x-1)^2 + \frac{1}{12}(x-1)^3 + \dots \right]$$



$$y'' + e^x y = 0 \quad y(0) = 1, y'(0) = 0 \quad \text{H.W}$$

$$\text{Let } y = \sum_{j=0}^{\infty} a_j x^j = a_0 x^0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

3rd approximation $\left\{ \sin \quad \cos \quad e^x \quad \frac{1}{x} \right\} \rightarrow \text{series}$

