Mathematical Methods In Chemical Engineering

Mathematical Methods
In chemical Engineering

يعطيكم العافية يارب، معاكم الطالب نزار موسى سنة ثالثة هندسة كيميائية،

عملت هاذ الملخص بهدف التسهيل عليكم، وجبت فيه الزبدة من شرح

الدكتور نعيم الفقير، وحاطط فيه حلول اسئلتوا وحلول اسئلة السنوات.

أتمنى يسهل عليكم ويفيدكم بالمادة ولو بشيء بسيط

طلب صغير بس، بتمنى تدعوا لأخوي [redacted] الصغير محمد

بالشفاء من مرض السكري الله يعافينا منه يارب.

Chapter 1

1st ODE

* classification of DE:

① order ⟶ Ex: $y'' + y' = x^2$ 2nd ord
$e^x y' + y = 0$ 1st ord

② linearity ! لازم تكون قوة y, y', y'' تساوي ① ولازم ما تحتوي على $\sin y$, e^y, $\sqrt{y}$ ، ولا علية ضرب بين $y y'$

Ex: $y' + y = x$ ⟶ linear ✓
$y' + y^2 = \sin x$ ⟶ non linear ✗ بسبب y^2

③ Hom. or non Hom.

~~y' + p(x)~~

→ case ① $y' + p(x) y = g(x)$ → zero → Hom
≠ 0 → non Hom

→ case ② form ادا كان case ①

نعوض λx و λy مكان كل x و y
اذا ضلت نفسها بتكون Hom
والعكس صحيح

* General solution ⟶ بدو الحل
* particular solution ⟶ معطيك IVP و بدو بالحل قيمة C

w

1st order ODE

① separable

② Non separable → sep. نقدر نحليه
Reduction of sep.

③ Exact

④ NoN Exact → can be exact

⑤ Standard form

⑥ Bernolli → can be Standard

① separable

* $\frac{dy}{dx} = f(x) f(y)$

* الفكرة انه الx بطرف والy بطرف

* لازم ال dx، dy في البسط

② non separable

* المسائل لي بنقدر نحولها ل sep ذاكات

case ① $y' = f\left(\frac{y}{x}\right)$ $\underbrace{\qquad}_{\text{form شكل}}$ $\longrightarrow v = \frac{y}{x} \rightarrow y = vx \rightarrow y' = v + v'x$

case ② $y' = f(ax+by)$

$v = ax+by \longrightarrow v' = a + by' \longrightarrow y' = v' - a$

③ Exact

$M\,dx + N\,dy = 0$

$M_y = N_x$ Exact ✓

$\neq$ not exact but can be exact

Ex: $\underbrace{(2xy^2+2)}_{M}dx + \underbrace{(2x^2y+4y)}_{N}dy=0$

$M_y = 4xy$ Exact (2)
$N_x = 4xy$

$\frac{dF}{dx} = 2xy^2+2$

$\int dF = \int 2xy^2+2\, dx$ → original

$F = x^2y^2+2x+C(y)$

$\frac{dF}{dy} = 2x^2y + C'(y)$ → اشتق نسبة (y) $\cancel{2x^2y}+\underline{4y} = \cancel{2x^2y}+\underline{C'(y)}$ Integrate (y)

↓

(N) نفسها → $x^2y^2+2x+2y^2=C-2x$ $2y^2 = C(y)$

$y^2(x^2+2) = C-2x$

$y = \sqrt{\frac{C-2x}{x^2+2}}$

(4) non Exact

اذا كان $M_y \neq N_x$ نقدر نخليها تساوي بعض

عن طريق ال I.F

Integration futor (I.F) $\begin{cases} e^{\int \frac{M_y-N_x}{N}dx} \rightarrow R(x) & \text{الناتج بدلالة x} \\ e^{\int \frac{N_x-M_y}{M}dy} \rightarrow R(y) & \text{الناتج بدلالة y} \end{cases}$

(5) stan. form linear 1st ODE

* form $y' + P(x)y = q(x)$

مشان I.F لازم معامل (1)

G.S → $y = \frac{1}{\mu_x}[\int \mu_x q(x)dx + C]$, $\mu_x = e^{\int P(x)dx}$

(6) Bernoulli (non linear that can be linear)

* form $\quad y' + p(x) y = q(x) y^n$, if $n=0$ it's linear

steps: $\quad * \; v = y^{1-n}$ $\qquad$ if $n=1$, it's separable

$$v' = (1-n) y^{-n} y'$$

بعدين نعوّض المعادلة بعامل الـ y' وبصير المعادلة linear

Ex: $\quad y' = 2y - 3y^2 \quad \div y^2$ $\qquad$ non linear

$$\cancel{y^{-2} y' = 2y^{-1} - 3}$$

$$v = y^{1-2} = y^{-1}$$

$$v' = -y^{-2} y'$$

$$\boxed{y^{-2} y' = -v'}$$

$$\underbrace{y^{-2} y'}_{-v'} = 2\underbrace{y^{-1}}_{v} - 3$$

$$-v' = 2v - 3$$

$$v' + 2v = 3 \leftarrow \text{linear} ✓$$

$$\mu_x = e^{\int 2\,dx} = \boxed{e^{2x}}$$

$$v = \frac{1}{e^{2x}}\left[\int e^{2x} \cdot 3\,dx + c\right]$$

$$v = \frac{1}{e^{2x}}\left[\frac{3}{2} e^{2x} + c\right.$$

$$v = \frac{3}{2} + c e^{-2x}$$

$$y^{-1} = \frac{3}{2} + c e^{-2x}$$

$$y = \frac{1}{\frac{3}{2} + c e^{-2x}} \quad ✓ \quad \text{explicit solution}$$

Chapter 2

2nd ODEs

* general form : $p(x)y'' + q(x)y' + r(x)y = g(x)$ $\begin{cases} = 0 & \text{hom} \\ \neq & \text{non hom} \end{cases}$

* the solution of homo. 2nd order ODE is:

$$y = c_1 \underset{\substack{\uparrow \\ \text{1st solu.}}}{y_1} + c_2 \underset{\substack{\uparrow \\ \text{2nd solu.}}}{y_2}$$

second order (solving)

- non linear (Reduction of order)
  - y-missing
  - x-missing
- linear
  - Hom
    - without const. coff.
    - with constant coff.
      - G.S (حل)
      - X-missing (Reduction of order) (اذا طلب)
  - non homo
    - with const. coff.
    - without const. coff.

* فكرة هاي الطريقة انه نحول ال 2nd order لـ 1st order

* في حال ماكان فيه با لسؤال x, y الحل استخدام ال y-missing

(*) Reduction of order

x-missing / y-missing

$V = y'$ $\quad$ $V = y'$

$V\frac{dv}{dy} = y''$ $\quad$ $V' = y''$

Ex: $y'' + y' = 0$

$V = y'$

$V' = y''$

$V' + V = 0$

$\frac{dv}{dx} = -V$

$\int \frac{dv}{-V} = \int dx$

$-\ln|V| = x + c$

$-\ln|y'| = x + c$

$\frac{1}{y'} = e^{x+c}$

$\int y' = \int \frac{1}{c} e^{-x}$

$y = -\frac{1}{c} e^{-x} + c_2$

[redacted]

(*) Hom. 2nd order ODE linear

$y'' + p(x)y' + Q(x)y = g(x)$

$g(x)$: Zero for Hom

* Wronskin

$$W(f,g) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x)$$

if $W = 0$ linearly dependent

if $W \neq 0$ linearly indep.

* if y_1 given then y_2 found it by:

(1) $y_2 = y_1 \cdot V$

where $V = \int \frac{1}{y_1^2} \cdot e^{-\int p(x)dx}$

(2) $\boxed{\text{or}}$ $y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$

(2) 2nd order linear Hom with Const. coff.

* $ay'' + by' + cy = 0$

$\boxed{a, b, c \text{ constants}}$

* G.S: $y = e^{rx}$

* Charachteretic equation

$ar^2 + br + c = 0$

we use it to find (r)

Real and distinc — case (1)

if $D = b^2 - 4ac > 0$

(r_1, r_2)

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Repeated roots — case (2)

if $D = 0$

$r_1 = r_2$

$y = c_1 e^{r_1 x} + c_2 x e^{r_2 x}$

Complex roots — case (3)

if $D < 0$

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* اذا كان عنا معادلة وهون فرق بين مربعين بنروح ع المميز

$$y = c_1 e^{\lambda x}\cos\beta x + c_2 e^{\lambda x}\sin\beta x$$

$$r_1 = \lambda + \beta j$$

$$r_2 = \lambda - \beta j$$

where $j = \sqrt{-1}$

Ex: $y'' + 11y' + 24y = 0$ $\quad y(0) = 0$, $y'(0) = -7$

$$r^2 + 11r + 24 = 0$$

$$(r+3)(r+8)$$

$$r = -3, -8$$

$$y(x) = c_1 e^{-3x} + c_2 e^{-8x}$$

$$y'(x) = -3c_1 e^{-3x} - 8c_2 e^{-8x}$$

$$y(0) = \boxed{0 = c_1 + c_2} \quad ①$$

$$y'(0) = \boxed{-7 = -3c_1 + -8c_2} \quad ②$$

معادلتين بمجهولين

$$c_1 + c_2 = 0$$

$$c_1 = -c_2$$

$$-7 = -3c_1 + 8c_1$$

$$-7 = 5c_1 \rightarrow c_1 = \boxed{\frac{-7}{5}}$$

$$\text{so } c_2 = \boxed{\frac{7}{5}}$$

$$\text{so } y(x) = \frac{-7}{5}e^{-3x} + \frac{7}{5}e^{-8x}$$

* non Homo. 2nd order linear ODE

* form: $y'' + p(x)y' + q(x)y = g(x) \neq 0$ (non Homo)

~~* There are two methods~~

* G.S: $y = y_c + y_p$

y_c: compilimentary solution

y_p: particular solution

[redacted]

* We have two methods to find y_p

① undetermined coefficient

* بنستخدم هاي الطريقة لما يكون عنا ال $g(x)$ أحد الاقترانات التالية

① Expo. (e^x)

② Poly.

③ sin or cos

④ عملية ضرب بين اي فوق

⑤ // جمع // // // //

* ايضا نستخدمها لما يكون عنا ال $p(x), q(x)$ constants وليست functions

$y(0) = \frac{18}{7}, \; y'(0) = \frac{-1}{7}$ IVP

Ex 1: $y'' - 4y' - 12y = 3e^{5x}$ → $\boxed{y_p = Ae^{5x}}$

① y_c

$r^2 - 4r - 12 = 0$

$(r-6)(r+2) = 0$

$r = +6, -2$

$$\boxed{y_c(x) = c_1 e^{-2x} + c_2 e^{6x}}$$

② y_p

$y_p = Ae^{5x}, \; y'_p = 5Ae^{5x}, \; y''_p = 25Ae^{5x}$

بعدين نعوضهم بالمعادلة الرئيسية $y'' - 4y' - 12y = 3e^{5x}$

$25Ae^{5x} - 20Ae^{5x} - 12Ae^{5x} = 3e^{5x}$

$-7A\cancel{e^{5x}} = 3\cancel{e^{5x}}$

$$\boxed{A = \frac{-3}{7}}$$

G.S

$$\boxed{y(x) = c_1 e^{-2x} + c_2 e^{6x} - \frac{3}{7} e^{5x}}$$

$y'(x) = -2c_1 e^{-2x} + 6c_2 e^{6x} - \frac{15}{7} e^{5x}$

$y(0)$ ① $\frac{18}{7} = c_1 + c_2 - \frac{3}{7}$

$y'(0)$ ② $\frac{-1}{7} = -2c_1 + 6c_2 - \frac{15}{7}$

$c_1 = 2, c_2 = 1$

so $\boxed{y(x) = 2e^{-2x} + e^{6x} - \frac{3}{7} e^{5x}}$

* عند فرض y_p بالاعتماد على نوع $g(x)$ بنفرض كالتالي :

	$g(x)$	y_p	Ex
①	Ae^{rx} ← معامل السؤال	$y_p = Ae^{rx}$	$\frac{1}{3}e^{-2x}$ so $y_p = Ae^{-2x}$
②	$a_n x^n + \dots$	$y_p = Aa x^n \dots$	* $3x^2+2$ $y_p = Ax^2+Bx+C$ * x $y_p = Ax+B$ * x^3 $y_p = Ax^3+Bx^2+Cx+D$
③	$a\sin\beta x$ $b\cos\beta x$ $a\sin\beta x + b\cos\beta x$	$y_p = A\cos\beta x + B\sin\beta x$ same	* $-3\cos 2x$ $y_p = A\cos 2x + B\sin 2x$ * $\sin x + 4\cos x$ $y_p = A\cos x + B\sin x$

* Modification rule y_p

لـ نبي نطلع الـ y_c ونشوف انه مش متشابه والـ g نبي نضرب بـ x بـ y_p ،
اذا هينا تفي المشكلة برضو نضربه بـ x لحد ما تنقلع المشكلة

$$\boxed{\text{الامثلة موجودة بـ slides تبعت الدكتور}}$$

* Fact: If y_{p_1} is a particular solution for $y'' + p(x)y' + q(x)y = g_1(x)$
and y_{p_2} " " " " " " $y'' + p(x)y' + q(x)y = g_2(x)$
then $y_{p_1} + y_{p_2}$ is a particular solution for

$$y'' + p(x)y' + q(x)y = g_1(x) + g_2(x)$$

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② Variation of Parameters

لازم يكون المعامل ① سواء بهاي ال method أو اي قبل

$$P(x)y'' + q(x)y' + r(x)y = g(x) \neq 0$$

$$G.S = y = y_c + y_p$$

$$y_p = -y_1 \int \frac{y_2\, g(x)}{W}\, dx + y_2 \int \frac{y_1\, g(x)}{W}\, dx$$

where $W = W(y_1, y_2) \neq 0$

* نستخدمها لما يكون معامل ال y' و y functions وليست constant

* يستخدم لما يكون ال $g(x)$ ال guess تبعو مش موجود في ال und.coe زي مثلاً sec

Examples in slides of doctor

Matrices

* عبارة عن صفوف واعمدة من الاعداد والمتغيرات وتكون موجودة داخل [ ] ، ( )

* يكتب : $A = (a_{ij})_{n \times m}$

value, row, column; num of rows (n), num of columns (m)

Ex: $A \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

* لما $n \neq m$ يسمى rectangular
* لما $n = m$ يسمى square

. size: 3×3

. a_{22}: 5

. a_{21}: 4

* لما نجمع ونطرح لازم يكون عدد الـ $n \times m$ بالـ two matrices متساوي

* الـ Identity matrix (I_n) بكون $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ، $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ، اذا ضربناه بـ matrix ثاني حيطلع نفس الـ matrix الثاني

* الـ Diagonal بكون زي هيك $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

بكون هاد ارقام والباقي اصفار

* عند الضرب او القسمة بعدد ثابت نضرب أو نقسم كل وحدة على حدة

$A = \begin{bmatrix} 3 & 1 \\ 7 & 8 \\ 2 & \end{bmatrix}$ $\quad 3A = \begin{bmatrix} 9 & 3 \\ 21 & 24 \\ 6 & \end{bmatrix}$

* عند ضرب الـ Matrices ببعض يتحقق بشرط وهو انه يكون عدد الأعمدة بـ A يساوي عدد الصفوف بـ B

Ex: $\underset{2 \times 3}{\overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}} \cdot \underset{3 \times 2}{\overset{B}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}}} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

2×2 الناتج

* using Determinants to solve equation

- Cramer's rule:

The solution (x, y) of the system

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

can be found using determinants

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

row vector ← $[a_1\ a_2\ a_3]$ *

column vector ← $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ *

* Transposition

Ex: $A \rightarrow A^T$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 4 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 4 \end{bmatrix} \quad , \quad B^T = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

ال Diagonal يظل زي ما هو

يعني عبارة عن تبديل الـ rows و columns

rows ← تصير column

columns ← تصير row

Eigenvalue problem

$* \; A x = \lambda x$

eigenvalue

square matrix

unknown scalar constant

unknown vector should be nonzero (مش كلو صفر)

(eigenvector)

$x \neq 0$

~~Let A = [a_jk]~~

$(A - \lambda I)x = 0 \longrightarrow$ to find x

Identity matrix

$\det(A - \lambda I) = 0 \longrightarrow$ to find λ

Ex: $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

أولاً: نوجد الـ λ عن طريق:

$\det(A - \lambda I) = 0$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$$

$= \lambda^2 + 7\lambda + 6 = 0 \longrightarrow \boxed{\lambda_1 = -1} \quad \boxed{\lambda_2 = -6}$

$(\lambda + 6)(\lambda + 1) = 0$

to find the eigenvectors

1 E.V of A corrs. to λ_1

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$-4x_1 + 2x_2 = 0$

$2x_1 - x_2 = 0$

$x_2 = 2x_1$ so $\begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}$ so $\boxed{x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

2 E.V of A corr. to λ_2

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 + 2x_2 = 0$

$2x_1 + 4x_2 = 0$

$x_2 = -x_1/2$

مانحب نشوف كسر فنضرب ② ×

so $\begin{bmatrix} x_1 \\ -\frac{x_1}{2} \end{bmatrix}$ so $\begin{bmatrix} 2x_1 \\ -x_1 \end{bmatrix}$ so $\boxed{x_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}}$

1. A differential equation is considered to be ordinary if it has
(2 Points)

c. One independent variable

d. More than one independent variable

b. More than one dependent variable

a. One dependent variable

2. Find the 2nd order differential equation whose general solution is given by

1. y(x) = Ae^(-2x)+Be^(x)
2. y(x) = Ae^(-2x)+Be^(x)+x(^2)

(Non-anonymous question ⓘ)

(4 Points)

3. Find the general solution of: $y'' - 4y' + 4y = 12x^{-4} e^{2x}$ (Non-anonymous question ⓘ)

(5 Points)

4. The differential equation y''+3y'+2xsin(5y) = e(-2x) is considered to be *
(2 Points)

- c. Nonlinear with variable coefficients and nonhomogeneous
- b. Nonlinear with constant coefficients and homogenous
- a. Linear with constant coefficients and homogenous
- d. linear with constant coefficients and nonhomogenous

5. Find the general solution of : $y' - (1/x)\, y = 1/(2y)$ (Non-anonymous question ⓘ)

(5 Points)

6. Classify the following differential equation: e^(x) y' + 3y = yx^2
(3 Points)

- a. Separable and not linear.
- b. Linear and not separable
- c. Both separable and linear
- d. Neither separable nor linear

7. Solve the following equation: $xy'+2y = 4x$

(Non-anonymous question ⓘ)

(4 Points)

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8. Solve the following IVP: $y'' + 4y' = e^{(x)}$;
$y(0) = 0$ $y'(0) = -4$ (Non-anonymous question ⓘ)
(5 Points)

2021

1st semester

past papers

① ODE is considred tobe ordinary if it has:

ans: one Indep. Variable

---

② find the 2nd order diff eq whose general solution is

ⓐ $y(x) = Ae^{-2x} + Be^{x}$ ⟶ ans: $y'' + y' - 2y = 0$

ⓑ $y(x) = Ae^{-2x} + Be^{x} + x^2$ ⟶ ans: $y'' + y' - 2y = 2 + 2x - 2x^2$

---

③ find the general solution of: $y'' - 4y' + 4y = 12x^{-4}e^{2x}$

ans: by variation of parameters

$$y = c_1 e^{2x} + c_2 x e^{2x} + \frac{2e^{2x}}{x^2}$$

---

④ the ODE $y'' + 3y' + 2x\sin(5y) = e^{-2x}$ is

ans: nonlinear, non Hom., variable coeff.

---

⑤ find the G.S of: $y' - \frac{1}{x}y = \frac{1}{2y}$

ans: $y = \sqrt{-x + cx^2}$ by Barnoli

---

⑥ $e^{x}y' + 3y = yx^2$

ans: separable and not linear

7) solve: $xy' + 2y = 4x$

ans; $y' + \frac{2}{x}y = 4$

$\mu(x) = e^{\int \frac{2}{x}dx} = \boxed{x^2}$

$x^2y' + \frac{2}{x}y = 4x^2$

$\int (x^2y)' = \int 4x^2$

$x^2y = \frac{4}{3}x^3 + c$

$$\boxed{y = \frac{4x}{3} + \frac{c_1}{x^2}}$$

---

8) solve the IVP: $y'' + 4y' = e^x$, $y(0) = 0$, $y'(0) = -4$

ans: 1) y_c

$r^2 + 4r = 0$

$r(r+4) = 0$

$\boxed{r=0}, \boxed{r=-4}$

$y_c(x) = c_1 + c_2e^{-4x}$

2) $y_p = Ae^x$, $y_p' = Ae^x$, $y_p'' = Ae^x$

$Ae^x + 4Ae^x = e^x$

$5A\cancel{e^x} = \cancel{e^x}$

$$\boxed{A = \frac{1}{5}}$$

$y(x) = c_1 + c_2e^{-4x} + \frac{1}{5}e^x \longrightarrow y(0) = 0 = c_1 + c_2 + \frac{1}{5}$

$y'(x) = -4c_2e^{-4x} + \frac{1}{5}e^x \longrightarrow y'(0) = -4 = -4c_2 + \frac{1}{5}$

The general solution is

$y(x) =$

$$y(x) = \frac{-5}{4} + \frac{21}{20}e^{-4x} + \frac{e^x}{5}$$

The form of the particular solution of $y'' - y' - 6y = x\sin(x)$ is

- $y_p = x(\cos(x) + \sin(x))$
- $y_p = (Ax + B)\cos(x) + (Cx + D)\sin(x)$
- $y_p = (Ax + B)\cos(x) + (Ax + B)\sin(x)$
- $y_p = (Ax + B)\cos(x)$

The integrating factor for the following equation $xy' + 2y = 10x^2$

- ○ $\mu = x^{-3}$
- ○ $\mu = x^3$
- ○ $\mu = \frac{1}{x^2}$
- ◉ $\mu = x^2$

Find the differential equation $y'' + by' - cy = 0$ given the basis of solutions $y_1 = 1$ and $y_2 = e^{3x}$

- $y'' + 3y' = 0$
- $y'' + 3y = 0$
- $y' + 3 = 0$
- $y'' + 3y' + 2 = 0$

Which of the following is the general solution to $y'' + 4y' = 0$

- [x] $y = c_1 \cos(2x) + c_2 \sin(2x)$
- [ ] $y = c_1 e^{4x} + c_2 e^{-4x}$
- [ ] $y = c_1 + c_2 e^{-4x}$
- [ ] $y = c_1 x + c_2 e^{-4x}$

The general solution to $y' - y^2 = 0$ has the from

- ○ $y = \frac{c}{x-1}$
- ◉ $y = \frac{1}{-x-c}$
- ○ $y = \frac{1}{x+c}$
- ○ $y = \frac{c}{x+1}$

The general solution to $y' + xy = 0$ has the form:

- ◉ $ce^{-(\frac{x^2}{2})}$
- ○ ce^{-x^2}
- ○ ce^{-x}
- ○ $ce^{(\frac{x^2}{2})}$

If the roots of the characteristic equation are $(-10,10)$,then the complementary solution is

- ○ $y_c = c_1 + c_2 \ln(x) e^{-10x}$
- ○ $y_c = (c_1 + c_2 x) e^{10x}$
- ○ $y_c = (c_1 + c_2 x) e^{-10x}$
- ◉ $y_c = c_1 e^{-10x} + c_2 e^{10x}$

Classify the following differential equation: $e^{x}y' + (3 - x^2)y^2 = 0$

- (selected) Separable and not linear.
- Both separable and linear.
- Linear and not separable.
- Neither separable nor linear.

Which of the following is an exact differential equation.

- ○ $(x+y)dx-dy=0$
- ○ $y^2dx+(2x-2y)dy=0$
- ◉ $2xydx+(x^2+3)dy=0$
- ○ $3ydx-3xdy=0$

The solution to $y' - 3x^2 - 9 = 0$ satisfying $y(1) = 2$ is given by

- ○ $y = x^2 + 9x - 8$
- ○ $y = x^3 + 8x - 9$
- ◉ $y = x^3 + 9x - 8$
- ○ $y = x^2 - 8x + 9$

The general solution of $y'' - 4y' + 4y = e^x$ when solved using method of undetermined coefficients is

- ○ $y = (c_1 + c_2 x)e^{2x} + 2e^x$
- ○ $y = c_1 e^{-2x} + c_2 e^{2x} + e^x$
- ◉ $y = (c_1 + c_2 x)e^{2x} + e^x$
- ○ $y = (c_1 + c_2 x)e^x + e^{2x}$

The differential equation $y'' + 3y' + 2xy^2 = e^{-2x}$ is considered to be :

- ○ Nonlinear with constant coefficients and homogeneous
- ◉ Nonlinear with variable coefficients and nonhomogeneous
- ○ Linear with constant coefficients and homogenous
- ○ Linear with variable coefficients and nonhomogeneous

The complementary solution to $y'' + 4y = \tan(200x)$ is

- (selected) $y_c = A\cos(2x) + B\sin(2x)$
- $y_c = A\cos(x) + B\sin(x)$
- $y_c = A\cosh(2x) + B\sinh(2x)$
- $y_c = A\cosh(x) + B\sinh(x)$

for 2021 2nd semester midterm

Past papers solutions

① the form of the particular solution $y'' - y' - 6y = x\sin x$

ans: $y_p = (Ax+B)\cos x + (Cx+D)\sin x$

---

② the integrating factor for $xy' + 2y = 10x^2$

ans: $y' + \frac{2}{x} y = 10x$

$$\mu_{(x)} = e^{\int \frac{2}{x} dx} = e^{\int 2x^{-1} dx} = \boxed{x^2}$$

---

③ find the D.E $y'' + by' - cy = 0$ given basis

$y_1 = 1$, $y_2 = e^{-3x}$

ans: $r_1 = 0$, $r_2 = -3$

$(r+0)(r+3) = 0$

$r^2 + 3r = 0$

$$\boxed{y'' + 3y' = 0}$$

---

④ the general solution to $y'' + 4y' = 0$

ans: $y = c_1 \cos(2x) + c_2 \sin(2x)$

---

⑤ the G.S to $y' - y^2 = 0$

ans: $y = \frac{1}{-x - c}$

(6) the G.S to $y' + xy = 0$ has the form!

ans! $ce^{-\left(\frac{x^2}{2}\right)}$

(7) the roots $(-10, 10)$, then the complementry solution is!

ans! $y_c = c_1 e^{-10x} + c_2 e^{10x}$

(8) classify the following d.E ; $e^x y' + (3 - x^2) y^2 = 0$

ans! separable and not linear

(9) which one is exact

ans! $2xy\,dx + (x^2 + 3)\,dy = 0$

(10) the solution to $y' - 3x^2 - 9 = 0$ satisfying $y(1) = 2$

ans! $y = x^3 + 9x - 8$

(11) the G.S of $y'' - 4y' + 4y = e^x$ with undeterminined coefficient's

ans! $y = (c_1 + c_2 x) e^{2x} + e^x$

(12) the D.E $y'' + 3y' + 2xy^2 = e^{-2x}$ is considred to be;

ans! nonlinear with variable coefficients and non Homogeneous

(14) $y'' + 4y = \tan(200x)$, the y_c is

ans! $y_c = A\cos(2x) + B\sin(2x)$

System of 1st ODEs (2x2)

1) Homogeneous system of 1st ODEs $\vec{Y}' = A\vec{Y}$, $y(t)=0$

* there are two ways to solve it:

1st one: direct method

G.S

Case 1: $\lambda_1 \neq \lambda_2$

$y(t) = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t}$

Case 2: $\lambda_1 = \lambda_2$

$y(t) = c_1 x_1 e^{\lambda t} + c_2 (t x_1 e^{\lambda t} + x_2 e^{\lambda t})$

Case 3: $\lambda = \alpha \pm \beta j$

Ex: $y_1' = 4y_1 - 2y_2$

$y_2' = y_1 + y_2$

Sol: $\underbrace{\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}}_{\vec{Y}'} = \underbrace{\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\vec{Y}}$

$y_1 = x_1 e^{\lambda t}$

$y_1' = x_1 \lambda e^{\lambda t}$

$y_2 = x_2 e^{\lambda t}$

$y_2' = x_2 \lambda e^{\lambda t}$

$$\begin{bmatrix} x_1 \lambda e^{\lambda t} \\ x_2 \lambda e^{\lambda t} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$

$$\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$

$$(A - \lambda I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(4-\lambda)(1-\lambda) + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = 3$$

$\lambda_1 = 2$

$$\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_1 - 2x_2 = 0$

$x_1 - x_2 = 0 \rightarrow x_1 = x_2$

$X_1 = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$

$$\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$$

$x_1 - 2x_2 = 0$

$x_1 = 2x_2$

$X_2 = k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

General solution: $\vec{Y} = \vec{Y}_1 + \vec{Y}_2 = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}$

Scanned with CamScanner

the second method is : Elimination method

Ex: $y_1' = 6y_1 + 5y_2$

$y_2' = y_1 + 2y_2$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cancel{e^{\lambda t}} = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cancel{e^{\lambda t}}$$

$$A - \lambda I \Rightarrow \begin{bmatrix} 6-\lambda & 5 \\ 1 & 2-\lambda \end{bmatrix}$$

$\lambda^2 - 8\lambda + 12 - 5 = 0$

$\lambda^2 - 8\lambda + 7 = 0$

$\Delta = b^2 - 4ac$ بمعنى تأكد من المميز

$64 - 4 \cdot 7 > 0$

Real

$(\lambda - 1)(\lambda - 7) = 0$

$\lambda_1 = 1$ $\lambda_2 = 7$

Note:

$\Delta > 0$ Real

$\Delta < 0$ Complex

$\Delta = 0$ $\lambda_1 = \lambda_2$

$\boxed{\lambda_1 = 1}$

$$\begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$5x_1 + 5x_2 = 0$

$x_1 = -x_2$

$$\boxed{X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

So the G.S

$\vec{Y} = c_1 X_1 e^{\lambda_1 t} + c_2 X_2 e^{\lambda_2 t}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{7t}$$

$\lambda_2 = 7$

$$\begin{bmatrix} -1 & 5 \\ 1 & -5 \end{bmatrix}$$

$-x_1 + 5x_2 = 0$ so $x_1 = 5x_2$ so $\boxed{X_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}}$

Case: complex eigenvalue

$$\text{Ex: } \begin{aligned} y_1' &= -3y_1 - 2y_2 \\ y_2' &= 5y_1 - y_2 \end{aligned}$$

Sol: $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$A - \lambda I = 0 \rightarrow \begin{bmatrix} -3-\lambda & -2 \\ 5 & -1-\lambda \end{bmatrix}$

$\lambda^2 + 4\lambda + 13 = 0$

$\Delta = 16 - 4 \cdot 13 < 0$

complex

$$\boxed{\lambda_{1,2} = -2 \pm 3j}$$

for $\lambda = -2+3j$

$\begin{bmatrix} -1-3j & -2 \\ 5 & 1-3j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(-1-3j)x_1 = (2)x_2$

تبادل

$x_1 \rightarrow 2$, $x_2 \rightarrow (-1-3j)$

$\vec{Y}_1 = \begin{bmatrix} 2 \\ -1-3j \end{bmatrix} e^{(-2+3j)t} \rightarrow e^{\theta j} = \cos\theta + j\sin\theta$, $e^{(-2+3j)t} = e^{-2t} \cdot e^{(3t)j} = e^{-2t}(\cos 3t + j\sin 3t)$

$\vec{Y}_1 = \begin{bmatrix} 2 \\ -1-3j \end{bmatrix} \cdot e^{-2t}(\cos 3t + j\sin 3t)$

$\vec{Y}_1 = e^{-2t} \begin{bmatrix} 2\cos 3t + j2\sin 3t \\ -\cos 3t - j\sin 3t - 3j\cos 3t + 3\sin 3t \end{bmatrix}$

$\vec{Y}_1 = e^{-2t} \underbrace{\begin{bmatrix} 2\cos 3t \\ -\cos 3t + 3\sin 3t \end{bmatrix}}_{X_1} + j \underbrace{\begin{bmatrix} 2\sin 3t \\ -\sin 3t - 3\cos 3t \end{bmatrix}}_{X_2}$

for $\lambda_2 = -2-3j \longrightarrow$ same answer

G.S: $\vec{Y} = c_1 \begin{bmatrix} 2\cos 3t \\ -\cos 3t + 3\sin 3t \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2\sin 3t \\ -\sin 3t - 3\cos 3t \end{bmatrix} e^{-2t}$

case: repeated eigenvalues (double) ~~$y(t) = c_1 x_1 e^{\lambda t} + c_2$~~

Ex: $y_1' = 4y_1 - y_2$
$y_2' = y_1 + 2y_2$

$y(t) = c_1 x_1 e^{\lambda t} + c_2 (t x_1 e^{\lambda t} + \cancel{x_2} e^{\lambda t}$

Sol: $\det \begin{bmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix} = (4-\lambda)(2-\lambda) + 1 = 0$

$\lambda_1 = \lambda_2 = 3$ double

$(A - \lambda I)\bar{x} = 0$

$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 = 0$
$x_1 = x_2$

$\boxed{x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\vec{Y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

to find $\vec{Y}_2$

$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$u_1 - u_2 = 1$

$u_2 = u_1 - 1$

$u = \begin{bmatrix} u_1 \\ u_1 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ if $u_1 = 2$

$\vec{Y}_2 = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}$

G.S

$\vec{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}}_{\vec{Y}_1} + c_2 \underbrace{\left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}\right)}_{\vec{Y}_2}$

② Nonhomogeneous linear system of ODEs

non Hom $\neq 0$

$$\vec{Y}' = A\vec{Y} + g(t)$$

$$G.S: \vec{Y}(t) = \vec{Y}_c(t) + \vec{Y}_p(t)$$

* Method of undetermined coefficients

باقي ال EX في ال slides

$$\boxed{\begin{aligned} Ex: \frac{dy_1}{dt} &= 2y_1 + y_2 \\ \frac{dy_2}{dt} &= 3y_2 + te^t \end{aligned}}$$

sol: $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ te^t \end{bmatrix}$

then find $\det(A - \lambda I) = 0$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(3-\lambda) = 0$$

$$\lambda_1 = 2, \ \lambda_2 = 3$$

So x_1:

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0x_1 + x_2 = 0$$

$$\boxed{x_2 = 0}$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for x_2:

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{Y}_c = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}}$$

for $\vec{Y}_p$:

$$g(t) = \begin{bmatrix} 0 \\ te^t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} e^t$$

$$\begin{bmatrix} y_{p_1} \\ y_{p_2} \end{bmatrix} = (B + Ct)e^t = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} te^t$$

$$\begin{bmatrix} y_{p_1}' \\ y_{p_2}' \end{bmatrix} = \left(\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + t \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right) e^t$$

على الصفحة

then subst. in $\vec{Y}' = A\vec{Y} + g(t)$

$$\left(\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + t \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right) \cancel{e^t} = A \left(\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} t \right) e^t + \begin{bmatrix} 0 \\ t \end{bmatrix} e^t$$

$$B + C + tC = AB + ACt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$$

t^1: $C = AC + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow ①$

t^0: $B + C = AB \longrightarrow ②$

① $(A - I)C = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\boxed{C_1 = 1/2} \quad \boxed{C_2 = -1/2}$$

② $(A - I)B = C$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

$$B_1 = 3/4, \ B_2 = -1/4$$

$$\vec{Y}_p = \begin{bmatrix} 3/4 \\ -1/4 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} te^t$$

$$G.S: \vec{Y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 3/4 \\ -1/4 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} te^t$$

* the 2nd method is: Variation of parameters

steps the procedure:

1. find $\vec{Y}_1$ and $\vec{Y}_2$ and then the construct $\vec{Y} = [\vec{Y}_1 / \vec{Y}_2]$
2. Compute $\vec{Y}^{-1} \cdot g(t)$ which is $u'(t)$
3. compute $u(t) = \int \vec{Y}^{-1} \cdot g(t)\, dt$
4. then find $y_p = \vec{Y} \cdot u(t)$ and compute it

$$y' = Ay + g(t)$$
$$\vec{Y} = y_c + y_p$$
$$y_p = \vec{Y} \int \vec{Y}^{-1} g(t)\, dt$$

Ex) $y_1' = -3y_1 + y_2 - 6e^{-2t}$

$y_2' = y_1 - 3y_2 + 2e^{-2t}$

Hom لازم نطلع $\vec{Y}_2$ and $\vec{Y}_1$ ليبي

$\vec{Y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$, $\vec{Y}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$

$$\vec{Y} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}$$

$$\vec{Y}^{-1} = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix}$$ → we can doit from symbolab

$$u' = \vec{Y}^{-1} \cdot g(t) = \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix} \begin{bmatrix} -6e^{-2t} \\ 2e^{-2t} \end{bmatrix} = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix}$$

$$u(t) = \int_0^t \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix} dt = \begin{bmatrix} \int_0^t -2\, dt \\ \int_0^t -4e^{2t}\, dt \end{bmatrix} = \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

$$y_p = \vec{Y} \cdot u(t) = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix} = \begin{bmatrix} -2te^{-2t} + e^{-4t}(-2e^{2t} + 2) \\ -2te^{-2t} - e^{-4t}(-2e^{2t} + 2) \end{bmatrix}$$

$$= \begin{bmatrix} -2t \\ -2t \end{bmatrix} e^{-2t} + \begin{bmatrix} -2 \\ +2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-4t}$$

ترتيب كمان

$$= -2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

G.S: $y(t) = y_p + y_c = -2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$

Laplace Transforms

① Laplace Transform

* constant $\longrightarrow \dfrac{a}{s}$

* $e^{-at} \longrightarrow \dfrac{1}{s+a}$
* $e^{at} \longrightarrow \dfrac{1}{s-a}$

* $t^n \longrightarrow \dfrac{n!}{s^{n+1}}$

* $\sin(at) \longrightarrow \dfrac{a}{s^2+a^2}$, $t\sin(at) \longrightarrow \dfrac{2as}{(s^2+a^2)^2}$

* $\cos(at) \longrightarrow \dfrac{s}{s^2+a^2}$, $t\cos(at) \longrightarrow \dfrac{s^2-a^2}{(s^2+a^2)^2}$

* $\sinh(at) \longrightarrow \dfrac{a}{s^2-a^2}$

* $\cosh(at) \longrightarrow \dfrac{s}{s^2-a^2}$

* $e^{\pm at}\sin(bt) \longrightarrow \dfrac{b}{(s\mp a)^2+b^2}$

* $e^{\pm at}\cos(bt) \longrightarrow \dfrac{(s\mp a)}{(s\mp a)^2+b^2}$

* $e^{\pm at}\sinh(bt) \longrightarrow \dfrac{b}{(s\mp a)^2-b^2}$

* $e^{at}\cosh(bt) \longrightarrow \dfrac{(s-a)}{(s-a)^2-b^2}$

(2) Inverse laplace Transform

EX: $\dfrac{1-3s}{s^2+8s+21}$ → تربيعي → صفر → $\Delta = 8^2 - 4\cdot1\cdot21$

$= 64 - 84$

$\Delta < 0$

Complex roots

we use

$s^2 + d_1 s + d_0 = \left(s + \frac{d_1}{2}\right)^2 + \left(d_0 - \frac{d_1^2}{4}\right)$

$\left(s + \frac{8}{2}\right)^2 + \left(21 - \frac{64}{4}\right) = (s+4)^2 + 5$

shifting

$F(s) = \dfrac{1-3s}{(s+4)^2+5} = \dfrac{1-12+12-3s}{(s+4)^2+5}$

$= \dfrac{13-12-3s}{(s+4)^2+5} = \dfrac{13 \cdot \sqrt{5}/\sqrt{5}}{(s+4)^2+5} - \dfrac{3(s+4)}{(s+4)^2+5}$

$\dfrac{13}{\sqrt{5}} e^{-4t} \sin\sqrt{5}t - 3e^{-4t} \cos\sqrt{5}\, t$

* partial fraction

- $(ax+b) \rightarrow \left(\dfrac{A}{(ax+b)}\right)$
- $(ax+b)^K \longrightarrow \dfrac{A_K}{(ax+b)^K} + \cdots + \dfrac{A_2}{(ax+b)^2} + \dfrac{A_1}{(ax+b)}$ repeated roots
- $(ax^2+bx+c) \longrightarrow \dfrac{Ax+B}{ax^2+bx+c}$ complete roots conjegate
-

Ex: $F(s) = \dfrac{s+7}{(s+2)(s-5)}$

$F(s) = \dfrac{A}{s+2}\Big|_{s=-2} + \dfrac{B}{s-5}\Big|_{s=5} = \dfrac{s+7}{(s+2)(s-5)}$

A: $A = \dfrac{s+7}{s-5} = -\dfrac{5}{7}$

B: $B = \dfrac{s+7}{s+2} = \dfrac{12}{7}$

$L^{-1}\left[\dfrac{A}{s+2}\right] + L^{-1}\left[\dfrac{B}{s-5}\right] = Ae^{-2t} + Be^{5t}$

$= -\dfrac{5}{7}e^{-2t} + \dfrac{12}{7}e^{5t}$

Ex: $f(s) = \dfrac{2-5s}{(s-6)(s^2+11)}$ → $\Delta = 0 - 4 \cdot 1 \cdot 11$ → $\Delta < 0$ complex

sol: $\dfrac{A}{s-6}\Big|_{s=6} + \dfrac{Bs+C}{s^2+11}$

A: $A = \dfrac{2-5s}{s^2+11} = \boxed{\dfrac{-28}{47}}$

$2-5s = A(s^2+11) + (Bs+C)(s-6)$

$(A+B)s^2 + (-6B+C)s + [11A-6C] = 2-5s$

s^2: $A+B=0$ → $B = \dfrac{28}{47}$

s^1: $-6B+C=-5$ → $C = -67/47$

s^0: $11A - 6C \overset{?}{=} 2$

$F(s) = \dfrac{A}{s-6} + \dfrac{Bs+C}{s^2+11}$

$L(t) = Ae^{6t} + L^{-1}\left[\dfrac{Bs+C}{s^2+11}\right]$

$L^{-1}\left[\dfrac{Bs+C}{s^2+11}\right] = \dfrac{Bs}{s^2+11} + \dfrac{C\cdot\sqrt{11}/\sqrt{11}}{s^2+11}$

$L^{-1}\left[\dfrac{Bs+C}{s^2+11}\right] = B\cos\sqrt{11}\,t + \dfrac{C}{\sqrt{11}}\sin\sqrt{11}\,t$

③ Solving IVP with Laplace

- $L[f'_{(t)}] = s\,F(s) - f(0)$ $\qquad y = F(s)$

- $L[f''_{(t)}] = s^2 F(s) - s f(0) - f'(0)$

Ex: $y'' - y = t$, $y(0) = y'(0) = 1$

$L[y'' - y] = L[t]$

$[s^2 F(s) - s f(0) - f'(0) - F(s)] = \frac{1}{s^2}$

$s^2 F(s) - F(s) = s f(0) + f'(0) + \frac{1}{s^2}$

$F(s)(s^2-1) = \frac{s+1+\frac{1}{s^2}}{(s^2-1)} = \frac{s+1}{(s^2-1)} + \frac{1}{s^2(s^2-1)}$

$F(s) = \frac{\cancel{s+1}}{\cancel{(s+1)}(s-1)} + \frac{1}{s^2(s^2-1)}$

$L^{-1}\left[F(s) = \frac{1}{s-1} + \left[\frac{1}{s^2-1} - \frac{1}{s^2}\right]\right]$

$\boxed{= e^{t} + \sinh t - t}$

Question 1/2 (10 p.)

Use Laplace transforms to solve

$$y'' + 4y = 8t, \quad y(0) = 3 \quad y'(0) = 0$$

Answer:

P

0 WORDS POWERED BY TINY

SUBMIT ANSWER

testportal

Time left to complete the test: 0 h 22 min. 12 sec.

Question 2/2 (10 p.)

Solve the following system of differential equations

$$\frac{dy_1}{dt} = y_1 + y_2$$

$$\frac{dy_2}{dt} = 4y_1 - 2y_2$$

Answer:

Paragraph

P 0 WORDS POWERED BY TINY

SUBMIT ANSWER

Q1

$$y'' + 4y = 8t \qquad y(0) = 3, \quad y'(0) = 0$$

sol:

$$L[y'' + 4y] = \frac{8}{s^2}$$

$$[s^2 F(s) - s f(0) - \underbrace{\cancel{f'(0)}}_{0} + 4F(s)] = \frac{8}{s^2}$$

$$[s^2 F(s) - 3s + 4F(s)] = \frac{8}{s^2}$$

$$(s^2+4)F(s) = \frac{8}{s^2} + 3s$$

$$F(s) = \frac{8}{s^2(s^2+4)} + \frac{3s}{s^2+4}$$

$$= \left[\frac{2}{s^2} - \frac{2}{s^2+4}\right] + \frac{3s}{s^2+4}$$

$$f(t) = 2L^{-1}\left[\frac{1}{s^2}\right] - 2L^{-1}\left[\frac{1}{s^2+4}\right] + 3L^{-1}\left[\frac{s}{s^2+4}\right]$$

$$= 3\cos 2t + 2t - 2 \cdot \frac{1}{2}\sin 2t$$

$$= 3\cos 2t + 2t - \sin 2t$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Q 2

$$\begin{bmatrix} x_1 \lambda e^{\lambda t} \\ x_2 \lambda e^{\lambda t} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$

$$\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cancel{e^{\lambda t}} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cancel{e^{\lambda t}}$$

$$(A - \lambda I) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2 + 2\lambda - \lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\lambda_1 = 2 \; , \; \lambda_2 = -3$$

for $\lambda_1 = 2$

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0 \rightarrow x_1 = x_2$$

$$4x_1 - 4x_2 = 0$$

$$x_1 = K_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = -3$

$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1 + x_2 = 0$$

$$4x_1 + x_2 = 0$$

$$\boxed{x_2 = K_2 \begin{bmatrix} -1 \\ 4 \end{bmatrix}}$$

So the general solution

$$\vec{Y} = \vec{Y}_1 + \vec{Y}_2 = K_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + K_2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} e^{-3t}$$