Mathematical Summary

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First order differental equation $\Rightarrow \frac{dy}{dx} = F(x,y)$

1) Seperble first order D.E :-

$\frac{dy}{G(y)} = F(x)\,d(x) \rightarrow$ seperble.

2) Reduction to seperable :-

A] $\frac{dy}{dx} = f\left(\frac{y}{x}\right) \rightarrow \frac{y}{x} = u \Rightarrow y = ux$

B] $\frac{dy}{dx} = \sqrt{f(x+y)} \rightarrow x+y = u^2$

C] $\frac{dy}{dx} = (f(x+y))^n \rightarrow x+y = u$

D] $\frac{dy}{dx} = \sin(f(x+y)) \rightarrow f(x+y) = u$

any trigonometric

E] $\frac{dy}{dx} = f(ax+by+K) \rightarrow ax+by+K = u$

F] If $(a_1x+b_1y+c_1)\,dx + (a_2x+b_2y+c_2)\,dy = 0$

① $\frac{a_2}{a_1} = \frac{b_2}{b_1}$, let $a_1x + b_1y = Z$

② $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$, let $x = X + h$, $y = Y + K$

✓ $a_1h + b_1K + c_1 = 0$ —①
✓ $a_2h + b_2K + c_2 = 0$ —②
$\rightarrow$ then insert h and K into the assumption

3) exact D.E :-

$\frac{dy}{dx} = f(x,y) \rightarrow M(x,y)\,dx + N(x,y)\,dy = 0$

M N

• If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow 1^{\circ}$ O.D.E is exact.

• If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow 1^{\circ}$ O.D.E is not exact.

* to obtain solution $\rightarrow$

① A) find $u(x,y) = \int M\,dx + K_y$

B) $K_y \rightarrow \frac{\partial u}{\partial y} = N(x,y) \rightarrow \frac{\partial}{\partial y}\left[\int M\,dx\right] + \frac{dK}{dy} = N$

C) $u(x,y) = \int M\,dx + K_y = C$

② another approach (to find solution for exact) :-

A) $u(x,y) = \int N\,dy + L_x$

B) $\frac{\partial u}{\partial x} = M \rightarrow \frac{\partial}{\partial x}\left(\int N\,dy\right) + \frac{dL}{dx} = M$

C) $u(x,y) = \int N\,dy + L_x = C$

• to obtain the integrating factor $\rightarrow$

* Try first ① $R_1 = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$, if R_1 is function of (x) only the $I.F = e^{\int R_1\,dx}$.

* Try second ② $R_2 = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$, if R_2 is function of (y) only the $I.F = e^{\int R_2\,dy}$.

4) Linear first O.D.E :- $\frac{dy}{dx} + P(x) = r(x)$

* $y_g = C_1 e^{-\int p(x)\,dx}$ $\rightarrow$ homogeneous

* $y = e^{-\int p(x)\,dx}\left[\int\left(e^{\int p(x)\,dx} \cdot r(x)\right)dx + C_1\right]$ $\rightarrow$ non homo

• Reduction of Non-linear to linear $\rightarrow$

$\frac{dy}{dx} + p(x)y = g(x)(y)^a \rightarrow$ Bernoili equation.

✓ If $a=0 \rightarrow \frac{dy}{dx} + p(x)y = g(x) \rightarrow$ non homo

$\rightarrow y = e^{-\int p(x)\,dx}\left[\int e^{\int p(x)\,dx} \cdot g(x)\,dx + C_1\right]$

✓ If $a=1 \rightarrow \frac{dy}{dx} + (p(x) - g(x))y = 0 \rightarrow$ homo

$\rightarrow y = C_1 e^{-\int (p(x) - g(x))\,dx}$

✓ If $a \neq 0$, $a \neq 1 \rightarrow \frac{dy}{dx} + p(x)y = g(x)y^a$, let $u = y^{1-a}$

$\rightarrow \frac{du}{dx} + (1-a)p(x)u = (1-a)g(x)$

$\rightarrow u = e^{-\int p(x)\,dx}\left[\int e^{\int p(x)\,dx} \cdot G(x)\,dx + C_1\right]$ #

Second order differential equation ⇒

• $\ddot{y} + p(x)\dot{y} + q(x)y = r(x)$

✓ if: $r(x) = \text{zero}$, then: 2^{nd} O.O.D.E homogeneous.

✓ if: $r(x) \neq \text{zero}$, then: 2^{nd} O.O.D.E non-homogeneous.

✓ if: $p(x)$ or $q(x)$ are not constant ⇒ then: 2^{nd} O.O.D.E linear with variable coefficients.

✓ if: $p(x)$ and $q(x)$ are constant ⇒ then: 2^{nd} O.O.D.E linear with constant coefficients.

* for a homogeneous linear D·equation, any linear combination of two solution on an open interval is a gain a solution.

* since y_1 & y_2 are solution, then y_g is also a solution ✓

(This is not true for a non-homogeneous or non-linear.)

* General solution: $y_g = C_1y_1 + C_2y_2$, y_1 & y_2 (part of solution)

↳ $\frac{y_1}{y_2} = \text{constant}$, y_1 and y_2 are proportional

↳ linear dependent ↳ a soln الاخر مضروب ب factor

↳ $\frac{y_1}{y_2} \neq \text{constant}$, y_1 and y_2 are not proportional.

↳ linear independent, basis of solution of D.E

↳ general solution can be constant from linearly independent solution only.

* Reduction of a 2^{nd} O.O.D.E to a 1^{st} O.O.D.E :-

✓ Case 1: if in a 2^{nd} O.O.D.E the dependent variable (y) does not appear explicitly, the equation will be at the form $f(x,\dot{y},\ddot{y}) = 0$, then we can reduce this to a 1^{st} O.O.D.E by assume $\dot{y} = \frac{dy}{dx} = z$

✓ Case 2: if in a 2^{nd} O.O.D.E, the independent variable (x) does not appear explicitly in the D.E $f(y,\dot{y},\ddot{y}) = 0$, then we can assume $\ddot{y} = \left(\frac{dz}{dy}\right)\cdot z$, where $z = \frac{dy}{dx}$

* Homogeneous 2^{nd} O.D.E with constant coefficient $\ddot{y} + a\dot{y} + by = 0$:-

to solve: let $y = e^{\lambda x}$, $\dot{y} = \lambda e^{\lambda x}$, $\ddot{y} = \lambda^2 e^{\lambda x}$

$\Rightarrow (\lambda^2 e^{\lambda x}) + a(\lambda e^{\lambda x}) + b(e^{\lambda x}) = 0$

↳ $\underbrace{(\lambda^2 + a\lambda + b)}_{=\text{zero}}\underbrace{e^{\lambda x}}_{\neq \text{zero}} = 0 \rightarrow \lambda^2 + a\lambda + b = 0$

↳ $\lambda = \frac{-a \mp \sqrt{a^2 - 4c}}{2}$ < $\lambda_1 = \frac{-a}{2} + \sqrt{\frac{a^2-4c}{4}}$, $\lambda_2 = \frac{-a}{2} - \sqrt{\frac{a^2-4c}{4}}$

• Case (1): if $(a^2 - 4b) > 0$, then $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$

↳ $\frac{y_1}{y_2} = \frac{e^{\lambda_1 x}}{e^{\lambda_2 x}} = e^{(\lambda_1 - \lambda_2)x} = e^{\lambda_3 x}$ → not constant, independent

$\Rightarrow y_g = C_1y_1 + C_2y_2 = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$

• Case (2): if $(a^2 - 4b) = 0$, $\lambda_1 = \lambda_2 = \frac{-a}{2}$

let $y_2 = uy_1$, drive and substitute

$\Rightarrow y_g = (C_1 + C_2x)e^{(-a/2)x}$

• Case (3): if $(a^2 - 4b) < 0$, $\lambda = \frac{-a}{2} \pm \sqrt{\frac{a^2-4b}{4}}$ (-ve)

↳ $= \frac{-a}{2} \pm \underbrace{\sqrt{\frac{4b-a^2}{4}}}_{\omega}\cdot\underbrace{\sqrt{-1}}_{i}$ (+ve)

$\Rightarrow \lambda = \frac{-a}{2} \pm \omega i \rightarrow \lambda_1 = \frac{-a}{2} + \omega i$

↳ $\lambda_2 = \frac{-a}{2} - \omega i$

↳ $y_g = e^{-\frac{a}{2}x}\left(C_1\cos(\omega x) + C_2\sin(\omega x)\right)$

②

* Homogeneous 2nd O.O.D.E with variable coefficient $x^2\ddot{y}+ax\dot{y}+by=0$:- (Euler equation)

to solve: let $y=x^m$, $\dot{y}=mx^{m-1}$, $\ddot{y}=(m^2-m)x^{m-2}$

$\Rightarrow x^2((m^2-m)x^{m-2})+ax(mx^{m-1})+bx=0$

↳ $\underbrace{(m^2+(a-1)m+b)}_{=\text{zero}}\underbrace{x^m}_{\neq\text{zero}}=0 \to m^2+(a-1)m+b=0$

↳ $m=\dfrac{-(a-1)\pm\sqrt{(a-1)^2-4b}}{2}$

• Case (1): if $((a-1)^2-4b)>0$, $y_1=x^{m_1}$, $y_2=x^{m_2}$

$m_1=\frac{-(a-1)}{2}+\sqrt{\frac{(a-1)^2-4b}{4}}$, $m_2=\frac{-(a-1)}{2}-\sqrt{\frac{(a-1)^2-4b}{4}}$

→ $\frac{y_1}{y_2}=\frac{x^{m_1}}{x^{m_2}}=x^{(m_1-m_2)}$ → $m_1-m_2\neq$ zero → not constant, independent

$\Rightarrow y_g=c_1y_1+c_2y_2$

• Case (2): if $((a-1)^2-4b)=0$, $y_1=x^{m_1}=x^{m_2}=x^{\frac{1-a}{2}}$

let $y_2=uy_1$, drive and substitute

$y_g=(c_1+c_2\ln(x))x^{m_1}$

• Case (3): if $((a-1)^2-4b)<0$, $m=\frac{1-a}{2}\pm\sqrt{\frac{(a-1)^2-4b}{4}}$ → -ve

↳ $\underbrace{\frac{1-a}{2}}_{\mu}\pm\underbrace{\sqrt{\frac{4b-(a-1)^2}{4}}}_{v}\cdot\sqrt{-1}$ → $m_1=\mu+iv$, $m_2=\mu-iv$

↳ $\frac{-a}{2}$

$\Rightarrow y_g=x^{\mu}[c_1\cos(v\ln(x))+c_2\sin(v\ln x)]$

* uniqueness Theory (wronskian): $y_g=c_1y_1+c_2y_2$

$W=\begin{vmatrix}y_1 & y_2\\ \dot{y}_1 & \dot{y}_2\end{vmatrix}=y_1\dot{y}_2-y_2\dot{y}_1$

• if W = zero → dependent → we can't construct y_g.

• if W ≠ zero → independent → we can construct y_g.

* Reduction of order ⇒

$\ddot{y}+p(x)\dot{y}+q(x)y=0$, $(p(x),q(x))$ → any function

→ $y_2=uy_1$ → $y_2=\left(\int\left[\frac{e^{-\int p(x)\,dx}}{y_1^2}\right]dx\right)y_1$

* 2nd O.O.D.E Non homogeneous :- $y_g=y_h+y_p$

$\ddot{y}+p(x)\dot{y}+q(x)y=r(x)$, $r(x)\neq$ zero

✓ y_h: solution of the homogeneous, $\ddot{y}+p(x)\dot{y}+q(x)=0$

↳ $y_h=c_1y_1+c_2y_2$

✓ y_p: solution of the nonhomogeneous, no arbitrary const.

↳ can obtained : 1) undetermined coefficients.
2) variation of parameters.

① • undetermined coefficients → $\ddot{y}+a\dot{y}+by=r(x)$

1) constant coefficients, 2) special forms (exp, poly, sin ...)

A) if r(x) is one of the following :-

$r(x)$	y_p
• $Ke^{\gamma x}$	$Ce^{\gamma x}$
• Kx^n	$Ax^n+Bx^{n-1}\dots Mx^0$
• $K\cos(wx)$ • $K\sin(wx)$	→ $A\cos(wx)+B\sin(wx)$
• $Ke^{\gamma x}\cos(wx)$	$e^{\gamma x}[M\cos(wx)+N\sin(wx)]$
• $Ke^{\gamma x}[x^2+3x]$	$e^{\gamma x}[Mx^2+Nx+Zc]$

B) if we have repetition, then multiply choice of y_p by (x) or (x^n).

• Variation of parameters → $\ddot{y}+p(x)\dot{y}+q(x)y=r(x)$

نلجأ لها لما r(x) تختلف عن ال case الأولى ، $y_g=y_h+y_p$

→ $y_h=\ddot{y}+p(x)\dot{y}+q(x)y=0$, $y_h=c_1y_1+c_2y_2$

$y_p=-y_1\int\frac{y_2}{W}\cdot r(x)\,dx+y_2\int\frac{y_1}{W}\cdot r(x)\,dx$

• $W=\begin{vmatrix}y_1 & y_2\\ \dot{y}_1 & \dot{y}_2\end{vmatrix}$, $W_1=\begin{vmatrix}0 & y_2\\ 1 & \dot{y}_2\end{vmatrix}$, $W_2=\begin{vmatrix}y_1 & 0\\ \dot{y}_1 & 1\end{vmatrix}$

$W_1=-y_2$, $W_2=y_1$

$=y_1\dot{y}_2-\dot{y}_1y_2$