$\frac{25}{30}$

The University of Jordan
Faculty of Engineering & Technology
Chemical Engineering Department

(0905231) Mathematical Methods in Chemical Engineering — Summer Semester – 2015/2016

Second Exam

Name: [illegible] ID # 0147452

1. (6 points) Convert the following ODE to a system of first order ODEs and write it in the form of $\vec{y'} = \vec{A}\vec{y}$. Do NOT solve it

$$y'''' + y''' - 2y'' + y' - y = 0$$

$$y'''' = -y''' + 2y'' - y' + y$$

6

$y_1 = y$ $\quad y_1' = y_2$

$y_2 = y'$ $\quad y_2' = y_3$

$y_3 = y''$ $\quad y_3' = y_4$

$y_4 = y'''$ $\quad y_4' = -y_4 + 2y_3 - y_2 + y_1$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

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2. [8 points] Solve the following ODE. Show the details of your work.

$$y''' + 2y'' - y' - 2y = 1 - 4x^3$$

$y''' + 2y'' - y' - 2y = 0$

$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$

λ^3	λ^2	λ	c
1	2	-1	-2
$\boxed{1}$	1	3	2
1	3	2	0

$(\lambda - 1)(\lambda^2 + 3\lambda + 2) = 0$

$(\lambda - 1)(\lambda + 1)(\lambda + 2) = 0$

$\lambda_1 = 1 \quad \lambda_2 = -1 \quad \lambda_3 = -2$

$y_h = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x}$

$y_p = k_3 x^3 + k_2 x^2 + k_1 x + k_0$

$y_p' = 3k_3 x^2 + 2k_2 x + k_1$

$y_p'' = 6k_3 x + 2k_2$

$y_p''' = 6k_3$

$\boxed{k_3 = 2}$

$\boxed{7}$

$\boxed{k_2 = -3}$

$\boxed{k_1 = +30}$

$\boxed{k_0 = 30}$

$y_p = 2x^3 - 3x^2 + 30x + 30$

$\boxed{y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + 2x^3 - 3x^2 + 30x + 30}$

$6k_3 + 12k_3 x + 4k_2 - 3k_3 x^2 - 2k_2 x - k_1 - 2k_3 x^3 - 2k_2 x^2 - k_1 x - k_0 = 1 - 4x^3$

$6k_3 + 4k_2 - k_1 - k_0 = 1$

$12k_3 - 2k_2 - k_1 = 0$

$-3k_3 - 2k_2 = 0$

$-2k_3 = -4$

$$\vec{y} = c_1\begin{bmatrix}1\\0.5\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.03t}$$

$$y_1 = c_1 + c_2 e^{-0.03t}$$

$$y_2 = c_1 0.5 - c_2 e^{-0.03t}$$

$$y_1(0) = 90 \text{ lb}$$

$$y_2(0) = 150 \text{ lb}$$

$$90 = c_1 + c_2$$

$$150 = 0.5c_1 - c_2$$

$$\boxed{c_1 = 160}$$

$$\boxed{c_2 = -70}$$

$$y_1 = 160 - 70e^{-0.03t}$$

$$y_2 = 80 + 70e^{-0.03t}$$

$\cancel{t = ?? \quad y_1 = 0}$

$\cancel{-80 = 70e^{-0.03t}}$

$\cancel{t = ?? \quad y_1 = 0}$

$\cancel{-160 = -70e^{-0.03t}}$

$\cancel{e^{-0.03t} = 2.28}$

$t = ?? \quad y_2 = 100$?!

2

$$20 = 70e^{-0.03t}$$

$$e^{-0.03t} = 0.285$$

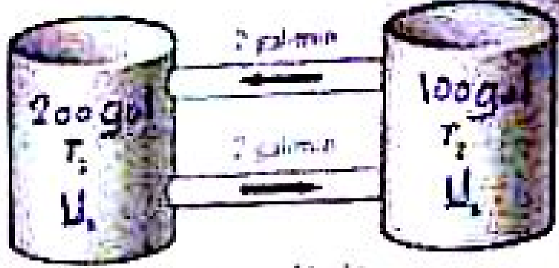
$$-0.03t = -1.25$$

$$\boxed{t = 41.6 \text{ min}}$$

$\boxed{8}$

3. [10 points]

Two tanks T_1 and T_2 are connected as shown in the figure. Tank T_1 contains initially 200 gal of water with 90 lb fertilizer dissolved in it, whereas T_2 contains initially 150 lb of fertilizer dissolved in 100 gal of water. By circulating liquid at a rate of 2 gal/min and stirring (to keep the mixture uniform) the amounts of fertilizer $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t. How long should we let the liquid circulate so that T_1 will contain as much fertilizer as there will be left in T_2?



System of tanks

OCC = in - out

$$y_1' = \frac{y_2}{100}\times 2 - \frac{y_1}{200}\times 2$$

$$y_1' = 0.02y_2 - 0.01y_1$$

$$y_2' = \frac{y_1}{200}\times 2 - \frac{y_2}{100}\times 2$$

$$y_2' = 0.01y_1 - 0.02y_2$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -0.01 & 0.02 \\ 0.01 & -0.02 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} -0.01 & 0.02 \\ 0.01 & -0.02 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -0.01-\lambda & 0.02 \\ 0.01 & -0.02-\lambda \end{bmatrix}$$

$$\det|A-\lambda I| = \lambda^2 + 0.03\lambda + 0.0002 - 0.0002$$

$$\lambda^2 + 0.03\lambda = 0$$

$$\lambda(\lambda + 0.03) = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = -0.03$$

$$\begin{bmatrix} -0.01-\lambda & 0.02 \\ 0.01 & -0.02-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-0.01-\lambda)x_1 + 0.02x_2 = 0$$

$$0.01x_1 + (-0.02-\lambda)x_2 = 0$$

$\lambda = 0$

$$-0.01x_1 + 0.02x_2 = 0$$

$$0.01x_1 - 0.02x_2 = 0$$

$\boxed{x_1 = 1}$ $\boxed{x_2 = 0.5}$

$\lambda_2 = -0.03$

$$0.02x_1 + 0.02x_2 = 0$$

$$0.01x_1 + 0.01x_2 = 0$$

$\boxed{x_1 = 1}$ $\boxed{x_2 = -1}$

4. [6 points] Find the particular solution of the following system of ODEs, if its homogeneous solution is as written below:

$$\left.\begin{aligned} y_1' &= 2y_1 + 2y_2 + e^t \\ y_2' &= -2y_1 - 3y_2 + e^t \end{aligned}\right\} \qquad \overline{y_h} = c_1\begin{bmatrix}1\\-2\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}-2\\1\end{bmatrix}e^t$$

$\boxed{4}$

$$y^{(p)} = \begin{bmatrix}a_1\\a_2\end{bmatrix}e^t + \begin{bmatrix}b_1\\b_2\end{bmatrix}te^t \qquad \begin{bmatrix}y_1'\\y_2'\end{bmatrix} = \begin{bmatrix}2 & 2\\-2 & -3\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}e^t$$

$$y^{(p)'} = \begin{bmatrix}a_1\\a_2\end{bmatrix}e^t + \begin{bmatrix}b\\b\end{bmatrix}te^t + \begin{bmatrix}b\\b\end{bmatrix}e^t$$

$$y' = Ay + g$$

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}e^t + \begin{bmatrix}b\\b\end{bmatrix}te^t + \begin{bmatrix}b\\b\end{bmatrix}e^t = \begin{bmatrix}2 & 2\\-2 & -3\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}e^t + \begin{bmatrix}2 & 2\\-2 & -3\end{bmatrix}te^t\begin{bmatrix}b\\b\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}e^t$$

$$\begin{bmatrix}a_1\\a_2\end{bmatrix} + \begin{bmatrix}b\\b\end{bmatrix} = \begin{bmatrix}2a_1 + 2a_2\\-2a_1 - 3a_2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}$$

$$a_1 + b_1 = 2a_1 + 2a_2 + 1$$
$$a_2 + b_2 = -2a_1 - 3a_2 + 1$$

$$-a_1 - 2a_2 + b = 1$$
$$2a_1 + 4a_2 + b = 1$$

$$2b_1 = 2$$
$$b = 1$$
$$3b = 3$$
$$\boxed{b = 1}$$

$$y^{(p)} = \begin{bmatrix}1\\-\frac{1}{2}\end{bmatrix}e^t + \begin{bmatrix}1\\1\end{bmatrix}te^t$$

$\boxed{a_1 = 1}$ $\boxed{a_2 = \frac{-1}{2}}$