

Second Exam

Name: Eng. Moustafa Al-Saleh

ID # 0147452

1. [6 points] Convert the following ODE to a system of first order ODEs and write it in the form of $\vec{y}' = \vec{A}\vec{y}$. Do NOT solve it

$$y''' + y'' - 2y' + y = 0$$

$$\underline{y}_1 = y$$

$$\underline{y}_2 = y'$$

$$\underline{y}^5 = -y''' + 2y'' - y' + y$$

$$\underline{y}_3 = y$$

$$\underline{y}_4 = y'$$

$$\underline{y}_5 = -y''' + 2y'' - y' + y$$

$$\underline{y}_6 = y$$

$$\underline{y}_7 = -y''' + 2y'' - y' + y$$

$$\underline{y}_8 = y$$

$$\underline{y}_9 = -y''' + 2y'' - y' + y$$

$$\underline{y}_{10} = y$$

$$\underline{y}_{11} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{12} = y$$

$$\underline{y}_{13} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{14} = y$$

$$\underline{y}_{15} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{16} = y$$

$$\underline{y}_{17} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{18} = y$$

$$\underline{y}_{19} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{20} = y$$

$$\underline{y}_{21} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{22} = y$$

$$\underline{y}_{23} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{24} = y$$

$$\underline{y}_{25} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{26} = y$$

$$\underline{y}_{27} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{28} = y$$

$$\underline{y}_{29} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{30} = y$$

$$\underline{y}_{31} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{32} = y$$

$$\underline{y}_{33} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{34} = y$$

$$\underline{y}_{35} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{36} = y$$

$$\underline{y}_{37} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{38} = y$$

$$\underline{y}_{39} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{40} = y$$

$$\underline{y}_{41} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{42} = y$$

$$\underline{y}_{43} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{44} = y$$

$$\underline{y}_{45} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{46} = y$$

$$\underline{y}_{47} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{48} = y$$

$$\underline{y}_{49} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{50} = y$$

$$\underline{y}_{51} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{52} = y$$

$$\underline{y}_{53} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{54} = y$$

$$\underline{y}_{55} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{56} = y$$

$$\underline{y}_{57} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{58} = y$$

$$\underline{y}_{59} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{60} = y$$

$$\underline{y}_{61} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{62} = y$$

$$\underline{y}_{63} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{64} = y$$

$$\underline{y}_{65} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{66} = y$$

$$\underline{y}_{67} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{68} = y$$

$$\underline{y}_{69} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{70} = y$$

$$\underline{y}_{71} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{72} = y$$

$$\underline{y}_{73} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{74} = y$$

$$\underline{y}_{75} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{76} = y$$

$$\underline{y}_{77} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{78} = y$$

$$\underline{y}_{79} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{80} = y$$

$$\underline{y}_{81} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{82} = y$$

$$\underline{y}_{83} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{84} = y$$

$$\underline{y}_{85} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{86} = y$$

$$\underline{y}_{87} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{88} = y$$

$$\underline{y}_{89} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{90} = y$$

$$\underline{y}_{91} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{92} = y$$

$$\underline{y}_{93} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{94} = y$$

$$\underline{y}_{95} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{96} = y$$

$$\underline{y}_{97} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{98} = y$$

$$\underline{y}_{99} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{100} = y$$

$$\underline{y}_{101} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{102} = y$$

$$\underline{y}_{103} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{104} = y$$

$$\underline{y}_{105} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{106} = y$$

$$\underline{y}_{107} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{108} = y$$

$$\underline{y}_{109} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{110} = y$$

$$\underline{y}_{111} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{112} = y$$

$$\underline{y}_{113} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{114} = y$$

$$\underline{y}_{115} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{116} = y$$

$$\underline{y}_{117} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{118} = y$$

$$\underline{y}_{119} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{120} = y$$

$$\underline{y}_{121} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{122} = y$$

$$\underline{y}_{123} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{124} = y$$

$$\underline{y}_{125} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{126} = y$$

$$\underline{y}_{127} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{128} = y$$

$$\underline{y}_{129} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{130} = y$$

$$\underline{y}_{131} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{132} = y$$

$$\underline{y}_{133} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{134} = y$$

$$\underline{y}_{135} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{136} = y$$

$$\underline{y}_{137} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{138} = y$$

$$\underline{y}_{139} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{140} = y$$

$$\underline{y}_{141} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{142} = y$$

$$\underline{y}_{143} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{144} = y$$

$$\underline{y}_{145} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{146} = y$$

$$\underline{y}_{147} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{148} = y$$

$$\underline{y}_{149} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{150} = y$$

$$\underline{y}_{151} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{152} = y$$

$$\underline{y}_{153} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{154} = y$$

$$\underline{y}_{155} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{156} = y$$

$$\underline{y}_{157} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{158} = y$$

$$\underline{y}_{159} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{160} = y$$

$$\underline{y}_{161} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{162} = y$$

$$\underline{y}_{163} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{164} = y$$

$$\underline{y}_{165} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{166} = y$$

$$\underline{y}_{167} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{168} = y$$

$$\underline{y}_{169} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{170} = y$$

$$\underline{y}_{171} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{172} = y$$

$$\underline{y}_{173} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{174} = y$$

$$\underline{y}_{175} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{176} = y$$

$$\underline{y}_{177} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{178} = y$$

$$\underline{y}_{179} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{180} = y$$

$$\underline{y}_{181} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{182} = y$$

$$\underline{y}_{183} = -y''' + 2y'' - y' + y$$

$$\underline{y}_{184} = y$$

$$\underline{y}_{185} = -y''' + 2y'' - y' + y$$

2. [8 points] Solve the following ODE. Show the details of your work.

$$y''' + 2y'' - y' - 2y = 1 - 4x^3$$

$$y''' + 2y'' - y' - 2y = 0$$

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

$$\begin{array}{cccc} \lambda & \lambda & \lambda & C \\ 1 & 2 & -1 & -2 \end{array}$$

$$\begin{array}{cccc} 1 & 3 & 2 \\ 1 & 3 & 2 & 0 \end{array}$$

$$(\lambda - 1)(\lambda^2 + 3\lambda + 2) = 0$$

$$(\lambda - 1)(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -2$$

$$y_p = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$$

$$y_p' = k_3 x^3 + k_2 x^2 + k_1 x + k_0$$

$$y_p'' = 3k_3 x^2 + 2k_2 x + k_1$$

$$y_p''' = 6k_3 x + 2k_2$$

$$y_p''' = 6k_3$$

$$6k_3 + 12k_2 x + 4k_1 - 3k_3 x^2 - 2k_2 x - k_1 - 2k_3 x^3 - 2k_2 x^2 - k_1 x - k_0 = 1 - 4x^3$$

$$6k_3 + 4k_1 - k_1 - k_0 = 1$$

$$12k_3 - 2k_2 - k_1 = 0$$

$$-3k_3 - 2k_2 = 0$$

$$-2k_3 = -4$$

$$k_3 = 2$$

N

$$k_1 = -3$$

$$k_1 = +30$$

$$k_0 = 30$$

$$y_p = 2x^3 - 3x^2 + 30x + 30$$

~~$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x} + 2x^3 - 3x^2 + 30x + 30$$~~

$$\vec{y} = C_1 \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-0.03t}$$

$$y_1 = C_1 + C_2 e^{-0.03t}$$

$$y_1 = C_1 0.5 + C_2 e^{-0.03t}$$

$$y_1(0) = 90 \text{ lt}$$

$$y_1(0) = 150 \text{ lt}$$

$$90 = C_1 + C_2$$

$$150 = 0.5C_1 - C_2$$

$$C_2 = 160$$

$$C_1 = -70$$

$$y_1 = 160 - 70e^{-0.03t}$$

$$y_1 = 80 + 70e^{-0.03t}$$

~~$$160 - 70e^{-0.03t} = 80 + 70e^{-0.03t}$$

$$160 = 80 + 140e^{-0.03t}$$

$$80 = 140e^{-0.03t}$$

$$e^{-0.03t} = 0.5714$$~~

~~$$160 - 80 = 70e^{-0.03t}$$~~

~~$$80 = 70e^{-0.03t}$$~~

~~$$e^{-0.03t} = 1.143$$~~

$t = ??$ $y_1 = 100$??

$$20 = 70 e^{-0.03t}$$

$$e^{-0.03t} = 0.285$$

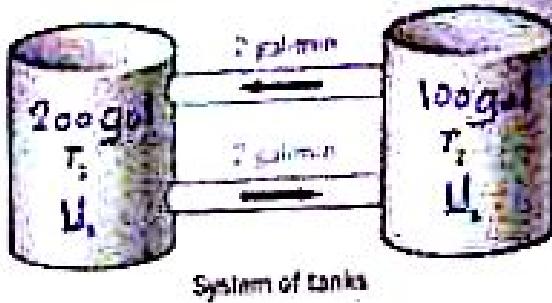
$$-0.03t = -1.25$$

$$t = 41.6 \text{ min}$$

✓

3. [10 points]

Two tanks T_1 and T_2 are connected as shown in the figure. Tank T_1 contains initially 200 gal of water with 90 lb fertilizer dissolved in it, whereas T_2 contains initially 150 lb of fertilizer dissolved in 100 gal of water. By circulating liquid at a rate of 2 gal/min and stirring (to keep the mixture uniform) the amounts of fertilizer $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t . How long should we let the liquid circulate so that T_1 will contain as much fertilizer as there will be left in T_2 ?



$$OCC = \text{in} - \text{out}$$

$$\dot{y}_1' = \frac{y_1}{100} \times 2 - \frac{y_1}{200} \times 2$$

$$\dot{y}_1' = 0.02y_1 - 0.01y_1$$

$$\dot{y}_2' = \frac{y_2}{100} \times 2 - \frac{y_2}{150} \times 2$$

$$\dot{y}_2' = 0.01y_2 - 0.01y_2$$

$$\begin{bmatrix} \dot{y}_1' \\ \dot{y}_2' \end{bmatrix} = \begin{bmatrix} -0.01 & 0.02 \\ 0.01 & -0.01 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -0.01 & 0.02 \\ 0.01 & -0.01 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -0.01 - \lambda & 0.02 \\ 0.01 & -0.01 - \lambda \end{bmatrix}$$

$$\det |A - \lambda I| = \lambda^2 + 0.03\lambda + 0.0007 -$$

$$0.0007$$

$$\lambda^2 + 0.03\lambda = 0$$

$$\lambda(\lambda + 0.03) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = -0.03$$

~~Method~~

$$\begin{bmatrix} 0.01 - \lambda & 0.02 \\ 0.01 & -0.01 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-0.01 - \lambda)x_1 + 0.02x_2 = 0$$

$$0.01x_1 + (-0.01 - \lambda)x_2 = 0$$

$$\lambda = 0$$

$$-0.01x_1 + 0.02x_2 = 0$$

$$0.01x_1 - 0.01x_2 = 0$$

$$x_1 = 1 \quad x_2 = 0.5$$

$$\lambda_1 = -0.03$$

$$0.01x_1 + 0.01x_2 = 0$$

$$0.01x_1 + 0.01x_2 = 0$$

$$x_1 = 1 \quad x_2 = -1$$

4. [6 points] Find the particular solution of the following system of ODEs, if its homogeneous solution is as written below:

$$\begin{aligned} \dot{y}_1 &= 2y_1 + 2y_2 + e^t \\ \dot{y}_2 &= -2y_1 - 3y_2 + e^t \end{aligned} \quad \boxed{\text{14}}$$

$$y_h = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^t$$

$$y^p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t e^t$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$$

$$y' = A y + g$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^t = \begin{bmatrix} 2 & 2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} 2 & 2 \\ -2 & -3 \end{bmatrix} t e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2a_1 + 2a_2 \\ -2a_1 - 3a_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$a_1 + b_1 = 2a_1 + 2a_2 + 1$$

$$a_2 + b_2 = -2a_1 - 3a_2 + 1$$

$$-a_1 - 2a_2 + b = 1$$

$$2a_1 + 4a_2 + b = 1$$

$$2b_1 = 2$$

$$b = 1$$

$$3b = 3$$

$$\boxed{b = 1}$$

$$\boxed{a_1 = 1}$$

$$\boxed{a_2 = -\frac{1}{2}}$$

$$y^p = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^t$$