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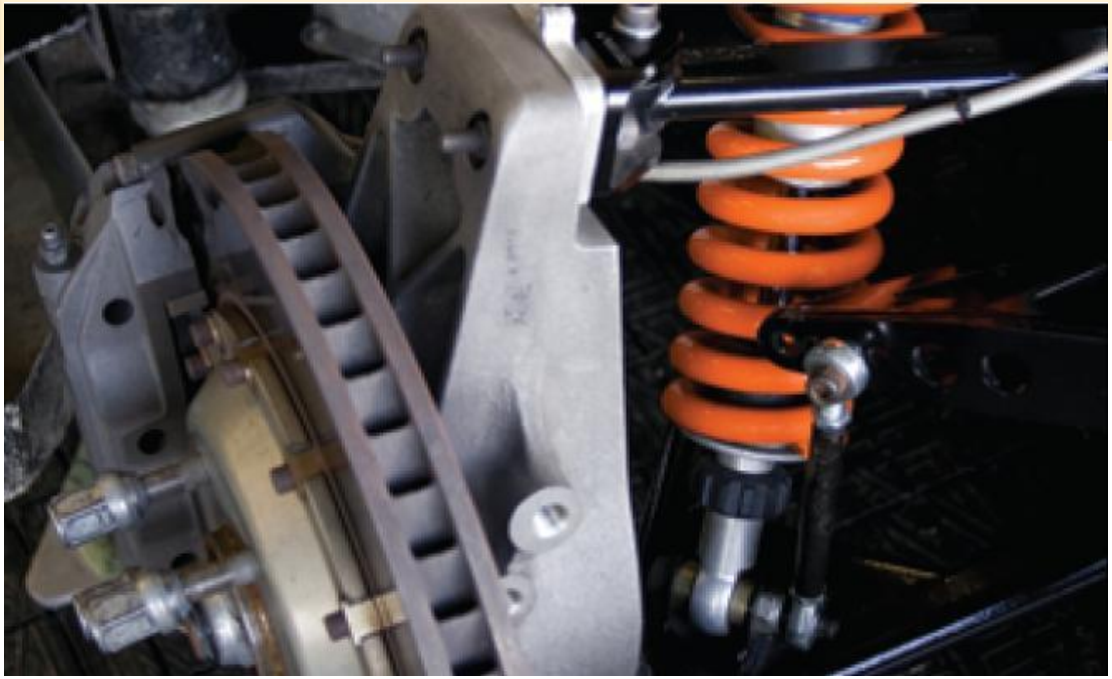
Second-Order Differential Equations



17.4 Series Solutions

Series Solutions

Many differential equations can't be solved explicitly in terms of finite combinations of simple familiar functions.

This is true even for a simple-looking equation like

$$y'' - 2xy' + y = 0 \tag{1}$$

But it is important to be able to solve equations such as Equation 1 because they arise from physical problems and, in particular, in connection with the Schrödinger equation in quantum mechanics.

Series Solutions

In such a case we use the method of power series; that is, we look for a solution of the form

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

The method is to substitute this expression into the differential equation and determine the values of the coefficients $c_0, c_1, c_2, \ldots$. This technique resembles the method of undetermined coefficients.

Series Solutions

Before using power series to solve Equation 1, we illustrate the method on the simpler equation $y'' + y = 0$ in Example1.

But it's easier to understand the power series method when it is applied to this simpler equation.

Example 1

Use power series to solve the equation $y'' + y = 0$.

Solution:

We assume there is a solution of the form

$$\boxed{2} \qquad y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

We can differentiate power series term by term, so

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

Example 1 – *Solution*

cont'd

3 $$y'' = 2c_2 + 2 \cdot 3c_3x + \cdots = \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2}$$

In order to compare the expressions for y and y'' more easily, we rewrite y'' as follows:

4 $$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$$

Substituting the expressions in Equations 2 and 4 into the differential equation, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} c_nx^n = 0$$

Example 1 – *Solution*

cont'd

or

5 $$\sum_{n=0}^{\infty} [(n + 2)(n + 1)c_{n+2} + c_n]x^n = 0$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore the coefficients of x^n in Equation 5 must be 0:

$$(n + 2)(n + 1)c_{n+2} + c_n = 0$$

6 $$c_{n+2} = -\frac{c_n}{(n+1)(n+2)} \qquad n = 0, 1, 2, 3, \ldots$$

Example 1 – *Solution*

cont'd

Equation 6 is called a *recursion relation.* If c_0 and c_1 are known, this equation allows us to determine the remaining coefficients recursively by putting $n = 0, 1, 2, 3, \ldots$ in succession.

Put $n = 0$: $$c_2 = -\frac{c_0}{1 \cdot 2}$$

Put $n = 1$: $$c_3 = -\frac{c_1}{2 \cdot 3}$$

Put $n = 2$: $$c_4 = -\frac{c_2}{3 \cdot 4} = \frac{c_0}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{c_0}{4!}$$

Example 1 – *Solution*

cont'd

Put n = 3: $$c_5 = -\frac{c_3}{4 \cdot 5} = \frac{c_1}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{c_1}{5!}$$

Put n = 4: $$c_6 = -\frac{c_4}{5 \cdot 6} = -\frac{c_0}{4!\,5 \cdot 6} = -\frac{c_0}{6!}$$

Put n = 5: $$c_7 = -\frac{c_5}{6 \cdot 7} = -\frac{c_1}{5!\,6 \cdot 7} = -\frac{c_1}{7!}$$

Example 1 – *Solution*

cont'd

By now we see the pattern:

For the even coefficients, $c_{2n} = (-1)^n \dfrac{c_0}{(2n)!}$

For the odd coefficients, $c_{2n+1} = (-1)^n \dfrac{c_1}{(2n + 1)!}$

Putting these values back into Equation 2, we write the solution as

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

Example 1 – *Solution*

cont'd

$$= c_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots\right)$$

$$+ c_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$$

$$= c_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Notice that there are two arbitrary constants, c_0 and c_1.

Series Solutions

Note 1:

We recognize the series obtained in Example 1 as being the Maclaurin series for cos x and sin x. Therefore we could write the solution as

$$y(x) = c_0 \cos x + c_1 \sin x$$

But we are not usually able to express power series solutions of differential equations in terms of known functions.

Example 2

Solve $y'' - 2xy' + y = 0$.

Solution:

We assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

as in Example 1.

Example 2 – *Solution*

cont'd

Substituting in the differential equation, we get

$$\sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2}x^n - 2x \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2}x^n - \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n + 2)(n + 1)c_{n+2} - (2n - 1)c_n]x^n = 0$$

Example 2 – *Solution*

cont'd

This equation is true if the coefficient of x^n is 0:

$$(n + 2)(n + 1)c_{n+2} - (2n - 1)c_n = 0$$

7 $$c_{n+2} = \frac{2n - 1}{(n + 1)(n + 2)} c_n \qquad n = 0, 1, 2, 3, \ldots$$

We solve this recursion relation by putting $n = 0, 1, 2, 3, \ldots$ successively in Equation 7:

Put $n = 0$: $$c_2 = \frac{-1}{1 \cdot 2} c_0$$

Example 2 – *Solution*

cont'd

Put n = 1: $$c_3 = \frac{1}{2 \cdot 3} c_1$$

Put n = 2: $$c_4 = \frac{3}{3 \cdot 4} c_2 = -\frac{3}{1 \cdot 2 \cdot 3 \cdot 4} c_0 = -\frac{3}{4!} c_0$$

Put n = 3: $$c_5 = \frac{5}{4 \cdot 5} c_3 = \frac{1 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5} c_1 = \frac{1 \cdot 5}{5!} c_1$$

Put n = 4: $$c_6 = \frac{7}{5 \cdot 6} c_4 = -\frac{3 \cdot 7}{4!\, 5 \cdot 6} c_0 = -\frac{3 \cdot 7}{6!} c_0$$

Example 2 – *Solution*

cont'd

Put $n = 5$: $$c_7 = \frac{9}{6 \cdot 7} c_5 = \frac{1 \cdot 5 \cdot 9}{5!\, 6 \cdot 7} c_1 = \frac{1 \cdot 5 \cdot 9}{7!} c_1$$

Put $n = 6$: $$c_8 = \frac{11}{7 \cdot 8} c_6 = -\frac{3 \cdot 7 \cdot 11}{8!} c_0$$

Put $n = 7$: $$c_9 = \frac{13}{8 \cdot 9} c_7 = \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!} c_1$$

Example 2 – *Solution*

cont'd

In general, the even coefficients are given by

$$c_{2n} = -\frac{3 \cdot 7 \cdot 11 \cdot \cdots \cdot (4n - 5)}{(2n)!} c_0$$

and the odd coefficients are given by

$$c_{2n+1} = \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n - 3)}{(2n + 1)!} c_1$$

The solution is

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

Example 2 – *Solution*

cont'd

$$= c_0\left(1 - \frac{1}{2!}x^2 - \frac{3}{4!}x^4 - \frac{3 \cdot 7}{6!}x^6 - \frac{3 \cdot 7 \cdot 11}{8!}x^8 - \cdots\right)$$

$$+ c_1\left(x + \frac{1}{3!}x^3 + \frac{1 \cdot 5}{5!}x^5 + \frac{1 \cdot 5 \cdot 9}{7!}x^7 + \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!}x^9 + \cdots\right)$$

or

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$$y = c_0\left(1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)}{(2n)!}x^{2n}\right)$$

$$+ c_1\left(x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!}x^{2n+1}\right)$$

Series Solutions

Note 2:

In Example 2 we had to *assume* that the differential equation had a series solution. But now we could verify directly that the function given by Equation 8 is indeed a solution.

Note 3:

Unlike the situation of Example 1, the power series that arise in the solution of Example 2 do not define elementary functions.

Series Solutions

The functions

$$y_1(x) = 1 - \frac{1}{2!}x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)}{(2n)!} x^{2n}$$

and

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} x^{2n+1}$$

are perfectly good functions but they can’t be expressed in terms of familiar functions.

We can use these power series expressions for y_1 and y_2 to compute approximate values of the functions and even to graph them.

Series Solutions

Figure 1 shows the first few partial sums T_0, T_2, T_4, . . . (Taylor polynomials) for $y_1(x)$, and we see how they converge to y_1.

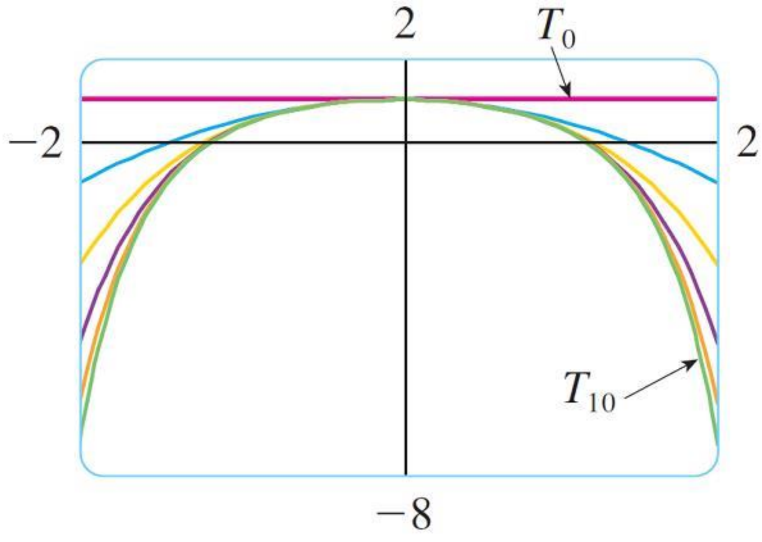

Figure 1

Series Solutions

In this way we can graph both y_1 and y_2 in Figure 2.

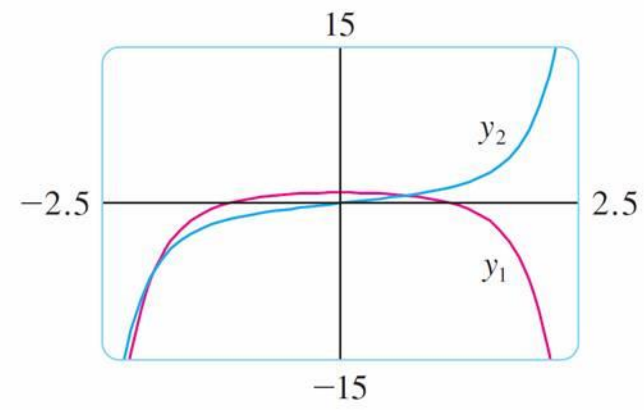

Figure 2

Note 4:
If we were asked to solve the initial-value problem

$$y'' - 2xy' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 1$$

Series Solutions

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \qquad |x - a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

we would observe from this Theorem that

$$c_0 = y(0) = 0 \qquad c_1 = y'(0) = 1$$

Series Solutions

This would simplify the calculations in Example 2, since all of the even coefficients would be 0. The solution to the initial-value problem is

$$y(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n - 3)}{(2n + 1)!} x^{2n+1}$$