University of Jordan

Faculty of Engineering and Technology
Chemical Engineering Department
0905301 Numerical Methods in Chemical Engineering

Chapter 3

Systems of Linear and Non-Linear Equations

We deal with the case of determining the values $x_1, x_2, \ldots\ldots, x_n$ that simultaneously satisfy a set of equations

$$f_1(x_1, x_2, \ldots\ldots\ldots\ldots, x_n) = 0$$
$$f_2(x_1, x_2, \ldots\ldots\ldots\ldots, x_n) = 0$$
$$f_3(x_1, x_2, \ldots\ldots\ldots\ldots, x_n) = 0$$
$$\bullet$$
$$\bullet$$
$$f_n(x_1, x_2, \ldots\ldots\ldots\ldots, x_n) = 0$$

Such systems can be **linear** or **nonlinear**.

Linear Systems:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots\ldots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots\ldots + a_{2n}x_n = c_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots\ldots\ldots + a_{3n}x_n = c_3$$
$$\bullet$$
$$\bullet$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots\ldots\ldots + a_{nn}x_n = c_n$$

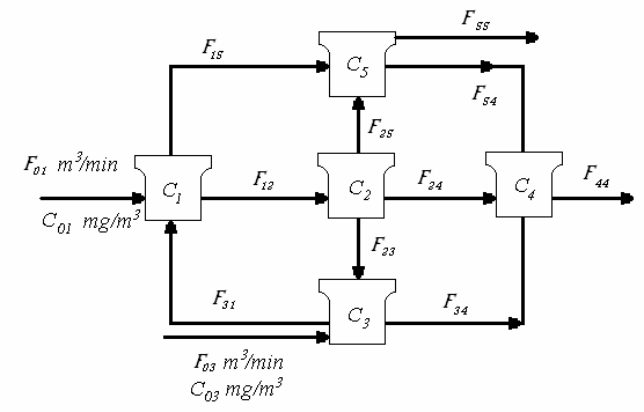
Where: *a's* are the coefficients.
c's are constants.
n is the number of equations.

Nonlinear Systems:

Algebraic and transcendental equations that do not fit the above form are called nonlinear equation.

$$x^2 + xy = 10$$
$$y + 3xy^2 = 57$$

Application of System of Linear Equations



- This figure shows a series of chemical reactors. We need to know the concentration in each of the five reactors at steady state.
- There are 5 unknowns [c_1, c_2, c_3, c_4, and c_5]. 5 equations are required to obtain the concentration in each reactor.

$$F_{01}C_{01} + F_{03}C_{03} = F_{55}C_5 + F_{44}C_4$$
$$F_{01}C_{01} + F_{31}C_3 = F_{12}C_1 + F_{15}C_1$$
$$F_{24}C_2 + F_{54}C_5 + F_{34}C_3 = F_{44}C_4$$
$$F_{03}C_{03} + F_{23}C_2 = F_{34}C_3 + F_{31}C_3$$
$$F_{15}C_1 + F_{25}C_2 = F_{55}C_5 + F_{54}C_5$$

- These 5 equations can be solved simultaneously to find the unknown concentrations. In this chapter we will see how we solve such systems using numerical techniques.

Matrix Notation

A matrix consists of a rectangular array of elements represented by a single symbol.

- *[A]* is the standard notation for the matrix.
- a_{ij} is designated an individual element of the matrix.

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13}.....a_{1n} \\ a_{21} & a_{22} & a_{23}.....a_{2n} \\ a_{31} & a_{32} & a_{33}.....a_{3n} \\ \bullet & & \\ \bullet & & \\ a_{m1} & a_{m2} & a_{m3}.....a_{mn} \end{bmatrix}$$

m is the number of rows.
n is the number of columns.
If *m* = *n*, we have **square matrix**.

- Matrices with row dimension *m = 1* are called **row vector**.

$$B_{1\times n} = [b_1 \quad b_2 \quad b_3.....b_n]$$

- Matrices with column dimension *n = 1* are called **column vector**.

$$C_{m\times 1} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \bullet \\ \bullet \\ c_m \end{bmatrix}$$

1) Two matrices A_{mn} and B_{kl} can be **added** to each other (or **subtracted** from each other) if they have the same number of rows ($m = k$) and the same number of columns ($n = l$).

$$A_{mn} + B_{kl} = C_{mn \quad or \quad kl}$$

2) Two matrices can be **multiplied** with each other, if they are **comfortable** ($n = k$)

$$A_{mn} \cdot B_{kl} = C_{ml}$$

Assuming A and B are square and of the same order:

$$A_{mn} \cdot B_{kl} \neq B_{kl} \cdot A_{mn}$$

3) The **distribution law** for multiplication applies to matrices

$$A(B + C) = AB + AC$$

4) The **associative law** of multiplication is also valid.

$$A(BC) = (AB)C$$

5) If the rows of an (mxn) matrix are written as columns, a new matrix of order (nxm) is formed. This new matrix is called the **transpose** of the original matrix.
Transpose of the product of two matrices is given by

$$(AB)^T = B^T A^T$$
$$(A + B)^T = A^T + B^T$$

A symmetric matrix is one that obeys the equation:

$$A = A^T$$

6) **Diagonal matrix** is one with nonzero elements on the principal diagonal and zero elements every where else.

$$D = \begin{bmatrix} d_{11} & 0 & 0.....0 \\ 0 & d_{22} & 0.....0 \\ 0 & 0 & d_{33}.....0 \\ \bullet & & \\ \bullet & & \\ 0 & 0 & 0.....d_{nn} \end{bmatrix}$$

7) A **unit matrix** (or identity) matrix is a diagonal matrix whose elements are unity.

$$I = \begin{bmatrix} 1 & 0 & 0.....0 \\ 0 & 1 & 0.....0 \\ 0 & 0 & 1.....0 \\ \bullet & & \\ \bullet & & \\ 0 & 0 & 0.....1 \end{bmatrix}$$

8) **Symmetric matrix** is a matrix which has a mirror image about its diagonal.

$$[A] = \begin{bmatrix} 5 & 1 & 2 & 6 \\ 1 & 3 & 7 & 9 \\ 2 & 7 & 8 & 4 \\ 6 & 9 & 4 & 11 \end{bmatrix}$$

9) **Banded matrix** has all elements equal to zero, with the exception of a band centered on the main diagonal.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}$$

10) An **upper triangular matrix** is one that has all zero elements below the principal diagonal.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

A **lower triangular matrix** is one where all elements above the main diagonal are zero.

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{31} & a_{32} & a_{33} & a_{3n} \end{bmatrix}$$

11) **Singular matrix** is a matrix which has a determinant equal to zero, [*det.= 0*]
Nonsingular matrix is a matrix which has a determinant other than zero, [*det.* $\neq 0$]

12) If a matrix [*A*] is square and **nonsingular**, there is another matrix $[A]^{-1}$ called the inverse of [*A*].

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

13) **Determinant** exists for **square** matrices only:
- If all elements of any row or column of a matrix are zero then *det = 0.*
- If the corresponding rows and columns of a matrix are interchanged, its *det* is unchanged.

- If two rows or two columns of a matrix are interchanged the sign of the *det* changes.
- If the elements of two rows or two columns of a matrix are equal, the *det* of the matrix is zero.
- If the elements of any row or column of a matrix are multiplied by a scalar, this is equivalent to multiplying the *det* by a scalar.

14) **Augmented matrix** is useful because several of the techniques for solving linear systems perform identical operations on a row of coefficients and the corresponding right-hand side constants.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \mathrm{b} = \begin{Bmatrix} \mathrm{c}_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

$$A|B = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & \mathrm{c}_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$$

15) **Rank of matrix** *A* is defined as the order of the largest nonsingular square matrix within *A*. Consider

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13}.....a_{1n} \\ a_{21} & a_{22} & a_{23}.....a_{2n} \\ a_{31} & a_{32} & a_{33}.....a_{3n} \\ \bullet & & \\ \bullet & & \\ a_{m1} & a_{m2} & a_{m3}.....a_{mn} \end{bmatrix}$$

Where $n \geq m$.

- The largest square sub-matrix within *A* of order (*m*xm).
- If the *det* of (*mxm*) $\neq 0$, the rank of *A* is *m* ($r = m$).
- If *det* of (*mxm*) $= 0$, the rank of *A* is less than *m* ($r < m$)

System of Linear Equations:

Solving Small Number of Equations ($n \leq 3$)

1) Graphical method

- For **two equations**: by plotting them on Cartesian coordinate with one axis corresponding to x_1 and the other to x_2. The point where the two lines intersect would represent the solution.
- For **three simultaneous equations**: each equation would be represented by a plane in a three dimensional coordinate system. The point where the three planes intersect would represent the solution.
- **Beyond** three equations, graphical method breaks down.

2) Elimination of unknowns.

3) Cramer's rule

This rule states that each unknown in a system of linear algebraic equations may be expressed as a function of two determinants with denominator *D* and with numerator obtained from *D* by replacing the column of coefficient of the unknown in question by the constants $c_1, c_2, \ldots\ldots\ldots, c_n$.

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}}{\det}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}}{\det}, \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}}{\det}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Singular and Ill-Conditioned Systems

- If the system is singular (*det = 0)* there will be no solution or there will be an infinite number of solutions.
- Systems that are very close to being singular ($det \approx 0$) can cause problems.
- These systems are said to be ill-conditioned.
  - It is difficult to identify the exact point which the lines intersect at.
  - They will be extremely sensitive to round-off errors.

Solving Large Number of Equations ($n \geq 3$)

Direct Methods	Iterative Methods
Gaussian Elimination	Jacobi
Thomas Method	Gauss-Seidel
Gauss Jordan	
LU decomposition	

Direct Methods

1) Gaussian Elimination

The **two phases** of Gaussian elimination are:

Forward elimination

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right] \downarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ 0 & a'_{22} & a'_{23} & c'_2 \\ 0 & 0 & a''_{33} & c''_3 \end{array}\right]$$

Back substitution

$$x_3 = \frac{c_3''}{a_{33}''}$$

$$x_2 = \frac{c_2' - a_{23}' x_3}{a_{22}'}$$

$$x_1 = \frac{c_1 - a_{13} x_3 - a_{12} x_2}{a_{11}}$$

Objections that we should eliminate:

1) If there is **zero in the pivot equation** and the pivot itself is zero, (division by zero).
2) **Round-off errors**: this is due to the fact that every result is dependent on previous results.
3) **Ill-conditioned** systems
   - ***Well-conditioned*** systems are those where a small change in one or more of the coefficients results in a similar small change in the solution.
   - ***Ill-conditioned*** systems are those where small changes in coefficients results in large changes in the solution.

Gaussian elimination with row pivoting

1) The method fails if the pivot element at any stage of the elimination is zero.
2) When the pivot element is close to zero because if the magnitude of the pivot element is small compared to the other elements, then round-off errors can be introduced.
3) Before each row is normalized, it is advantageous to determine the largest available coefficient. The rows can then be switched so that the largest element is the pivot element.

Scaling

- Scaling is the operation of adjusting the coefficients of a set of equations so that they are all of the same order of magnitude.
- In some instances, a set of equations may involve relationships between quantities measured in widely different units. This may result in some of the equations having very large numbers and other very small.

2) Thomas Method

- This method is used for **tridiagonal** (banded) systems
- In many applications, the linear system to be solved has a banded structure.
- Suppose that you have the following system:

$$
\begin{aligned}
d_1 x_1 + a_1 x_2 &= r_1 \\
b_2 x_1 + d_2 x_2 + a_2 x_3 &= r_2 \\
b_3 x_2 + d_3 x_3 + a_3 x_4 &= r_3 \\
&\bullet \\
&\bullet \\
b_{n-1} x_{n-2} + d_{n-1} x_{n-1} + a_{n-1} x_n &= r_{n-1} \\
b_n x_{n-1} + d_n x_n &= r_n
\end{aligned}
$$

1) For the first equation: $A_1 = \frac{a_1}{d_1}, \qquad R_1 = \frac{r_1}{d_1}$
2) For each of the equations from *i = 2* to *n-1*:

$$A_i = \frac{a_i}{d_i - b_i A_{i-1}}, R_i = \frac{r_i - b_i R_{i-1}}{d_i - b_i A_{i-1}}$$

3) For the last equation

$$R_n = \frac{r_n - b_n R_{n-1}}{d_n - b_n A_{n-1}}$$

4) Solve by back substitution

$$x_n = R_n$$
$$x_i = R_i - A_i x_{i+1}, \; i = n-1, n-2, ..., 2, 1$$

Note: Thomas algorithm requires that $d_1 \neq 0$ and that $d_i - b_i A_{i-1} \neq 0$ for each *i*.

3) Gauss-Jordan

- Gauss-Jordan is very similar to Gauss elimination.
- When an unknown is eliminated in the Gauss Jordan method, it is eliminated from all other equations rather than just the subsequent ones.
- All rows are normalized by dividing them by their pivot elements.
- The elimination step results in an identity matrix rather than triangular matrix.
- Not necessary to employ back substitution to obtain the solution.

$$
\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right] \xrightarrow{\text{elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1^n \\ 0 & 1 & 0 & c_2^n \\ 0 & 0 & 1 & c_3^n \end{array}\right] \xrightarrow{\text{Solve}} \left\{\begin{array}{l} x_1 = c_1^n \\ x_2 = c_2^n \\ x_3 = c_3^n \end{array}\right\}
$$

- c_i^n means that the element of the right hand side vector have been modified n times, (for the above matrix 3 times).
- Gauss Jordan provides a straightforward method for obtaining the matrix inverse.
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Matrix Inversion by Gauss Jordan

- The coefficient matrix is augmented with an identity matrix.
- Then the Gauss-Jordan is applied in order to reduce the coefficient matrix to an identity matrix.

- When this is accomplished, the right hand side of the augmented matrix will contain the inverse.

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{elimination}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a_{11}^{-1} & a_{12}^{-1} & a_{13}^{-1} \\ 0 & 1 & 0 & a_{21}^{-1} & a_{22}^{-1} & a_{23}^{-1} \\ 0 & 0 & 1 & a_{31}^{-1} & a_{32}^{-1} & a_{33}^{-1} \end{array}\right]$$

$$[\mathrm{A}] \qquad [I] \qquad\qquad [I] \qquad [A]^{-1}$$

The superscript $^{-1}$'s denote that the original values have been converted to the matrix inverse.

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

$$[A]\{x\} = \{c\} \xrightarrow{\text{To solve}} \{x\} = \frac{\{c\}}{[A]} = [A]^{-1}\{c\}$$

4) LU Factorization

- LU decomposition method can be used to find a solution for systems of linear equations.
- LU decomposition method can be used to find the determinant large matrices.

There are 3 steps to find a solution for the system:

Step1. Decomposition of the original matrix

$$[A]\{x\} = \{c\}$$
$$[A] = [L][U]$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}, \quad [U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{32} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

Step2. Using the forward substitution solve the following equation to find *{D}*.

$$[L]\{D\} = \{c\}$$

Step3. Using the backward substitution solve the following equation to get the solution of the system *{x}*.

$$[U]\{x\} = \{D\}$$

Step4. Determinant of the original matrix *Det [A]* = $u_{11}u_{22}u_{33}u_{44}$

LU decomposition method employs fewer operations to find the inverse of a matrix compared with Gauss-Jordan.

- To find an inverse of a matrix do the same steps as above but instead of *{c}* put *{I}*.

$$[A]\{x\} = \{I\}$$
$$[A] = [L][U]$$

- The resulted {x} is the inverse of the original matrix.

Iterative Methods

- Systems of linear equations for which numerical solutions are needed are very large, making the computational effort of general, direct methods, such as Gauss elimination, expensive.
- For systems that have coefficient matrices with the appropriate structure (large, spare system) iterative techniques may be preferable.

1) Gauss-Seidel

- Most commonly used iterative method.
- Assume that we are given a set of n equations:

$$[A]\cdot\{x\}=\{c\}$$

- If the diagonal elements are all nonzero, the first equation can be solved for x_1, the second for x_2, and so on.

Procedure:

1) Rearrange the equation in the following form:

$$x_1=\frac{c_1-a_{12}x_2-a_{13}x_3-.....-a_{1n}x_n}{a_{11}}$$

$$x_2=\frac{c_2-a_{21}x_1-a_{23}x_3-.....-a_{2n}x_n}{a_{22}}$$

2) $$x_3=\frac{c_3-a_{31}x_1-a_{32}x_2-.....-a_{3n}x_n}{a_{33}}$$

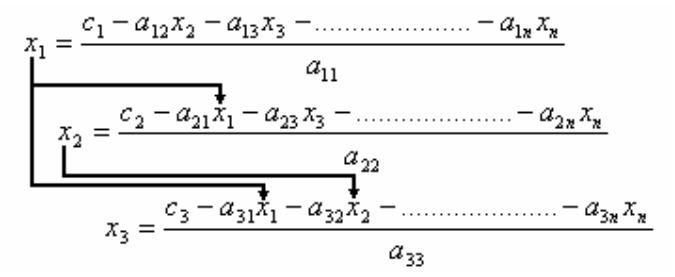
-
-

$$x_n=\frac{c_n-a_{n1}x_1-a_{n2}x_2-.....-a_{n,n-1}x_{n-1}}{a_{nn}}$$

3) Start the solution process by using guesses for the *x's*. If there are no guesses available assume that they are all zero.
4) Substitute the guesses in the above equation to obtain x_1.
5) Then substitute the new estimated x_1 along with the previous guesses for x_3, x_4, ……, x_n into equation above to compute a new value of x_2.
6) The process is repeated for each of the equations until x_n.
7) Convergence can be checked using the criterion

$$\left|\frac{x_i^j-x_i^{j-1}}{x_i^j}\right|*100\%\langle\varepsilon$$

for all *i*, where *j* and *j-1* are the present and previous iterations.



The diagonal element must be greater than the off-diagonal element for each row:

$$|a_{ii}| \rangle \sum_{\substack{j=1 \\ j \neq j}}^{n} |a_{ij}|$$

2) Jacobi

This method is similar to Gauss Seidel except that as new values for the variables are generated, they are not immediately used but rather are retained for the next iteration.

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$
$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$
$$x_3 = \frac{c_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

To second iteration

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$
$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$
$$x_3 = \frac{c_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

To third iteration

System of Nonlinear Equations:

$$f_1(x, y, z.....,m) = 0$$
$$f_2(x, y, z.....,m) = 0$$
$$f_3(x, y, z.....,m) = 0$$
-
-
$$f_n(x, y, z.....,m) = 0$$

1) Newton's Method

To solve a system of nonlinear equations using Newton's method follow the following procedure:

Step1. Find the Jacobian matrix (matrix of partial derivatives)

$$J(x, y, z, \ldots\ldots, m) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \ldots\ldots\ldots & \frac{\partial f_1}{\partial m} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \ldots\ldots\ldots & \frac{\partial f_2}{\partial m} \\ \bullet & & & & \\ \bullet & & & & \\ \frac{\partial f_n}{\partial x} & \frac{\partial f_n}{\partial y} & \frac{\partial f_n}{\partial z} & \ldots\ldots\ldots & \frac{\partial f_n}{\partial m} \end{bmatrix}$$

Step2. Using Gaussian elimination solve to find $\Delta x, \Delta y, \Delta z, \ldots\ldots\ldots, \Delta m$.

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_i} & \frac{\partial f_1}{\partial y_i} & \frac{\partial f_1}{\partial z_i} & \ldots\ldots\ldots & \frac{\partial f_1}{\partial m_i} \\ \frac{\partial f_2}{\partial x_i} & \frac{\partial f_2}{\partial y_i} & \frac{\partial f_2}{\partial z_i} & \ldots\ldots\ldots & \frac{\partial f_2}{\partial m_i} \\ \bullet & & & & \\ \bullet & & & & \\ \frac{\partial f_n}{\partial x_i} & \frac{\partial f_n}{\partial y_i} & \frac{\partial f_n}{\partial z_i} & \ldots\ldots\ldots & \frac{\partial f_n}{\partial m_i} \end{bmatrix} \begin{Bmatrix} \Delta x_i \\ \Delta y_i \\ \bullet \\ \bullet \\ \Delta m_i \end{Bmatrix} = - \begin{Bmatrix} f_1(x_i, y_i, z_i, \ldots\ldots, m_i) \\ f_2(x_i, y_i, z_i, \ldots\ldots, m_i) \\ \\ \\ f_n(x_i, y_i, z_i, \ldots\ldots, m_i) \end{Bmatrix}$$

Step3. Correct the values of the variables according to:

$$x_{i+1} = x_i + \Delta x_i$$
$$y_{i+1} = y_i + \Delta y_i$$
$$\bullet$$
$$m_{i+1} = m_i + \Delta m_i$$

Repeat the procedure until you get an acceptable error.

2) Fixed-Points

Suppose that we a system of 3 nonlinear simultaneous equations

$$f_1(x, y, z) = 0$$
$$f_2(x, y, z) = 0$$
$$f_3(x, y, z) = 0$$

- A fixed point for such a system can be rearranged as follows:

$$x = g_1(x, y, z)$$
$$y = g_2(x, y, z)$$
$$z = g_3(x, y, z)$$

- Iterate until you get the desired tolerance.

$$x_{i+1} = g_1(x_i, y_i, z_i)$$
$$y_{i+1} = g_2(x_i, y_i, z_i)$$
$$z_{i+1} = g_3(x_i, y_i, z_i)$$

Note: Use the following equations to check the convergence of the arrangements after finish the first iteration:

$$\left|\frac{\partial g_1}{\partial x}(x_1, y_1, z_1)\right| + \left|\frac{\partial g_1}{\partial y}(x_1, y_1, z_1)\right| + \left|\frac{\partial g_1}{\partial z}(x_1, y_1, z_1)\right| < 1$$

$$\left|\frac{\partial g_2}{\partial x}(x_1, y_1, z_1)\right| + \left|\frac{\partial g_2}{\partial y}(x_1, y_1, z_1)\right| + \left|\frac{\partial g_2}{\partial z}(x_1, y_1, z_1)\right| < 1$$

$$\left|\frac{\partial g_3}{\partial x}(x_1, y_1, z_1)\right| + \left|\frac{\partial g_3}{\partial y}(x_1, y_1, z_1)\right| + \left|\frac{\partial g_3}{\partial z}(x_1, y_1, z_1)\right| < 1$$