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Numerical Methods

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Define Variables

not $K=2.6$ } Default $K=2$ Real $K=2.6$
 $L=3$ } $L=3$ $L=3$

Def: $\rightarrow X=4.2$ } $\leftarrow$ Real $X=4$ 0

For each variable, the computer reserves a word

We have to explain to the system whether the number is real or an integer

each bit in a system can reserve a number.

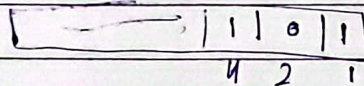
لا يمكن لكل البت أن يحفظ رقمًا عشوائيًا، بل يجب أن يكون له معنى

First bit is reserved for the Sign

In integer number, we move right to left in reserving places, (Binary)

$K=5$

$K=$

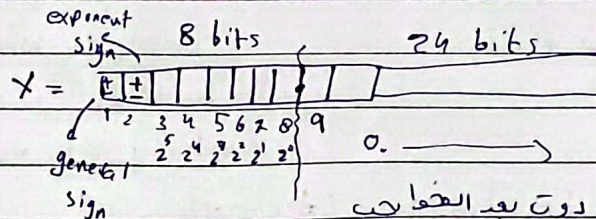


Real numbers

$$X = \pm q \times 2^n$$

conversion
of the number
to binary

(limitless)



أول دوت بعد الفاصلة
أنت كمان

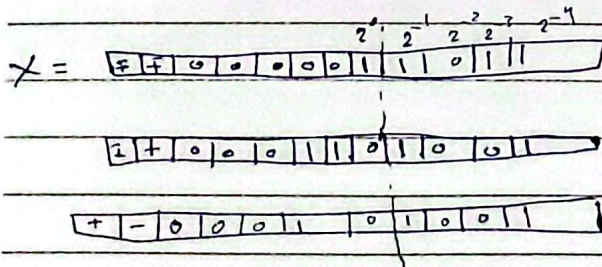
shift to the right 2^+

$$0.1011 \times 2^0$$

shift to the left 2^-

$$1.001 \rightarrow 0.1001 \times 2^1$$

$$0.01001 \rightarrow 0.001 \times 2^{-1}$$



$$0.1011 \times 2^0 \rightarrow \frac{1}{2} + 0 \times \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

Conversion from decimal numbers to Binary tertiary

(0,1,10) two → binary

N integer number

$$N = 2Q_0 + b_0 \quad \begin{cases} \text{if } Q_0 \text{ is even} \\ \text{if } Q_0 \text{ is odd} \end{cases}$$

$$Q_0 = 2Q_1 + b_1$$

$$Q_1 = 2Q_2 + b_2$$

Stop only if $Q_i = 0$

$$N = \leftarrow \begin{matrix} 2^3 & 2^2 & 2^1 & 2^0 \\ -b_3 & b_2 & b_1 & b_0 \end{matrix}$$

Example 23?

$$23 = 2 \times 11 + 1 \quad b_0$$

$$11 = 2 \times 5 + 1 \quad b_1$$

$$5 = 2 \times 2 + 1 \quad b_2$$

$$2 = 2 \times 1 + 0 \quad b_3$$

$$1 = 1 \times 0 + 1 \quad b_4$$

$$\begin{matrix} 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{matrix}$$

We change the base "2 for example" based on what system we want to use tertiary for example we use 3 and thus changing b_i values to "1,0,2"

Real to binary (decimal)

Real number N, R
 $N = \text{int}$ $R = \text{fraction}$

$$R = 0.d_1d_2d_3d_4d_5$$

$$2R = 0.1 \text{ fraction}$$

$$2F_1 = 0.1 \cdot F_2$$

$$2F_2 = 0.1 \cdot F_3$$

etc

$$\text{Real number} = b_1b_2b_3b_4d_1d_2d_3d_4$$

example | 3.5??

$$3 = 2 \times 1 + 1$$

$$1 = 2 \times 0 + 1$$

$$3 = \begin{matrix} 2^1 & 2^0 \\ 1 & 1 \end{matrix}$$

$$R = 0.5 \Rightarrow 2 \times R = 2 \times 0.5 = 1.0$$

$$2 \times F_1 = 2 \times 0 = 0.0$$

$$0.5 = 0.10000$$

$$3.5 = 11.1$$

$$\text{Real number} = \pm q \times 2^{\pm n}$$

$$3.5 = 11.1_{\text{two}} = 0.111 \times 2^{+2}$$

example | 16.75

base 3

$$16 = 3 \times 5 + 1$$

$$16 = \begin{matrix} 3^2 & 3^1 & 3^0 \\ 1 & 2 & 1 \end{matrix}$$

$$5 = 3 \times 1 + 2$$

$$1 = 3 \times 0 + 1$$

$$3 \times 0.75 = 2.25$$

$$0.75 = 0.2020 = 0.\overline{20}$$

$$3 \times 0.25 = 0.75$$

$$0.75 = 0.2020 = 0.\overline{20}$$

$$16.75 = 121.202020 \rightarrow \text{we normalise it}$$

$$\rightarrow = 0.12120 \times 3^3$$

$$\rightarrow = \begin{matrix} 2^2 & 2^1 & 2^0 & 2^{-1} & 2^{-2} & 2^{-3} & 2^{-4} & 2^{-5} & 2^{-6} & 2^{-7} \\ + & + & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 2 \end{matrix}$$

هذا هو الضرب والقسمة مراجعة ذاتية



operations:-

ex) $3 \rightarrow 11^2$

$5 \rightarrow 101^2$
8

1	1	1	1
0	0	1	1
0	1	0	1
1	1	1	0

$11 + 101 = 1000$

ex)

1	1	1	1	1	1
0	1	0	1	1	1
0	1	1	0	0	1
1	1	1	1	1	1

$\rightarrow 110000$

ex)

0	1	1
0	1	1
0	1	0

$\Rightarrow 10^2$

ex)

0	1	0	1	1
1	1	1	0	1
1	1	1	0	1
1	1	1	0	1
1	1	1	0	1

1	1	1	1	1
1	1	1	1	1
0	0	0	1	0

ex)

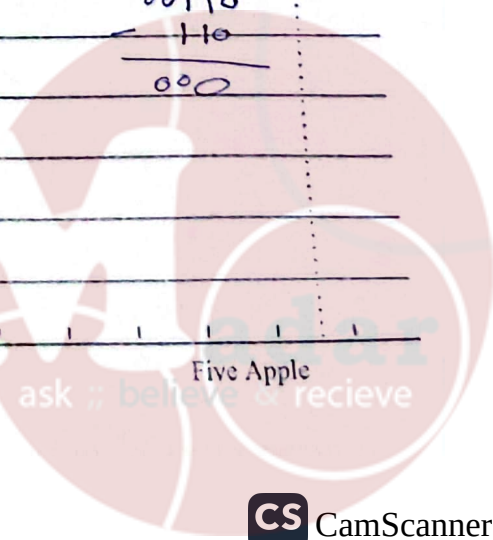
$\times 0100$
0011
0100
01000
01100

ex) $0101 \times$

1011
0101
01010
000000
0101000
0110111

0110
0111
10110
101100
011000
101010

111
1101010
- 1101
1101
00110
- 110
000



chapter 2

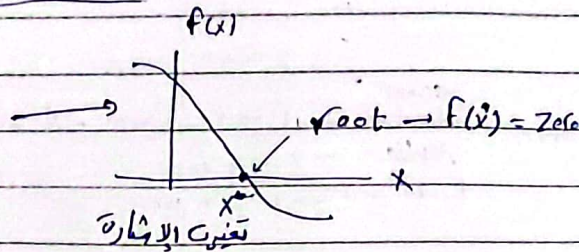
numerical

Singl Non-linear equation

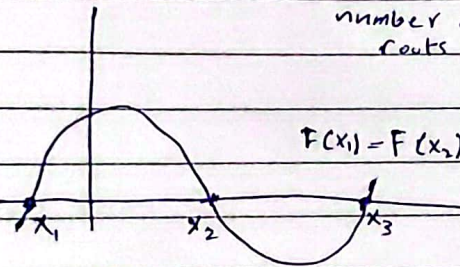
Solution = roots

$$f(x) = \text{---}$$

x	f(x)
1	



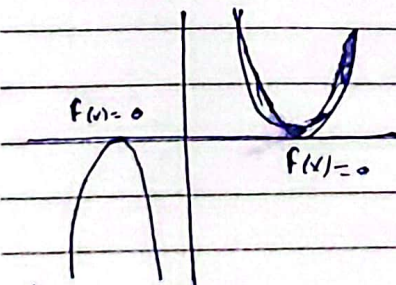
not even
number of
roots



$$f(x_1) = f(x_2) = f(x_3) = \text{Zero}$$

not all algorithms give the
same root

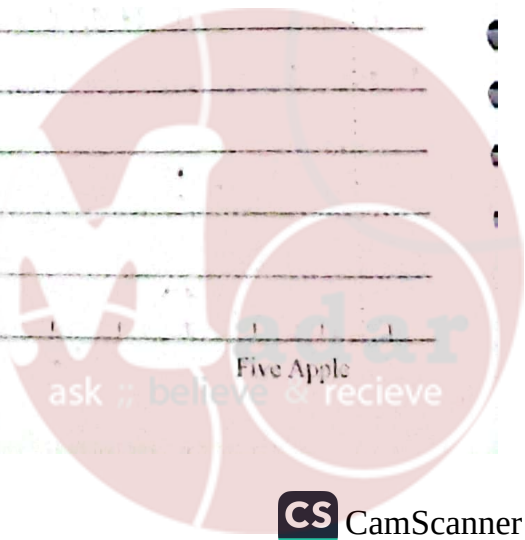
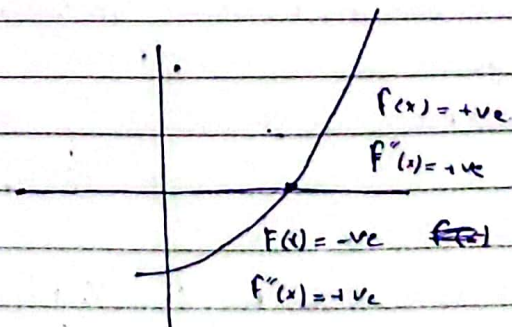
Some algorithms, the solution is
guarnteed while with some, it isn't

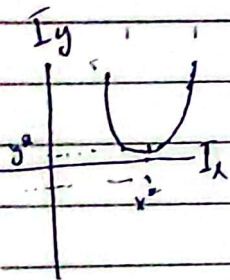


if concavity changes, it's not an
even number of roots

$f'(x) > 0$ $f(x) < 0$
even number of
roots

$$f(x) > 0 \quad f(x) > 0$$





$$X_{imag} = R \pm iI$$

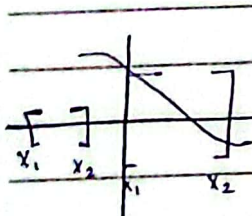
guarantee soln if you start with +ve and -ve

Methods
closed [bracketed]

open

bisection
False Position

secant → 2 initial guesses
Newton's → 1 initial guess → doesn't matter if before or after the root
Myller's → 3 initial guesses
Fixed point → 1 initial guess



① 2 initial guesses
 x_0, x_1

② $f(x_0)$ the sign $\neq$ sign $f(x_1)$

they're not suitable when we have an even number of roots

modified newton's method

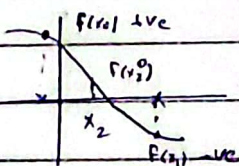
we use it when we have an even number of roots

① Bisection method:-

very slow and no functions are involved

2 initial guesses

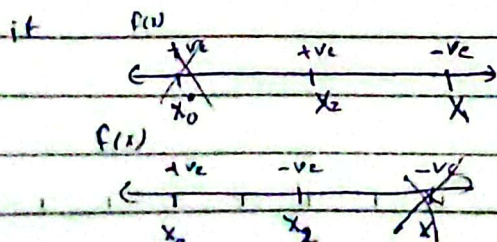
sign $f(x_0) \neq f(x_1)$



1st iteration $\rightarrow f(x) = 0$

the expected value of the root $x_2 = \frac{x_0 + x_1}{2}$

$f(x_2) = ?$, if $|f(x_2)| \leq 0.00001$, then x_2 is the root, otherwise it isn't
we do a 2nd iteration with x_2 and use an x of the opposite sign



② False position:-

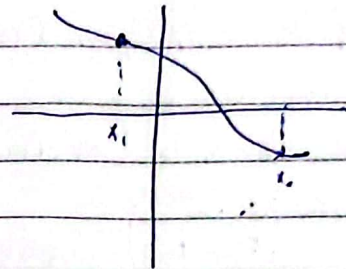
same conditions as previous method but a different way of calculation

★ x_0, x_1

★ opposite signs of $f(x_0)$ and $f(x_1)$

★ slope = $\frac{f(x_2) - f(x_0)}{x_2 - x_0}$

$$\text{slope} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



∴ $\frac{f(x_2) - f(x_0)}{x_2 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ with some rearrangement we find x_2

$$x_2 = x_0 - f(x_0) \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right)$$

example | $f(x) = 3x + \sin(x) - e^{+x}$

$$f(x_0) = 3x_0 + \sin(x_0) - e^{x_0} = -1 \text{ (-ve)}$$

$$f(x_1) = 3x_1 + \sin(x_1) - e^{x_1} = 1.23189 \text{ (+ve)}$$

1st iteration

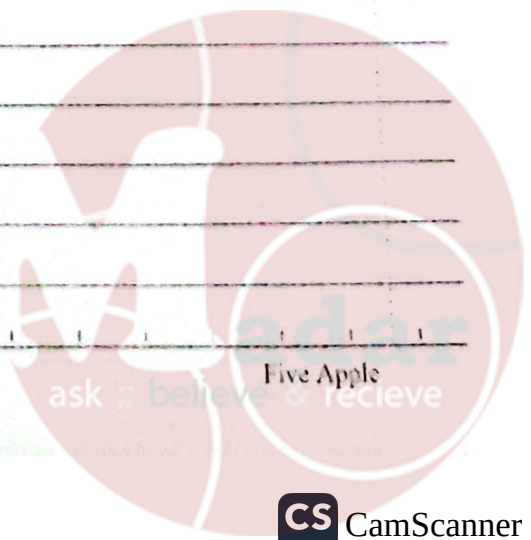
$$x_2 = \frac{0 + 1}{2} = 0.5 \quad f(x_2) = 1.37289$$

$$\text{2nd} = \frac{0.5 + 0}{2} = 0.25 \quad f(x_2) = 0.218603$$

False position:-

$$x_2 = x_0 - f(x_0) \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right) = 0 - (-1) \left(\frac{0 - 1}{-1 - 1.23189} \right) = 0.4709896$$

$$\text{2nd} = x_2 = 0 - (-1) \left(\frac{0 - 0.4709896}{-1 - 0.265158} \right)$$



open method:-

① Secant method

1- x_0, x_1

* $\text{Sign}(f(x_0)) = \text{sign}(f(x_1))$ and $\text{sign}(f(x_0)) \neq \text{sign}(f(x_1))$, it's OK

* No guarantee there will be a soln

* To proceed the abs value of $f(x_0) > |f(x_1)|$ and calculate x_2^0

$$x_2^0 = x_0 - f(x_0) \left(\frac{x_0 - x_1}{f(x_0) - f(x_1)} \right)$$

$F(x_2^0) = \epsilon \leq \text{T.p}$: x_2 is the root

$> \text{T.p}$: new iteration

for the new iterations

$$x_0^{\text{new}} = x_1$$

$$x_1^{\text{new}} = x_2^0$$

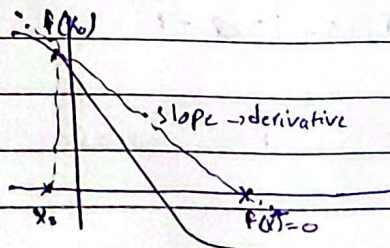
* if $|f(x_0)| < |f(x_1)|$, then swap the values

② Simple newton method

* Initial guess

$$f'(x) = \frac{f(x^0) - f(x_0)}{x^0 - x_0}$$

$$x^0 = x_0 - \frac{f(x_0)}{f'(x_0)}$$



* only works for single root and odd number of roots

Example:- Secant method

$$f(x) = 3x + \sin x - e^x$$

$$x_0 = 0$$

$$x_1 = -1$$

$$f(x_0) = f(0) = -1$$

$$f(1) = 1.123189$$

we have
to re-arrange
the values

$$x_1 = \frac{0}{-1} \quad x_0 = -1$$

$$f(0) = -1$$

$$f(-1) = 1.123189$$

$$x_2^0 = 1 - (1.123189) \left(\frac{1 - 0}{1.123189 - (-1)} \right) = 0.4209895$$

$$f(x_2^0) = |0.2651588| > 1 \times 10^{-4}$$

$$x_0 = 0, \quad x_1 = x_2^0$$

$$x_2^0 = 0 - (-1) \left(\frac{0 - 0.4209895}{1.123189 - (-1 - (0.2651588))} \right) \rightarrow \text{calculator}$$

example/ Newton method:-

$$f'(x) = 3 + \cos(x) - e^x$$

$$x_0 = 0$$

$$f(0) = -1 \quad f'(0) = 3$$

1st iteration:-

$$x_1^0 = x_0 - \frac{f(x_0)}{f'(x_0)} = - \frac{(-1)}{3} = 0.333$$

$$f(0.333) = |-0.068422| > 1 \times 10^{-4}$$

2nd iteration:-

$$x_0 = 0.333$$

$$f(x_0) = -0.068422$$

$$f'(x_0) = 2.549345$$

$$x_1^0 = 0.333 - \left(\frac{-0.068422}{2.549345} \right) = 0.360171$$

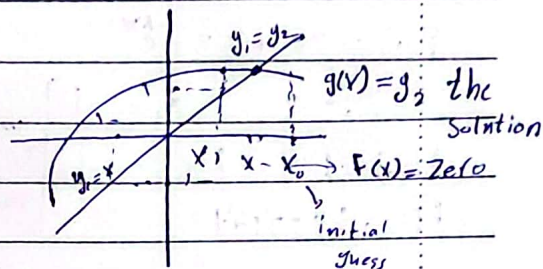
$$f(x_1^0) = -0.00063$$



⑤ Fixed point iteration

all other methods $f(x)=0$

fixed point $x = g(x)$
 $y_1 = x$ $y_2 = g(x)$



$$f(x) = x^2 - 2x - 3$$

$$① 2x = x^2 - 3 \Rightarrow x = \frac{x^2 - 3}{2} \rightarrow g_1(x)$$

$$② x^2 = 3 + 2x \Rightarrow x = \sqrt{3 + 2x} \rightarrow g_2(x)$$

$$③ x(x-2) = 3 \Rightarrow x = \frac{3}{x-2} \rightarrow g_3(x)$$

$$x = \sqrt{3 + 2x}$$

$$x_0 = 4$$

1st iteration

$$x_1^{(1)} = g_2(x_0) = \sqrt{3 + 2 \times 4} = 3.31662 \neq 4$$

2nd

Will Converge

$$x_2^{(1)} = g_2(x_1^{(1)}) = \sqrt{3 + 2 \times 3.316} = 3.10375$$

$$\text{Relative error} = \frac{|x_1^{(1)} - x_0^{(1)}|}{x_1^{(1)}} \Rightarrow \frac{|3.10375 - 3.31662|}{3.10375} \times 100\% = 6.8\%$$

$$x_1^{(2)} = 3.0345$$

$$g_1(x) = x \Rightarrow x = \frac{x^2 - 3}{2} \quad x_0 = 4$$

$$x_1^{(1)} = 3.01144$$

$$x_1^{(1)} = \frac{16 - 3}{2} = 6.5$$

$$x_1^{(2)} = 3.000141$$

$$x_1^{(2)} = 11.625$$

$$x_1^{(3)} =$$

$$x_1^{(3)} = 191.02 \quad \text{will diverge}$$

$$x_1^{(4)} = 18252$$

$$g_3(x) = \frac{3}{x-2} \quad x_0 = 4$$

$$x_1^{(1)} = \frac{3}{4-2} = 1.5$$

don't give up yet

$$x_1^{(2)} = -6$$

$$x_1^{(3)} = -0.325$$

$$\xrightarrow{\text{roots}} x = \frac{2 \pm \sqrt{4+12}}{2} = 1 \pm 2$$

$$x_1^{(4)} = -1.263145$$

$$x_1^{(5)} = -0.91935$$

if $|g'(x)| < 1$, proceed

$$x_1^{(6)} = -1.022624$$

$$g(x) = \frac{3}{x-2}$$

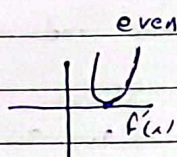
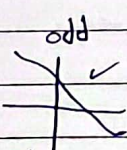
$$g'(x) = \frac{-3}{(x-2)^2}$$

$$g'(4) = -0.75 < 1 \quad \text{and there was a solution!!!}$$

do so with other G_s

→ Modified Newton's method

$$\text{Simple} \rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$



Problem with even number of roots

$$f'(\text{before}) = f'(\text{after}) \Rightarrow \text{Sign} \Rightarrow f'(\text{before}) \neq f'(\text{after})$$

$$\text{Sign} \Rightarrow f(\text{before}) = f(\text{after})$$

$$\text{Sign} \Rightarrow f''(\text{before}) \neq f''(\text{after})$$

Modified Newton's

$$x_1 = x_0 - \frac{f(x_0) \cdot f'(x_0)}{f'(x_0) - f(x_0) \cdot f''(x_0)}$$

$$f(x) = 3x^2 + 7 - 10x \quad x_0 = 0$$

$$T.P = 1 \times 10^{-8}$$

Try S.N ^{straight forward} not 1, 1,

graph it from 0 → 5

M.N

System of linear and non-linear

$$2x_1 + 3x_2 + 4x_3 = 5$$

$$3x_1 - 2x_2 + 5x_3 = 7$$

$$8x_1 - 10x_2 + 11x_3 = 12$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & -2 & 5 \\ 8 & -10 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$$

In this chapter, we will only deal with square matrices

$$A = \begin{bmatrix} 2 & 3 & 6 \\ 4 & 5 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} \quad \text{المربعين متساويين}$$

$$\rightarrow A(B+C) = AB + AC$$

* check the rule "no. ^{rows} columns of A = no. columns of B"

$$\rightarrow A(BC) = (AB)C$$

→ Transpose → rows to columns and vice versa

$$[A] \rightarrow [A]^T$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$[A, B]^T \rightarrow A^T \cdot B^T$$

→ Symmetric → when a matrix and its transpose are similar

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

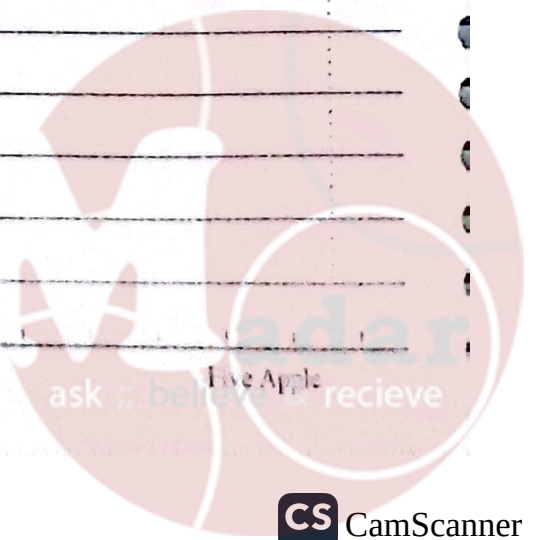
$$B = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \rightarrow B^T = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

Diagonal:-

$$[A] = \begin{bmatrix} \overset{\text{main}}{a_{11}} & \overset{\text{upper}}{a_{12}} & a_{13} \\ \underset{\text{lower}}{a_{21}} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$

$$[B] = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$[C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



صفر نہ ہونے کے افسار
كلها ادر ولا صفر

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{banded matrix or Tri-diagonal}$$

Zero

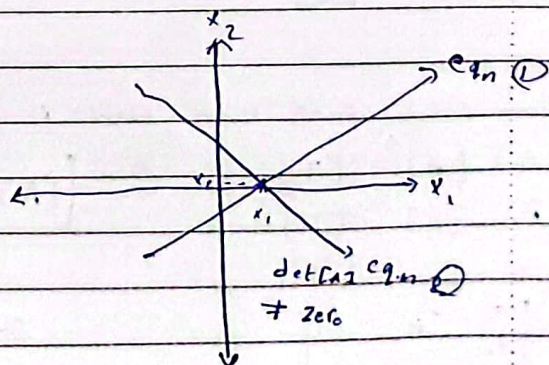
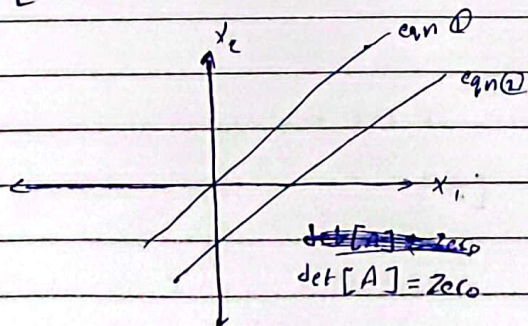
$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \text{at least one element isn't Zero}$$

* upper triangle matrix

$$[B] = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{lower triangle matrix}$$

Singular matrix $\rightarrow$ every matrix that has a determinant equal to Zero.
 $\rightarrow$ means that it's dependent and can't be solved
 اعتمادی نہیں ہوتا ہے

$$\begin{bmatrix} 2 & -4 \\ 4 & -8 \end{bmatrix} \rightarrow 16 - 16 = 0 \quad \text{no solution}$$



$$x_1 + x_2 = 2 \quad \text{All point are (well)}$$

$$x_1 + x_2 = 2 \quad \text{a solution}$$

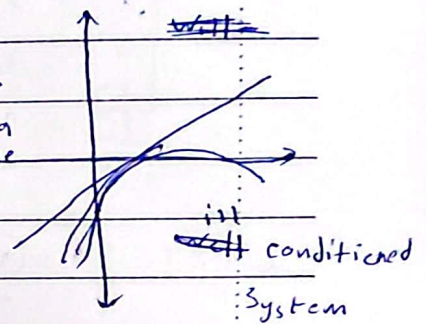
* Avoid all singular systems



$$\begin{bmatrix} 1000 & 2000 \\ 3000 & 5000 \end{bmatrix} \rightarrow \text{Det} = 0.2$$

(well)

Soln is
over a
c range



$$\begin{bmatrix} 1 & 0.3 \\ 5 & 0.4 \end{bmatrix} \Rightarrow \text{Det} = 0.3$$

مربوب و غير مستقل
ill

$$\begin{bmatrix} 1 & 0.3 \\ 5 & 0.4 \end{bmatrix} \rightarrow \begin{matrix} x_1 = 2.1 \\ x_2 = 5.3 \end{matrix} \quad \begin{matrix} x_1 = 2.09 \\ x_2 = 5.32 \end{matrix}$$

if the system is ill,

it's very susceptible (a change in founding of anything will)
make a dramatic change in the Soln

→ Inverse of the matrix

$$[A] \rightarrow [A]^{-1}$$

there's no division by matrix

$$\{X\} = \frac{\{C\}}{[A]} = [A]^{-1} \{C\}$$

How do I check if the matrix is an inverse of the original?

$$\textcircled{1} [A] \cdot [A]^{-1} = [I]$$

$$\textcircled{2} [A]^{-1} \cdot [A] = [I]$$

$$\rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

إذا كان أحد المصفوفات كان كل عناصرها صفراً فإن $\text{Det} = 0$

$$\rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

→ Det stays the same

$$\rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

→ 2 rows are the same elements

Det = 0

← إذا ضرب عناصر صف أو عمود بـ scalar فإن Det لا يتغير، بمعنى أنه

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

↓

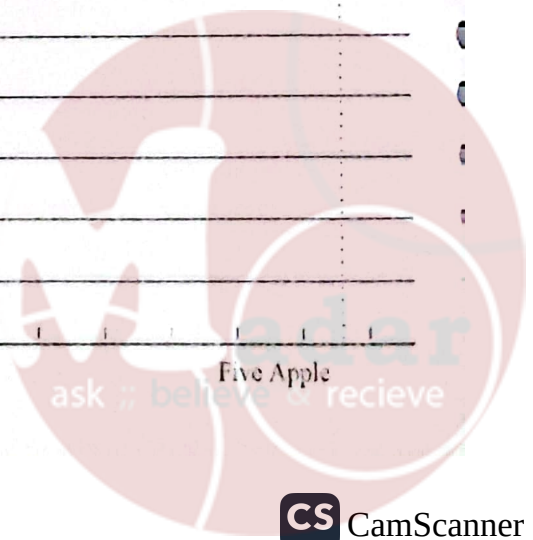
أي تغيير في
بعض عناصر الصف
أو

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & : & c_1 \\ a_{21} & a_{22} & a_{23} & : & c_2 \\ a_{31} & a_{32} & a_{33} & : & c_3 \end{bmatrix}$$

Rank of a matrix → only for ~~square~~ ^{square} matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3 \times 4}$$

فإن matrix 3x3 من matrix ونحتاج إيجاد واحد من الأمثليات يكون
of Det



Types of errors

① True $\rightarrow$ = True ans - approx. ans.

$$\textcircled{2} \text{ True \% relative error} = \frac{\text{True} - \text{approx.}}{\text{True}} \times 100\% = \frac{\text{True error}}{\text{True ans}} \times 100\%$$

$$\textcircled{3} \text{ Approximate \% relative error} = \frac{\text{present approx.} - \text{prev. approx.}}{\text{pres. approx.}} \times 100\%$$

④ prespecified \% tolerance

$$\epsilon = (0.5 \times 10^{-n})\% \quad n \text{ significant figures}$$

Solution of single N.I. equation

$$f(x) = x^3 - 5x^2 + 7x - 3 \quad x_0 = 0$$

$$f'(x) = 3x^2 - 10x + 7$$

$$f(0) = -3 \quad f'(0) = 7$$

S.N1

$$\text{1st} \rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-3}{7} = 0.42857$$

$$f(x_1^{(0)}) = |-0.8396548| > 1 \times 10^{-5}$$

$$f'(x_1^{(0)}) = 3.265316$$

$$\text{2nd} \rightarrow x_1^{(1)} = 0.6852135$$

$$x_1^{(2)} = 0.8328646$$

$$x_1^{(3)} = 0.913336798$$

$$x_1^{(4)} = 0.95528$$

$$x_1^{(5)} = 0.972657$$

Modified

$$f''(x) = 6x - 10$$

$$x_0 = 0 \quad f(0) = -3, \quad f'(0) = 2 \quad f''(0) = -10$$

$$1^{st} \quad x_1^0 = 0 - \frac{f(x_0) \cdot f'(x_0)}{(f'(x_0))^2 - f(x_0) \cdot f''(x_0)} = 1.105263158$$

$$f(x_1^0) = 0.02099, \quad f'(x_1^0) = -0.3828116 \quad f''(x_1^0) = -3.368421$$

$$x_1^{(2)} = 1.00308166$$

$$x_1^{(3)} = 1.000002382 \quad f(x_1^{(3)}) = 1.13 \times 10^{-11} < 1 \times 10^{-5}$$

Muller's method

open method but soln isn't guaranteed.

① 3 initial guesses are required

if only 2 guesses are given then $x_3 = \frac{x_0 + x_1}{2}$

② arrange the values as follows

$$\begin{array}{ccc} | & | & | \\ x_2 & < & x_0 < x_1 \\ f(x_2) & & f(x_0) \quad f(x_1) \end{array}$$

③ calculate the following constants

$$C_1 = \frac{f(x_1) - f(x_2)}{x_1 - x_2}, \quad C_2 = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$d_1 = \frac{C_2 - C_1}{x_0 - x_2}, \quad S = C_2 + d_1(x_0 - x_1)$$

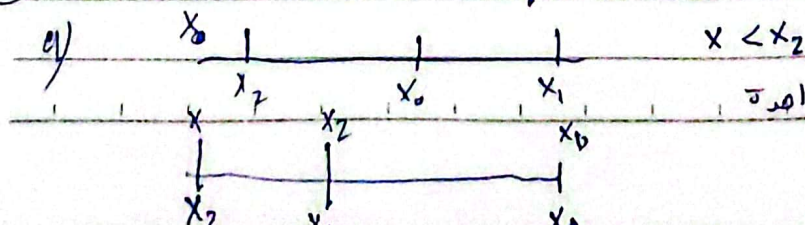
④ calculate the expected root

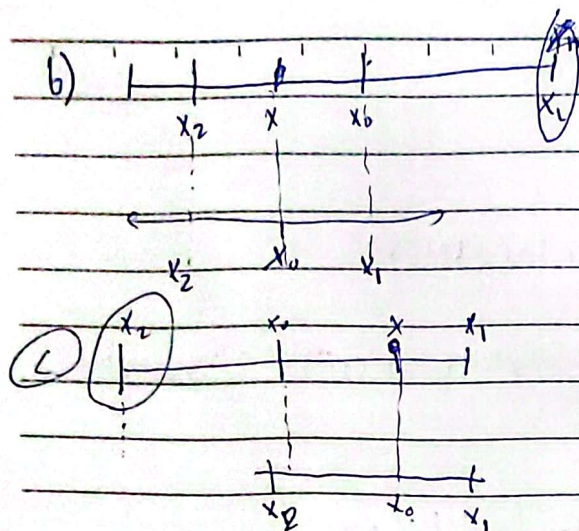
$$X = x_0 - \frac{2f(x_0)}{[5 + (\text{sign of}) \sqrt{5^2 - 4f(x_0) \cdot d_1}]}$$

⑤ if $|f(x)| \leq \text{Top}$, then x is the root

if $|f(x)| > \text{Top}$, then another iteration is required

⑥ If another iteration is required then





example

$$f(x) = x^2 - 2, \quad x_0 = 0.5, \quad x_1 = 1.0$$

$$x_2 = \frac{1 + 0.5}{2} = 0.75 \quad f(x_2) = -1.822021$$

$$f(x_1) = -1$$

$$f(x_0) = -1.984375$$

$$0.5 \quad 0.75 \quad 1 \quad C_1 = 1.96875 \quad A$$

$$x_2 \quad x_0 \quad x_1 \quad C_2 = 3.288084 \quad B$$

$$d_1 = 5.277336 \quad C$$

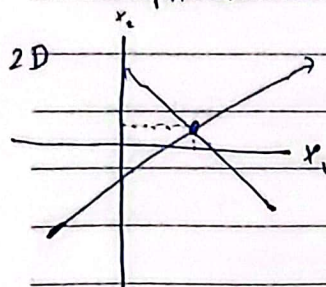
$$S = 1.96875 \quad D$$

$$x^0 = 1.579132$$

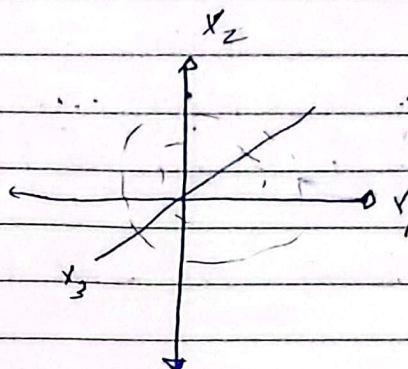


Solving small systems

① Graphical solution



3D



② Elimination of unknowns

$$2x_1 + 3x_2 = 5$$

$$x_2 = \frac{5 - 2x_1}{3}$$

$$x_1 - 4x_2 = 10$$

$$x_1 - 4\left(\frac{5 - 2x_1}{3}\right) = 10 \quad \text{up to 2 unknowns}$$

③ Cramer's rule (up to 3 eqns)

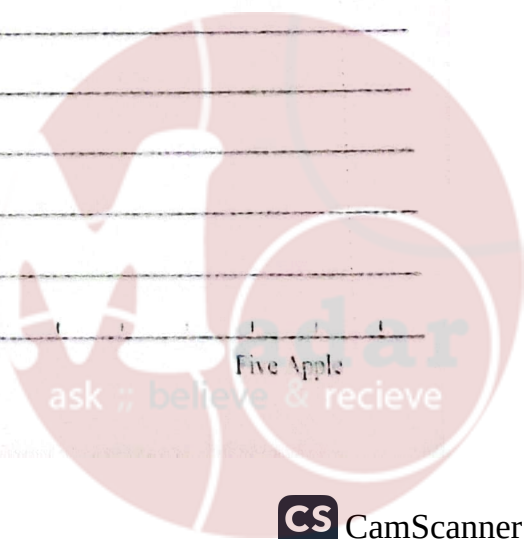
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

$$[A] \cdot \{x\} = \{C\}$$

① find $\det[A] = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

$$x_1 = \frac{\begin{bmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{bmatrix}}{\det[A]}$$

x_2 ~~and~~ x_3 the same with changing the column with C for the suitable x



Example $7x_1 - 3x_2 + 3x_3 = -49$

$x_1 - 2x_2 - 5x_3 = 5$

$3x_1 - 6x_2 + 10x_3 = -84$

$$\begin{bmatrix} 7 & -3 & 3 \\ 1 & -2 & -5 \\ 3 & -6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -49 \\ 5 \\ -84 \end{bmatrix}$$

* we check the dependency by
finding the det.

So
the calc $\det[A] = -275 \rightarrow$ not dependent
yourself (non Singular)

$$\det[A_1] = \begin{bmatrix} -49 & -3 & 3 \\ 5 & -2 & -5 \\ -84 & -6 & 10 \end{bmatrix} \Rightarrow \det[A_{x_1}] = 746$$

$$\det[A_2] = \begin{bmatrix} 7 & -49 & 3 \\ 1 & 5 & -5 \\ 3 & -84 & 10 \end{bmatrix} \Rightarrow \det[A_{x_2}] = -1662$$

$$\det[A_{x_3}] = 1089$$

$$x_1 = \frac{746}{-275} = -2.712727$$

$$x_2 = \frac{-1662}{-275} = 6.043636$$

$$x_3 = \frac{1089}{-275} = -3.96$$



Solving a system

① direct method :- Solution will be obtained from the 1st iteration
(non-singular only)

② iterative methods

→ direct methods

① Gauss method (Elimination) $[A]\{X\} = \{C\}$

② Gauss-Jordan [inverse] $\rightarrow \{X\} = \frac{\{C\}}{[A]} = [A]^{-1}\{C\}$

③ L.U Factorisation

$$\underbrace{[A]}_{LU} \{X\} = \{C\}$$

④ (special case) (elements only in the diagonal) Thomas method [Banded matrix]

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

→ iterative methods :-

We need initial guesses X_1^0, X_2^0, X_3^0

① Jacobi

② Gauss-Seidel



Gauss Elimination:-

* Don't switch columns

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

→ switching columns causes changes in unlike switching rows

$$\begin{array}{c|ccc|c} R_1 & a_{11} & a_{12} & a_{13} & c_1 \\ R_2 & a_{21} & a_{22} & a_{23} & c_2 \\ R_3 & a_{31} & a_{32} & a_{33} & c_3 \end{array}$$

Forward Elimination →

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \quad 3 \text{ unknowns}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2 \quad 2 \text{ unknowns}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3 \quad 1 \text{ unknown}$$

→ we eliminate all elements

below the main diagonal

→ example to clarify:- we use the row to eliminate in a column

$$\text{after elimination} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & c_1 \\ 0 & a_{22} & a_{23} & c_2 \\ 0 & a_{32} & a_{33} & c_3 \end{bmatrix}$$

تبقی کواثری و ستون

Elimination

$$\begin{array}{c|ccc|c} \text{2nd} \\ \text{eli} \rightarrow & a_{11} & a_{12} & a_{13} & c_1 \\ & 0 & a_{22} & a_{23} & c_2 \\ & 0 & 0 & a_{33} & c_3 \end{array} \leftarrow \text{upper triangle matrix}$$

② Backward subtraction:-

$$\text{eqn ① } 0x_1 + 0x_2 + a_{33}x_3 = c_3$$

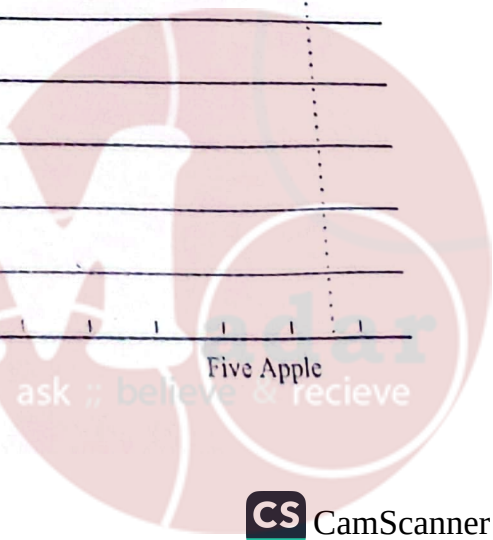
$$x_3 = \frac{c_3}{a_{33}}$$

$$x_2 = \left[\right]$$

eqn ②

$$\text{eqn ② } 0x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$

$$\text{eqn ③ } a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$



Normalised line $\rightarrow$

$$\left\{ \frac{a_{11}}{a_{11}} \quad \frac{a_{12}}{a_{11}} \quad \frac{a_{13}}{a_{11}} \quad \frac{c_1}{a_{11}} \right\} \rightarrow \left\{ 1 \quad \frac{a_{12}}{a_{11}} \quad \frac{a_{13}}{a_{11}} \quad \frac{c_1}{a_{11}} \right\} \text{ multiply by } -a_{21}$$

Pivot
row

$$-a_{21} \left\{ 1 \quad \frac{a_{12}}{a_{11}} \quad \frac{a_{13}}{a_{11}} \quad \frac{c_1}{a_{11}} \right\} \rightarrow \text{we multiply this by the row of the element we wanna eliminate and then add it to the elements next to it}$$

(کے لئے پوائے لائن سے گزرتے ہوئے)

$$3x_1 - 2x_2 + x_3 = -10$$

$$2x_1 + 6x_2 - 4x_3 = 44$$

$$-x_1 - 2x_2 + 5x_3 = -26$$

an excel $\rightarrow$ undeterm

normalised line

$$\begin{bmatrix} 3 & -2 & 1 & -10 \\ 2 & 6 & -4 & 44 \\ -1 & -2 & 5 & -26 \end{bmatrix} \xrightarrow{a_{21}} \left[\frac{3}{3} \quad \frac{-2}{3} \quad \frac{1}{3} \quad \frac{-10}{3} \right]$$

we do individual ones specifically for each element

$$\left\{ -2 \quad \frac{+4}{3} \quad \frac{-2}{3} \quad \frac{+20}{3} \right\}$$

next matrix $\rightarrow$

$$\begin{bmatrix} 3 & -2 & 1 & -10 \\ 0 & 7.333 & -4.666 & 50.666 \\ 0 & 2.666 & 5.555 & -29.333 \end{bmatrix}$$

there's still this element

$$2.666 \left[\frac{0}{7.333} \quad \frac{7.333}{7.333} \quad \frac{-4.666}{7.333} \quad \frac{50.666}{7.333} \right]$$

$$\begin{bmatrix} 3 & -2 & 1 & -10 \\ 0 & 7.333 & -4.666 & 50.666 \\ 0 & 0 & 3.636 & -10.909 \end{bmatrix}$$

addition

$$3.6363 x_3 = -10.9091 \quad x_3 = \frac{-10.9091}{3.6363}$$

do this until we find all x_s

$$7.333 x_2 - \frac{10.9091 \times -4.666}{3.6363} = 50.666$$

② Gauss Jordan (Elimination with a dif. twist)

original $\rightarrow$ identity

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

Gauss Elimination

① normalise each row

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_1' \\ c_2' \\ c_3' \end{Bmatrix}$$

divide the 1st row by a_{11}

$$\textcircled{1} \begin{bmatrix} \frac{a_{11}}{a_{11}} & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} \\ 0 & a_{22}-n_1 & a_{23}-n_1 \\ 0 & a_{32}-n_2 & a_{33}-n_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} \frac{c_1}{a_{11}} \\ c_2-n_1 \\ c_3-n_2 \end{Bmatrix} \rightarrow \text{multiply by } a_{21}$$

$$n_2 = [-a_{31} \quad -a_{31} \frac{a_{12}}{a_{11}} \quad -\frac{a_{12} a_{31}}{a_{11}}]$$

② we normalise the following rows and the exact same steps

$$\begin{bmatrix} 1 & a_{12}' & a_{13}' \\ 0 & 1 & \frac{a_{22}'}{a_{22}} \\ 0 & a_{32}' & a_{33}' \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_1' \\ \frac{c_2'}{a_{22}} \\ c_3' \end{Bmatrix}$$

Example $\left[\begin{array}{ccc|c} 3 & -2 & 1 & -10 \\ 2 & 6 & -4 & 44 \\ -1 & -2 & 5 & -26 \end{array} \right] \xrightarrow{+} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -10 \\ 0 & 10 & -6 & 50 \\ 0 & -2 & 6 & -26 \end{array} \right] \xrightarrow{\times} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -10 \\ 0 & 1 & -0.6 & 5 \\ 0 & -2 & 6 & -26 \end{array} \right] \xrightarrow{\times} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -10 \\ 0 & 1 & -0.6 & 5 \\ 0 & 0 & 0 & -16 \end{array} \right]$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.27 \\ 0 & 1 & -0.6 & 6.90909 \\ 0 & 0 & 3.6 & -10.90909 \end{array} \right] \xrightarrow{\times} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.27 \\ 0 & 1 & -0.6 & 6.90909 \\ 0 & 0 & 1 & -3.02777 \end{array} \right] \xrightarrow{\text{add to row 1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.27 \\ 0 & 1 & -0.6 & 6.90909 \\ 0 & 0 & 1 & -3.02777 \end{array} \right] \xrightarrow{\text{add to row 2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.27 \\ 0 & 1 & 0 & 5.11111 \\ 0 & 0 & 1 & -3.02777 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{array} \right] \xrightarrow{0.40909 \times \{\text{row 3}\}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{array} \right] \xrightarrow{0.6363 \times \{\text{row 3}\}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{array} \right]$$

#

Gauss-Jordan

$$[A]\{x\} = \{c\} \Rightarrow \{x\} = \frac{\{c\}}{[A]} = [A]^{-1}\{c\}$$

*matrix must be constant

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array} \right]$$

find the inverse

$[A][I] \rightarrow$ and this will be converted to $[A]^{-1}$

by Elimination and normalization $\rightarrow$ will become an identity matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a_{11}^{-1} & a_{12}^{-1} & a_{13}^{-1} \\ 0 & 1 & 0 & a_{21}^{-1} & a_{22}^{-1} & a_{23}^{-1} \\ 0 & 0 & 1 & a_{31}^{-1} & a_{32}^{-1} & a_{33}^{-1} \end{array} \right] \rightarrow \text{multiple soln} \Rightarrow$$

$$\begin{matrix} 3 \times 3 & 1 \times 3 \\ \times & \end{matrix} \quad [A]\{c\} = \{c\}[A]^{-1}$$

Example

$$[A]_s = \left[\begin{array}{ccc|ccc} 3 & -2 & 1 & 1 & 0 & 0 \\ 2 & 6 & -4 & 0 & 1 & 0 \\ -1 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad C = \begin{pmatrix} -10 \\ 44 \\ -26 \end{pmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2/3 & 1/3 & 1/2 & 0 & 0 \\ 0 & 7/3 & -11/3 & -1/6 & 1 & 0 \\ 0 & -2/3 & 5/3 & 0 & 1 & 1 \end{array} \right] \quad \begin{matrix} -2 \{ \text{row 1 (norm)} \} \\ 1 \{ \text{row 1 (norm)} \} \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -0.4 & 0.27 & 0.04 & 0 \\ 0 & 1 & -0.686 & -0.09 & 0.126 & 0 \\ 0 & 0 & 3.65 & 0.09 & 0.363 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.275 & 0.1 & 0.025 \\ 0 & 1 & 0 & -0.075 & 0.2 & 0.175 \\ 0 & 0 & 1 & 0.025 & 0.1 & 0.275 \end{array} \right] \quad \begin{matrix} I \\ [A]^{-1} \end{matrix}$$

to find the soln
we multiply $[A]^{-1}\{c\}$

$$\begin{bmatrix} -10 \\ 44 \\ -26 \end{bmatrix} \begin{bmatrix} 0A^{-1} \end{bmatrix}$$

$$\rightarrow [A]^{-1} \{C\} \rightarrow \{ \} [A]^{-1}$$

$3 \times 3 \quad 3 \times 1 \quad 3 \times 1 \quad 3 \times 3$

Example 1

$$\begin{bmatrix} 3 & -2 & 1 & -10 \\ 2 & 6 & -4 & 44 \\ -1 & -2 & 5 & -26 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 3 & -2 & 1 & -10 \\ 2 & 6 & -4 & 44 \\ -1 & -2 & 5 & -26 \end{bmatrix} \xrightarrow{+2/3} \begin{bmatrix} 1 & -2/3 & 1/3 & -10/3 \\ 2 & 6 & -4 & 44 \\ -1 & -2 & 5 & -26 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2/3 & 1/3 & -10/3 \\ 0 & 7.3333 & -4.6 & 50.6667 \\ 0 & -2.6 & 5.3 & -29.3333 \end{bmatrix} \xrightarrow{-2(1)} \begin{bmatrix} 1 & -2/3 & 1/3 & -10/3 \\ 0 & 7.3333 & -4.6 & 50.6667 \\ 0 & -2.6 & 5.3 & -29.3333 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -0.0909 & 1.2778 \\ 0 & 1 & -0.6364 & 6.9091 \\ 0 & 0 & 3.1818 & -10.9091 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1.2778 \\ 6.9091 \\ -10.9091 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$$

$$x_1 = 1$$

$$x_2 = 5$$

$$x_3 = -3$$

Thomas

Banded matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix}$$

for Row no.1

$$A_1 = \frac{a_1}{d_1}, R_1 = \frac{r_1}{d_1}$$

row 2 $\rightarrow n-1$

$$A_i = \frac{a_i}{d_i - b_i A_{i-1}} \quad R_i = \frac{r_i - b_i R_{i-1}}{d_i - b_i A_{i-1}}$$

For 2nd row

$$\frac{A_2 = a_1}{d_2 - b_2 A_1} \quad \frac{R_2 = r_2 - b_2 R_1}{d_2 - b_2 A_1}$$

for the last row:

$$A_n = 0 \quad R_n = r_n - b_n A_{n-1}$$

$n = 4$

$$A_4 = 0 \quad R_4 = r_4 - b_4 R_3$$

$$X_4 = X_n = R_4$$

backward substitution

$$X_i = R_i - A_i X_{i+1}$$

$$x_3 = R_3 - A_3 x_4, \quad x_2 = R_2 - A_2 x_3, \quad x_1 = R_1 - A_1 x_2$$

LU Factorization

Step 1 :- Decomposition of the original matrix

$$[A] \rightarrow [L][U] \neq [U][L]$$

$$L_{21} = \frac{-a_{21}}{a_{11}}$$

$$L_{31} = \frac{-a_{31}}{a_{11}}$$

$$L_{41} = \frac{-a_{41}}{a_{11}}$$

$$L_{32} = \frac{-a_{32}}{a_{22}}$$

$$L_{42} = \frac{-a_{42}}{a_{22}}$$

$$L_{43} = \frac{-a_{43}}{a_{33}}$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Zero by elimination

$$[U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$u_{44} = a_{44}$$

$$u_{34} = a_{34}$$

$$u_{33} = a_{33}$$

$[L][U]$ must be equal to the original matrix

Example

$$\begin{bmatrix} 3 & -2 & 1 \\ 2 & 6 & -4 \\ -1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 44 \\ -26 \end{bmatrix}$$

$$L_{21} = \frac{2}{3}$$

$$L_{31} = \frac{-1}{3}$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & 1 \\ 0 & 7.33 & -4.67 \\ 0 & -2.67 & 5.33 \end{bmatrix} \rightarrow \text{we find } L_{32}$$

$$L_{32} = \frac{-2.67}{7.33}$$

$$[U] = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 7.33 & -4.67 \\ 0 & 0 & 3.63 \end{bmatrix}$$

Step 2: to find a solution

$$[L]\{D\} = \{C\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.66 & 1 & 0 \\ -0.33 & -0.33 & 1 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 44 \\ -26 \end{bmatrix}$$

forward

substitution

$$D_1 = -10$$

$$D_2 = 50.666$$

$$D_3 = -10.9091$$

intermediate
intermediate

$$[L]^{-1}[C] = \{D\}$$

Five Apple

ask :: believe & recieve

Step 2

$$[U][X] = \{D\}$$

$$\begin{bmatrix} 3 & -2 & 1 \\ 0 & 2.32 & -4.66 \\ 0 & 0 & 3.62 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -10 \\ 50.111 \\ -10.909 \end{Bmatrix} \xrightarrow{\text{Backward Sub}} \begin{aligned} x_3 &= \frac{-10.909}{3.62} = -3 \\ x_2 &= +5 \\ x_1 &= 1 \end{aligned}$$

Step 3:- To find the inverse of the matrix $[A]$

$$[I] \{D^0\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} D_{11} \\ D_{21} \\ D_{31} \end{Bmatrix}$$

$$[I] \{D^1\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} D_{12} \\ D_{22} \\ D_{32} \end{Bmatrix}$$

$$[I] \{D^2\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \rightarrow \begin{Bmatrix} D_{13} \\ D_{23} \\ D_{33} \end{Bmatrix}$$

$$[U] \begin{Bmatrix} x_{11}^{-1} \\ x_{21}^{-1} \\ x_{31}^{-1} \end{Bmatrix} = \begin{Bmatrix} D_{11} \\ D_{21} \\ D_{31} \end{Bmatrix} \text{ and we do this for each column}$$

Continue
Example $\begin{bmatrix} 1 & 0 & 0 \\ 0.66 & 1 & 0 \\ -6.32 & 0 & 1 \end{bmatrix} \begin{Bmatrix} D_{11} \\ D_{21} \\ D_{31} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$

$$D_{11} = 1 \quad D_{21} = -0.666 \quad D_{31} = 0.0909$$

$$\begin{bmatrix} 3 & -2 & 1 \\ 0 & 2.32 & -4.66 \\ 0 & 0 & 3.62 \end{bmatrix} \begin{Bmatrix} x_{11}^{-1} \\ x_{21}^{-1} \\ x_{31}^{-1} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.66 \\ 0.0909 \end{Bmatrix} \xrightarrow{\text{Back Sub}} \begin{aligned} x_{31}^{-1} &= \dots \\ x_{21}^{-1} &= \dots \\ x_{11}^{-1} &= \dots \end{aligned}$$

and we do this with the rest of the columns

Iterative methods

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3 \end{aligned} \right\} [A][x] = \{c\}$$

$$\left. \begin{aligned} x_1 &= \frac{c_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \\ x_2 &= \frac{c_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} \\ x_3 &= \frac{c_3 - a_{31}x_1 - a_{32}x_2}{a_{33}} \end{aligned} \right\}$$

$x_1^0, x_2^0, x_3^0 \rightarrow$ initial guesses

1st iteration $x_1^0, x_2^0, x_3^0 \rightarrow$ it's a soln if the relative error for ALL of them is less than the T.P

$$\left| \frac{x_1^0 - x_1'}{x_1^0} \right| \leq \text{T.P and we do this for } x_2 \text{ and } x_3$$

if anyone of them was higher than T.P, we do another iteration

① Jacobi

x_1^0, x_2^0, x_3^0 all of them are required here

$$x_1^1 = \frac{c_1 - a_{12}x_2^0 - a_{13}x_3^0}{a_{11}} \rightarrow \text{and we do this with } x_2^0 \text{ and } x_3^0$$

For a 2nd iteration, we use x^0 values

② Gauss-Seidel

x_1^0, x_2^0, x_3^0 we don't need all of them

$$x_1^0 = \frac{c_1 - a_{12}x_2^0 - a_{13}x_3^0}{a_{11}}$$

$$x_2^0 = \frac{c_2 - a_{21}x_1^0 - a_{23}x_3^0}{a_{22}}$$

we didn't use x_1^0 but instead x_1^1

} Speed of Convergence

$$\begin{cases} |a_{11}| > |a_{12}| + |a_{13}| \\ |a_{22}| > |a_{21}| + |a_{23}| \\ |a_{33}| > |a_{31}| + |a_{32}| \end{cases}$$

بترتيب
هذا الحل ونرتب على
هذا الأسس

Example /

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$x_1 = \frac{71.4 + 0.2x_2 - 10x_3}{0.3}$$

$$x_2 = \frac{7.85 + 0.1x_1 + 0.2x_3}{-0.1} \quad \text{i.g.} \rightarrow x_1^0 = 0 \quad x_2^0 = 0 \quad x_3^0 = 0$$

$$x_3 = \frac{-19.3 - 0.1x_1 - 7x_2}{-0.3}$$

Jacobi

1st iteration:-

$$x_1^0 = \frac{71.4}{0.3} = 238$$

$$x_2^0 = -78.5$$

$$x_3^0 = 64.33$$

2nd iteration:-

$$x_1^1 = \frac{71.4 + 0.2 \times 78.5 - 10 \times 64.33}{0.3} = -1958$$

لازم ترتيب المعادلات

$$x_1^0 =$$

$$x_2^0 =$$

Gauss-seidel /

$$x_1^0 = \frac{71.4}{0.3} = 238$$

$$x_2^0 = \frac{7.85 - 3 \times 238 + 0.2 \times 0}{-0.1} = 706$$

$$x_3^0 =$$

لم نجد
أي حل
بحسب أن ترتيب

من مرتب المعادلات

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

نعمل iteration ونطالع من الحل

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3}$$

على عكس قبل شو كى

$$x_1 =$$

$$x_3 =$$

Continue the rest by yourself

System of non-linear equations

$$\begin{cases} f_1(x,y,z) = 0 \\ f_2(x,y,z) = 0 \\ f_3(x,y,z) = 0 \end{cases} \quad \begin{cases} \text{if the system is non-linear} \\ \text{① Newton's method} \\ \text{② fixed point iteration} \end{cases}$$

① Newton's method:-

1- find the Jacobian matrix

$$J(x,y,z) = \begin{bmatrix} \frac{df_1}{dx} & \frac{df_1}{dy} & \frac{df_1}{dz} \\ \frac{df_2}{dx} & \frac{df_2}{dy} & \frac{df_2}{dz} \\ \frac{df_3}{dx} & \frac{df_3}{dy} & \frac{df_3}{dz} \end{bmatrix} \quad \text{مصفوفة باقى } x_0, y_0, z_0$$

Step ②

$$J(x,y,z) \cdot \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{Bmatrix} = - \begin{Bmatrix} f_1(x_0, y_0, z_0) \\ f_2(x_0, y_0, z_0) \\ f_3(x_0, y_0, z_0) \end{Bmatrix}$$

collections (guess elimination)

Step ③ Find the expected value of x, y, z

$$x^0 = x_0 + \Delta x$$

$$y^0 = y_0 + \Delta y$$

$$z^0 = z_0 + \Delta z$$

→ we check the relative error for all values and they all must be lower than T.P

→ initial guesses are the key factor in determining convergence or divergence

after the 1st iteration, you have x^0, y^0, z^0

$$\left\{ \left| \frac{df_1}{dx} \right| + \left| \frac{df_2}{dy} \right| + \left| \frac{df_3}{dz} \right| \right\}_{x^0, y^0, z^0} < 1 \rightarrow \text{and we do the same with } f_2 \text{ and } f_3$$

and they all must be < 1

if one of them isn't less than 1, we change the initial guesses

Example

$$f_1(x, y) = y^2 e^x - \sin x \quad x_0 = 0.8 \quad y_0 = 0.3$$

$$f_2(x, y) = x^2 e^y - \cos y$$

$$\text{Step 1} \quad J(x, y) = \begin{bmatrix} 2y^2 e^x - \cos x & 2y e^{2x} \\ 2x e^y & 3x^2 e^y + \sin y \end{bmatrix}$$

$$J(x_0, y_0) = \begin{bmatrix} 0.194839 & 2.971819 \\ 3.9353 & 5.012518 \end{bmatrix}$$

$$D = - \begin{bmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{bmatrix} = \begin{bmatrix} 0.271583 \\ -0.61881 \end{bmatrix}$$

$$\begin{bmatrix} 0.194839 & 2.971819 \\ 3.9353 & 5.012518 \end{bmatrix} \begin{bmatrix} \Delta x^0 \\ \Delta y^0 \end{bmatrix} = \begin{bmatrix} 0.271583 \\ -0.61881 \end{bmatrix}$$

Solve using Gauss elimination

$$\Delta x^0 = -0.299874$$

$$\Delta y^0 = 0.110972$$

$$\therefore x^0 = x_0 + \Delta x^0$$

$$y^0 = y_0 + \Delta y^0$$

② Fixed point

$$f_1(x, y, z) = 0 \rightarrow g_1(x, y, z) = x$$

$$f_2(x, y, z) = 0 \rightarrow g_2(x, y, z) = y$$

$$f_3(x, y, z) = 0 \rightarrow g_3(x, y, z) = z$$

1st iteration :-	<u>Stör</u>	<u>Jacobi</u>
$x^0 = g_1(x_0, y_0, z_0)$	{	$x^0 = g_1(x_0, y_0, z_0)$
$y^0 = g_2(x^0, y_0, z_0)$		$y^0 = g_2(x_0, y_0, z_0)$
$z^0 = g_3(x^0, y^0, z_0)$		$z^0 = g_3(x_0, y_0, z_0)$

$$\left| \frac{x^0 - x_0}{x^0} \right| \times 100\% \leq T.P$$

$$\left| \frac{y^0 - y_0}{y^0} \right| \times 100\% \leq T.P$$

$$\left| \frac{z^0 - z_0}{z^0} \right| \times 100\% \leq T.P$$

We go to a 2nd iteration
if one of them fails

after the 1st iteration:-

$$\left| \frac{dg_1}{dx} \right| + \left| \frac{dg_1}{dy} \right| + \left| \frac{dg_1}{dz} \right| \left| \frac{x^0}{y^0} \right| < 1 \quad \text{and do the same with } g_2 \text{ and } g_3$$

→ to check if the g s are stable
give a soln

Example/

$$f_1(x, y) = x^3 + 10y - y - 5 = 0$$

$$f_2(x, y) = x + y^3 - 10x + 1 = 0$$

$x = \frac{(y+5-x^3)}{10} = g_1(x, y)$	$x = (y+5-10x)^{\frac{1}{3}} = g_1(x, y)$	} alternatives (you can choose what you want
$y = \frac{(x+y^3+1)}{10} = g_2(x, y)$	$y = (10y-x-1)^{\frac{1}{3}} = g_2(x, y)$	

$$x_0 = y_0 = 0.6$$

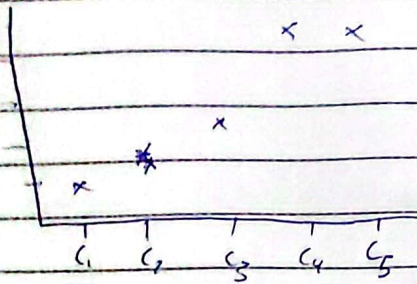
$$1st \text{ iteration:- } x^0 = g_1(0.6, 0.6) = \frac{0.6 + 5 - 0.6^3}{10} = 0.5384$$

$$y^0 = g_2(0.5384, 0.6) = 0.17544$$

$$2nd \text{ iteration:- } x^0 = g_1(0.5384, 0.17544) = 0.50194$$

$$y^0 = g_2(0.50194, 0.17544) = 0.15073$$

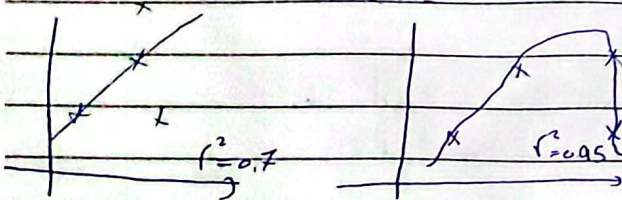
Interpolation and curve fitting



Purpose: How to interpret the model
and draw a curve with less errors

Minimization of summations of squares of error $\rightarrow$ least squares method

$$0 \leq r^2 \leq 1$$



Curve fitting

- ① Linear or polynomial $\rightarrow$ least squares method
- ② Simple or non-linear
- ③ General non-linear

① $\epsilon = (y_i - y(x_i))$ $\therefore$ y_i is the given value
error $y(x_i)$ is the calculated value (from model)

② $S = \sum_{i=0}^n (\epsilon)^2 = \sum_{i=0}^n (y_i - y(x_i))^2$

③ minimise the value of $\frac{dS}{d \text{ Parameters}} = 0$



① Linear ~~val~~ model

$$y(x_i) = \underbrace{(A + Bx_i)}_{\text{Parameters}}$$

$$\textcircled{1} \sigma = (y_i - A - Bx_i)$$

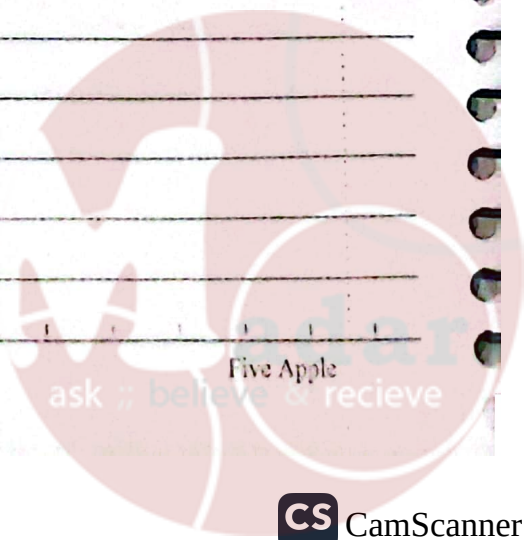
$$\begin{array}{c|c} x_i & y_i \\ \hline x_0 & y_0 \\ x_1 & y_1 \end{array}$$

$$\textcircled{2} \xi = \sum (y_i - A - Bx_i)^2$$

$$\textcircled{3} \frac{d\xi}{dA} = 2 \sum_{i=0}^n (y_i - A - Bx_i) (-1) = 0$$

$$\frac{d\xi}{dB} = 0 = 2 \sum_{i=0}^n (y_i - A - Bx_i) (-x_i) = 0$$

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$



Least squares method

$A \rightarrow$ linear, non linear, Poly, $\rightarrow \frac{dS}{d\text{Parameter}} \neq \text{any one of parameters}$

$$y = a e^{bx}$$

$$S = \sum_{i=0}^n [y_i - a e^{bx_i}]^2$$

$$\frac{dS}{da} = 0 = 2 \sum_{i=0}^n y_i - a e^{bx_i} (-e^{bx_i}) x$$

$$\frac{dS}{db} = 0 = 2 \sum_{i=0}^n (y_i - a e^{bx_i}) (-a x_i e^{bx_i})$$

$$y(x_i) = A + b x_i$$

$$S = \sum_{i=0}^n (y_i - A - b x_i)^2$$

$$\frac{dS}{dA} = 0 = \sum y_i - \sum a - \sum b x_i$$

$$\frac{dS}{dB} = 0 = \sum x_i y_i - \sum x_i a - \sum b x_i^2$$

anything related to x, y is known
 A, B are not

$$a \sum_{i=0}^n (1) + b \sum_{i=0}^n x_i = \sum y_i$$

$$a \sum_{i=0}^n x_i + b \sum_{i=0}^n x_i^2 = \sum x_i y_i$$

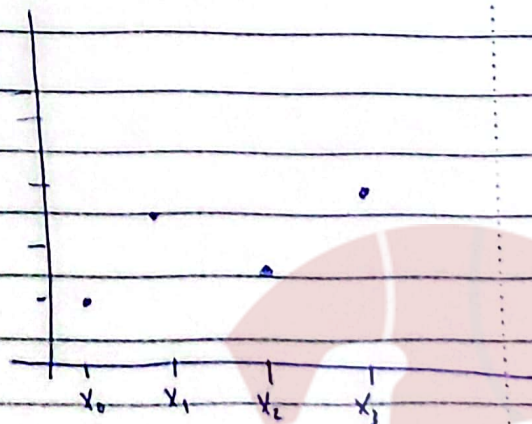
$$a n + b \sum x_i = \sum y_i$$

$$a \sum x_i + b \sum x_i^2 = \sum x_i y_i$$

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

we can solve by Cramer's rule, Gauss elimination or any method.

x_i	$y(x_i)$	x_i^2	$x_i y_i$
x_0	$y(x_0)$	x_0^2	$x_0 y(x_0)$
x_1	$y(x_1)$	x_1^2	$x_1 y(x_1)$
x_2	$y(x_2)$	x_2^2	$x_2 y(x_2)$
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum x_i y_i$



Least squares method

② simple non linear

non linear $\rightarrow$ linear

$$y = a e^{bx} \quad \leftarrow \text{غير خطي ان تقابل على الجانبي وتكون خطيا}$$

$$\ln y = \ln a + bx$$

$$Y = A + bx_i$$

$$b = \frac{Y_i - A}{x_i}$$

$$S = \sum_{i=0}^n (Y_i - A - bx_i)^2$$

$$\frac{dS}{dA} = 2 \sum (Y_i - A - bx_i)(-1)$$

$$\frac{dS}{db} = 2 \sum (Y_i - A - bx_i)(-x_i)$$

$$A \sum (1) + b \sum x_i = \sum Y_i$$

$$A \sum x_i + b \sum x_i^2 = \sum x_i Y_i$$

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{Bmatrix} A \\ b \end{Bmatrix} = \begin{Bmatrix} \sum Y_i \\ \sum x_i Y_i \end{Bmatrix}$$

$$A = \ln a$$

$$\rightarrow y = e^{ax+b} \quad \ln y = ax + b$$

$$Y_i = ax_i + b$$

$$\rightarrow y = \frac{1}{a+bx} \Rightarrow \frac{1}{y} = a+bx$$

$$Y_i = a+bx_i$$

$$F = C_1 + C_2 x + C_3 y$$

$$b = (F_i - C_1 - C_2 x - C_3 y)$$

$$S = \sum (F_i - C_1 - C_2 x - C_3 y)^2$$

$$\frac{dS}{dC_1} = 0 = 2 \sum (F_i - C_1 - C_2 x - C_3 y)(-1)$$

$$\frac{dS}{dC_2} = 0 = 2 \sum (F_i - C_1 - C_2 x - C_3 y)(-x_i)$$

$$\frac{dS}{dC_3} = 0 = 2 \sum (F_i - C_1 - C_2 x_i - C_3 y_i)(-y_i)$$

$$\begin{bmatrix} n & \sum x_i & \sum y_i \\ \sum x_i & \sum x_i^2 & \sum x_i y_i \\ \sum y_i & \sum x_i y_i & \sum y_i^2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} \sum F_i \\ \sum x_i F_i \\ \sum y_i F_i \end{Bmatrix}$$

$$F_i = C_1 + C_2 x_i + C_3 y_i + C_4 x_i^2 + C_5 y_i^2 + C_6 x_i y_i$$

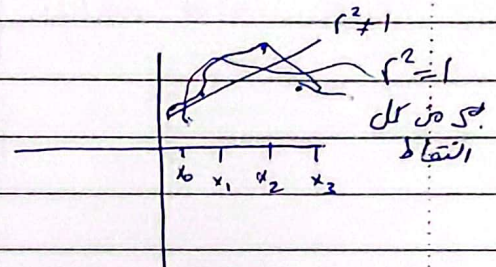
Coefficient of Determination (r^2)

$$0 \leq r^2 \leq 1$$

$$r^2 = \frac{S_t - S_r}{S_t}$$

$$S_t = \sum (y_i - \bar{y})^2$$

$$S_r = \sum (y_i - y_{\text{calculated}})^2$$



[3] non-linear regression

$$y = a e^{bx}$$

We need initial guesses

$$a_0 =$$

$$b_0 =$$

Step 1:- find the Z matrix

each column a_i represents a diff

$$[Z] = \begin{bmatrix} e^{bx_0} & x_0 e^{bx_0} & \dots & x_0^{n-1} e^{bx_0} \\ e^{bx_1} & x_1 e^{bx_1} & \dots & x_1^{n-1} e^{bx_1} \\ \vdots & \vdots & \ddots & \vdots \\ e^{bx_n} & x_n e^{bx_n} & \dots & x_n^{n-1} e^{bx_n} \end{bmatrix}$$

$n \times \text{parameters}$

$$\frac{dy}{dx} = e^{bx} \quad \eta \begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}$$

column = no. of parameters

rows = depends on given

data

$$\frac{dy}{db} = ax e^{bx}$$

$[Z]$ will be changed from each iteration to another

Step 2: find vector $\{D\}$ where $\{D\} = \begin{Bmatrix} f(x_0) - a_0 e^{b_0 x_0} \\ f(x_1) - a_0 e^{b_0 x_1} \\ f(x_2) - a_0 e^{b_0 x_2} \\ f(x_3) - a_0 e^{b_0 x_3} \end{Bmatrix}$

4×1

Step 3: find the following multiplication

$$[Z]_{4 \times 4}^T [Z]_{4 \times 2} = [Z]_{4 \times 2}^T [Z]_{4 \times 2}$$

$$[Z]_{4 \times 4}^T \{D\}_{4 \times 1} = \{[Z]_{4 \times 2}^T \{D\}_{4 \times 1}\}_{2 \times 1}$$

Step 4: solve the following system using Gauss elimination

$$[Z]_{4 \times 4}^T [Z]_{4 \times 2} \begin{Bmatrix} \Delta a \\ \Delta b \end{Bmatrix} = \{[Z]_{4 \times 2}^T \{D\}_{4 \times 1}\}_{2 \times 1}$$

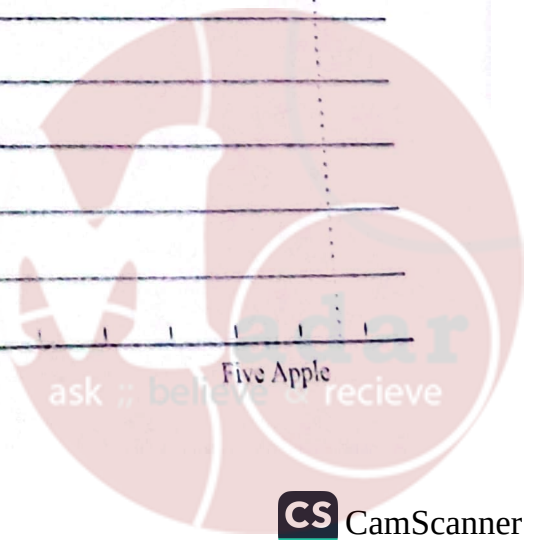
Step 5 correct the initial guesses

$$a_1 = a_0 + \Delta a$$

$$b_1 = b_0 + \Delta b$$

Stop only if $\left| \frac{a_1 - a_0}{a_0} \right| \leq \text{Tolerance}$

$$\left| \frac{b_1 - b_0}{b_1} \right| \leq \text{Tolerance}$$



Interpolation

① Lagrange

$$f_n(x) = \sum_{i=0}^n L_i(x) \cdot f(x_i)$$

x_i	$f(x_i)$
x_0	$f(x_0)$
x_1	$f(x_1)$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

If $n=1$ $L_0(x) = \frac{x - x_1}{x_0 - x_1}$

$$f_1(x) = \sum_{i=0}^1 L_i(x) \cdot f(x_i) = L_0(x) f(x_0) + L_1(x) \cdot f(x_1)$$

$$f_1(x) = \left(\frac{x - x_1}{x_0 - x_1} \right) \cdot f(x_0)$$

$n=2$ 2nd order

$$f_2(x) = \sum_{i=0}^2 f(x_i) \cdot L_i(x)$$

x_i	$f(x_i)$
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$

$$f_2(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

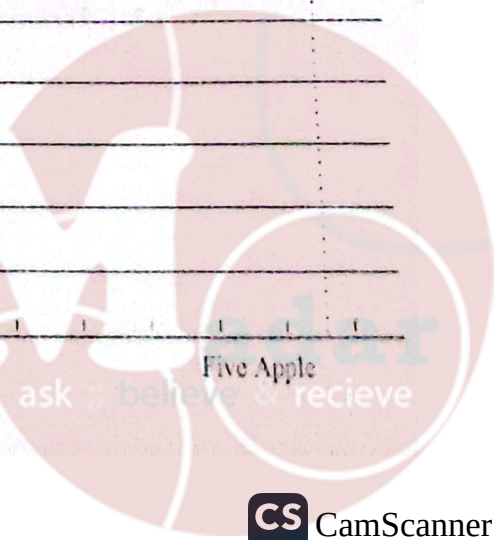
$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^2 \frac{x - x_j}{x_0 - x_j} = \frac{x - x_1}{x_0 - x_1} \cdot \left(\frac{x - x_2}{x_0 - x_2} \right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^2 \frac{x - x_j}{x_1 - x_j} = \left(\frac{x - x_0}{x_1 - x_0} \right) \cdot \left(\frac{x - x_2}{x_1 - x_2} \right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j \neq 2}}^2 \frac{x - x_j}{x_2 - x_j} = \left(\frac{x - x_0}{x_2 - x_0} \right) \cdot \left(\frac{x - x_1}{x_2 - x_1} \right)$$

$$f_2(x) = \left(\frac{x - x_1}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right) f(x_1) + \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right) f(x_2)$$

$$\boxed{\frac{x - x_j}{x_i - x_j} f(x_i)}$$



$$f_4(x) = \left(\frac{x-x_1}{x_0-x_1} \right) \left(\frac{x-x_2}{x_0-x_2} \right) \left(\frac{x-x_3}{x_0-x_3} \right) \left(\frac{x-x_4}{x_0-x_4} \right) f(x_0) +$$

$$+ \left(\frac{x-x_0}{x_1-x_0} \right) \left(\frac{x-x_2}{x_1-x_2} \right) \left(\frac{x-x_3}{x_1-x_3} \right) \left(\frac{x-x_4}{x_1-x_4} \right) f(x_1) +$$

$$+ \left(\frac{x-x_0}{x_2-x_0} \right) \left(\frac{x-x_1}{x_2-x_1} \right) \left(\frac{x-x_3}{x_2-x_3} \right) \left(\frac{x-x_4}{x_2-x_4} \right) f(x_2) +$$

$$+ \left(\frac{x-x_0}{x_3-x_0} \right) \left(\frac{x-x_1}{x_3-x_1} \right) \left(\frac{x-x_2}{x_3-x_2} \right) \left(\frac{x-x_4}{x_3-x_4} \right) f(x_3) +$$

$$+ \left(\frac{x-x_0}{x_4-x_0} \right) \left(\frac{x-x_1}{x_4-x_1} \right) \left(\frac{x-x_2}{x_4-x_2} \right) \left(\frac{x-x_3}{x_4-x_3} \right) f(x_4)$$

x_i	$f(x_i)$
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$
x_4	$f(x_4)$

$$x_0 \leq x \leq x_4$$

Newton's method

$$f_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_n(\quad) \dots$$

1st order

$$f_1(x) = a_0 + a_1(x-x_0)$$

2nd:-

$$f_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$

$$f_1(x) = a_0 + a_1(x-x_0)$$

$$\text{at } x = x_0 \rightarrow f(x_0) = f(x_0)$$

$$f(x_0) = a_0 + a_1(x-x_0)$$

$$f_1(x) = f(x_0) + a_1(x-x_0)$$

$$\text{at } x = x_1, f_1(x) = f(x_1) \rightarrow \text{take}$$

$$f(x_1) = f(x_0) + a_1(x_1-x_0)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\therefore f_1(x) = f(x_0) + \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) (x - x_0)$$

x_i	$f(x_i)$
x_0	$f(x_0)$
x_1	$f(x_1)$

$$f_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$

at $x = x_0$, $f_2(x) = f(x_0)$

$$f(x_0) = a_0 + a_1(x_0-x_0) + a_2(x_0-x_0)(x_0-x_1)$$

$$\rightarrow f_2(x) = f(x_0) + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$

at $x = x_1$, $f_2(x) = f(x_1)$

$$f(x_1) = f(x_0) + a_1(x_1-x_0) + a_2(x_1-x_0)(x_1-x_1)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\rightarrow f_2(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x-x_0) + a_2(x-x_0)(x-x_1)$$

at $x = x_2$, $f_2(x) = f(x_2)$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

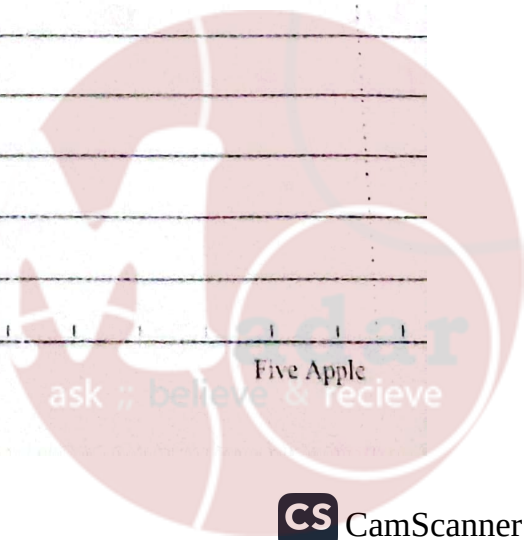
$$a_2 = \left\{ \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right\} - \left\{ \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right\}$$

$$\rightarrow f_2(x) = f_0(x) + \left\{ \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right\} (x-x_0) + \left\{ \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right\} (x-x_0)(x-x_1)$$

$a_0 = f(x_0)$, $a_1 = \text{slope of the first segment}$

③ Divided difference Table

x_i	$f(x_i)$	1 st D.D.	end D.D.
x_0	$f(x_0)$	$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$	We need two values
x_1	$f(x_1)$	a_1	



x_i	$f(x_i)$	1st D.O	2nd D.O	المقادير (x_0, x_1, x_2)
x_0	$f(x_0)$	$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$	$\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}$	
x_1	$f(x_1)$	a_1		
x_2	$f(x_2)$	$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$		

كل ما زاد الأعداد، يزيد عدد عناصر الأعمدة الخاصة به

1st
2nd
3rd -- P.D

Example 1

1st order lagrange $f(2)$

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

المقادير
number
of data
points

x_i	$f(x_i)$
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$

$$f_1(x) = \frac{x - 4}{1 - 4} (0) + \frac{x - 1}{4 - 1} (1.3862944)$$

$$f_1(2) = \frac{2 - 4}{1 - 4} (0) + \frac{2 - 1}{4 - 1} (1.3862944) = 0.46209813$$

$$f(2) = 0.69314 \text{ (true)}$$

$$f_2(x) = \frac{x - 4}{1 - 4} \frac{(x - 6)}{1 - 6} (0) + \frac{(x - 1)}{(4 - 1)} \frac{(x - 6)}{(4 - 6)} (1.3862944) + \frac{(x - 1)}{(6 - 1)} \frac{(x - 4)}{(6 - 4)} (1.7917595)$$

$$f_2(x) = x^2 -$$

$$f_2(x) = 0.56584432$$

Newton's

$$x_0 = 1 \quad f(x_0) = 0$$

$$f_1(x) = a_0 + a_1(x - x_0)$$

$$x_1 = 4 \quad f(x_1) = 1.3862944$$

$$a_0 = f(x_0) = 0$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1.3862944 - 0}{4 - 1} = 0.46209813$$

$$f_1(x) = 0 + 0.46209813(x - x_0)$$

$$f_1(2) = 0.46209813$$

$$f_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$

الأرقام من الجدول

$$a_0 = f(x_0) = 0$$

$$a_1 = 0.462098133$$

$$a_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} =$$

Example

x	-2	-1	0	1	2	3	4
y	-15	0	3	0	-3	0	15

$$-2 \leq x \leq 4$$

$$x^* = 1.6$$

حيث أقل مسافة عن أي من الحروف هو الصحيح والأبعد

$$f(x) = \frac{(x+1)(x-0)(x-1)(x-3)(x-4)}{(-2+1)(-2-0)(-2-1)(-2-3)(-2-4)}(-15) + \frac{(x+2)(x+1)(x+1)(x-2)(x-3)(x-4)}{(0+2)(0+1)(0-1)(0-2)(0-3)(0-4)}(3) + \frac{(x+2)(x+1)(x)(x-1)(x-3)(x-4)}{(-1+2)(-1+1)(-1)(-1-3)(-1-4)}(0)$$

~~The value of parameter has a big effect~~

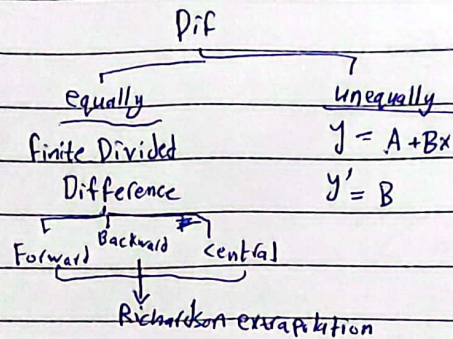
آتشه ای آن عدد از x ليس بالفرة أن يكون دليلاً على عدد الـ q



Differentiation and integration

x_i | $f(x_i)$

التمثيل العددي على الحاسوب



Differentiation

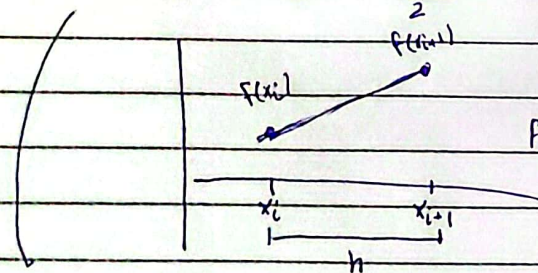
Equally spaced data points

Ⓐ Forward direction:-

Taylor Series

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2}h^2 + \dots$$

x_i	$f(x_i)$
Δx_1 [1	
= [2	
Δx_2 [3	
= [4	
Δx_3 [5	



$$f'(x_i) = \text{slope} = \frac{f(x_{i+1}) - f(x_i)}{h}$$

$h \rightarrow$ step size

Eliminating $f(x_i)$ and all other higher derivatives

$$\therefore f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

$$f'_i(1.4) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x}$$

Step Size

x	$f(x)$
x_i 1.4	
x_{i+1} 1.8	
x_{i+2} 2.2	

$$\therefore f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2}h^2$$

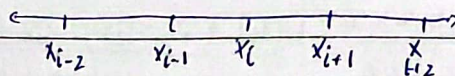
$$f'(x_i) = -\frac{f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2) \rightarrow \text{higher accuracy}$$

- if the error is $O(h)$, then halving the step size will halve the error

كلما قللت قيمة step size إلى النصف فإن error إلى 1.50 يقل

- if the error is $O(h^2)$, then halving the step size will quarter the error
بدي أن عمل فلط من أقل الخطأ

Forward



$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

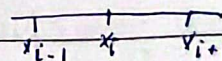
$$f'(x_i) = -\frac{f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2)$$

Backward | $f'(x_i) = \frac{f(x_i) - f(x_{i-1}))}{h} + O(h)$

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2}))}{2h} + O(h^2)$$

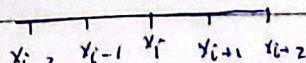
Central

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1}))}{2h} + O(h^2)$$



بدي أن يكون Backward

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2}))}{12h} + O(h^4)$$



شروط: 1- مسافات متساوية

2- عدم القدرة على إيجاد مشتقة لأي نقطة غير موجودة

unequally spaced data points

① you have to do Fitting:-

If $f'(x)$ is required, so at least a 2nd order function must be obtained {Lagrange, Newton's, least squares}

$$f_2(x) = \left(\frac{x-x_1}{x_0-x_1} \right) \left(\frac{x-x_2}{x_0-x_2} \right) f(x_0) + \left(\frac{x-x_0}{x_1-x_0} \right) \left(\frac{x-x_2}{x_1-x_2} \right) f(x_1) + \left(\frac{x-x_0}{x_2-x_0} \right) \left(\frac{x-x_1}{x_2-x_1} \right) f(x_2)$$

x_0	$f(x_0)$	x_0
x_1	$f(x_1)$	x_1
x_2	$f(x_2)$	x_2

$$f_2(x) = \frac{x^2 - x_2x - x_1x + x_1x_2}{x_0^2 - x_2x_0 - x_1x_0 + x_1x_2} f(x_0) + \frac{x^2 - x_2x - x_0x + x_0x_2}{x_0^2 - x_2x_0 - x_1x_0 + x_1x_2} f(x_1) + \frac{x^2 - x_1x - x_0x + x_0x_1}{x_0^2 - x_2x_0 - x_1x_0 + x_1x_2} f(x_2)$$

$f'(x_i) \rightarrow$ 2nd order lagrange $+ O(h^2)$

Example | Forward and Backward $\{O(h), O(h^2)\}$

Central $\{O(h^2), O(h^4)\}$ for $f'(\frac{\pi}{4})$ of the function $f(x) = \sin x$, $\Delta x = \frac{\pi}{12}$

Soln | $y' = \cos x$ $y'(\frac{\pi}{4}) = 0.7071$

Forward

$$f'(\frac{\pi}{4}) = \frac{f(x_{i+1}) - f(x_{i-1})}{h} + O(h)$$

$$f'(\frac{\pi}{4}) = \frac{0.866 - 0.7071}{\frac{\pi}{12}} + O(h) = 0.607 + O(h)$$

x	$f(x)$
$\frac{\pi}{12}$	0.259
$\frac{\pi}{6}$	0.5
$\frac{\pi}{4}$	0.7071
$\frac{\pi}{3}$	0.866
$\frac{5\pi}{12}$	0.9659

بطلهم بالذات
الخطية

$$\text{error} = \left| \frac{0.7071 - 0.607}{0.7071} \right| \times 100\% =$$

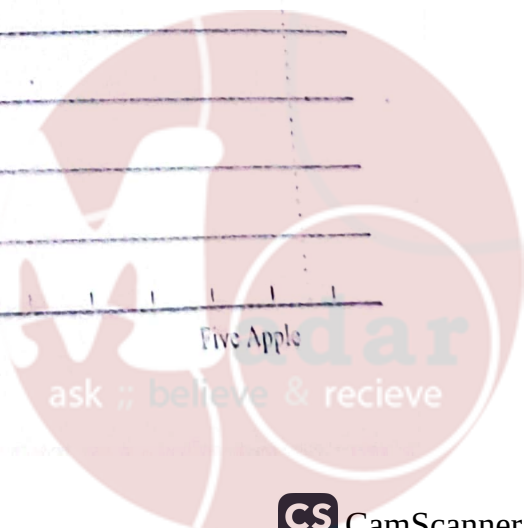
$$f'(\frac{\pi}{4}) = \frac{-0.9659 + 4(0.866) - 3(0.7071)}{2(\frac{\pi^2}{12})} + O(h^2) =$$

Backward

$$f'(\frac{\pi}{4}) = \frac{f(x_i) - f(x_{i+1})}{h} + O(h)$$

لا تكتب بالذات

$$f'(\frac{\pi}{4}) = \frac{3(0.7071) - 4(0.5) + 0.259}{2(\frac{\pi}{12})} + O(h^2) =$$



Central

$$f'(\pi) = \frac{f(x_{i+1}) - f(x_{i-1}))}{2h} + O(h^2) = \frac{0.866 - 0.5}{2 \times \frac{\pi}{12}} + O(h^2)$$

$$f'(\frac{\pi}{4}) = \frac{1}{12(\frac{\pi}{12})} [-0.9659 + 8(0.866) - 8(0.5) + 0.259] + O(h^4) = 0.70699 + O(h^4)$$

Example

$f'(0)$ of $O(h^2)$

$$f_2(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)}(0.1) + \frac{(x-0)(x-3)}{(1-0)(1-3)}(0.4)$$

$$+ \frac{(x-0)(x-1)}{(3-0)(3-1)}(0.9)$$

$$f_2(x) = \frac{(x^2 - 3x - x + 3)}{3}(0.1) + \frac{(x^2 - 3x)}{-2}(0.4) + \frac{(x^2 - x)}{6}(0.9)$$

$$f_2'(x) = \frac{(2x-4)}{3}(0.1) + \frac{(2x-3)}{-2}(0.4) + \frac{2x-1}{6}(0.9)$$

x_i	C
0	0.1
1	0.4
3	0.9

Differentiation

→ Richardson Extrapolation :-

$f'(x_i) \rightarrow$ target point

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h_i} + o(h)$$

less accurate $h_i = 4$

$$h_2 = \frac{1}{2} h_1 = 2$$

$$f'(x_0) = \frac{f(x_{i+2}) - f(x_i)}{2} + o(h)$$

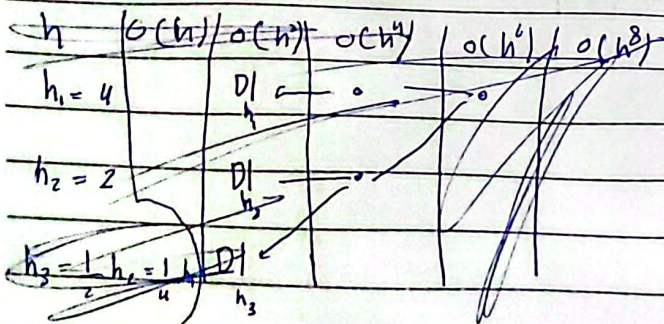
more accurate $h_2 = 2$

Δx	x_i	$f(x_i)$
$\Delta x = 1$	x_0	
$\Delta x = 1$	x_{i+1}	
$\Delta x = 1$	x_{i+2}	
$\Delta x = 1$	x_{i+3}	
$\Delta x = 1$	x_{i+4}	

$h_1 = 4$

D_1 less D_2 more
 h_1 h_2

$$D_{best} = D_2 + \frac{1}{\left(\frac{h_1}{h_2}\right)^2 - 1} [D_2 - D_1] + o(h^2)$$



h	$o(h)$	$o(h^2)$	$o(h^4)$	$o(h^6)$	$o(h^8)$
$h_1 = 4$	D_1				
$h_2 = 2$	D_2				
$h_3 = \frac{1}{2}h_1$	D_3				
$h_4 = \frac{1}{8}h_1$	D_4				

$$\frac{D_2 - D_1}{\frac{1}{4} - 1} = \frac{D_3 - D_2}{\frac{1}{16} - 1}$$



~~$h_1 = \frac{1}{4}$~~
 ~~$h_2 = \frac{1}{2}$~~
 ~~$h_3 = \frac{1}{4}$~~

Example use the richardson extrapolation to estimate $f'(x)$
 $y = \ln(x)$ at $x=4$ find $y'(4)$ using a step size of $\frac{1}{2}$, employ
 central difference of $O(h^2)$ to get $y'(x)$ at $O(h^4)$

$h_1=2$	2	0.693
$h_2=1$	3	1.099
$h_3=0.5$	4	1.257
$h_4=0.25$	5	1.386
$h_5=0.125$	6	1.504
$h_6=0.0625$	7	1.609
$h_7=0.03125$	8	1.791

h	$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
$h_1=2$	$D_1 = 0.2747$	$0.255 + \frac{1}{3}(0.255 - 0.2747)$ $= 0.2455$	$0.257 + \frac{1}{15}(0.255 - 0.2455)$ $= 0.2507$	
$h_2=1$	$D_1 = 0.255$	$0.2513 + \frac{1}{3}(0.255 - 0.2513)$ $= 0.2507$		
$h_3=0.5$	$D_1 = 0.243$			

$$D_1 = \frac{1.79176 - 0.693}{2 \times 2} = 0.2747 + O(h^2)$$

$$D_1 = \frac{1.6904 - 1.099}{2 \times 1} = 0.255 + O(h^2)$$

$$D_1 = \frac{1.5041 - 1.257}{2 \times \frac{1}{2}} = 0.2513 + O(h^2)$$

Integration | (Area under certain curve under certain range)

Range, أنسب بندا

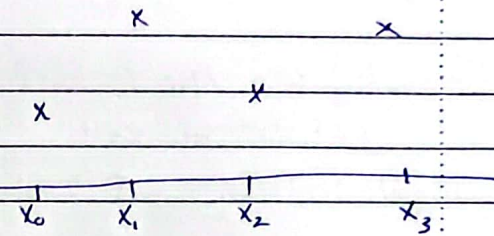
إذا بدلت نوب الدقة أفكار نقاط أخرى

$f(x_i)$

$$f_A(x) = f_1(x) = A + Bx$$

$$f_2(x) = \left(\frac{x - x_1}{x_0 - x_1} \right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0} \right) f(x_1)$$

$$f_i(x) = a_0 + a_1(x - x_0) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0)$$

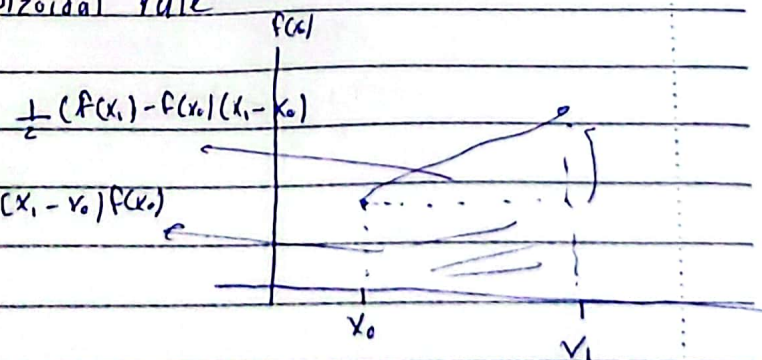


$$\int_{x_0}^{x_n} f_n(x) dx = \int_{x_0}^{x_n} f_i(x) dx = \int_{x_0}^{x_n} \left[f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) \right] dx$$

$$f(x_0)(x_1 - x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} \left\{ \frac{1}{2} x_1^2 - x_1 x_0 - \frac{1}{2} x_0^2 + x_0 x_0 \right\}$$

$$\int_{x_0}^{x_1} f_i(x) dx = \left[\frac{f(x_1) + f(x_0)}{2} \right] (x_1 - x_0)$$

$$= \frac{h}{2} [f(x_0) + f(x_1)] \text{ Trapezoidal rule}$$



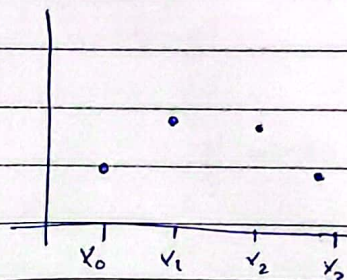
$$\frac{1}{2} (x_1 - x_0) f(x_0) \dots b/c$$



Multiple application of trapezoidal rule

$$\int_{x_0}^{x_3} f(x) dx = \left(\frac{x_3 - x_0}{2} \right) (f(x_0) + f(x_3))$$

$$\int_{x_0}^{x_3} f(x) dx = I_{x_0, x_1} + I_{x_1, x_2} + I_{x_2, x_3}$$



$$= \left(\frac{x_1 - x_0}{2} \right) (f(x_0) + f(x_1)) + \frac{x_2 - x_1}{2} (f(x_1) + f(x_2)) + \frac{x_3 - x_2}{2} (f(x_2) + f(x_3))$$

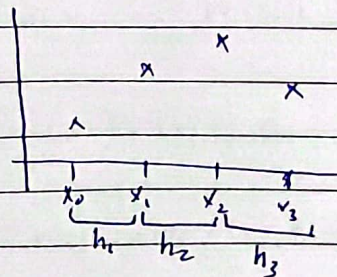
What if all h s were equal??

$$I_1 = \frac{h_1}{2} (f(x_0) + f(x_1))$$

$$I_2 = \frac{h_2}{2} (f(x_1) + f(x_2))$$

$$I_3 = \frac{h_3}{2} (f(x_2) + f(x_3))$$

$$\int_{x_0}^{x_3} f(x) dx = \frac{h}{2} [f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3)]$$



→ If we have equally spaced data

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} [f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n)] + o(h^2)$$

$$f_n(x) = f_2(x) = q_0 + q_1(x - x_0) + q_2(x - x_0)(x - x_1)$$

$$q_0 = f(x_0)$$

$$q_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$q_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\text{MUST! } \begin{cases} h_1 [x_0 \\ x_1] \\ h_2 [x_1 \\ x_2] \end{cases}$$



$$\int_{x_0}^{x_2} \left[f(x) + \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) (x - x_0) + a_2 (x - x_0) (x - x_1) \right] dx$$

$$\int_{x_0}^{x_2} f_2(x) dx = \frac{h}{3}$$

Integration

$$\int_{x_0}^{x_n} f_n(x) dx = \int_{x_0}^{x_n} f_n(x) dx ; n = 1, 2, 3, 4, \dots$$

$$n=1 \quad f(x) = f_1(x) = a_0 + a_1(x - x_0)$$

$$\int_{x_0}^{x_1} f_1(x) dx = \frac{h}{2} (f(x_0) + f(x_1))$$

$$n=2, \quad f(x) = f_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_1)(x - x_2)$$

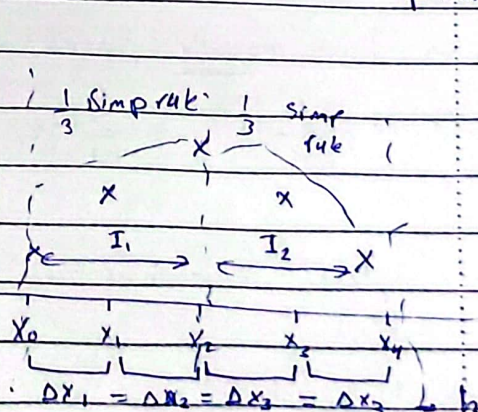
$$\int_{x_0}^{x_2} f_2(x) dx = \frac{h}{3} (f(x_0) + 4f(x_1) + f(x_2)) \quad \frac{1}{3} \text{ Simpson rule} \rightarrow \text{odd number of data points}$$

$$I_1 = \int_{x_0}^{x_2} \frac{1}{3} S.R. = \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + f(x_2))$$

$$I_2 = \int_{x_2}^{x_4} \frac{1}{3} S.R. = \frac{\Delta x}{3} (f(x_2) + 4f(x_3) + f(x_4))$$

$$I = \int_{x_0}^{x_n} = \int_{x_0}^{x_2} + \int_{x_2}^{x_4}$$

$$= \frac{\Delta x}{3} [f(x_0) + 4(f(x_1) + f(x_3)) + 2f(x_2) + f(x_4)]$$



* In General, for multiple application:-

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{3} \left[f(x_0) + f(x_n) + 4 \sum_{\substack{l=1 \\ l=2,4,6,\dots}}^{l=n-1} f(x_l) + 2 \sum_{l=2,4,6,\dots}^{l=n-2} f(x_l) \right]$$

→ odd number of points
→ even number of segments

$$n=3, \int_{x_0}^{x_3} f_3(x) dx, \quad f_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$\int_{x_0}^{x_3} f_3(x) dx = \frac{3}{8} h [f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] + O(h^5)$$

$\frac{3}{8}$ Simp rule

x_i	$f(x_i)$
x_0	
x_1	
x_2	
x_3	

	①	①	①	①
odd or even	①	①	①	①
number of data points	3	4	5	6
has no meaning				

→ Multiple application of $\frac{3}{8}$ S.R

$$I = \frac{3}{8} h \left[f(x_0) + 3 \sum_{i=1,2,4,5,8} f(x_i) + 2 \sum_{i=3,6,9} f(x_i) + f(x_n) \right]$$

→ Boole's Rule :- (5 Data points that are equally spaced)

6 data points

الانقل في ادارة النقاط المتساوية

	x_i	
1	1	1 S.R
1	2	2
1	3	1 T.R
2	4	2
1	5	3 S.R
1	6	4
1	7	5
1	8	6
2	9	7
5	10	8
5	15	9
5	20	10

→ Minimize number of formulas and maximize number of data points



Integrate the following function:- $\sqrt[3]{x^3+1}$ from 0 to 3 using

① 10 subintervals' $h = \frac{3-0}{10} = 0.3$

Gaussian Quadrature (Gauss Legendre) →

Table in slides is set to be used

	Point	weighing factor	function
$I = \int_a^b f(x) dx$	2	$C_0 = 1$	$x_0 = -0.57735$
step ① choose $C_1, C_2, \dots$		$C_1 = 1$	$x_1 = +0.57735$
$x_0, x_1, x_2, \dots$			

$C_0 = 1, C_1 = 1, x_0 = -0.577, x_1 = +0.577$ —
 تقريب الارقام

step ② $I = \int_a^b f(x) dx \approx C_0 f(x_0) + C_1 f(x_1)$

$I = C_0 f(-0.57735) + C_1 f(0.57735)$

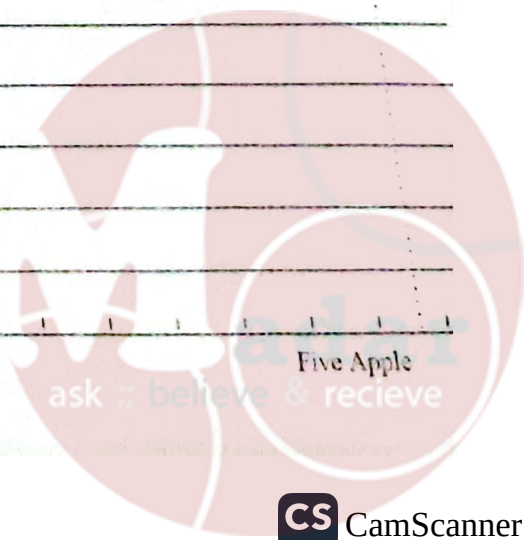
step ③ $x = \frac{(b-a)t}{2} + \frac{b+a}{2}, dx = \frac{(b-a)}{2} dt$
 must substitute

step ④ $dx = \frac{(b-a)}{2} dt$

step ⑤ $\int_a^b f(x) dx = \int_{-1}^{+1} f(t) dt = \int_{-1}^{+1} f\left(\frac{(b-a)t}{2} + \frac{b+a}{2}\right) \frac{(b-a)}{2} dt$
 $f(t) \rightarrow x_0, x_1, x_2, \dots$

$x = \frac{(b-a)t}{2} + \frac{b+a}{2}$

$a = \frac{(b-a)t}{2} + \frac{b+a}{2} \Rightarrow \frac{a}{2} - \frac{b}{2} = \frac{b-a}{2} t \quad t = -1$



$$\int_0^3 x e^{2x} dx$$

2 points $C_0 = C_1 = 1$

$$x_0 = -0.577 \dots \quad x_1 = +0.5775$$

$$\int_0^3 x e^{2x} dx \approx \int_{-1}^1 f(t) dt = (1) f(-0.577) + (1) f(0.577)$$

$$x = \frac{(b-a)t}{2} + b + \frac{a-b}{2} = \frac{(3-0)t}{2} + 3 + \frac{0-3}{2} = 1.5t + 1.5$$

$$dx = 1.5 dt$$

$$\int_{-1}^1 (1.5t + 1.5) e^{2(1.5t + 1.5)} \cdot 1.5 dt \quad ; \quad f(t) = 1.5(1.5t + 1.5) e^{2(1.5t + 1.5)}$$

$$3 = b \Rightarrow 3 = \frac{(3-0)t}{2} + 3 + \frac{0-3}{2} \Rightarrow 1.5t + 1.5$$

$$\int_0^3 f(x) dx = (1) f(-0.577) + (1) f(0.577) \quad 1.5 = 1.5t$$

$$= (1) \left[1.5(1.5(-0.577) + 1.5) e^{2(1.5(-0.577) + 1.5)} \right] + (1) \left[1.5(1.5(0.577) + 1.5) e^{2(1.5(0.577) + 1.5)} \right] =$$

$$(1.5)(1) [268.6102 + 225.287] = 493.897 \dots$$

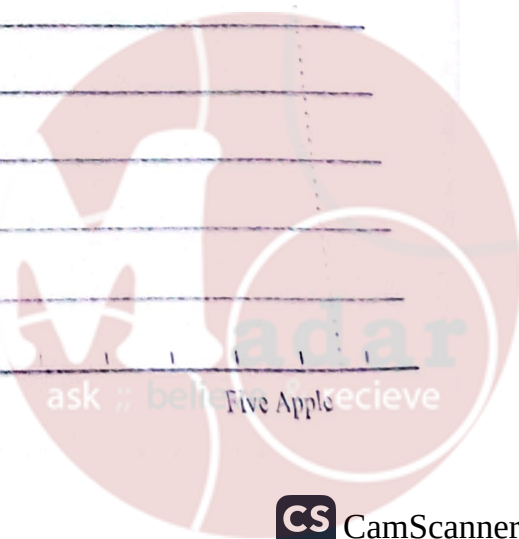
$$3 \text{ points} \quad C_0 = 0.555 \quad C_1 = 0.888 \quad C_2 = 0.555$$

$$x_0 = -0.77459669 \quad x_1 = 0 \quad x_2 = +0.77459669$$

$$\int_0^3 f(x) dx = \text{Substitute into formula}$$

check ≈ 495.82

Try at home $\int_0^3 \frac{e^x \sin x}{1+x^2} dx$



Romberg integration

$$I_{best} = I_{more} + \frac{1}{\left(\frac{h_1}{h_2}\right)^2 - 1} [I_{more} - I_{less}]$$

h	$O(h^2)$	$O(h^4)$	$O(h^6)$
h_1	.	.	.
$h_2 = \frac{1}{2} h_1$	.	.	.
$h_3 = \frac{1}{2} h_2$	.	.	.

Example 1

$$\int_0^3 x e^x dx$$

obtain the integration
 $O(h^4)$

at $h_1 = 3$ $\int_0^3 f(x) dx$ using T.R

$$I_1 = \frac{3}{2} [6 + 1210.286] = 1815.429$$

$$\text{at } h_2 = 1.5 \quad \int_0^3 f(x) dx$$

$$= \int_0^{1.5} f(x) dx + \int_{1.5}^3 f(x) dx$$

$$= \frac{1.5}{2} [\quad] = 952.902$$

h	$O(h^2)$	$O(h^4)$	$O(h^6)$
$h_1 = 3$	.	.	.
$h_2 = 1.5$	.	.	.
$h_3 = 0.75$	.	.	.

x	
0	0
0.75	3.34127
1.5	30.1283
2.25	202.3325
3	1210.286

$$952.902 + \frac{1}{2} [952.902 - 1815.429] = 665.3997 + O(h^4)$$

Ordinary differential equation

IVP

BVP

Direct solution

$$y(0), y'(0), y''(0)$$

$$y(0), y(1)$$

→ واحدة من

$$y(2.5), y'(2.5), y''(2.5)$$

$$y(0), y(1), y'(0)$$

القيمة

← متى تكون المعادلة مصنفة كـ IVP و BVP يجب أن تكون Second order و كلاهما معادلات أولية

عبارة عن IVP

IVP

1st O.D.E

higher order
بالترتيب
بالمرتبة

1st order ordinary differential equation

- ① Euler
- ② Polygon
- ③ Heun's
- ④ Runge - Kutta

التقريب بالخط

Euler

$$y_{i+1} = y_i + f(x_i, y_i) h$$

$f(x, y)$ is the differential equation evaluated at

x_i, y_i, h is the step size

$$x=0 \quad \text{to} \quad x=4 \quad h_1 = 1$$

$$f'(x, y) = \text{---} \quad y(0) = \text{given (2)}$$

at $x=1$

$$y(1) = y(0) + f'(0, y(0)) h$$

$$= 2 + f'(0, 2)(1) = 3.5 \quad (\text{نقطة})$$

at $x=2$

$$y(2) = y(1) + f'(1, y(1)) h$$

$$= 3.5 + f'(1, 3.5)(1) = 4 \quad (\text{نقطة})$$

example

$$f'(x) = -2x^3 + 12x^2 - 20x + 8.5$$

$$f(0) = 1$$

$$(x=0 \text{ to } x=4), h=1$$

at $x=1$

$$y(1) = y(0) + f'(0,1)x(1)$$

$$f'(0,1) = 8.5 \rightarrow \text{تعويض}$$

$$y(1) = 1 + 8.5(1) = 9.5$$

at $x=2$

$$y(2) = y(1) + f'(1,9.5)(1)$$

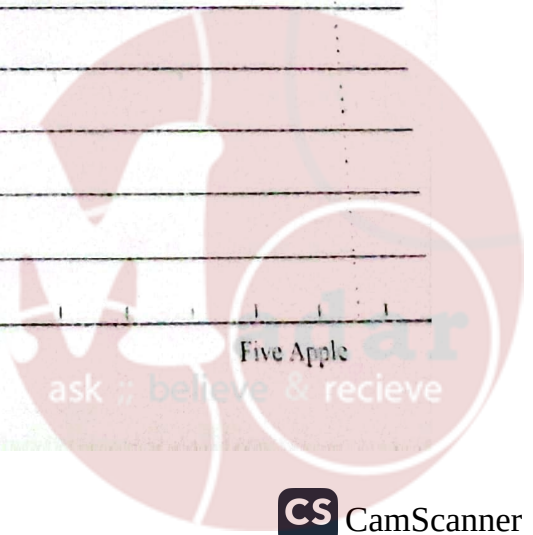
$$f'(1,9.5) = -1.5$$

$$= 9.5 - 1.5(1) = 8$$

at $x=3$

$$y(3) = y(2) + f'(2,8)(1)$$

if step size is large, there's gonna be big deviation



Ordinary Differential equations

1) First O.O.D.E

A - Euler Method

$$y(x_{i+1}) = y(x_i) + f'(x_i, y(x_i)) \times h$$

B - ~~Poly~~ Heun's method

$$y(x_{i+1}) = \left\{ \frac{y(x_i) + f'(x_i, y(x_i)) + f'(x_{i+1}, y(x_{i+1}))}{2} \right\} h$$

$$y^p(x_{i+1}) = y(x_i) + f'(x_i, y(x_i)) \times h$$

C - improved polygon method:- Euler's

$$y(x_{i+1}) = y(x_i) + y'(x_{i+\frac{1}{2}}, y(x_{i+\frac{1}{2}})) \times h$$

$$y(x_{i+\frac{1}{2}}) = y(x_i) + f'(x_i, y(x_i)) \frac{h}{2}$$

$$\frac{dx}{dy} = 4e^{0.8x} - 0.5y$$

$$y(0) = 2$$

$$y' + 0.5y = 4e^{0.8x}$$

$$y = e^{-\frac{0.5x}{1}} \left[e^{\frac{0.5x}{1}} \int 4e^{0.8x} dx + C \right]$$

Heun's method:-

$$y' = 4e^{0.8x} - 0.5y$$

$$y(0) = 2 \quad h = 1$$

$$y(1) = y(0) + f'(0, 2) \times h$$

From $x = 0$ to $x = 4$

$$= 2 + f'(0, 2) \times 1$$

$$f'(0, 2) = 4e^{0.8 \times 0} - 0.5(2) = 4 - 1 = 3$$

$$y(1) = 2 + 3(1) = 5$$

$$y(1) = y(0) + \left\{ \frac{f'(0, 2) + f'(1, y(1))}{2} \right\} \times (1)$$

$$y^p(1) = y(0) + f'(0, 2) \times (1)$$

$$f'(0,2) = 3$$

$$f \rightarrow y(1) = 2 + (3)(1) = 5$$

$$\therefore f'(1,5) = 4e^{0.8(1)} - 0.5(5) = 6.4021637$$

$$y(1) = 2 + \left\{ \frac{3 + 6.4021637}{2} \right\} (1) = 6.70108185$$

$$\text{true } y(1) = 6.19463138$$

$$x=2$$

$$y(2) = y(1) + f'(1, 6.70108185)$$

$$f'(x_1, y_1) = 4e^{0.8x_1} - 0.5(6.70108185) = 5.55162278$$

$$y(2)^f = 6.70108185 + 5.55162278(1) = 12.2527$$

$$f'(2, y(2)) = 4e^{0.8(2)} - 0.5(12.2527) = 13.685772$$

$$\therefore y(2) = 6.70108185 + \left\{ \frac{5.55 + \dots + 13.68}{2} \right\} (1) = 16.3197893$$

true =

$$\text{Polygon} \quad \text{at } x=1, y(1) = y(0) + \hat{y}(x_i + \frac{1}{2} \Delta x, y(x_i + \frac{1}{2} \Delta x)) \Delta x$$

$$y(0.5) = y(0) + f'(0,2) \times \frac{1}{2}$$

$$f'(0,2) = 4e^{0.8(0)} - 0.5(2) = 3$$

$$y(0.5) = 2 + 3 \times \frac{1}{2} = 3.5$$

$$y(1) = 2 + f'(0.5, 3.5)(1)$$

$$y(1) = 2 + 4.217298 = 6.217298$$



at $x=2$

$$y(2) = y(1) + f(1.5, y(1.5)) \times 1$$

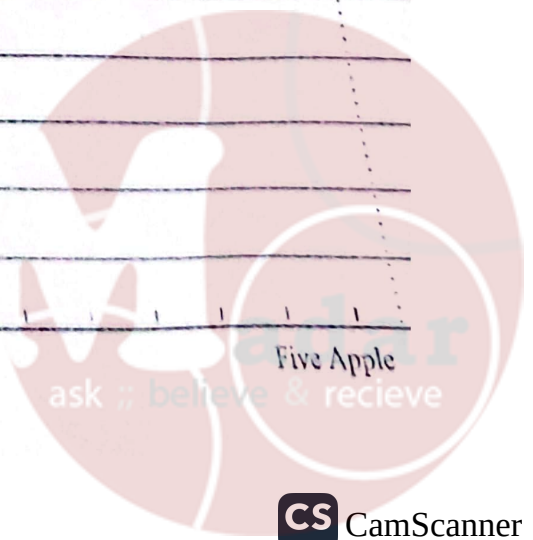
$$y(1.5) = y(1) + f(1.5, y(1.5)) \times \frac{1}{2}$$

$$f' = 5.793515$$

$$y(1.5) = 6.217298 + \frac{1}{2} \times 5.793515 = 9.114055$$

$$f(1.5, y(1.5)) = 8.7234399$$

$$y(2) = \overset{6.217298}{y(1)} + (8.7234399) \times 1 = 14.94073$$



First O.O.D.E

Runge-Kutta methods:-

$$y(x_{i+1}) = y(x_i) + \Phi(x_i, y(x_i), h) \cdot h$$

$$y(x_{i+1}) = y(x_i) + f'(x_i, y(x_i)) \cdot h$$

$$\Phi = a_1 K_1 + a_2 K_2 + a_3 K_3 + \dots + a_n K_n$$

a 's are constants

$$K_1 = f'(x_i, y_i)$$

1st order

$$\Phi = a_1 K_1 = a_1 f'(x_i, y_i)$$

$$y(x_{i+1}) = y(x_i) + a_1 f'(x_i, y_i) \cdot h$$

if $a_1 = 1$, then R.K (1st order)

$$y(x_{i+1}) = y(x_i) + f'(x_i, y_i) \cdot h \quad \text{Euler equation}$$

$$K_2 = f'(x_i + p_1^* h, y(x_i) + q_{11} K_1 h)$$

$$K_3 = f'(\underbrace{x_i + p_2^* h}_{x_{\text{new}}}, \underbrace{y(x_i) + q_{21} K_1 h + q_{22} K_2 h}_{y_{\text{new}}})$$

If $n=2$ $\therefore$ 2nd R-K

$$y(x_{i+1}) = y(x_i) + \Phi(x_i, y_i, h) \cdot h$$

$$\Phi = a_1 K_1 + a_2 K_2$$

if $a_1 + a_2 = 1$, $a_1 p_1 = 0.5$, $a_2 q_{11} = 0.5$ 4 unknowns

Case ① if we let $a_2 = \frac{1}{2}$

$$a_1 + \frac{1}{2} = 1 \quad \therefore a_1 = \frac{1}{2}$$

$$\frac{1}{2} p_1 = \frac{1}{2} \quad \therefore p_1 = 1$$

$$\frac{1}{2} q_{11} = \frac{1}{2} \quad \therefore q_{11} = 1$$

$$\Phi = \frac{1}{2} K_1 + \frac{1}{2} K_2$$

$$K_1 = f'(x_i, y(x_i))$$

$$K_2 = f'(x_i + (1)h, y(x_i) + (1) \cdot f'(x_i, y(x_i)) \cdot h)$$

$$\phi = \frac{1}{2} K_1 + \frac{1}{2} K_2$$

$$\therefore y(x_{i+1}) = y(x_i) + \left[\frac{1}{2} f'(x_i, y_i) + \frac{1}{2} [f'(x_i + h, y_i) + f'(x_i, y_i)] \cdot h \right] h$$

Heun's

$$y(x_{i+1}) = y(x_i) + \left(\frac{f'(x_i, y_i) + f'(x_{i+1}, y_{i+1})}{2} \right) h$$

$$y(x_{i+1}) = y(x_i) + f'(x_i, y_i) \cdot h$$

~~2nd~~ 2nd order R.K when $a_2 = \frac{1}{2}$, this will reduce it to heun's method

if $a_2 = 1$

$$a_1 + a_2 = 1 \quad \therefore a_1 = 0$$

$$p_1 a_2 = \frac{1}{2} \quad \therefore p_1 = \frac{1}{2}$$

$$q_1 a_2 = \frac{1}{2} \quad \therefore q_1 = \frac{1}{2}$$

$$\therefore y(x_{i+1}) = y(x_i) + \left(\frac{1}{2} K_2 \right) h$$

$$K_1 = f'(x_i, y_i)$$

$$K_2 = f'(x_i + \frac{1}{2}h, y(x_i) + \frac{1}{2}K_1 h)$$

$$y(x_{i+1}) = y(x_i) + \frac{h}{2} [f'(x_i + \frac{1}{2}h, y(x_i) + \frac{1}{2}f'(x_i, y_i) \cdot h)]$$

polygen $\therefore y(x_{i+1}) = y(x_i) + f'(x_i + \frac{1}{2}h, y(x_i + \frac{1}{2}h)) h$

$$y(x_{i+1}) = y(x_i) + f'(x_i + \frac{1}{2}h, y(x_i) + f'(x_i, y_i) \cdot \frac{h}{2}) h$$



Runge-Kutta

$$y(x_{i+1}) = y(x_i) + \phi(x_i, y(x_i), h) \cdot h$$

$$a_2 = \frac{2}{3} \quad \text{Rattson}$$

$$y(x_{i+1}) = y(x_i) + \left(\frac{1}{2} K_1 + \frac{2}{3} K_2 \right) h$$

$$\text{where } K_1 = f'(x_i, y(x_i))$$

$$K_2 = f'\left(x_i + \frac{3}{4}h, y(x_i) + \frac{3}{4}K_1h\right)$$

3rd order Runge-Kutta

$$K_1 = f'(x_i, y(x_i))$$

$$\phi = a_1 K_1 + a_2 K_2 + a_3 K_3$$

$$K_2 = f'\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1h\right)$$

$$K_3 = f'(x_i + h, y_i + 2K_2h - K_1h)$$

$$y(x_{i+1}) = y(x_i) + \frac{1}{6} [K_1 + 4K_2 + K_3]h$$

4th order

$$a_1 = a_4 = \frac{1}{6}$$

$$a_2 = a_3 = \frac{2}{6}$$

$$\phi = a_1 K_1 + a_2 K_2 + a_3 K_3 + a_4 K_4$$

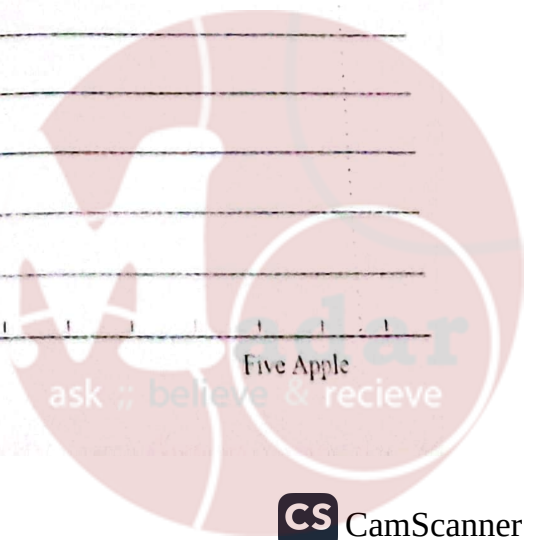
$$K_1 = f'(x_i, y_i)$$

$$K_2 = f'\left(x_i + \frac{1}{2}h, y(x_i) + \frac{1}{2}K_1h\right)$$

$$K_3 = f'\left(x_i + \frac{1}{2}h, y(x_i) + \frac{1}{2}K_2h\right)$$

$$K_4 = f'(x_i + h, y(x_i) + K_3h)$$

$$y(x_{i+1}) = y(x_i) + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4]h$$



5th order / Butcher's

$$y(x_{i+1}) = y(x_i) + \frac{h}{90} [7K_1 + 32K_2 + 12K_3 + 32K_4 + 7K_5]$$

$$\phi = a_1 K_1 + a_2 K_2 + a_3 K_3 + a_4 K_4 + a_5 K_5 + a_6 K_6$$

→ look at the sheet for K's

System of 1st ODE

$$y_1' = f_1(x, y_1, y_2, \dots, y_n) \quad y_1(x_0) = \checkmark$$

$$y_2' = f_2(x, y_1, y_2, \dots, y_n) \quad y_2(x_0) = \checkmark$$

$$\vdots$$
$$y_n' = f_n(x, y_1, y_2, \dots, y_n) \quad y_n(x_0) = \checkmark$$

$$x_0 \xrightarrow{t_0} x_1$$

at x_1

$$y_1(x_1) = y_1(x_0) + y_1'(x_0, y_1(x_0), y_2(x_0), \dots, y_n(x_0))h$$

$$y_2(x_1) = y_2(x_0) + y_2'(x_0, y_1(x_0), y_2(x_0), \dots, y_n(x_0))h$$

$\vdots$

$$y_n(x_1) = y_n(x_0) + y_n'(x_0, y_1(x_0), y_2(x_0), \dots, y_n(x_0))h$$

at $x = x_2$

$$y_1(x_2) = y_1(x_1) + y_1'(x_1, y_1(x_1), y_2(x_1), \dots, y_n(x_1))h$$

$$y_2(x_2) = y_2(x_1) + y_2'(x_1, y_1(x_1), y_2(x_1), \dots, y_n(x_1))h$$

$$y_n(x_2) = y_n(x_1) + y_n'(x_1, y_1(x_1), y_2(x_1), \dots, y_n(x_1))h$$

31/12/2023

$$\frac{dy}{dx} = 4e^{0.8x} - 0.5y, \quad y(0) = 2 \quad \text{from } x=0 \text{ to } x=4 \quad h=1$$

Runge-Kutta $n=1$

$$y(x_{i+1}) = y(x_i) + (a, K_1) h$$

$$K_1 = f'(x_i, y_i)$$

$$y(1) = y(0) + K_1(1)$$

$$K_1 = f'(0, 2) = 4e^{0.8(0)} - 0.5(2) = 3$$

$$y(1) = 2 + 3(1) = 5$$

$$x_2 = 2 \quad y(2) = y(1) + K_1(1)$$

$$K_1^{\circ} = f'(1, 5) = 4e^{0.8(1)} - 0.5(5) = 6.4021637$$

$$y(2) = 5 + 6.4021637(1) = 11.4021637$$

Runge-Kutta $a_2 = \frac{1}{2}$, 2nd order R-K

$$y(x_{i+1}) = y(x_i) + \frac{1}{2}(K_1 + K_2)h$$

$$K_1 = f'(x_i, y_i), \quad K_2 = f'(x_i + h, y_i + K_1 h)$$

at $x=1$

$$y(1) = y(0) + \frac{1}{2}(K_1 + K_2)h \quad \therefore K_1 = f'(0, 2) = 3$$

$$\therefore K_2 = f'(\overset{x_{\text{new}}}{0+1}, 2+3 \times 1) = f'(1, 5) = 6.4021637$$

$$y(1) = 2 + \frac{1}{2}(3 + 6.4021637) = 6.7010818$$

$\therefore$ at $x=2$

$$y(2) = y(1) + \frac{1}{2}(K_1^{\circ} + K_2^{\circ})h \quad K_1 = f'(1, 6.7010818) = 5.5516228$$

$$K_2 = f'(1+1, 6.7010818 + 5.5516228(1)) = 13.68522$$

$$y(2) = 6.7010818 + \frac{1}{2}(5.5516228 + 13.68522)(1) = 16.31688$$

2nd order R-K $a_2 = 1$

$$y(x_{i+1}) = y(x_i) + K_2 h$$

$$K_1 = f'(x_i, y_i), \quad K_2 = f'(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h)$$

at $x=1$

$$\therefore K_1 =$$

$$y(1) = y(0) + K_2(1)$$

$$\therefore K_2 = f'(0.5, 3.5) = 4.2172987$$

$$y(1) = 2 + 4.2172987 = 6.2172987$$

at $x=2$

$$y(2) = y(1) + K_2^{\circ}(1)$$

$$K_1^{\circ} = f'(1, 6.2172987) = 5.8156702$$

$$K_2^{\circ} = f'(1 + \frac{1}{2}(1), 6.2172987 + \frac{1}{2} \cdot 5.8156702(1)) = f'(1.5, 9.1250438) = 8.7179207$$

$$y(2) = 6.2172987 + 8.7179207(1) = 14.9352194$$

3rd order R-K | $y(x_{i+1}) = y(x_i) + \frac{1}{6} [K_1 + 4K_2 + K_3](h)$

$$K_1 = f'(x_i, y_i), K_2 = f'(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1h), K_3 = f'(x_i + h, y_i + K_1h)$$

at $x=1$

$$y(1) = y(0) + \frac{1}{6} [K_1 + 4K_2 + K_3](h) \quad \therefore K_2 = f'(0 + \frac{1}{2}(1), 2 + \frac{1}{2}(3)(1)) = f'(0.5, 3.5) = 4.2172987$$

$$\therefore K_3 = f'(0 + 1, 2 + (3)(1) + 4.2172987(1)) = f'(1, 7.4345974) = 5.8156702$$

$$y(1) = 2 + \frac{1}{6} [3 + 4(4.2172987) + 5.8156702](1) = 5.84864924$$

$$= 6.175676681$$

at $x=2$

$$y(2) = 6.175676681 + \frac{1}{6} [5.814325 + 4(8.735548) + 10.89489] \quad \therefore K_1^{\circ} = f'(1, 6.175676681) = 5.814325$$

$$\therefore K_2^{\circ} = f'(1 + \frac{1}{2}(1), 6.175676681 + \frac{1}{2}(5.814325)(1)) = f'(1.5, 9.089839) = 8.735548$$

$$= f'(1.5, 9.089839) = 8.735548$$

$$\therefore K_3^{\circ} = f'(1 + 1, 6.175676681 + 5.814325(1) + 2(8.735548)(1)) = f'(2, 17.832447) = 10.89489$$

$$= f'(2, 17.832447) = 10.89489$$

System of 1st o.o.D.E

$$\frac{dy_1}{dx} = -0.5y_1$$

$$y_1(0) = 4$$

$$y_2(0) = 6$$

$$x=0 \text{ to } x=4$$

$$\frac{dy_2}{dx} = 4 - 0.3y_2 - 0.1y_1$$

$$h=1$$

Euler's method

$$x=1$$

$$y_1(1) = y_1(0) + y_1'(0, 4, 6)$$

$$y_1'(0, 4, 6) = -0.5y_1 = -0.5(4) = -2$$

$$\therefore y_1(1) = 4 + (-2)(1) = 2$$

$$y_2(1) = y_2(0) + y_2'(0, 4, 6)(1)$$

$$\cancel{y_2(0)}$$

$$y_2'(0, 4, 6) = 4 - 0.3(6) - 0.1(4) = 1.8$$

$$y_2(1) = 6 + (1.8)(1) = 7.8$$

$$y_2(1) = 7.8$$

$$y_1(1) = 2$$

$$\text{at } x=2$$

$$y_1(2) = y_1(1) + y_1'(1, 2, 7.8)(1)$$

$$y_1'(1, 2, 7.8) = -0.5(2) = -1$$

$$\therefore y_1(2) = 2 + (-1)(1) = 1$$

$$y_2(2) = y_2(1) + y_2'(1, 2, 7.8)(1)$$

$$y_2'(1, 2, 7.8) = 4 - 0.3(7.8) - 0.1(2) = 1.46$$

$$y_2(2) = 7.8 + 1.46(1) = 9.26$$

Same example with 4th R-K

$$y(x_{i+1}) = y(x_i) + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4](h)$$

$$K_1 = f'(x_i, y_1(x_i), y_2(x_i))$$

$$K_1^R \quad K_1^O$$

$$K_2 = f'(x_i + \frac{1}{2}h, y_1(x_i) + \frac{1}{2}K_1^O h, y_2(x_i) + \frac{1}{2}K_1^O h)$$

$$K_2^R \quad K_2^O$$

$$K_3 = f'(x_i + \frac{1}{2}h, y_1(x_i) + \frac{1}{2}K_2^O h, y_2(x_i) + \frac{1}{2}K_2^O h)$$

$$K_3^O \quad K_4^O$$

$$K_4 = f'(x_i + h, y_1(x_i) + K_3^O h, y_2(x_i) + K_3^O h)$$

$$y_1' = -0.5y_1$$

$$y_2' = 4 - 0.3y_2 - 0.1y_1$$

$$y_1(0) = 4$$

$$y_2(0) = 6$$

K_1^0

K_1^0

الحقبة الأولى

$$K_2^0 = y_1'(0 + \frac{1}{2}, 4 + \frac{1}{2}(-2)(1), 6 + \frac{1}{2}(1.8)(1)) = -0.5(3) = -1.5$$

$$K_2^0 = y_2'(1/2, 3, 6.9) = 4 - 0.3(6.9) - 0.1(3) = 1.63$$

$$K_3^0 = y_1'(0 + \frac{1}{2}, 4 + \frac{1}{2}(-1.5)(1), 6 + \frac{1}{2}(1.63)(1)) = y_1'(1/2, 3.25, 6.815) = -1.625$$

$$K_3^0 = y_2'(1/2, 3.25, 6.815) = 1.6305$$

$$K_4^0 = y_1'(0 + 1, 4 + (-1.625)(1), 6 + 1.6305(1)) = y_1'(1, 2.375, 7.6305) = -1.1875$$

$$K_4^0 = y_2'(1, 2.375, 7.6305) = 1.47335$$

$$y_1(1) = y_1(0) + \frac{1}{6} [K_1^0 + 2K_2^0 + 2K_3^0 + K_4^0]$$

$$= 4 + \frac{1}{6} [-2 + 2(-1.5) + 2(-1.625) + (-1.1875)] (1) = 2.427083$$

$$y_2(1) = 6 + \frac{1}{6} [1.8 + 2(1.63) + 2(1.6305) + 1.47335] (1) = 7.632391667$$

Higher O.D.E

$$y'' = g(x, y, y') + \frac{y(0)}{y'(0)}$$

$$y''' = g(x, y, y', y'')$$

let $u = y \rightarrow u' = y' = v$

$$v' = y'' \rightarrow v' = y'' = g(x, u, v)$$

→ cannot solve as BVP

$$u' = v$$

$$y(0) = u(0)$$

$$v' = g(x, u, v)$$

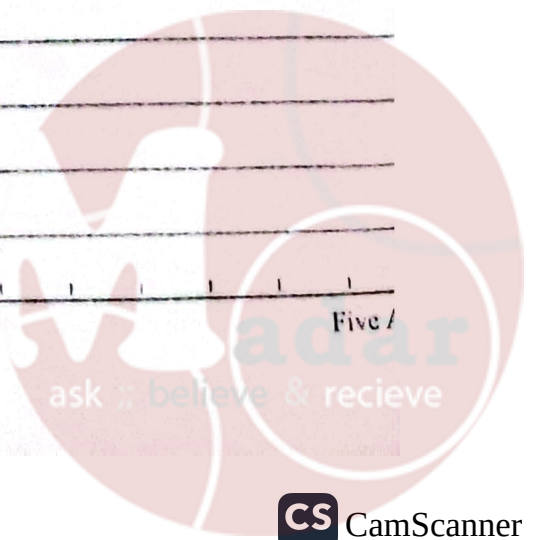
$$y'(0) = v(0)$$

$$u = y \rightarrow u' = y' = v$$

$$v = y' \rightarrow v' = y'' = w$$

$$w = y'' \rightarrow w' = y'''$$

$$= g(x, u, v, w)$$



Higher O.D.E

$$y'' = f(x, y, y')$$

$$\text{let } u = y$$

$$u' = y'$$

$$y(0) = 1$$

$$y'(0) = 1$$

$$v = y'$$

$$v' = y'' = f(x, y, y')$$

$$v(0) = 1$$

$$y'' = t + y = 0 \Rightarrow y'' = t - y, \quad y(0) = 2$$

$$u = y \Rightarrow u' = y'$$

$$y'(0) = 0$$

$$v = y' \Rightarrow v' = y'' = t - u$$

$$u(0) = 2, \quad v(0) = 0$$

from 0 to 0.1

$$t = 0.1$$

$$h = 0.1$$

$$u(0.1) = u(0) + u'(0, 2, 0)h$$

$$v(0.1) = v(0) + v'(0, 2, 0)(0.1)$$

$$= 2 + (0)(0.1) = 2$$

$$v'(0, 2, 0) = 0 - 2 = -2$$

$$v(0.1) = 0 + (-2)(0.1)$$

$$v(0.1) = -0.2$$

$$t = 0.2$$

$$u(0.2) = u(0.1) + u'(0.1, 2, -0.2)(0.1)$$

$$v(0.2) = v(0.1) + v'(0.1, 2, -0.2)$$

$$u'(0.1, 2, -0.2) = -0.2$$

$$v'(0.1, 2, -0.2) = 0.1 - 2 = -1.9$$

$$u(0.2) = 2 + (-0.2)(0.1) = 1.98$$

$$v(0.2) = -0.2 + (-1.9)(0.1)$$

$$v(0.2) = -0.39$$

Using Heun's method

$$y(x_{i+1}) = y(x_i) + \frac{y'(x_i, y(x_i)) + y'(x_{i+1}, y(x_{i+1}))}{2} (h)$$

$$t = 0.1 \quad u(0.1) = u(0) + \frac{0}{2} + \frac{-0.2}{2} (0.1)$$

$$u(0.1) = u(0) + u'(0, 2, 0)(0.1)$$

$$v(0.1) = v(0) + v'(0, 2, 0)(0.1)$$

$$u' = 0, \quad u''(0.1) = 2$$

$$v'(0, 2, 0) = -2, \quad v(0.1) = 0 + (-2)(0.1) = -0.2$$

$$\therefore u'(0.1, 2, -0.2) = -0.2$$

$$u(0.1) = 2 + \left(\frac{0 - 0.2}{2} \right) (0.1) = 2 - 0.1 \times 0.1 = 1.99$$

$$V(0.1) = V(0) + \overset{-2}{V'(0, 2, 0)} + \overset{-1.94}{V'(0.1, 2, -0.2)}(0.1)$$

$$V'(0, 2, 0) = 0 - 2 = -2$$

$$V'(0.1, 2, -0.2) = 0.1 - 2 = -1.94$$

$$\therefore V(0.1) = \underline{\underline{0.145}}$$

Runge Kutta 4th

$$y'' + 0.5y' + 5y = 0$$

$$y(0) = 4, y'(0) = 0$$

$$\text{From } x=0 \text{ to } x=1 \quad h=0.5$$

$$y'' = -0.5y' - 5y$$

$$y(0) = 4, y'(0) = 0$$

$$\text{Let } u=y \Rightarrow u' = y' = V$$

$$V = y' \Rightarrow V' = y'' = -0.5V - 5u$$

$$F(V, u)$$

$$t=0.5$$

$$u(0.5) = u(0) + \frac{1}{6} [K_1^u + 2K_2^u + 2K_3^u + K_4^u](0.5)$$

$$K_1^u = u'(0, 4, 0) = 0$$

$$K_2^u = u'(0 + \frac{1}{2}(0.5), 4 + \frac{1}{2}(0)(0.5), 0 + \frac{1}{2}(-2)(0.5))$$

$$(0.5)$$

$$K_2^u = u'(0.25, 4, -5) = -5$$

$$K_3^u = u'(0 + \frac{1}{2}(0.5), 4 + \frac{1}{2}(-5)(0.5), 0 + \frac{1}{2}(-5)(0.5)) = u'(0.25, 2.75, -4.375) = -4.375$$

$$K_3^V = V'(0.25, 2.75, -4.375) = -0.5(-4.375) - 5(2.75) = -11.5625$$

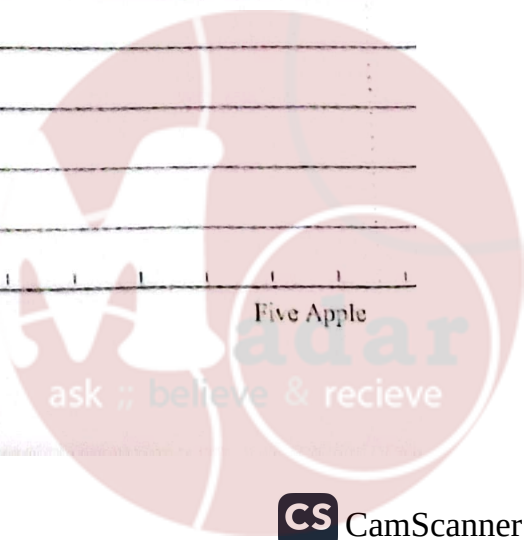
$$K_4^u = u'(0 + 0.5, 4 + (-4.375)(0.5), 0 + (-11.5625)(0.5)) = -5.78125$$

$$K_4^V = V'(0.5, 1.8125, -5.78125) = -6.171875$$

$$u(0.1) = 4 + \frac{0.5}{6} [0 + 2(-5) + 2(-4.375) + 5.78125] = 1.95578167$$

$$\cancel{V(0.1) = 0}$$

$$V(0.1) = 0 + \frac{0.5}{6} [-20 + 2(-17.5) + 2(-11.5625) + -6.171875] = -2.024739$$



2nd order and above and no given initial values

ex) $y'' + 2y' + 3y = e^{2x}$ $y(0) = 2$ $y(2) = 3$ $h = 0.5$

let $y'(0) = 1.5$

$y(2) = 4.25$

$y'(0) = 0.9$

$y(2) = 1.3$

shooting method

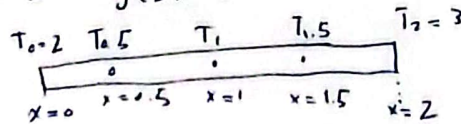
trial and error

probably not coming
in exam

due to not being
straight forward

Finite difference

$y(0) = 2$ $y(2) = 3$



13.3.1
first order
 $y'' + 2y' + 3y = e^{2x}$

$$\left. \frac{dy}{dx} \right|_{x=0.5} = \frac{T_1 - T_0}{2(0.5)}$$

at $x = 0.5$

$$\frac{T_1 - T_0}{2(0.5)} + 3T_{0.5} = e^{2x} \dots T_1$$

$$\left. \frac{dy}{dx} \right|_{x=1} = \frac{T_{1.5} - T_{0.5}}{2(0.5)}$$

at $x = 1$

$$\frac{T_{1.5} - T_{0.5}}{2(0.5)} + 3T_{1.5} = e^{2x} \dots T_2$$

at $x = 1.5$

$$\frac{T_2 - T_1}{2(0.5)} + 3T_{1.5} = e^{2(1.5)}$$

$$\left. \frac{dy}{dx} \right|_{x=1.5} = \frac{T_2 - T_1}{2(0.5)}$$

