

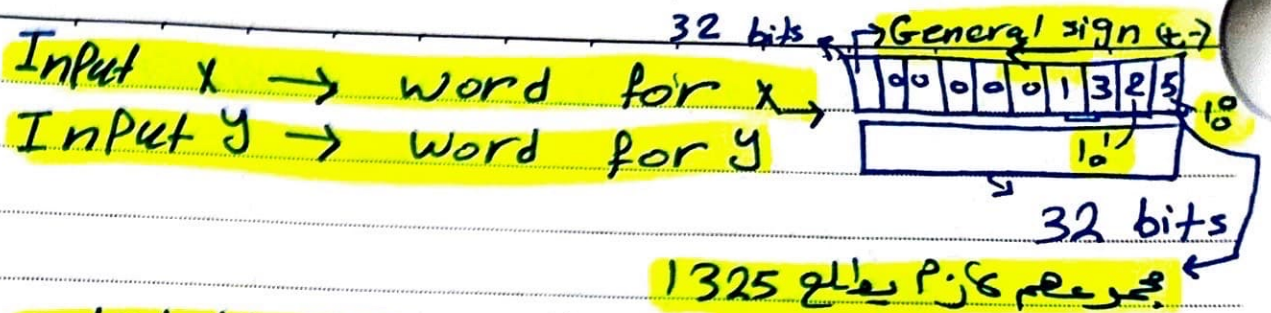
Numerical Methods



20
23

Perpared
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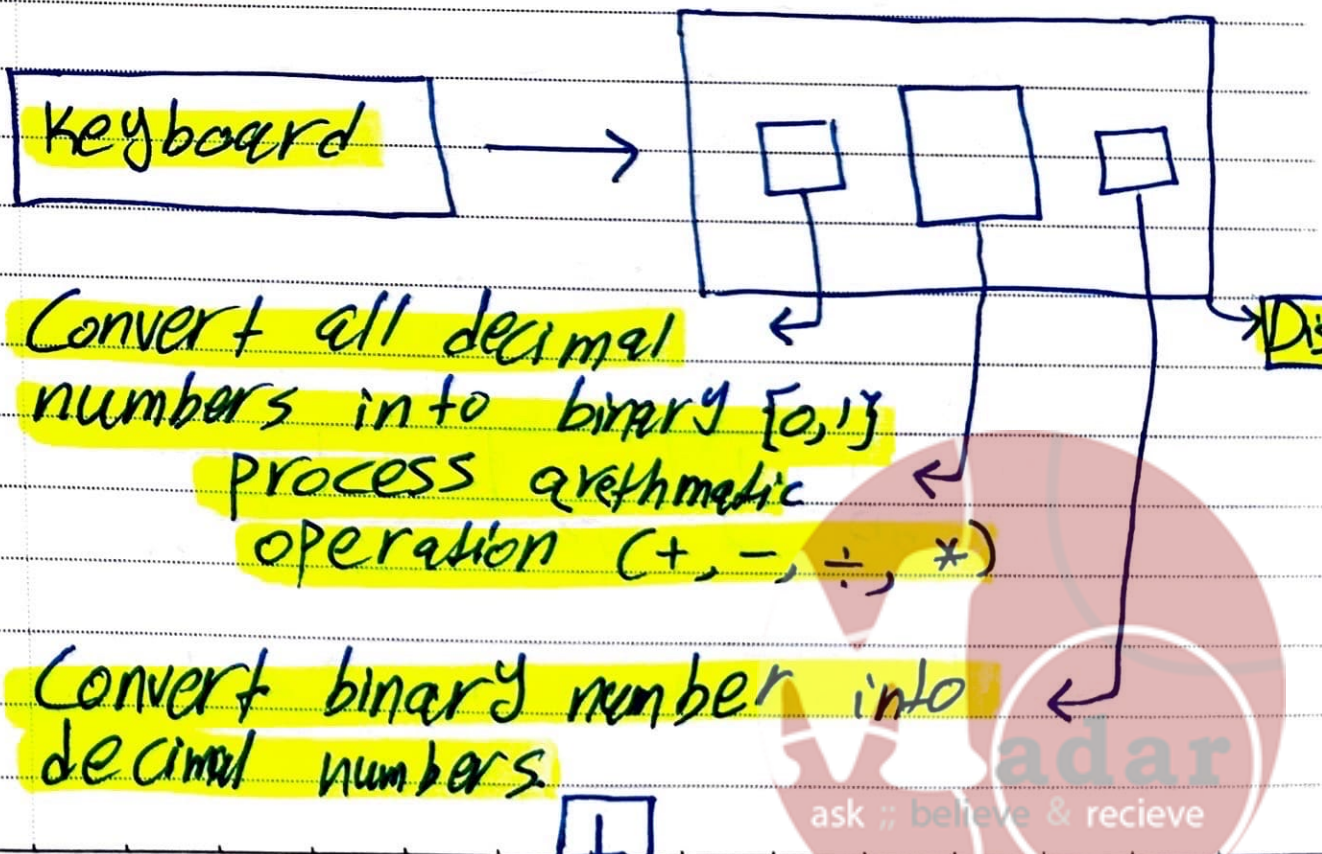
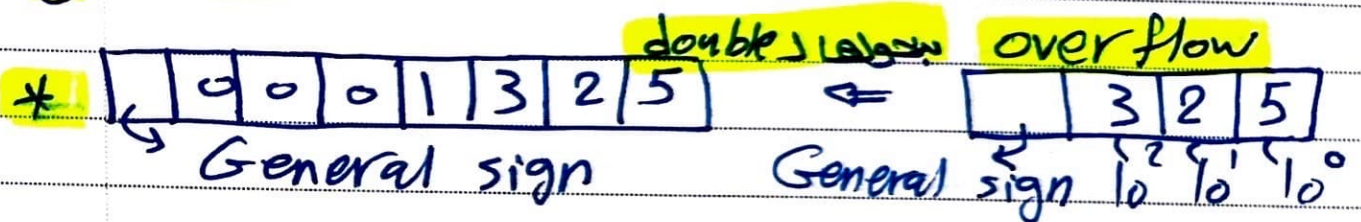
Dr:
Yousef
Mubarak



calculate: $z = x + y \rightarrow$ word for z

- * Human $\rightarrow \{0, 1, 2, \dots, 9\}$ (decimal)
- * Binary number $\rightarrow \{0, 1\}$
- * Base 4 $\rightarrow \{0, 1, 2, 3\}$ (العدد)

- ① Integer $\rightarrow 1325 \rightarrow$
- ② Real



* N is an integer number

← تابع القسمة المتتالية

$$\begin{aligned} N &= 2Q_0 + b_0 \\ Q_0 &= 2Q_1 + b_1 \\ Q_1 &= 2Q_2 + b_2 \\ Q_2 &= 2Q_3 + b_3 \end{aligned}$$

① Convert all decimal integer number to binary number

$$\begin{array}{cccccc} b_4 & b_3 & b_2 & b_1 & b_0 \\ 1 & 1 & 0 & 1 & 1 \end{array}$$

{ we only stop when the value of $Q_i = 0$ }

* b_0 (باقی قسمت) → 0
→ 1

* Ex: $N = 27$

$$\begin{aligned} 27 &= 2 \times 13 + \textcircled{1} b_0 \\ 13 &= 2 \times 6 + \textcircled{1} b_1 \\ 6 &= 2 \times 3 + \textcircled{0} b_2 \\ 3 &= 2 \times 1 + \textcircled{1} b_3 \\ 1 &= 2 \times 0 + \textcircled{1} b_4 \\ 0 &= 2 \times 0 + \textcircled{0} b_5 \end{aligned}$$

$$\begin{array}{cccccc} b_5 & b_4 & b_3 & b_2 & b_1 & b_0 \\ \boxed{0} & \boxed{1} & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{1} \\ 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \end{array}$$

$$0 + 16 + 8 + 0 + 2 + 1 = 27$$

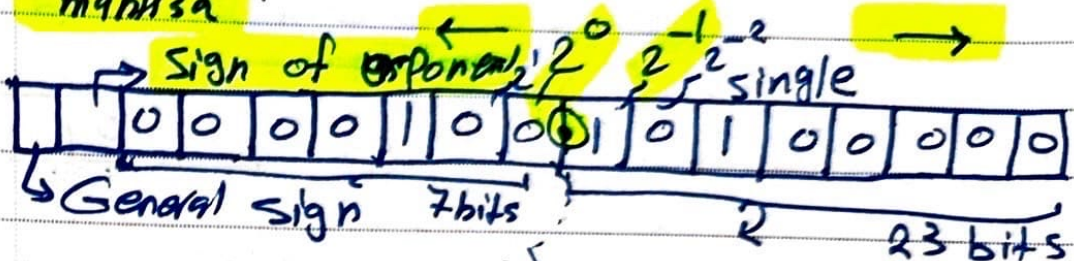
2

ask, believe & receive

* For real number $\rightarrow$ float-point number

$$X = \pm m \times 2^{\pm n} \quad \leftarrow \text{exponent}$$

mantissa $\leftarrow$



* Ex: $X = 0 \odot 101 \times 2^2$

$\leftarrow$ mantissa

$\leftarrow$ mantissa

$$1 \times \frac{1}{2} + 0 \times \frac{1}{4} + 1 \times \frac{1}{8} = 0.625 \times 2^0 = 2.5$$

* Computing and computers:

2.0 [integer]

$\rightarrow$ 2.1 [real]

$\downarrow \downarrow$

Ex: $101.001 = \pm 0.1 \dots \times 2^{\text{Power of exponent}}$

$\leftarrow$ mantissa

* Ex: 0.25

integer $\rightarrow$ fraction (F) [الجزء من العدد (الجزء العشري)]

$$2F_0 = 2 \times 0.25 = 0.5$$

$$2F_1 = 2 \times 0.5 = 1.0 \rightarrow F_1$$

$$2F_2 = 2 \times 0 = 0.0 \rightarrow F_2$$

$$2F_3 = 2 \times 0 = 0.0 \rightarrow F_3 \text{ (stop)}$$

[3]

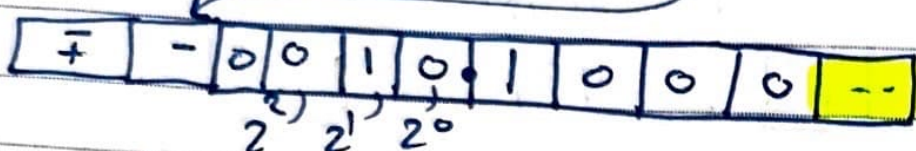
ask :: believe & recieve

$$0. d_0 d_1 d_2 d_3 d_4 = 0.0100002^0$$

$\frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16} \quad \frac{1}{32}$

* Normalization (القيمة العائمة)

$$0.100000 \times 2^0$$



$$1 \times \frac{1}{2} + 2^{-1} = 1 \times \frac{1}{2} + 0 \times \frac{1}{2} + 0 \times \frac{1}{4}$$

$$= 0.25$$

* Ex:

$$x = \boxed{13}.\boxed{4367}$$

integer ← → fraction

$$\begin{aligned}
 13 &= 2 \times 6 + 1 & 2^2 & 2^1 & 2^0 \\
 6 &= 2 \times 3 + 0 & 13 &= 11 & 0 & 1 \\
 3 &= 2 \times 1 + 1 & &= 1 \times 1 & + 0 \times 2 & + 1 \times 4 \\
 1 &= 2 \times 0 + 1 & & & + 8 \times 1
 \end{aligned}$$

$$= 0.4367 \times 2 = 0.8734$$

$$\begin{aligned}
 0.8734 \times 2 &= 1.7468 \rightarrow D_0 \\
 0.7468 \times 2 &= 1.4936 \rightarrow D_1 \\
 0.4936 \times 2 &= 0.9872 \rightarrow D_2 \\
 0.9872 \times 2 &= 1.9744 \rightarrow D_3
 \end{aligned}$$

$$\begin{aligned}
 \therefore 13.4367 &= 1101.01101 \times 2^0 \\
 &= 0.110101101 \times 2^4
 \end{aligned}$$

4

ask, believe & recieve

+	+	0	1	0	0	0	0	1	1	0	1	0	1	1	0	1
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

$$= \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{128} + \frac{1}{512} \right] \times 16 =$$

13.40625

approximate

* Ex: 13.4367 $\rightarrow$ decimal number

$$\begin{array}{l} 13 = 4 \times 3 + 1 \\ 3 = 4 \times 0 + 3 \end{array} \quad \downarrow \quad \begin{array}{l} 31 \rightarrow 031 \\ \text{Base 4} \end{array}$$

(2) $\rightarrow$ 4 bits

$$1 \times 1 + 3 \times 4 = 13 \text{ number}$$

$$0.4367 \times 4 = 1.7468$$

$$0.7468 \times 4 = 2.9872$$

$$0.9872 \times 4 = 3.9488$$

$$0.9488 \times 4 = 3.7952$$

$$\therefore 0.4367 = 0.1233_{\text{base 4}} = 0.4335$$

$$\begin{array}{l} * X = 101.11 \\ Y = 11.01 \end{array} \quad \left. \begin{array}{l} \\ \end{array} \right\} Z = X + Y$$

(normalize the result)

$$Z = 101.11 + 11.01 = 0.10111 \times 2^3 +$$

$$= 0.10111 \times 2^3 + 0.1101 \times 2^2$$

0	1	0	1	1	1				
0	0	1	1	0	1				
1	0	0							

کمزیم پاور
نقص الپاور

$$1.00100 \times 2^{+3}$$

$$= 0.100100 \times 2^{+4} = \left[\frac{1}{2} + \frac{1}{8} \right] \times 16$$

$$\frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16}$$

{ كل كسيف بالصف الاول عن واحد بالصف الثاني
 وعند قسم بالصف الثاني عن واحد بالصف
 الثالث وهكذا }

$$* X - Y =$$

0	.	1	0	1	1	1	$\times 2^{+3}$
0	.	0	1	0	1		$\times 2^{+3}$
0	.	0	0	0	1	0	$\times 2^{+3}$

{ بخط اوسجيل امت الحصة اليسار }

$$* X * Y =$$

No.

7



FIVE APPLE

* Solution of single non linear eq:

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

* if the function cross or touch the x-axis at point $\Rightarrow$ at this point the roots of this function, $f(x) = 0$, other we have imaginary roots.

* To find the roots, ① by using ^{matlab} ^{polymath} ^{graphical solution}

* If the sign of $f'(x)$ remain constant before and after the root then we have single or even # of roots

$\left\{ \begin{array}{l} \text{if } f'(x) \text{ variable} \Rightarrow \text{single root} \\ \text{if } f'(x) \text{ remain constant} \Rightarrow \text{even \# of roots} \end{array} \right.$

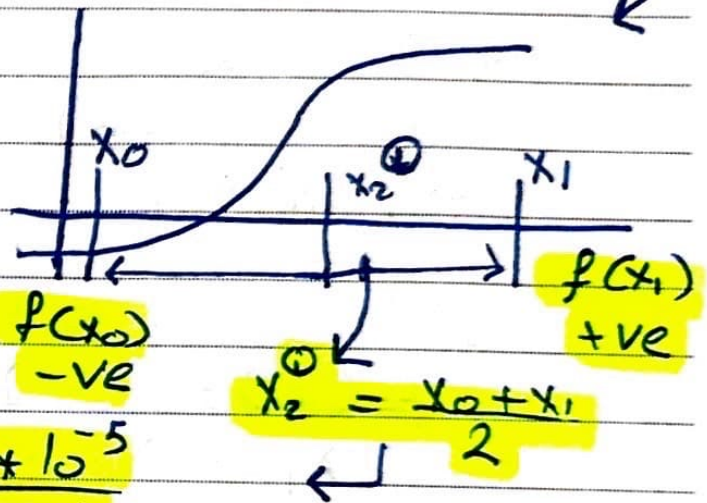
and if the sign of $f''(x)$ not remain constant (the same) $\Rightarrow$ we have odd # of roots.

* Bracketed methods: (closed)

① Bisection method:

② Two initial guesses are required.

③ $\text{sign } f(x_0) \neq \text{sign } f(x_1)$



$$f(x_2^{(1)}) > \frac{1 \times 10^{-5}}{\text{Tol.}}$$

$\therefore x_2^{(1)}$ is not a root.

$$\Rightarrow x_2^{(2)} = \frac{x_0 + x_2^{(1)}}{2}$$

(sign) (الفعلية) x_1 (التي)

$$\Rightarrow f(x_2^{(2)}) > 1 \times 10^{-5} \dots \text{وحتى}$$

② False position method: (faster than Bi section method)

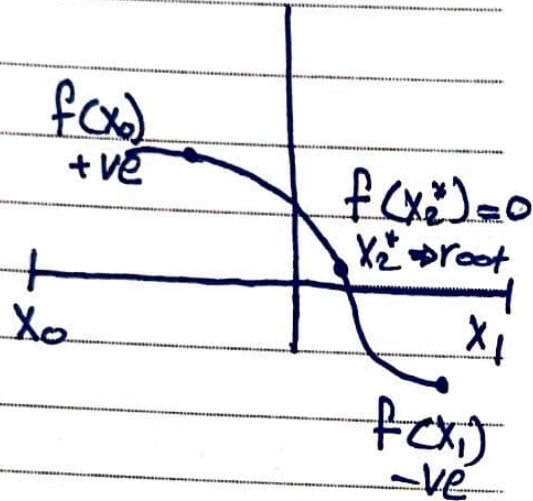
- ① Two initial guesses are required
 ② sign $f(x_0) \neq \text{sign } f(x_1)$
 (for each iteration)

$\therefore \text{slope ①} =$

$$\frac{f(x_2^*) - f(x_0)}{x_2^* - x_0}$$

$\text{slope ②} =$

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

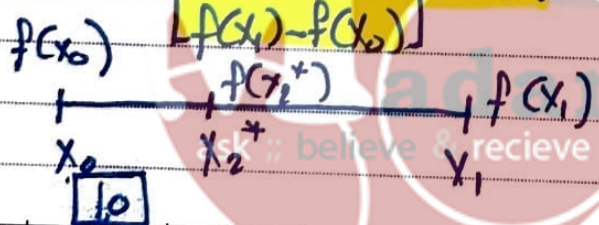


$$\rightarrow \frac{f(x_2^*) - f(x_0)}{x_2^* - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\therefore x_2^* - x_0 = - \frac{f(x_0)}{\frac{f(x_1) - f(x_0)}{x_1 - x_0}}$$

$$x_2^* = x_0 - \left[\frac{x_0 - x_1}{f(x_0) - f(x_1)} \right] f(x_0)$$

$$x_2^* = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right] f(x_1)$$



* open method :

- ① Secant method
- ② Newton's method
- ③ Muller's method
- ④ Fixed method

□ Secant method : (faster than the previous two methods)

② Two initial guesses are required, (x_0, x_1) .

③ $\text{sign } f(x_0) = \text{sign } f(x_1)$

$\Rightarrow$ we can proceed.

$\text{sign } f(x_0) \neq \text{sign } f(x_1)$

$f(0.5) = -2.3 \Rightarrow$ we can proceed (convergence).

$\swarrow$ $\searrow$
 $f(0.5) = -2.3$ $f(1.5) = +0.6$
 $\checkmark$

0.5	1.5
x_0	x_1

 $\Rightarrow$ we must arrange x_0 and x_1 in a way that guaranteed $|f(x_0)| > |f(x_1)|$

* To calculate the expected value

$$x_2^* = x_0 - \left[\frac{x_0 - x_1}{f(x_0) - f(x_1)} \right] f(x_0)$$

or

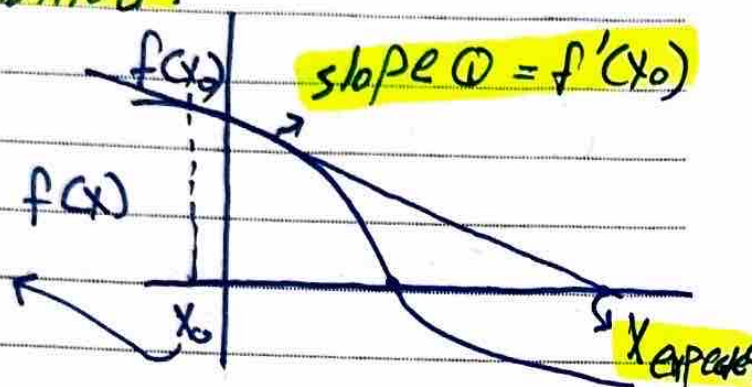
$$x_2^* = x_1 - \left[\frac{x_1 - x_0}{f(x_1) - f(x_0)} \right] f(x_1)$$



[renamed the value as $(x_0)_{\text{new}} = (x_1)_{\text{old}}$ and $(x_1)_{\text{new}} = (x_2^*)_{\text{new}}$] and always eliminate $(x_0)_{\text{old}}$ and keep $(x_1)_{\text{old}}$ and $(x_2^*)_{\text{new}}$.

2) Newton's method:

@ only we need one initial guess.



The slope of the extended line, x_1

$$\text{slope ②} = \frac{f(x_1^*) - f(x_0)}{x_1^* - x_0}$$

$$\text{slope ①} = \text{slope ②}$$

$$f'(x_0) = \frac{f(x_1^*) - f(x_0)}{x_1^* - x_0}$$

$$x_1^* - x_0 = \frac{-f(x_0)}{f'(x_0)} \Rightarrow x_1^* = x_0 - \frac{f(x_0)}{f'(x_0)}$$

secant method * if $|f(x_1^*)| \leq \text{tol.}$, then x_1 is the root.
 * if $|f(x_1^*)| > \text{tol.}$, then another iteration is required.

* For the next iteration eliminate $(x_0)_{\text{old}}$ and keep x_1^* , then rename $(x_0)_{\text{old}} = x_1^*$

$$* X_1^{(2)} = (X_0)_{\text{new}} - \frac{f(X_0)_{\text{new}}}{f'(X_0)_{\text{new}}}$$

$$* \text{Ex: } f(x) = 3x + \sin(x) - e^x \quad / \quad X_0 = 0, X_1 = 1$$

$$* \text{Bisection method: } f(X_0) = f(0) = -1 \quad \text{-ve}$$

$$f(X_1) = f(1) = 1.1231891 \quad \text{+ve}$$

sign of $f(X_0) \neq$ sign of $f(X_1)$

$$1^{\text{st}} \text{ iteration: } X_2^{(1)} = \frac{X_0 + X_1}{2} = \frac{0 + 1}{2} = 0.5$$

$$f(0.5) = 3(0.5) + \sin(0.5) - e^{0.5}$$

$$= 0.3307042679 \quad | \times 10^{-5}$$

Tol. (مقدار التوقف)

$$2^{\text{nd}} \text{ iteration: } X_2^{(2)} = \frac{0 + 0.5}{2} = 0.25$$

$\begin{array}{ccc} \text{-ve} & & \text{+ve} \\ \downarrow & & \downarrow \\ 0 & & 0.5 \end{array}$

$$f(0.25) = -0.2866214574 \quad | \times 10^{-5}$$

لازم تكون الدالة الحاسبة degree من radian

$$* X_2^{(1)} = X_0 - \left[\frac{X_0 - X_1}{f(X_0) - f(X_1)} \right] f(X_0)$$

(False position method)

$$= 0 - \left[\frac{0 - 1}{-1 - 1.1231891} \right] \times -1 = 0.470989595$$

$$f(0.470989595) = 0.2651588161 \quad \text{Tol. +ve}$$

*Ex : $x \cos x - 2x^2 + 3x - 1 \rightarrow [0.1 - 1.5]$
 $x \cos x - 2x^2 + 3x - 1 = 0$ (فرض $f(x) = 0$)

$$f'(x) = \cos(x) - x \sin(x) - 4x + 3$$

① Bisection : $[0.1 - 0.4]$ (نطاقاً اولياً طويلاً)

$$x_0 = 0.1 \rightarrow f(0.1) = -0.6025 \text{ (-ve)}$$

$$x_1 = 0.4 \rightarrow f(0.4) = 0.248424 \text{ (+ve)}$$

$$x_2 = \frac{0.1 + 0.4}{2} = 0.25$$

$$f(x_2) = f(0.25) = -0.132771 > 1 \times 10^{-8} \text{ (-ve) Tol.}$$

until $f(x_2) \leq \text{Tol.}$...
 we stop. (25 iterations)

$f(x)$	-ve	-ve	+ve
	0.1	0.25	0.4
	x_0		x_1

② Fake Position:

$$x_2 = 0.1 - (-0.6025) \times \frac{0.1 - 0.4}{(-0.6025) - (0.248424)}$$

$$= +0.31423$$

$$f(x_2) = 0.4454 > 1 \times 10^{-8}$$

(6 iterations) -----

14

$f(x)$	-ve	+ve	+ve
	0.1	0.31423	0.4
	x_0	x_1	

ask ... believe & recieve
FIVE APPLE

③ Secant: $x_0 = 0.1$, $x_1 = 0.4$
 $f(x_0) = -0.6025$, $f(x_1) = 0.248424$

since $|f(x_0)| > |f(x_1)| \therefore$ keep $x_0 = 0.1$, $x_1 = 0.4$

$$x_2^{(1)} = 0.1 - (-0.6025) \left[\frac{0.1 - 0.4}{(-0.6025) - (0.248424)} \right]$$

$$= 0.31423$$

$$f(x_2^{(1)}) = |0.06405| > \text{Tol. [one check only]}$$

$\therefore$ eliminate x_0 and swap it with x_1
 ($i=1$) $\rightarrow x_0 = 0.4$ and $x_1 = x_2^{(1)} = 0.31423$

$$\therefore x_2^{(2)} = 0.4 - (0.248424) \left[\frac{0.4 - 0.31423}{(0.248424) - (0.06405)} \right]$$

$$= \dots \dots \dots \text{(6 iterations)}$$

④ Newton's: $x_0 = 0.1$, $f(0.1) = -0.625$

$$f'(x) = \cos(x) - x \sin(x) - 4x + 3$$

$$f'(x) = \cos(0.1) - 0.1 \sin(0.1) - 4(0.1) + 3$$

$$= 3.585020824$$

$$x_1^{(1)} = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.1 - \frac{(-0.6025)}{(3.585020824)}$$

$$x_1^{(1)} = 0.2680603906, f(x_1^{(1)}) = f(0.268 \dots) > \text{Tol.}$$

eliminate x_0 and swap it with x_1

$$x_0 = 0.2680603906, f(x_0), f'(x_0)$$

Bisection method ← 2 initial guesses (بين الـ 2 initial guesses)

No.

* Open methods:

3 Muller's method: (أول عدد من الـ iteration) (و أدق)

* 3 initial guesses are required (x_0, x_1, x_2)

(لـ 3 initial guesses) $x_2 < x_0 < x_1$
 $f(x_2) \quad f(x_0) \quad f(x_1)$

* calculate the following:

$$C_1 = \frac{f(x_1) - f(x_2)}{x_1 - x_2}, \quad C_2 = \frac{f(x_0) - f(x_2)}{x_0 - x_2}$$

$$d_1 = \frac{C_2 - C_1}{x_0 - x_2}, \quad S = C_2 + d_1(x_0 - x_1)$$

* calculate the expected root

$$x_{exp} = x_0 - \frac{2f(x_0)}{S}$$

$$S + (\text{sign of } S) * (\sqrt{S^2 - 4f(x_0)})$$

* if $|f(x_{exp})| < \text{Tol}$, $\rightarrow$ root

$x_{exp} < x_2 < x_0$ $\rightarrow$ further iteration.

$x_2 < x_0 < x_1$

$$\textcircled{2} \quad \begin{array}{cccc} x_2 & x_{\text{exp}} & x_1 & x_0 \end{array}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$x_2 < x_0 < x_1$$

بسط الی آخر
 x_{exp}

$$\textcircled{3} \quad \begin{array}{cccc} x_2 & x_0 & x_{\text{exp}} & x_1 \end{array}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$x_2 < x_0 < x_1$$

$$\textcircled{4} \quad \begin{array}{cccc} x_2 & x_0 & x_1 & x_{\text{exp}} \end{array}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$x_2 < x_0 < x_1$$

4 Fixed Point iteration:

* all other methods we solve $f(x)=0$

⇒ rewrite $f(x)$ in the form $x=g(x)$

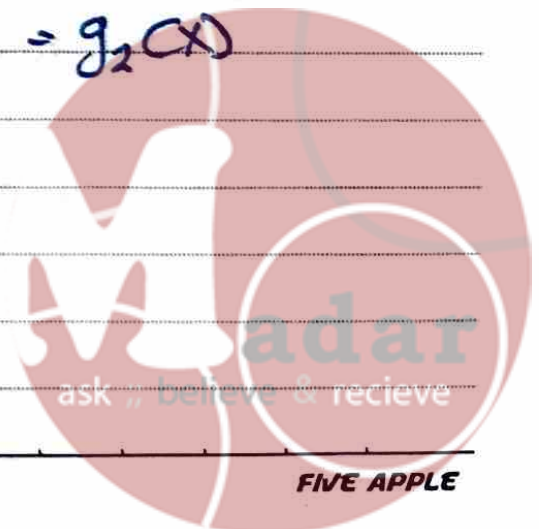
* Ex: $f(x) = x \cos(x) - 2x^2 + 3x - 1$

$$x = \frac{1 + 2x^2 - x \cos(x)}{3} = g_1(x)$$

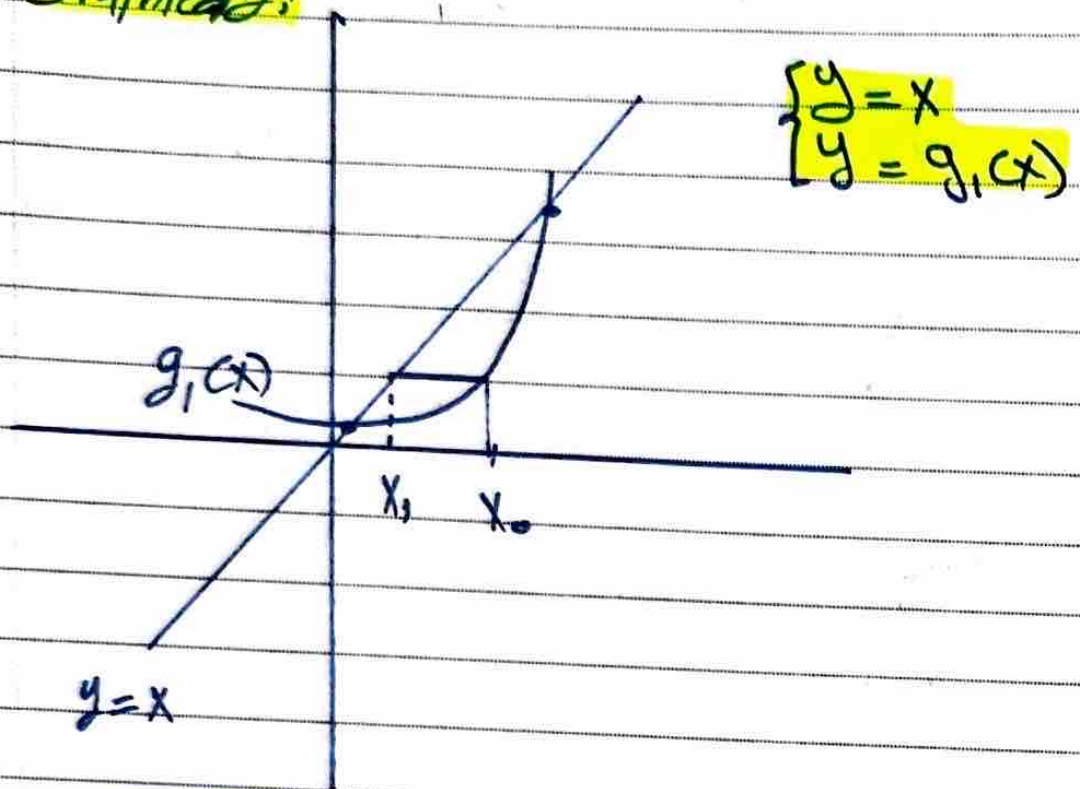
$$x = \frac{1 - x \cos(x)}{3 - 2x} = g_2(x)$$

many

7



* Graphically:



- ① قطع عمودي تحت الصب الـ Curve
- ② عمودي افقي تحت الصب الـ line
- ③ نجو من x_0 الى $f(x)$ اذا $\leq Tol.$
 دمج نفس الخطوات

* Numerically:

$$x = g_1(x) = \frac{1}{3}(1 + 2x^2 - x \cos(x))$$

1st iteration $x_1^{(1)} = g_1(x_0) = \frac{1}{3}(1 + 2x^2 - x \cos(x))$

if $|f(x_1^{(1)})| \leq Tol. \rightarrow stop$

$> Tol. \rightarrow$ another iteration

in the 2nd iteration: $x_0 = x_1^{(1)}$, (eliminate the old x_0 and swap it by $x_1^{(1)}$)

Then $x_1^{(0)} = g_1(x_0)$

The other check you may do $\left| \frac{x_0 - x_1}{x_0} \right| \begin{matrix} \text{stop} \swarrow \\ \text{R.E} \leftarrow \\ \text{(Relative error)} \end{matrix}$ $\begin{matrix} \text{another iteration} \downarrow \end{matrix}$

* if $|g'(x_0)| \leq 1$, then proceed

$$x_0 = 1 \Rightarrow g_1(x) = \frac{1}{3} (1 + 2x^2 - x \cos x)$$

$$x_1^{(0)} = g_1(x_0) = 0.8199$$

$$\text{R.E} = \left| \frac{1 - 0.8199}{1} \right| > 1 \times 10^{-8}$$

$$x_1^{(0)} = g_1(x_1^{(0)}) = 0.59502$$

$$\text{R.E} = \left| \frac{0.8199 - 0.59502}{0.8199} \right| > 1 \times 10^{-8}$$

$$x_1^{(0)} = g_1(x_1^{(0)}) = 0.40511$$

$$\text{R.E} = \left| \frac{0.59502 - 0.40511}{0.59502} \right| > 1 \times 10^{-8} \dots$$

Hw *

For 4 different $g(x)$, each $g(x)$ on excell.

$\begin{matrix} 0.5 \\ 1 \\ 1.5 \end{matrix} \left\{ \begin{matrix} \text{initial} \\ \text{guesses} \end{matrix} \right\}$

* Ex : Muller's method:

$$f(x) = 3x + \sin(x) - e^x, x_0 = 0.1, x_1 = 0.4$$

$$\rightarrow x_2 = \frac{x_0 + x_1}{2} = \frac{1}{4} \begin{array}{ccc|c} x_2 & x_0 & & x_1 \\ \hline 0.1 & 0.25 & & 0.4 \end{array}$$

$$\begin{array}{l} f(x_2) - f(x_0) = -0.705375 \\ f(x_1) - f(x_0.25) = -0.2866214 \\ f(x_1) - f(x_2) = 0.09759 \end{array}$$

$$C_1 = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{0.09759}{0.4 - 0.1} = 2.67655$$

$$C_2 = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{-0.2866214 - 0.09759}{0.25 - 0.1} = 2.561409333$$

$$d_1 = \frac{C_2 - C_1}{x_0 - x_2} = \frac{2.561409333 - 2.67655}{0.25 - 0.1} = -0.76760$$

$$S = C_2 + d_1(x_0 - x_1) = 2.67655$$

(Value of C_1 is not given) Iteration

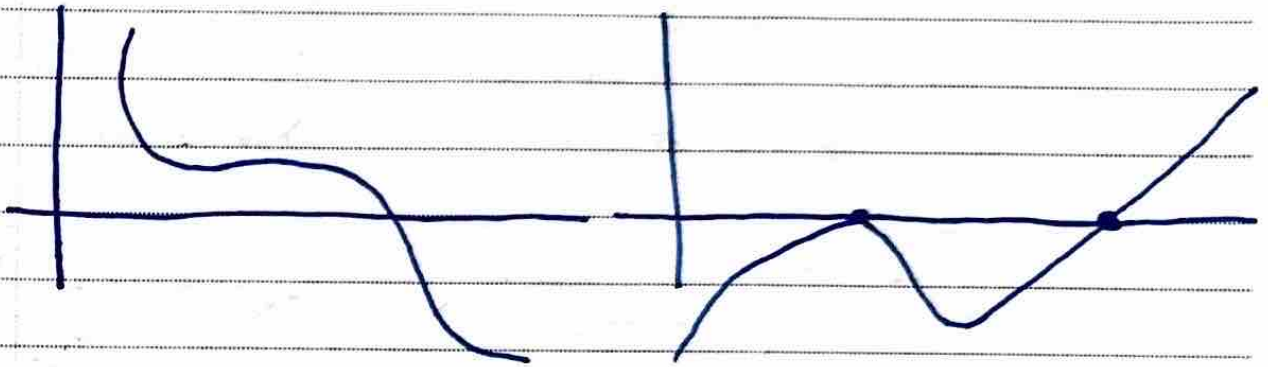
$$x_{exp}^0 = 0.25 - \left\{ \frac{2 \times -0.2866214}{2.67655} \right\}$$

$$x_{exp}^1 = 0.36059$$

$$|f(x_{exp}^0)| > T_0$$

2 nd :	x_2	x_{exp}	x_0	x_1
	0.1	0.3	0.25	0.4
	x_2	x_0	x_{exp}	

[5] Modified Newton's Method:



$$x_1 = x_0 - \left\{ \frac{f(x_0) \cdot f'(x_0)}{[f'(x_0)]^2 - f(x_0) \cdot f''(x_0)} \right\}$$

* Ex: $f(x) = x^3 - 5x^2 + 7x - 3$, $x_0 = 0$

* Simple Newton's: $f'(x) = 3x^2 - 10x + 7$

$$f(x_0) = f(0) = -3, \quad f'(x_0) = f'(0) = 7$$

$$x_1^{(1)} = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-3)}{7} = 0.42857$$

$$f(x_1^{(1)}) = -0.8396548, \quad f'(x_1^{(1)}) = 3.265316$$

$$x_0 = x_1^{(1)} - \frac{f(x_1^{(1)})}{f'(x_1^{(1)})} = 0.6857135$$

$$x_1^{(3)} = 0.83286496 \dots$$

* Modified Newton's : $f(x) = x^3 - 5x^2 + 7x - 3$
 $f'(x) = 3x^2 - 10x + 7$
 $f''(x) = 6x - 10$
 $f(0) = -3, f'(0) = 7, f''(0) = -10$
 $x_0 = 0$

$$x_1 = 0 - \frac{(-3)(7)}{7^2 - (-3)(-10)} = \frac{1.105263}{15}$$

$$f(x_1^{(1)}) = |-0.0209943| > \text{Tol},$$

$$f'(x_1^{(1)}) = -0.378116, f''(x_1^{(1)}) = -3.368421.$$

$$x_1^{(2)} = 1.00308166, f(x_1^{(2)}) = |-1.89639 \times 10^{-5}| > \text{Tol}.$$

$$x_1^{(3)} = 1.000002382, f(x_1^{(3)}) = |-1.3 \times 10^{-11}| < \text{Tol},$$

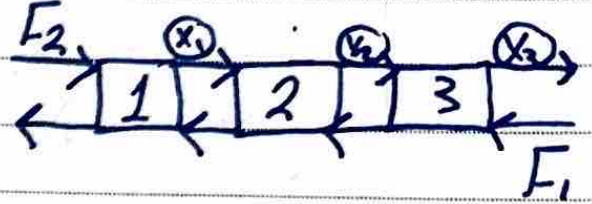
stop.

* System of linear and non-linear equations:

$$f_1(x_1, x_2, x_3) = 0$$

$$f_2(x_1, x_2, x_3) = 0$$

$$f_3(x_1, x_2, x_3) = 0$$



* Linear system:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = C_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = C_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = C_3$$

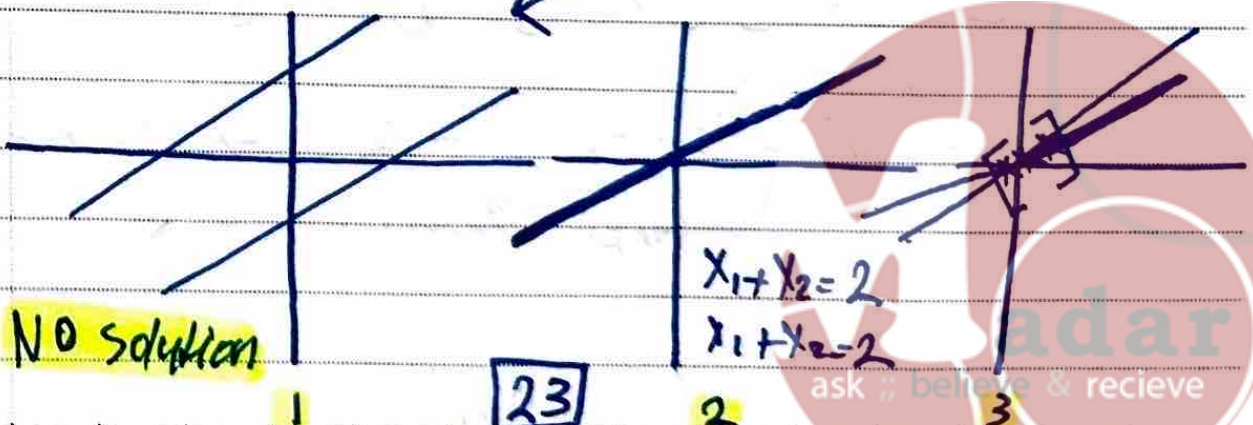
* a 's are coefficients
 C 's are constants

* Non-linear system:

$$(x, y) \Rightarrow x^2 + y^2 = 10 \quad \dots \textcircled{1}$$

$$y + 3xy = 57 \quad \dots \textcircled{2}$$

* well and ill: $x_1 + x_2 = 2$, $2x_1 + 2x_2 = 4$



لازم الجواب
 (Dependent)

$$1. [A]\{x\} = \{c\}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

↓

$$\text{Det}[A] = 2 - 2 = 0$$

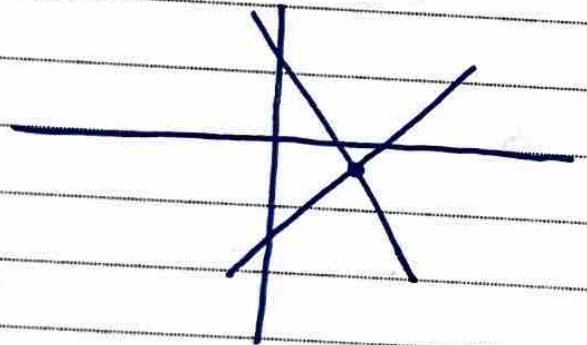
$$2. \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

↓ A

$$\text{Det}[A] = 1 - 1 = 0$$

3. ill conditioned system, $\text{Det}[A] \approx \text{zero}$

Well



$$x_1 + x_2 = 2$$

$$5x_1 + 3x_2 = -4$$

$$\begin{bmatrix} 1 & 1 \\ 5 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ -4 \end{Bmatrix}$$

↓ A

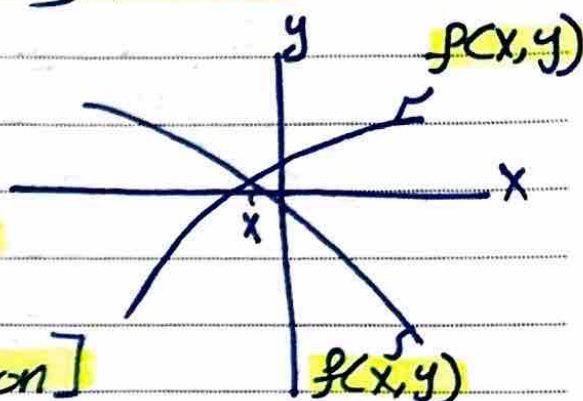
$$\text{Det}[A] = 3 - 5 = -2 \neq \text{Zero.}$$

independent.

* Solution of small systems:

① Graphical solution:

$\max = 3$ (حداکثر)



② Substitution: [No limitation]

$$2x_1 + 3x_2 + 4x_3 = 6$$

$$x_1 - 2x_2 - 6x_3 = 2$$

$$\begin{cases} 3x_1 - 5x_2 + 7x_3 = 9 \Rightarrow x_3 = \frac{9 - 3x_1 + 5x_2}{7} \\ \rightarrow x_1 - 2x_2 - 6 \left[\frac{9 - 3x_1 + 5x_2}{7} \right] = 2 \end{cases}$$

③ Cramer's Rule: [3x3 matrix قابل]

→ جدول $[A]$ متساوی

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$[A]_{3 \times 3} \{x\}_{3 \times 1} = \{c\}_{3 \times 1}$$

$$* \begin{Bmatrix} X \end{Bmatrix}_{3 \times 1} = \frac{\begin{Bmatrix} C \end{Bmatrix}_{3 \times 1}}{[A]_{3 \times 3}}$$

$$* \text{Det}[A] = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \neq \text{Zero}$$

$$* X_1 = \frac{\begin{vmatrix} C_1 & a_{12} & a_{13} \\ C_2 & a_{22} & a_{23} \\ C_3 & a_{32} & a_{33} \end{vmatrix}}{\text{Det}(A)}$$

$$X_2 = \frac{\begin{vmatrix} a_{11} & C_1 & a_{13} \\ a_{21} & C_2 & a_{23} \\ a_{31} & C_3 & a_{33} \end{vmatrix}}{\text{Det}[A]}$$

$$X_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & C_1 \\ a_{21} & a_{22} & C_2 \\ a_{31} & a_{32} & C_3 \end{vmatrix}}{\text{Det}[A]}$$

$$* \text{Ex: } \begin{aligned} 7X_1 - 3X_2 + 3X_3 &= -49 \\ X_1 - 2X_2 - 5X_3 &= 5 \\ 3X_1 - 6X_2 + 10X_3 &= -84 \end{aligned}$$

$$\begin{bmatrix} 7 & -3 & 3 \\ 1 & -2 & -5 \\ 3 & -6 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -49 \\ 5 \\ -84 \end{Bmatrix}$$

$$\text{Det } [A] = 7(-20 - 30) + 3(10 + 15) + 3(-6 + 6) = -275$$

$$x_1 = \frac{\begin{vmatrix} -49 & -3 & 3 \\ 5 & -2 & -5 \\ -84 & -6 & 10 \end{vmatrix}}{-275} = \frac{746}{-275} = -2.712727$$

$$x_2 = \frac{\begin{vmatrix} 7 & -49 & 3 \\ 1 & 5 & -5 \\ 3 & -84 & 10 \end{vmatrix}}{-275} = \frac{-1662}{-275} = 6.04363636$$

$$x_3 = \frac{\begin{vmatrix} 7 & -3 & -49 \\ 1 & -2 & 5 \\ 3 & -6 & -84 \end{vmatrix}}{-275} = \frac{1089}{-275} = -3.96$$

* ل. ا. excell: اذا اكتبنا هذه

→ enter

اذا اكتبنا اكنز من هذه

→ ctrl + shift + enter

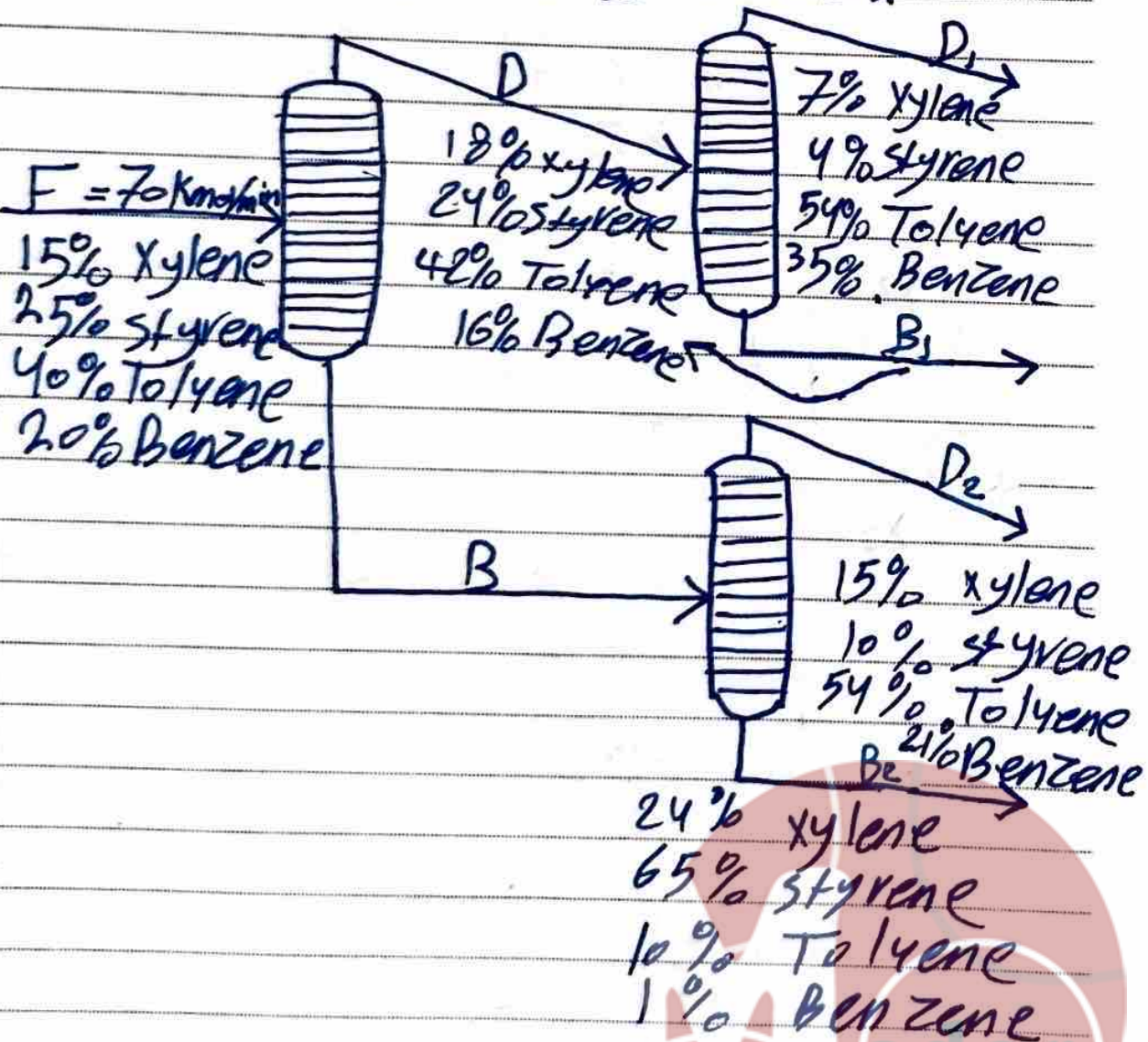
27

ask :: believe & recieve

* MDE function : To determine $[A]$ in exell.

* Large system of linear equations :

* Ex: Paraxylene, styrene, toluene and benzene are to be separated with the array of distillation columns show in the following figure. Calculate the molar flow rates of D_1, D_2, B_1 and B_2 .



$$\begin{aligned}
 \text{Xylene} &: 70 \times 0.15 = 0.07D_1 + 0.18B_1 + 0.15D_2 + 0.24B_2 \\
 \text{Styrene} &: 70 \times 0.25 = 0.04D_1 + 0.24B_1 + 0.1D_2 + 0.65B_2 \\
 \text{Toluene} &: 70 \times 0.4 = 0.54D_1 + 0.42B_1 + 0.54D_2 + 0.1B_2 \\
 \text{Benzene} &: 70 \times 0.2 = 0.35D_1 + 0.16B_1 + 0.21D_2 + 0.01B_2
 \end{aligned}$$

$$\begin{bmatrix} 0.07 & 0.18 & 0.15 & 0.24 \\ 0.04 & 0.24 & 0.1 & 0.65 \\ 0.54 & 0.42 & 0.54 & 0.1 \\ 0.35 & 0.16 & 0.21 & 0.01 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} 10.5 \\ 17.5 \\ 28 \\ 14 \end{Bmatrix}$$

① Gauss elimination :

*Step ①: Forward elimination: make the lower triangle all zero's

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix}$$

↖ main diagonal

↳ lower triangle
(Augmented matrix)

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & c_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & c_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & c_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & c_4 \end{array} \right] \rightarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & C_1 \\ 0 & a_{22} & a_{23} & a_{24} & C_2 \\ 0 & a_{32} & a_{33} & a_{34} & C_3 \\ 0 & a_{42} & a_{43} & a_{44} & C_4 \end{bmatrix} \quad (\text{Sub matrix 1})$$

$$\downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & C_1 \\ 0 & a_{22} & a_{23} & a_{24} & C_2 \\ 0 & 0 & a_{33} & a_{34} & C_3 \\ 0 & 0 & a_{43} & a_{44} & C_4 \end{bmatrix} \quad (\text{Sub matrix 2})$$

$$\downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & C_1 \\ 0 & a_{22} & a_{23} & a_{24} & C_2 \\ 0 & 0 & a_{33} & a_{34} & C_3 \\ 0 & 0 & 0 & a_{44} & C_4 \end{bmatrix} \quad (\text{Sub matrix 3})$$

[Which is forward matrix]

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= C_1 \quad \text{--- (1) 4 unknown} \\ 0x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= C_2 \quad \text{--- (2) 3 unknown} \\ 0x_1 + 0x_2 + a_{33}x_3 + a_{34}x_4 &= C_3 \quad \text{--- (3) 2 unknown} \\ 0x_1 + 0x_2 + 0x_3 + a_{44}x_4 &= C_4 \quad \text{--- (4) 1 unknown} \end{aligned}$$

Step ②: Backward substitution: Find the values of unknowns starting from the last.

From ④: $x_4 = C_4 / a_{44}$

From ③ : $q_{33}'' x_3 + q_{34}'' \left[\frac{C_4'''}{q_{44}'''} \right] = C_3''$

$$\therefore x_3 = \frac{C_3'' - q_{34}'' (C_4''' / q_{44}''')}{q_{33}''}$$

From ② : $q_{22}' x_2 + q_{23}' \left[\frac{C_3'' - q_{34}'' (C_4''' / q_{44}''')}{q_{33}''} \right] + q_{24}' \left[\frac{C_4'''}{q_{44}'''} \right] = C_2'$

$$\therefore x_2 = \frac{C_2' - \frac{q_{24}' * C_4'''}{q_{44}''}}{q_{22}'} - \frac{q_{23}' (C_3'' - q_{34}'' (C_4''' / q_{44}'''))}{q_{33}''}$$

From ① : $q_{11} x_1 + q_{12} \left[\downarrow \right] + q_{13} (x_3) + q_{14} x_4$
From ③ From ④

$$\therefore \begin{bmatrix} 0.07 & 0.18 & 0.15 & 0.24 & 10.5 \\ 0.04 & 0.24 & 0.1 & 0.65 & 17.5 \\ 0.54 & 0.42 & 0.54 & 0.1 & 28 \\ 0.35 & 0.16 & 0.21 & 0.01 & 14 \end{bmatrix}$$

→ STEP ①:

submatrix ①

$$\begin{bmatrix} 0.07 & 0.18 & 0.15 & 0.24 & 10.5 \\ 0 & 0.13714 & 0.01429 & 0.51286 & 11.5 \\ 0 & -0.96857 & -0.61714 & -1.75143 & -53 \\ 0 & -0.74 & -0.54 & -1.19 & -38.5 \end{bmatrix}$$

* Use the 1st row to eliminate all elements in the 1st column. $\downarrow$
each one $\leftarrow$

$$0.07 \times \frac{(0.04)}{0.07} \rightarrow \text{[pivot element]}$$

= For the second row: $\downarrow$ $\left(\begin{array}{cccc} 0.07 & 0.18 & 0.15 & 0.24 \\ & & & 10.5 \end{array} \right)$
 $\times \frac{-0.04}{0.07}$

$$= [-0.04 \quad -0.102857 \quad -0.08571 \quad -0.137143 \quad -6]$$

(pivot element) $\downarrow$ 3rd row $\downarrow$ 5th row

* Then add each one of them to the original matrix (2nd row).

For 3rd row: $\frac{-0.54}{0.07} \times \left(\begin{array}{ccc} 0.07 & 0.18 & 0.15 \\ & 0.24 & 10.5 \end{array} \right) \leftarrow$
 $= [-0.54 \quad -1.38857 \quad -1.15714 \quad -1.8543 -8]$

then add them to the 3rd row in the original matrix.

for the 4th row: $\frac{-0.35}{0.07} \times \left(\begin{array}{ccc} 0.07 & 0.18 & 0.15 \\ & 0.24 & 10.5 \end{array} \right)$
 $= [-0.35 \quad -0.9 \quad -0.75 \quad -1.2 \quad -52.5]$

then add them to the 4th row in the original matrix.

* Use row 2 to eliminate all elements in the 2nd column. ↴

$$\frac{+0.96857}{0.137143} * (0 \quad 0.137143 \quad 0.014286 \quad 0.512857 \quad 11.5)$$

$$= [0 \quad 0.96857 \quad 0.100893 \quad 3.622054 \quad 81.21875]$$

add them to row 3 in Submatrix 1, then write the results in Submatrix 2.

Submatrix 2

$$\begin{bmatrix} 0.07 & 0.18 & 0.15 & 0.24 & 6.5 \\ 0 & 0.137143 & 0.014286 & 0.512857 & 11.5 \\ 0 & 0 & -0.51625 & 1.870025 & 28.2 \\ 0 & 0 & -0.4629 & 1.5773 & 23.532 \end{bmatrix}$$

$$\frac{+0.74}{0.137143} * (0 \quad 0.137143 \quad 0.014286 \quad 0.512857 \quad 11.5)$$

$$= [0 \quad 0.74 \quad 0.077083 \quad 2.767292 \quad 62.0528]$$

add them into row 4 in Submatrix 1, and write the results in Submatrix 2.

* Use row 3 to eliminate all elements in Column 3.

* [The first 3 rows in submatrix 2 will be moved as is to submatrix 3]

Submatrix 3

$$\begin{bmatrix} 0.07 & 0.18 & 0.15 & 0.24 & 10.5 \\ 0 & 0.137143 & 0.014286 & 0.512857 & 11.5 \\ 0 & 0 & -0.51625 & 1.870625 & 28.22 \\ 0 & 0 & 0 & -0.10008 & -1.751 \end{bmatrix}$$

$$0.46292 \times (0 \ 0 \ -0.51625 \ 1.870625 \ 28.22) \\ -0.51625$$

$$= [0 \ 0 \ 0.46292 \ -1.67737 \ -25.3035]$$

add them to row 4 in submatrix 2 and write the results in submatrix 3.

→ step 1: $0.07 D_1 + 0.18 B_1 + 0.15 D_2 + 0.24 B_2 = 10.5$ — (1)
 $0 D_1 + 0.137143 B_1 + 0.014286 D_2 + 0.512857 B_2 = 11.5$ — (2)
 $0 D_1 + 0 B_1 - 0.51625 D_2 + 1.870625 B_2 = 28.21875$ — (3)
 $0 D_1 + 0 B_1 + 0 D_2 - 0.10008 B_2 = -1.75141$ — (4)

starting from equation 4:

$$B_2 = \frac{-1.75141}{-0.10008} = 17.5$$

34

from ③ $-0.51625D_2 + 1.876625(17.5) = 28.21875$
 $\therefore D_2 = 8.75$

from ② $0.137143B_1 + 0.014286(8.75) + 0.512857(17.5) = 11.5$

$\therefore B_1 = 17.5$

from ① $0.07D_1 + 0.18(17.5) + 0.15(8.75) + 0.24(17.5) = 6.5$
 $\therefore D_1 = 26.25$

* Thomas method:

المتجه الحسابي
 Row 1

$$\begin{bmatrix} d_1 & a_1 & & & \\ q_{11} & q_{12} & q_{13} & q_{14} & \\ b_2 & d_2 & a_2 & & \\ q_{21} & q_{22} & q_{23} & q_{24} & \\ b_3 & q_{31} & d_3 & a_3 & \\ q_{32} & q_{33} & q_{34} & & \\ b_4 & q_{41} & q_{42} & d_4 & \\ q_{43} & q_{44} & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

Upper
 Lower
 Main

Banded matrix (Tri diagonal)

* Step ① * For the first row $A_1 = \frac{q_1}{d_1}$, $R_1 = \frac{r_1}{d_1}$
 * For the second row and all other rows (except the first row)
 $A_i = \frac{q_i}{d_i - b_i A_{i-1}}$
 $R_i = \frac{r_i - b_i R_{i-1}}{d_i - b_i A_{i-1}}$

$$\rightarrow 2^{\text{nd}} \quad A_2 = \frac{a_2}{d_2 - b_2 A_1}, \quad R_2 = \frac{r_2 - b_2 R_1}{d_2 - b_2 A_1}$$

* Step ②: Solve by back substitution.

$$X_n = R_n \rightarrow X_4 = R_4$$

$$X_i = R_i - A_i X_{i+1}$$

$$X_3 = R_3 - A_3 X_4$$

$$X_2 = R_2 - A_2 X_3 \rightarrow X_1 = R_1 - A_1 X_2$$

* Gauss - Jordan :

making the upper and lower diagonal equal zero and main diagonal equal 1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} c_4 \\ c_5 \\ c_6 \end{Bmatrix} = \begin{Bmatrix} c_7 \\ c_8 \\ c_9 \end{Bmatrix}$$

$\nwarrow 70 \quad \nwarrow 90 \quad \nwarrow 50$

$$[A] \{x\} = \{c\}_{70} \Rightarrow [A]^{-1} \{c\}_{70}$$

$$[A] \{x\} = \{c\}_{90} \Rightarrow [A]^{-1} \{c\}_{90}$$

$$[A] \{x\} = \{c\}_{50} \Rightarrow [A]^{-1} \{c\}_{50}$$

کند تفریق
البرای

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & C_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & C_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & C_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & C_4 \end{bmatrix}$$

$$\begin{bmatrix} \frac{a_{11}}{a_{11}} & 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \frac{a_{14}}{a_{11}} & \frac{C_1}{a_{11}} \\ 0 & a_{22} - \frac{a_{21}a_{12}}{a_{11}} & & & \\ \vdots & & & & \end{bmatrix}$$

$$\begin{aligned} & -a_{21} - \frac{a_{21}a_{12}}{a_{11}} - \frac{a_{21}a_{13}}{a_{11}} \\ & -a_{21} \frac{a_{14}}{a_{11}} \dots \end{aligned}$$

$$\begin{bmatrix} 1 & a'_{12} & a'_{13} & a'_{14} & C'_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & C'_2 \\ 0 & a'_{31} & a'_{32} & a'_{33} & C'_3 \\ 0 & a'_{42} & a'_{43} & a'_{44} & C'_4 \end{bmatrix}$$

Submatrix 1

$$\begin{bmatrix} 1 & 0 & a''_{13} & a''_{14} & C''_1 \\ 0 & 1 & \frac{a'_{23}}{a'_{22}} & \frac{a'_{24}}{a'_{22}} & \frac{C'_2}{a'_{22}} \\ 0 & 0 & a''_{33} & a''_{34} & C''_3 \\ 0 & 0 & a''_{43} & a''_{44} & C''_4 \end{bmatrix}$$

$$\begin{aligned} & -a'_{21} (0 \mid \frac{a'_{23}}{a'_{22}} \frac{a'_{24}}{a'_{22}}) \\ & \frac{C'_2}{a'_{22}} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & a'''_{14} & C'''_1 \\ 0 & 1 & 0 & a'''_{24} & C'''_2 \\ 0 & 0 & 1 & a'''_{34} & C'''_3 \\ 0 & 0 & 0 & a'''_{44} & C'''_4 \end{bmatrix}$$

$$\begin{aligned} & -a'''_{13} (0 \mid 0 \mid a'''_{34} C'''_3) \\ & \text{(add to row 1 in Submatrix 2)} \end{aligned}$$

[I]

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

 $\begin{matrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{matrix}$

$$x_1 = C_1$$

$$x_2 = C_2$$

$$x_3 = C_3$$

$$x_4 = C_4$$

(Identity matrix)

(Backward Substitution)

*** Thomas Method:**

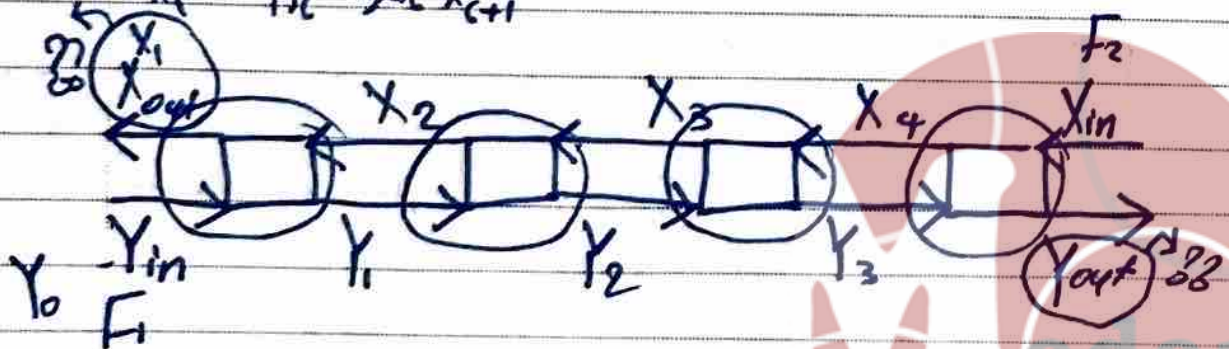
$$\begin{bmatrix} d_1 & a_1 & 0 & 0 \\ b_1 & d_2 & a_2 & 0 \\ 0 & b_2 & d_3 & a_3 \\ 0 & 0 & b_3 & d_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

$$A_1 = a_1/d_1, \quad R_1 = r_1/d_1$$

$$A_i = \frac{a_i}{d_i - b_i A_{i-1}}, \quad R_i = \frac{r_i - b_i R_{i-1}}{d_i - b_i A_{i-1}}$$

$$x_n = R_n$$

$$x_i = R_i - A_i x_{i+1}$$



$$F_1 Y_{i+1} + F_2 X_{i+1} = F_1 Y_i + F_2 X_i$$

$$k = \frac{X_i}{Y_i}$$

$$F_1 = 300, Y_{in} = 0.2, F_2 = 600, X_{in} = 0, k = 2$$

sol.

$$300(0.2) + (600)X_2 = X_1(600) + (300)Y_1$$

$$60 + 600X_2 = 600X_1 + 300Y_1$$

$$60 + 600(2Y_2) = 600(2Y_2) + 300Y_1$$

$$-1500Y_1 + 1200Y_2 = -60 \quad \dots \textcircled{1}$$

$$300Y_1 - 1500Y_2 + 1200Y_3 = 0 \quad \dots \textcircled{2}$$

$$300Y_2 - 1500Y_3 + 1200Y_4 = 0 \quad \dots \textcircled{3}$$

$$300Y_3 - 1500Y_4 = 0 \quad \dots \textcircled{4}$$

$$\begin{matrix} d_1 & a_1 \\ \begin{bmatrix} -1500 & 1200 & 0 & 0 \\ 300 & -1500 & 1200 & 0 \\ 0 & 300 & -1500 & 1200 \\ 0 & 0 & 300 & -1500 \end{bmatrix} & \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} & = & \begin{bmatrix} -60 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{matrix}$$

1st

$$\text{row: } A_1 = \frac{1200}{-1500} = -0.8, R_1 = \frac{-60}{-1500} = 0.04$$

2nd

$$\text{row: } A_2 = \frac{1200}{-1500 - 300(-0.8)} = -0.9523809$$

$$R_2 = \frac{0 - 300(0.04)}{-1500 - 300(-0.8)} = 0.0095238$$

39

ask :: believe & recieve

$$3^{\text{rd}} \text{ row: } A_3 = \frac{1200}{-1500 - 300(-0.9523809)}$$

$$= -0.98823529$$

$$R_3 = \frac{0 - 300(0.0095238)}{-1500 - 300(-0.9523809)}$$

$$= 0.00235294$$

$$\text{Last row: } A_4 = 0, R_4 = \frac{0 - 300(0.00235294)}{-1500 - 300(-0.98823)}$$

$$= 0.00058651$$

$$Y_4 = R_4 = 0.0005865 \rightarrow X_4 = 2Y_4$$

$$Y_3 = R_3 - A_3 X_4 = 0.00235294 + 0.98823529(0.0005865)$$

$$= 0.00293551 \rightarrow X_3 = 2Y_3$$

$$Y_2 = R_2 - A_2 R_3 = 0.0095238 + 0.9523809(0.00293551)$$

$$= 0.012316716 \rightarrow X_2 = 2Y_2$$

$$Y_1 = R_1 - A_1 R_2 = 0.04 + 0.8(0.012316716)$$

$$= 0.04985337$$

$$X_1 = 2Y_1$$

out

40

* Gauss - Jordan:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & : & c_1 \\ a_{21} & a_{22} & a_{23} & : & c_2 \\ a_{31} & a_{32} & a_{33} & : & c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} \Rightarrow \text{Solution}$$

$$\begin{aligned} 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 &= 71.4 \end{aligned}$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 & : & 7.85 \\ 0.1 & 7 & -0.3 & : & -19.3 \\ 0.3 & -0.2 & 10 & : & 71.4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.033\bar{3} & -0.066\bar{6} & : & 2.616666667 \\ 0 & 7.0033\bar{3} & -0.2933\bar{4} & : & -19.56166667 \\ 0 & -0.1900001 & 10.019998 & : & 70.615 \end{bmatrix}$$

$$\begin{aligned} -0.1 * [1 \quad -0.033\bar{3} \quad -0.066\bar{6} \quad : \quad 2.616666667] \\ = -0.1 \quad 0.0033\bar{3} \quad 0.0066\bar{6} \quad : \quad -0.261666667 \end{aligned}$$

$$\begin{aligned} -0.3 * [1 \quad -0.033\bar{3} \quad -0.066\bar{6} \quad : \quad 2.616666667] \\ = -0.3 \quad 0.00999 \quad 0.019998 \quad : \quad -0.785000001 \end{aligned}$$

[4]

ask :: believe & receive

$$\begin{bmatrix} 1 & 0 & -0.0668061 & 2.52366 \\ 0 & 1 & -0.0418858 & -2.793194 \\ 0 & 0 & 10.01204 & 70.08429 \end{bmatrix}$$

$$0.0333 \times [0 \ 1 \ -0.0418858 \ -2.793194] \\ = 0 \ 0.0333 \ -1.3948 \times 10^{-3} \ -0.09301$$

$$0.1900001 \times [0 \ 1 \ -0.0418858 \ -2.793194] \\ = 0 \ 0.1900001 \ -7.958 \times 10^{-3} \ -0.53071$$

$$\begin{bmatrix} 1 & 0 & 0 & 2.991303 \\ 0 & 1 & 0 & -2.499993 \\ 0 & 0 & 1 & 7.000000999 \end{bmatrix}$$

$$0.0668061 \times [0 \ 0 \ 1 \ 7.000000999] \\ = 0 \ 0 \ 0.0668061 \ 0.4676428$$

$$0.0418858 \times [0 \ 0 \ 1 \ 7.000000999] \\ = 0 \ 0 \ 0 \ 0.0418858 \ 0.293201$$

$$\therefore x_1 = 2.991303$$

$$x_2 = -2.499993$$

$$x_3 = 7.000000999$$

* To obtain $[A]^{-1}$:

$$\begin{bmatrix} 3 & -0.1 & -0.2 & | & 1 & 0 & 0 \\ 0.1 & 7 & -0.3 & | & 0 & 1 & 0 \\ 0.3 & -0.2 & 10 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -0.03\bar{3} & -0.06\bar{6} & | & 0.3\bar{3} & 0 & 0 \\ 0 & 7.00333 & -0.2934 & | & -0.03\bar{3} & 1 & 0 \\ 0 & -0.1901 & 10.0198 & | & -0.099 & 0 & 1 \end{bmatrix}$$

$$-0.1 * [1 \quad -0.03\bar{3} \quad -0.06\bar{6} \quad 0.3\bar{3} \quad 0 \quad 0] \\ = -0.1 \quad 0.003\bar{3} \quad 0.006\bar{6} \quad -0.03\bar{3} \quad 0 \quad 0$$

$$-0.3 * [1 \quad -0.03\bar{3} \quad -0.06\bar{6} \quad 0.3\bar{3} \quad 0 \quad 0] \\ = -0.3 \quad +0.0099 \quad +0.0198 \quad -0.099 \quad 0 \quad 0$$

$$\begin{bmatrix} 1 & 0 & -0.06462 & | & 0.3302 & 0.004712 & 0 \\ 0 & 1 & -0.04189 & | & -0.004712 & 0.1428 & 0 \\ 0 & 0 & 10.01184 & | & -0.09999 & 0.027 & 1 \end{bmatrix}$$

$$0.03\bar{3} * [0 \quad 1 \quad -0.04189 \quad -0.004712 \quad 0.1428 \quad 0] \\ = 0 \quad 0.03\bar{3} \quad 0.001382 \quad 0.000155 \quad 0.00472 \quad 0$$

$$0.1901 * [0 \quad 1 \quad -0.04189 \quad -0.004712 \quad 0.1428 \quad 0] \\ = 0 \quad 0.1901 \quad -0.00796 \quad -0.0008969 \quad 0.027 \quad 0$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0.330845 & 0.00489 & +0.00645 \\ 0 & 1 & 0 & | & 0.00486 & 0.142913 & 0.004185 \\ 0 & 0 & 1 & | & -0.009987 & 0.002697 & 0.0999 \end{bmatrix}$$

$$0.06462 \times \begin{bmatrix} 0 & 0 & 1 & -0.009987 & 0.002697 & 0.0999 \end{bmatrix}$$

$$= 0 \quad 0 \quad 0.06462 \quad 0.000645 \quad 0.000174 \quad 0.000645$$

$$0.04189 \times \begin{bmatrix} 0 & 0 & 1 & -0.009987 & 0.002697 & 0.0999 \end{bmatrix}$$

$$= 0 \quad 0 \quad 0.04189 \quad 0.0004184 \quad 0.000113 \quad 0.0004185$$

$$\therefore [A]^{-1} = \begin{bmatrix} 0.330845 & 0.00489 & 0.000695 \\ -0.00486 & 0.142913 & 0.004185 \\ -0.009987 & 0.002697 & 0.0999 \end{bmatrix}$$

* In Gauss-Jordan:

To find $[A]^{-1}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & | & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & | & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & | & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow$$

و بجز حذف الخطوات بالزبط

$$[I] \quad \left\{ \begin{array}{ccc} a_{11}^{-1} & a_{12}^{-1} & a_{13}^{-1} \\ a_{21}^{-1} & a_{22}^{-1} & a_{23}^{-1} \\ a_{31}^{-1} & a_{32}^{-1} & a_{33}^{-1} \end{array} \right\}$$

* To check if $[A]^{-1}$ is correct!

$$\begin{aligned} \rightarrow [A][A]^{-1} &= [I], & [A]^{-1}[A] &= [I] \\ \rightarrow [A]^{-1} & \end{aligned}$$

44

{x} = [A]^{-1} [c]

* In excel: (To check) := MIV(array)

* (To multiplication the matrix's) := MM [Ctrl+shift+entre]

* LU factorization:

$$\{x\} [A] = \{c\}$$

$$[A] = [L][U] \neq [U][L]$$

Diagram illustrating LU factorization of a 4x4 matrix A into a lower triangular matrix L and an upper triangular matrix U .

Matrix A (original matrix):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Matrix L (Lower triangular matrix):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}$$

Matrix U (Upper triangular matrix):

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

The process is labeled "Gauss Elimination".

we obtain it from original matrix

$$L_{21} = a_{21}/a_{11}, \quad L_{31} = a_{31}/a_{11}, \quad L_{41} = a_{41}/a_{11}$$

فقط لأول عمود ما شتر

الباقي ليس احد

$$* \text{Ex: } \begin{bmatrix} 3 & -2 & 1 \\ 2 & 6 & -4 \\ -1 & -2 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -10 \\ 44 \\ -26 \end{Bmatrix}$$

$[A]$

STEP ①

$$* U \begin{bmatrix} 3 & -2 & 1 \\ 0 & 7.33\bar{3} & -4.6\bar{6} \\ 0 & -2.6\bar{6} & 5.3\bar{3} \end{bmatrix} \rightarrow -2 \left[\frac{3}{3} \quad -\frac{2}{3} \quad \frac{1}{3} \right]$$

$$* [L] = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix} \quad 1 \left[\frac{3}{3} \quad -\frac{2}{3} \quad \frac{1}{3} \right]$$

$$\frac{-2.66\bar{6}}{7.33\bar{3}}$$

$$[U] \begin{bmatrix} 3 & -2 & 1 \\ 0 & 7.33\bar{3} & -4.6\bar{6} \\ 0 & 0 & 3.63\bar{6} \end{bmatrix} \rightarrow \text{Divide } \left(\frac{0}{7.33\bar{3}} \quad \frac{7.33\bar{3}}{7.33\bar{3}} \quad \frac{-4.6\bar{6}}{7.33\bar{3}} \right)$$

$$\therefore [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 0.66\bar{6} & 1 & 0 \\ 0.3\bar{3} & -0.36\bar{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 0 & 7.33\bar{3} & -4.6\bar{6} \\ 0 & 0 & 3.63\bar{6} \end{bmatrix}$$

$$* \text{Step ②: } [L]\{D\} = \{C\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} \Rightarrow \begin{cases} 1D_1 + 0D_2 + 0D_3 = C_1 \\ l_{21}D_1 + D_2 + 0D_3 = C_2 \\ l_{31}D_1 + l_{32}D_2 + D_3 = C_3 \end{cases}$$

* Step ③: Backward: $[U] \{X\} = \{D\}$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} \rightarrow u_{33}x_3 = D_3 \dots$$

* Matrix multiplication [note]:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

* LU Factorization:

* Step ①: Split matrix $[A] \rightarrow \begin{matrix} [U] \\ [L] \end{matrix}$

$$[U] \rightarrow \text{Gauss elimination} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix}$$

$$[L] \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \rightarrow [L][U] = [A]$$

* Step ②: D's vector {forward}:

$$[L]\{D's\} = \{C\} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}$$

* Step ③: Find x's {Backward}:

$$[U]\{x\} = \{D's\} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}$$

* Ex: $\begin{bmatrix} 30 & -20 & -10 \\ -20 & 55 & -10 \\ -10 & -10 & 50 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 80 \\ 0 \end{Bmatrix}$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ \frac{20}{30} & 1 & 0 \\ \frac{-10}{30} & \frac{-50/3}{125/3} & 1 \end{bmatrix}$$

Submatrix $\rightarrow = \begin{bmatrix} 30 & -20 & -10 \\ 0 & 125/3 & -5/3 \\ 0 & -50/3 & 140/3 \end{bmatrix}$

$$\frac{20}{30} \begin{bmatrix} 30 & -20 & -10 \end{bmatrix} = 20 \quad \frac{-400}{30} \quad \frac{-200}{30}$$

$$\frac{10}{30} \begin{bmatrix} 30 & -20 & -10 \end{bmatrix} = 10 \quad \frac{-200}{30} \quad \frac{-100}{30}$$

48

$$\therefore [L] = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & -\frac{2}{5} & 1 \end{bmatrix}$$

$$\therefore [U] = \begin{bmatrix} 30 & -20 & -10 \\ 0 & 125/3 & -50/3 \\ 0 & 0 & 40 \end{bmatrix}$$

* Step ②: { forward }:

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & -\frac{2}{5} & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 80 \\ 0 \end{Bmatrix} \Rightarrow \begin{aligned} D_1 &= 0 \\ -\frac{2}{3}D_1 + D_2 + 0D_3 &= 80 \\ \frac{1}{3}D_1 - \frac{2}{5}D_2 + D_3 &= 0 \end{aligned}$$

$$\therefore \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 80 \\ 32 \end{Bmatrix}$$

$$D_3 = 32$$

* Step ③: { Backward }:

$$\begin{bmatrix} 30 & -20 & -10 \\ 0 & 125/3 & -50/3 \\ 0 & 0 & 40 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 80 \\ 32 \end{Bmatrix}$$

$$40x_3 = 32 \Rightarrow x_3 = 4/5$$

$$\frac{125}{3}x_2 - \frac{50}{3}\left(\frac{4}{5}\right) = 80 \Rightarrow x_2 = \frac{56}{25}$$

$$30x_1 - 20\left(\frac{56}{25}\right) - 10\left(\frac{4}{5}\right) = 0 \Rightarrow x_1 = \frac{44}{25}$$

* How to find the inverse:

* STEP ①: Split the matrix $[A] \begin{matrix} \nearrow [U] \\ \searrow [L] \end{matrix}$

* STEP ②: Use forward substitution to find D's.

3x3 inverse ~~det~~ 3x3 upper $[A] \rightarrow [U]$ *
(depends on the (size) Step ② matrix)

$$[L] \{D\}^{(1)} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$[L] \{D\}^{(3)} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$[L] \{D\}^{(2)} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

* STEP ③: Use backward substitution to find X's (for each D)

$$[U] \{X\}^{(1)} = \{D\}^{(1)}, [U] \{X\}^{(2)} = \{D\}^{(2)}$$

$$[U] \{X\}^{(3)} = \{D\}^{(3)}$$

$$\therefore [A]^{-1} = \begin{bmatrix} X_1^{(1)} & X_1^{(2)} & X_1^{(3)} \\ X_2^{(1)} & X_2^{(2)} & X_2^{(3)} \\ X_3^{(1)} & X_3^{(2)} & X_3^{(3)} \end{bmatrix}$$

$$\begin{bmatrix} 30 & -20 & -10 \\ -20 & 55 & -10 \\ -10 & -10 & 50 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 80 \\ 0 \end{Bmatrix}$$

* STEP ①: $[A] \xrightarrow{[U]} [L]$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -1/3 & -2/5 & 1 \end{bmatrix}, [U] = \begin{bmatrix} 30 & -20 & -10 \\ 0 & 125/3 & -50/3 \\ 0 & 0 & 40 \end{bmatrix}$$

* Step ②: $\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -1/3 & -2/5 & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$

$D_1^{①} = 1$
 $D_2^{①} = 2/3$
 $D_3^{①} = 3/5$ ←

$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -1/3 & -2/5 & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$

$D_1^{②} = 0$
 $D_2^{②} = 1$
 $D_3^{②} = 2/5$ ←

$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -1/3 & -2/5 & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

$D_1^{③} = 0$
 $D_2^{③} = 0$
 $D_3^{③} = 1$ ←

* Step ③: $\begin{Bmatrix} [U] \\ [U] \\ [U] \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}$

$$\begin{bmatrix} 30 & -20 & -10 \\ 0 & 125/3 & -50/3 \\ 0 & 0 & 40 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2/3 \\ 3/5 \end{Bmatrix} \begin{matrix} \textcircled{1} \\ \textcircled{1} \\ \textcircled{1} \end{matrix} \rightarrow \begin{matrix} x_3^{\textcircled{1}} = 3/200 \\ x_2^{\textcircled{1}} = 0.022 \\ x_1^{\textcircled{1}} = 0.053 \end{matrix}$$

$$\begin{bmatrix} 30 & -20 & -10 \\ 0 & 125/3 & -50/3 \\ 0 & 0 & 40 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 2/5 \end{Bmatrix} \begin{matrix} \textcircled{2} \\ \textcircled{2} \\ \textcircled{2} \end{matrix} \rightarrow \begin{matrix} x_1^{\textcircled{2}} = 0.037 \\ x_2^{\textcircled{2}} = 0.032 \\ x_3^{\textcircled{2}} = 0.035 \end{matrix}$$

$$\begin{bmatrix} 30 & -20 & -10 \\ 0 & 125/3 & -50/3 \\ 0 & 0 & 40 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \begin{matrix} \textcircled{3} \\ \textcircled{3} \\ \textcircled{3} \end{matrix} \rightarrow \begin{matrix} x_1^{\textcircled{3}} = 0.015 \\ x_2^{\textcircled{3}} = 0.01 \\ x_3^{\textcircled{3}} = 0.025 \end{matrix}$$

$$\therefore [A]^{-1} = \begin{bmatrix} x_1^{\textcircled{1}} & x_1^{\textcircled{2}} & x_1^{\textcircled{3}} \\ x_2^{\textcircled{1}} & x_2^{\textcircled{2}} & x_2^{\textcircled{3}} \\ x_3^{\textcircled{1}} & x_3^{\textcircled{2}} & x_3^{\textcircled{3}} \end{bmatrix} = \begin{bmatrix} 0.053 & 0.037 & 0.015 \\ 0.022 & 0.032 & 0.01 \\ 0.015 & 0.035 & 0.025 \end{bmatrix}$$

* To check: $[A] \cdot [A]^{-1} = [I]$ ✓

* Ex: Find $[A]^{-1}$ for $[A] = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

$$\therefore [A]^{-1} = \begin{bmatrix} -6 & -5 & 3 \\ -1 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$$

* Iterative Methods:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= C_1 \Rightarrow x_1 = (C_1 - a_{12}x_2 - a_{13}x_3)/a_{11} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= C_2 \Rightarrow x_2 = (C_2 - a_{21}x_1 - a_{23}x_3)/a_{22} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= C_3 \Rightarrow x_3 = (C_3 - a_{31}x_1 - a_{32}x_2)/a_{33} \end{aligned}$$

1] Jacobi

↓

$$x_1^0, x_2^0, x_3^0$$

initial guesses

2] Gauss Seidel

$$\begin{aligned} x_1^{\textcircled{1}} &= [C_1 - a_{12}x_2^0 - a_{13}x_3^0]/a_{11} \\ x_2^{\textcircled{1}} &= [C_2 - a_{21}x_1^{\textcircled{1}} - a_{23}x_3^0]/a_{22} \\ x_3^{\textcircled{1}} &= [C_3 - a_{31}x_1^{\textcircled{1}} - a_{32}x_2^{\textcircled{1}}]/a_{33} \end{aligned}$$

Relative error : (for Both methods)

$$R.E_{x_1} = \left| \frac{x_1^{\textcircled{1}} - x_1^0}{x_1^{\textcircled{1}}} \right| \leq 10^{-5}$$

$$R.E_{x_2} = \left| \frac{x_2^{\textcircled{1}} - x_2^0}{x_2^{\textcircled{1}}} \right| \leq 10^{-5}$$

$$R.E_{x_3} = \left| \frac{x_3^{\textcircled{1}} - x_3^0}{x_3^{\textcircled{1}}} \right| \leq 10^{-5}$$

* If $R.E > 10^{-5}$ (at least one of x 's) :

$$\begin{aligned} x_1^0 &\rightarrow x_1^{\textcircled{1}} \\ x_2^0 &\rightarrow x_2^{\textcircled{1}} \\ x_3^0 &\rightarrow x_3^{\textcircled{1}} \end{aligned}$$

$$\begin{aligned} x_1^{\textcircled{1}} &\rightarrow x_1^{\textcircled{2}} \\ x_2^{\textcircled{1}} &\rightarrow x_2^{\textcircled{2}} \\ x_3^{\textcircled{1}} &\rightarrow x_3^{\textcircled{2}} \end{aligned}$$

2 Gauss Seidel:

$$\begin{aligned} X_1^{\oplus} &= [C_1 - a_{12}X_2^{\circ} - a_{13}X_3^{\circ}] / a_{11} \text{ (old)} \\ X_2^{\oplus} &= [C_2 - a_{21}X_1^{\oplus} - a_{23}X_3^{\circ}] / a_{22} \text{ (mix)} \\ X_3^{\oplus} &= [C_3 - a_{31}X_1^{\oplus} - a_{32}X_2^{\oplus}] / a_{33} \text{ (new)} \end{aligned}$$

* Ex: $3X_1 - 0.1X_2 - 0.2X_3 = 7.85$
 $0.1X_1 + 7X_2 - 0.3X_3 = -19.3$
 $0.3X_1 - 0.2X_2 + 10X_3 = 71.4$

$$X_1 = (19.3 - 7X_2 + 0.3X_3) / 0.1$$

$$X_2 = (7.85 - 3X_1 + 0.2X_3) / -0.1$$

$$X_3 = (71.4 - 0.3X_1 + 0.2X_2) / 10$$

$$X_1^{\circ} = X_2^{\circ} = X_3^{\circ} = 0 \quad [\text{Defy 1st}]$$

* Jacobi:

$$\begin{aligned} X_1^{\oplus} &= \frac{-19.3}{0.1} = -193 \\ X_2^{\oplus} &= \frac{7.85}{-0.1} = -78.5 \\ X_3^{\oplus} &= \frac{71.4}{10} = 7.14 \end{aligned}$$

* B.E for X 's equal 100% because the initial guesses equal 0.

* 2nd iteration:

$$\begin{aligned} x_1^{(2)} &= [-19.3 - 7(-78.5) + 0.3(71.4)] / 0.1 = \uparrow \\ x_2^{(2)} &= [7.85 - 3(193) + 0.2(71.4)] / -0.1 = -5882 \\ x_3^{(2)} &= [71.4 - 0.3(-193) + 0.2(-78.5)] / 10 = 11.36 \end{aligned}$$

تکثیر از مقادیر جدید و مقادیر قبلی
 Δ (دو بار با دو بار)

If |main diagonal element| > the summation of other |element's| in the same row

$$|0.1| < |7.3| \quad \times$$

$$|0.1| < |3.2| \quad \times$$

$$|10| > |0.5| \quad \checkmark$$

معادلات

$$\begin{aligned} \therefore 3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 &= 71.4 \end{aligned}$$

$$\begin{aligned} x_1 &= (7.85 + 0.1x_2 + 0.2x_3) / 3 = 2.61666 \\ x_2 &= (-19.3 - 0.1x_1 + 0.3x_3) / 7 = -2.757145 \\ x_3 &= (71.4 - 0.3x_1 + 0.2x_2) / 10 = 7.14 \end{aligned}$$

let $x_1^0 = x_2^0 = x_3^0 = 0$

2nd iteration:

$$x_1^{(2)} = [7.85 + 0.1(-2.757143) + 0.2(7.14)]/3 \\ = 3.0007619$$

$$x_2^{(2)} = [-19.3 - 0.1(2.61666) + 0.3(7.14)]/7 \\ = -2.488524$$

$$x_3^{(2)} = [7.14 - 0.3(2.61666) + 0.2(-2.757)]/10 \\ = 7.006357$$

$$R.E \ x_1 = \left| \frac{3.0007619 - 2.61666}{3.0007619} \right| = 12.7949\%$$

$$R.E \ x_2 = 10.7943\%, \ R.E \ x_3 = 1.90745\%$$

[5 iteration are required] ↑ دقة عالية

* System of Non-Linear equations:

$$f_1(x, y, z, \dots, m) = 0$$

$$f_2(x, y, z, \dots, m) = 0$$

$$f_n(x, y, z, \dots, m) = 0$$

$$m = n$$

(عدد المعادلات =
عدد المتغيرات)

$$f_1(x, y, z) = 0, \ f_2(x, y, z) = 0, \ f_3(x, y, z) = 0$$

1 Newton's Method :

* Step ① : Find Jacobian Matrix :

$$J(x, y, z) = \begin{bmatrix} \frac{df_1}{dx} & \frac{df_1}{dy} & \frac{df_1}{dz} \\ \frac{df_2}{dx} & \frac{df_2}{dy} & \frac{df_2}{dz} \\ \frac{df_3}{dx} & \frac{df_3}{dy} & \frac{df_3}{dz} \end{bmatrix}_{3 \times 3}$$

$\begin{matrix} \text{Eq. 1} \rightarrow & x \rightarrow & y \rightarrow & z \rightarrow \\ \text{Eq. 2} \rightarrow & x \rightarrow & y \rightarrow & z \rightarrow \\ \text{Eq. 3} \rightarrow & x \rightarrow & y \rightarrow & z \rightarrow \end{matrix}$

* Step ② : Using gauss elimination solve the system below to find $\Delta x, \Delta y, \Delta z$ {correctors}

$$J \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} f_1(x_0, y_0, z_0) \\ f_2(x_0, y_0, z_0) \\ f_3(x_0, y_0, z_0) \end{bmatrix}$$

$$\begin{bmatrix} \frac{df_1}{dx} & \frac{df_1}{dy} & \frac{df_1}{dz} \\ \frac{df_2}{dx} & \frac{df_2}{dy} & \frac{df_2}{dz} \\ \frac{df_3}{dx} & \frac{df_3}{dy} & \frac{df_3}{dz} \end{bmatrix}_{x_0, y_0, z_0} \begin{bmatrix} \Delta x^{(1)} \\ \Delta y^{(1)} \\ \Delta z^{(1)} \end{bmatrix} = - \begin{bmatrix} f_1(x_0, y_0, z_0) \\ f_2(x_0, y_0, z_0) \\ f_3(x_0, y_0, z_0) \end{bmatrix}$$

* Step ③ : Correct the values of the variables according to

$$\begin{aligned} x^{(1)} &= x^0 + \Delta x^{(1)} \\ y^{(1)} &= y^0 + \Delta y^{(1)} \\ z^{(1)} &= z^0 + \Delta z^{(1)} \end{aligned} \Rightarrow \begin{aligned} |f_1(x^{(1)}, y^{(1)}, z^{(1)})| &\leq Tol. \\ |f_2(x^{(1)}, y^{(1)}, z^{(1)})| &\leq Tol. \\ |f_3(x^{(1)}, y^{(1)}, z^{(1)})| &\leq Tol. \end{aligned}$$

$$② \quad \left| \frac{x^{\oplus} - x^{\circ}}{x^{\oplus}} \right| \leq \text{error}$$

$$\left| \frac{y^{\oplus} - y^{\circ}}{y^{\oplus}} \right| \leq \text{error}$$

$$\left| \frac{z^{\oplus} - z^{\circ}}{z^{\oplus}} \right| \leq \text{error}$$

2nd

$$[J]_{\begin{matrix} x^{\oplus} \\ y^{\oplus} \\ z^{\oplus} \end{matrix}}$$

$$- \begin{bmatrix} f_1(x^{\oplus}, y^{\oplus}, z^{\oplus}) \\ f_2(x^{\oplus}, y^{\oplus}, z^{\oplus}) \\ f_3(x^{\oplus}, y^{\oplus}, z^{\oplus}) \end{bmatrix} \Rightarrow \begin{matrix} \Delta x^{\oplus} \\ \Delta y^{\oplus} \\ \Delta z^{\oplus} \end{matrix}$$

$$\begin{aligned} * \text{Ex: } & \left. \begin{aligned} f_1(x, y) &= x^2 - 2x - y + 0.5 \\ f_2(x, y) &= x^2 + 4y^2 - 4 \end{aligned} \right\} \begin{aligned} x_0 &= 2 \\ y_0 &= 0.25 \end{aligned} \end{aligned}$$

$$* \text{Step ①: } J \begin{bmatrix} 2x-2 & -1 \\ 2x & 8y \end{bmatrix} \Leftarrow \text{General}$$

$$1^{\text{st}} \text{ iteration: } J \Big|_{\begin{matrix} x=2 \\ y=0.25 \end{matrix}} = \begin{bmatrix} 2(2)-2 & -1 \\ 2(2) & 8(0.25) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$$

$$f_1(2, 0.25) = (2)^2 - 2(2) - 0.25 + 0.5 = 0.25$$

$$f_2(2, 0.25) = (2)^2 + 4(0.25)^2 - 4 = 0.25$$

$$\begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} \begin{Bmatrix} \Delta x^{(1)} \\ \Delta y^{(1)} \end{Bmatrix} = - \begin{Bmatrix} 0.25 \\ 0.25 \end{Bmatrix}$$

$$2\Delta x^{(1)} - \Delta y^{(1)} = -0.25$$

$$4\Delta x^{(1)} + 2\Delta y^{(1)} = -0.25$$

$$\Rightarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \Delta x^{(1)} \\ \Delta y^{(1)} \end{Bmatrix} = \begin{Bmatrix} -0.125 \\ 0.25 \end{Bmatrix}$$

Gauss elimination

$$\Delta x^{(1)} = [-0.125 + \frac{1}{2}(0.0625)] / 1$$

$$\Delta y^{(1)} = \frac{0.25}{4} = 0.0625 \uparrow$$

$$\Delta x^{(1)} = -0.09375$$

$$\begin{aligned} * \text{ Step ③ : } x^{(1)} &= x^0 + \Delta x^{(1)} = 2 - 0.09375 \\ &= 1.90625 \end{aligned}$$

$$y^{(1)} = y^0 + \Delta y^{(1)} = 0.25 + 0.0625 = 0.3125$$

check

$$f_1(1.90625, 0.3125) = 0.008789 > 1 \cdot 10^{-5}$$

$$f_2(1.90625, 0.3125) = 0.024914 > 1 \cdot 10^{-5}$$

2nd iteration:

$$* \text{ Step ①: } J| = \begin{bmatrix} 1.8125 & 1 \\ 3.8125 & 2.5 \end{bmatrix}$$

1.90625
0.3125

$$\begin{bmatrix} 1.8125 & -1 \\ 3.8125 & 2.5 \end{bmatrix} \begin{Bmatrix} \Delta x^{\text{②}} \\ \Delta y^{\text{②}} \end{Bmatrix} = - \begin{Bmatrix} 0.008789 \\ 0.024914 \end{Bmatrix}$$

$$\Delta x^{\text{②}} = -0.004849$$

$$\Delta y^{\text{②}} = -0.001287557$$

$$x^{\text{②}} = x^{\text{①}} + \Delta x^{\text{②}} = 1.90625 - 0.004849 = 1.900691$$

$$y^{\text{②}} = y^{\text{①}} + \Delta y^{\text{②}} = 0.3125 - 0.001287557 = 0.311212$$

$$\text{Error in } x = \left| \frac{1.900691 - 1.90625}{1.900691} \right| \leftarrow \text{error}$$

$$\text{Error in } y = \left| \frac{0.311212 - 0.3125}{0.311212} \right| \leftarrow \text{error}$$

2 Fixed-points:

$$\left. \begin{aligned} f_1(x, y, z) &= 0 \\ f_2(x, y, z) &= 0 \\ f_3(x, y, z) &= 0 \end{aligned} \right\} \Rightarrow \begin{aligned} x &= g_1(x, y, z) \\ y &= g_2(x, y, z) \\ z &= g_3(x, y, z) \end{aligned}$$

1st iteration and find x^0, y^0, z^0 :

$$\left. \begin{aligned} x^0 &= g_1(x^0, y^0, z^0) \\ y^0 &= g_2(x^0, y^0, z^0) \\ z^0 &= g_3(x^0, y^0, z^0) \end{aligned} \right\} \text{Jacobi}$$

$$\left. \begin{aligned} x^0 &= g_1(x^0, y^0, z^0) \\ y^0 &= g_2(x^0, y^0, z^0) \\ z^0 &= g_3(x^0, y^0, z^0) \end{aligned} \right\} \text{Seidel} \\ \text{Recommended}$$

$$\left| \frac{\partial g_1}{\partial x} \right|_{x^0, y^0, z^0} + \left| \frac{\partial g_1}{\partial y} \right|_{x^0, y^0, z^0} + \left| \frac{\partial g_1}{\partial z} \right|_{x^0, y^0, z^0} < 1$$

$$\left| \frac{\partial g_2}{\partial x} \right|_{x^0, y^0, z^0} + \left| \frac{\partial g_2}{\partial y} \right|_{x^0, y^0, z^0} + \left| \frac{\partial g_2}{\partial z} \right|_{x^0, y^0, z^0} < 1$$

* Fixed point Iteration: $x_0 = y_0 = z_0 = 2$

$$f_1(x, y, z) = x^2 + 50x + y^2 + z^2 - 200 = 0$$

$$f_2(x, y, z) = x^2 + 20y + z^2 - 50 = 0$$

$$f_3(x, y, z) = -x^2 - y^2 + 40z + 75 = 0$$

$$\begin{aligned} \rightarrow x &= (200 - x^2 - y^2 - z^2) / 50 \rightarrow g_1(x, y, z) \\ \rightarrow y &= (50 - x^2 - z^2) / 20 \rightarrow g_2(x, y, z) \\ \rightarrow z &= (-75 + x^2 + y^2) / 40 \rightarrow g_3(x, y, z) \end{aligned}$$

$$\left| \frac{\partial g_1}{\partial x} \right|_{x_1, y_1, z_1} + \left| \frac{\partial g_1}{\partial y} \right|_{x_1, y_1, z_1} + \left| \frac{\partial g_1}{\partial z} \right|_{x_1, y_1, z_1} \leq 1$$

$$\left| \frac{\partial g_2}{\partial x} \right| + \left| \frac{\partial g_2}{\partial y} \right| + \left| \frac{\partial g_2}{\partial z} \right| \leq 1$$

$$\left| \frac{\partial g_3}{\partial x} \right| + \left| \frac{\partial g_3}{\partial y} \right| + \left| \frac{\partial g_3}{\partial z} \right| \leq 1$$

check b6

1st iteration:

$$x_1 = 3.76, y_1 = 2.1, z_1 = -1.675$$

then check b6

2nd iteration: $x_2 = 3.5729, y_2 = 1.6528, z_2 = -1.4113$

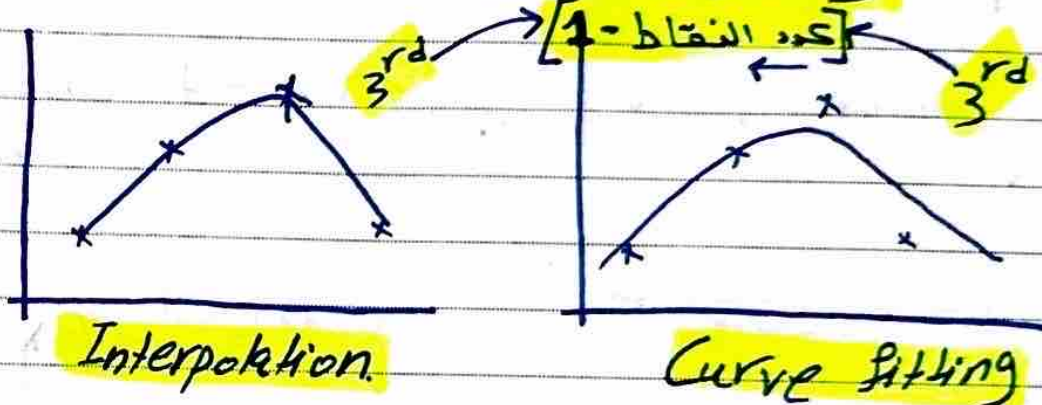
$$R.E.(x) = (3.5729 - 3.76) / 3.5729 \approx 100\%$$

$$\dots X_9 = 3.6328, y_9 = 1.7321, z_9 = -1.4701$$

$$\dots X_{10} = 3.6328, y_{10} = 1.7321, z_{10} = -1.4701$$

$$R.E. = 0 \quad \checkmark$$

* Interpolation and curve fitting:



* Fitting:

① * Least squares method:

1) Linear : $y = A + Bx$

2) Polynomial : $y = A + Bx + Cx^2$

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4$$

3) Simple Non-linear model:

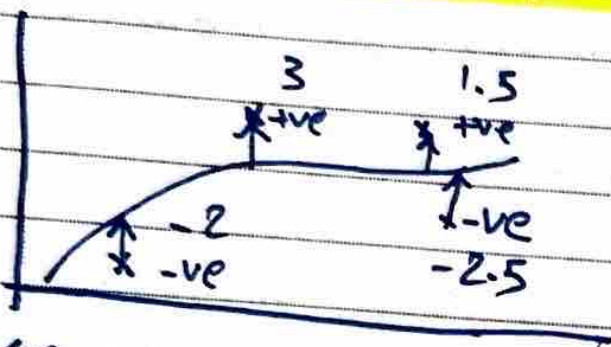
$$y = Ae^{bx} \Rightarrow \ln y = \ln A + bx$$

$$y = e^{bx} + \sin(Cx)$$

* If the function $y = e^{bx} + \sin(Cx)$

② Non-linear regression method

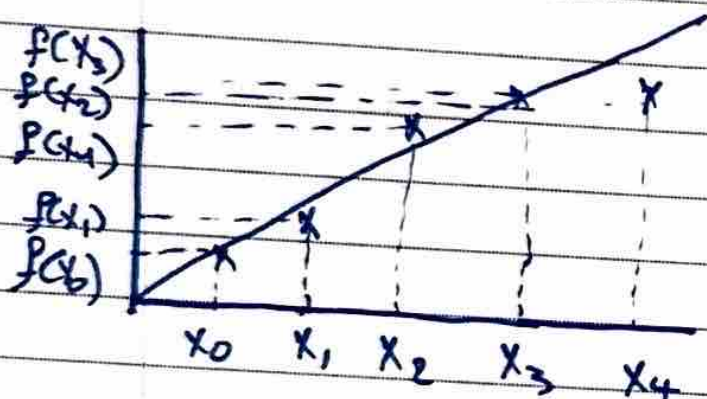
Least Squares method:



Error = given - calculated

(Summation of squares of error)

minimization



$$y = A + Bx$$

$$\text{Error}_i = f(x_i) - y(x_i)$$

$$\text{Error}_{x_0} = f(x_0) - A - Bx_0$$

$$\text{Error}_{x_1} = f(x_1) - A - Bx_1$$

$$\text{Error}^2 = [f(x_0) - A - Bx_0]^2$$

$$S = \sum_{i=0}^n (\text{Error}_i)^2 = \sum_{i=0}^n [f(x_i) - A - Bx_i]^2$$

[# of parameter = # of order + 1] Parameters

$$\frac{\partial S}{\partial A} = 0, \quad \frac{\partial S}{\partial B} = 0 \quad (\text{parameter})$$

$$\begin{aligned} \frac{\partial S}{\partial A} = 0 &= 2 \sum_{i=0}^n [f(x_i) - A - Bx_i](-1) \\ \frac{\partial S}{\partial B} = 0 &= 2 \sum_{i=0}^n [f(x_i) - A - Bx_i](-x_i) \\ \rightarrow 0 &= \sum f(x_i) - \sum A - \sum Bx_i \\ \rightarrow 0 &= \sum f(x_i)x_i - \sum Ax_i - \sum Bx_i^2 \end{aligned}$$

$$\begin{aligned} \sum A + \sum Bx_i &= \sum f(x_i) \\ \sum Ax_i + \sum Bx_i^2 &= \sum x_i \cdot f(x_i) \end{aligned}$$

$$\begin{aligned} A \sum 1 + B \sum x_i &= \sum f(x_i) \\ A \sum x_i + B \sum x_i^2 &= \sum x_i \cdot f(x_i) \end{aligned}$$

$$\begin{aligned} A(n) + B(\sum x_i) &= \sum f(x_i) \\ A(\sum x_i) + B(\sum x_i^2) &= \sum x_i \cdot f(x_i) \end{aligned}$$

x_i	$f(x_i)$	x_i^2	$x_i \cdot f(x_i)$
x_0	$f(x_0)$	x_0^2	$x_0 f(x_0)$
x_1	$f(x_1)$	x_1^2	$x_1 f(x_1)$
x_2	$f(x_2)$	x_2^2	$x_2 f(x_2)$
x_3	$f(x_3)$	x_3^2	$x_3 f(x_3)$
$\sum x_i$	$\sum f(x_i)$	$\sum x_i^2$	$\sum x_i \cdot f(x_i)$

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} \sum f(x_i) \\ \sum x_i \cdot f(x_i) \end{Bmatrix}$$

$$* y = A + Bx + Cx^2$$

$$\varepsilon = f(x_i) - A - Bx_i - Cx_i^2$$

$$S = \sum_{i=1}^n [f(x_i) - A - Bx_i - Cx_i^2]^2$$

$$\frac{\partial S}{\partial A} = 0 = -2 \sum_{i=1}^n [f(x_i) - A - Bx_i - Cx_i^2] (+1)$$

$$\frac{\partial S}{\partial B} = 0 = -2 \sum_{i=1}^n [f(x_i) - A - Bx_i - Cx_i^2] (-x_i)$$

$$\frac{\partial S}{\partial C} = 0 = -2 \sum_{i=1}^n [f(x_i) - A - Bx_i - Cx_i^2] (-x_i^2)$$

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} \sum f(x_i) \\ \sum x_i f(x_i) \\ \sum x_i^2 f(x_i) \end{Bmatrix}$$

x_i	$f(x_i)$	x_i^2	x_i^3	x_i^4	$x_i f(x_i)$	$x_i^2 f(x_i)$
$\sum x_i$	$\sum f(x_i)$					

$$* y = A + Bx + Cx^2 + Dx^3 + Ex^4$$

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \sum x_i^5 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \sum x_i^5 & \sum x_i^6 \\ \sum x_i^3 & \sum x_i^4 & \sum x_i^5 & \sum x_i^6 & \sum x_i^7 \\ \sum x_i^4 & \sum x_i^5 & \sum x_i^6 & \sum x_i^7 & \sum x_i^8 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \\ E \end{Bmatrix} = \begin{Bmatrix} \sum f(x_i) \\ \sum x_i f(x_i) \\ \sum x_i^2 f(x_i) \\ \sum x_i^3 f(x_i) \\ \sum x_i^4 f(x_i) \end{Bmatrix}$$

* Simple Non-linear :

$$* y = ab^x, y = ax^b, y = ae^{bx}$$

$$\downarrow$$

$$\ln y = \ln a + x \ln b$$

$$\downarrow$$

$$Y = A + Bx$$

x	y	Y = \ln y	X = \ln x
X	X		

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} \sum Y \\ \sum x_i Y \end{Bmatrix}$$

* Coefficient of Determination (r^2):

$$0 \leq r^2 \leq 1, r^2 = \frac{S_r - S_r}{S_T}$$

$\leftarrow \rightarrow$

$$\text{if } S_r = 0 \Rightarrow r^2 = S_r / S_T = 1 \quad [S_r \uparrow, r^2 \downarrow]$$

x_i	$f(x_i)$	$(f(x_i) - \bar{y})^2$	$y_{\text{calculated}}$	$(f(x_i) - y_{\text{calculated}})^2$
1	$f(x_1)$	$(f(x_1) - \bar{y})^2$	$A + Bx_1$	$(f(x_1) - (A + Bx_1))^2$
2	$f(x_2)$	$(f(x_2) - \bar{y})^2$	$A + Bx_2$	$(f(x_2) - (A + Bx_2))^2$
	$\sum f(x_i)$	$S_T = \sum \dots$		$S_r = \sum \dots$
	$\bar{y} = \frac{\sum f(x_i)}{n}$			

* Non-Linear Regression: [Iterative]

parameters = $m \leftarrow b_x$

$$y = a e^{bx}$$

x_i	$f(x_i)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$
n	

* Step ①: Find Z matrix

$n \times m$ parameters

$\frac{\partial y}{\partial a}$ $\frac{\partial y}{\partial b}$

det $\frac{\partial y}{\partial a}$

Z matrix

$$Z(a, b) = \begin{bmatrix} e^{bx_1} & a x_1 e^{bx_1} \\ e^{bx_2} & a x_2 e^{bx_2} \\ e^{bx_3} & a x_3 e^{bx_3} \\ \vdots & \vdots \\ e^{bx_n} & a x_n e^{bx_n} \end{bmatrix}$$

a_0 b_0

Initial guesses $a = a_0$
 $b = b_0$

* Step ②: Find $[D] \rightarrow [f(x) - y]$

$$[D] = \begin{bmatrix} f(x_1) - a_0 e^{b_0 x_1} \\ f(x_2) - a_0 e^{b_0 x_2} \\ f(x_3) - a_0 e^{b_0 x_3} \\ \vdots \\ f(x_n) - a_0 e^{b_0 x_n} \end{bmatrix}_{n \times 1}$$

* Step ③: (Transpose):

$$Z^T = \begin{bmatrix} e^{bx_1} & a_0 x_1 e^{bx_1} & e^{bx_2} & a_0 x_2 e^{bx_2} & e^{bx_3} & a_0 x_3 e^{bx_3} & \vdots & e^{bx_n} & a_0 x_n e^{bx_n} \end{bmatrix}_{m \times n}$$

$$* [Z]^T [Z] = [Z] [Z]^T$$

$$\begin{Bmatrix} Z \\ Z \end{Bmatrix}_{m \times n} \begin{Bmatrix} a \\ b \end{Bmatrix}_{n \times 1} = \begin{Bmatrix} [Z]^T [Z] \end{Bmatrix}_{m \times m}$$

* Step ④: Find the corrector:
 $\Delta a, \Delta b, \dots$

$$[Z]^T [Z] \begin{Bmatrix} \Delta a \\ \Delta b \end{Bmatrix}_{m \times 1} = [Z]^T [D]_{m \times 1}$$

* Step ⑤: Find the new values of the parameters:

$$a^{(1)} = a^0 + \Delta a^{(1)}$$

$$b^{(1)} = b^0 + \Delta b^{(1)}$$

* Step ⑥: Find the relative error for all parameters:

$$R.E.(a) = \left| \frac{a^{(1)} - a^0}{a^{(1)}} \right| * 100\% \leq \text{Given Error}$$

$$R.E.(b) = \left| \frac{b^{(1)} - b^0}{b^{(1)}} \right| * 100\% \leq \text{Given Error}$$

$$* [X]: y = A e^{Bx}$$

$$A_0 = 0.9, B_0 = 0.1$$

X_i	$f(X_i)$
0.1	2.1
0.9	3.1
2.3	6.3
5	24.4

$$Z = \begin{bmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \\ 0.9 & 0.1 \\ 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} e^{0.1(0.1)} \\ e^{0.1(0.9)} \\ e^{0.1(2.3)} \\ e^{0.1(5)} \end{bmatrix} = \begin{bmatrix} 1.01 & 0.09 \\ 1.09 & 0.88 \\ 1.26 & 2.61 \\ 1.65 & 7.42 \end{bmatrix}$$

$$Z^T = \begin{bmatrix} 1.01 & 1.09 & 1.26 & 1.65 \\ 0.09 & 0.88 & 2.61 & 7.42 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 2.1 - 0.9 e^{0.1(0.1)} \\ 3.1 - 0.9 e^{0.1(0.9)} \\ 6.3 - 0.9 e^{0.1(2.3)} \\ 24.4 - 0.9 e^{0.1(5)} \end{bmatrix} = \begin{bmatrix} 2.01 \\ 2.22 \\ 5.69 \\ 22.92 \end{bmatrix}$$

$$[Z][Z] = \begin{bmatrix} 1.01 & 1.09 & 1.26 & 1.65 \\ 0.09 & 0.88 & 2.61 & 7.42 \end{bmatrix}$$

$$* \begin{bmatrix} 1.01 & 0.09 \\ 1.09 & 0.88 \\ 1.26 & 2.61 \\ 1.65 & 7.42 \end{bmatrix}$$

[70]

$$[Z]^T [Z] = \begin{bmatrix} 1.03 & 1.18 & 1.51 & 2.34 \\ 1.18 & 1.98 & 3.70 & 8.40 \\ 1.51 & 3.70 & 8.40 & 21.44 \\ 2.34 & 8.40 & 21.44 & 57.78 \end{bmatrix}$$

$$[Z]^T [D] = \begin{bmatrix} 47.84 \\ 185.5 \end{bmatrix}$$

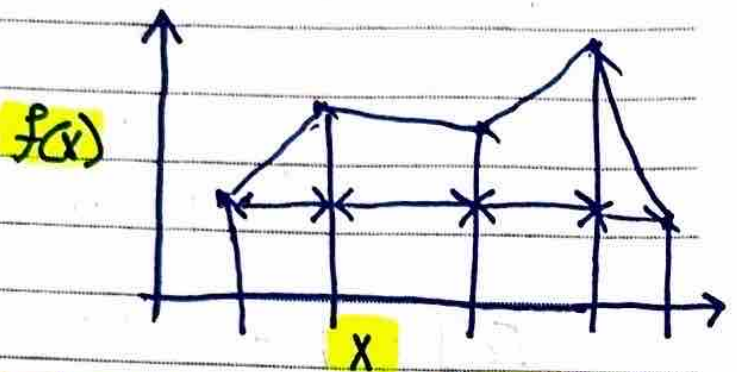
$$\begin{bmatrix} 1.03 & 1.18 & 1.51 & 2.34 \\ 1.18 & 1.98 & 3.70 & 8.40 \\ 1.51 & 3.70 & 8.40 & 21.44 \\ 2.34 & 8.40 & 21.44 & 57.78 \end{bmatrix} \begin{Bmatrix} \Delta A_0 \\ \Delta B_0 \end{Bmatrix} = \begin{bmatrix} 47.84 \\ 185.5 \end{bmatrix}$$

$$A_1 = A_0 + \Delta A_0 = 0.9 + =$$

$$B_1 = B_0 + \Delta B_0 = 0.1 + =$$

[Read Regression work sheet] ...

* Interpolation:



① Newton's

② Lagrange

③ Table difference

↓
1st, 2nd, higher order

* General formula for Lagrange:

$$f_n(x) = \sum_{i=0}^n L_i(x) \cdot f(x_i)$$

n : * of order

x_i	$f(x_i)$
x_0	$f(x_0)$
x_1	$f(x_1)$
$x_2 = 1$	$f(x_2)$
$x_3 = 5$	$f(x_3)$

$\Delta x = 4$

* How do you select the point in odd number of x's $\Rightarrow |x - x^*|$
 و بوی این

$$\begin{aligned} & * L_1(x) \cdot f(x_1) \\ & * L_2(x) \cdot f(x_2) \\ & L_3(x) \cdot f(x_3) \\ & L_n(x) \cdot f(x_4) \end{aligned}$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

x_i : x_i
 x_i : x_i

72

ask :: believe & recieve

FIVE APPLE

$$\text{Ex: } n=1, \quad f_1(x) = \sum_{i=0}^1 L_i(x) f(x_i)$$

$$f_1(x) = L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1)$$

$$L_i(x) = \prod_{j=0, j \neq i}^1 \frac{x - x_j}{x_i - x_j}$$

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^1 \frac{x - x_j}{x_0 - x_j} = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^1 \frac{x - x_j}{x_1 - x_j} = \frac{x - x_0}{x_1 - x_0}$$

$$\therefore f_1(x) = \left(\frac{x - x_1}{x_0 - x_1} \right) \cdot f(x_0) + \left(\frac{x - x_0}{x_1 - x_0} \right) \cdot f(x_1)$$

$$\text{q.t. } x = x_0:$$

$$f_1(x_0) = \frac{x_0 - x_1}{x_0 - x_1} \cdot f(x_0) + \frac{x_0 - x_0}{x_1 - x_0} \cdot f(x_1)$$

$$\text{q.t. } x = x_1$$

$$f_1(x_1) = \frac{x_1 - x_1}{x_0 - x_1} \cdot f(x_0) + \frac{x_1 - x_0}{x_1 - x_0} \cdot f(x_1)$$

$$\text{+ Ex: } f_2(x) = \sum_{i=0}^2 L_i(x) \cdot f(x_i)$$

x_i	$f(x_i)$
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$

$x_0 \quad x_1 \quad x_2$

$$f_2(x) = L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1) + L_2(x) \cdot f(x_2)$$

$$L_0(x) = \prod_{\substack{j=0 \\ j \neq 0}}^2 \left(\frac{x - x_j}{x_0 - x_j} \right) = \left(\frac{x - x_1}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j \neq 1}}^2 \left(\frac{x - x_j}{x_1 - x_j} \right) = \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j \neq 2}}^2 \left(\frac{x - x_j}{x_2 - x_j} \right) = \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right)$$

$$\therefore f_2(x) = \left(\frac{x - x_1}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right) \cdot f(x_0) \\ + \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right) \cdot f(x_1) \\ + \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right) \cdot f(x_2)$$

at $x = x_1$:

$$f_2(x_1) = \cancel{\left(\frac{x_1 - x_1}{x_0 - x_1} \right)} \left(\frac{x_1 - x_2}{x_0 - x_2} \right) \cdot f(x_0) \\ + \left(\frac{x_1 - x_0}{x_1 - x_0} \right) \left(\frac{x_1 - x_2}{x_1 - x_2} \right) \cdot f(x_1) \\ + \left(\frac{x_1 - x_0}{x_2 - x_0} \right) \left(\frac{x_1 - x_1}{x_2 - x_1} \right) \cdot f(x_2)$$

74

$$\therefore f_2(x_1) = f(x_2)$$

$$* \text{Ex: } f_4(x) = \sum_{i=0}^4 L_i(x) \cdot f(x_i)$$

$$= L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1) + L_2(x) \cdot f(x_2) + L_3(x) \cdot f(x_3) + L_4(x) \cdot f(x_4)$$

$$L_0(x) = \left(\frac{x-x_1}{x_0-x_1} \right) \left(\frac{x-x_2}{x_0-x_2} \right) \left(\frac{x-x_3}{x_0-x_3} \right) \left(\frac{x-x_4}{x_0-x_4} \right)$$

$$L_1(x) = \left(\frac{x-x_0}{x_1-x_0} \right) \left(\frac{x-x_2}{x_1-x_2} \right) \left(\frac{x-x_3}{x_1-x_3} \right) \left(\frac{x-x_4}{x_1-x_4} \right)$$

...

1] Newton's: Must give the same answer compared with Lagrange.

$$f_n(x) = a_0(x-x_0) + a_1(x-x_0)(x-x_1) + a_2(x-x_0)(x-x_1)(x-x_2) + a_3(x-x_0)(x-x_1)(x-x_2)(x-x_3) + \dots + a_n$$

a 's: Newton's constant

*Ex: 1st order:

$$f_1(x) = a_0 + a_1(x-x_0)$$

a_0, a_1 are constants

15

ask :: believe & recieve

at $x = x_0$, $f_1(x) = f(x_0)$

$$f(x_0) = a_0 + a_1(x_0 - x_0) \xrightarrow{\text{zero}} \Rightarrow a_0 = f(x_0)$$

at $x = x_1$, $f_1(x) = f(x_1)$

$$f(x_1) = f(x_0) + a_1(x_1 - x_0) \Rightarrow a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \xrightarrow{\text{slope}}$$

$$\therefore f_1(x) = f(x_0) + \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] (x - x_0)$$

*Ex: 2nd order Newton's:

$$f_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

at $x = x_0$, $f_2(x) = f(x_0)$

$$\boxed{f(x_0) = a_0}$$

at $x = x_1$, $f_2(x) = f(x_1)$

$$f(x_1) = f(x_0) + a_1(x_1 - x_0) + a_2 \frac{(x_1 - x_0)^2}{(x_1 - x_1)} \xrightarrow{\text{Zero}}$$

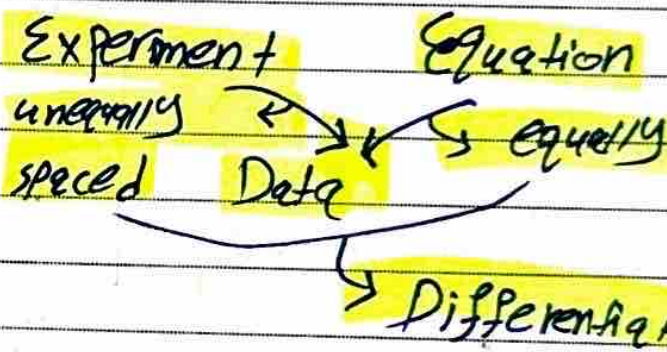
$$\therefore a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

at $x = x_2$, $f_2(x) = f(x_2)$

$$f(x_2) = f(x_0) + \left\{ \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right\} (x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$a_2 = \left\{ \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right\} - \left\{ \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right\}$$

* Integration and Differentiation: See page 81-85



1* For equally spaced:

Finite Difference Formulas

	x_i	$f(x_i)$
Δx	x_0	
$=$	x_1	
Δx	x_2	

2* For unequally spaced:

Curve fit.

- 11 1- Forward formula 2- Backward formula
3- Central formula (very powerful)
Richardson

1- Forward formula:

$$f(x_i+1) = f(x_i) + f'(x_i)h + f''(x_i) \frac{h^2}{2} + \dots$$

$$f(x_i+1) = f(x_i) + f'(x_i)h$$

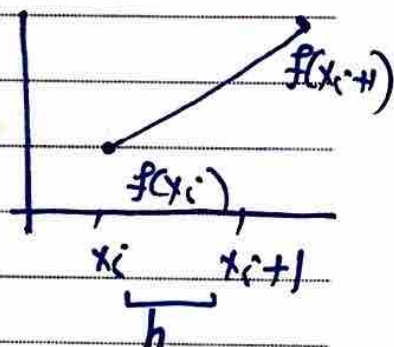
$$\therefore f'(x_i) = \frac{f(x_i+1) - f(x_i)}{h} + o(h)$$

x_i	$f(x_i)$
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$
x_4	$f(x_4)$

$$\text{Slope} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + o(h^2)$$

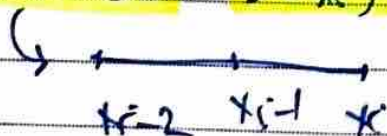
More Accurate.



Forward: $f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + o(h)$

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + o(h^2)$$

Backward: $f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + o(h)$



$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2}))}{2h} + O(h^2)$$

$$\rightarrow \text{Central: } f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1}))}{2h}$$

Formula's) 1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12) (13) (14) (15) (16) (17) (18) (19) (20) (21) (22) (23) (24) (25) (26) (27) (28) (29) (30) (31) (32) (33) (34) (35) (36) (37) (38) (39) (40) (41) (42) (43) (44) (45) (46) (47) (48) (49) (50) (51) (52) (53) (54) (55) (56) (57) (58) (59) (60) (61) (62) (63) (64) (65) (66) (67) (68) (69) (70) (71) (72) (73) (74) (75) (76) (77) (78) (79) (80) (81) (82) (83) (84) (85) (86) (87) (88) (89) (90) (91) (92) (93) (94) (95) (96) (97) (98) (99) (100)

$$f'(x_i) = \frac{-f(x_{i+2}) + 2f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))}{12h} + O(h^4)$$

$$f''(x_i) \approx O(h^2) :$$

Forward

$$f''(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 5f(x_i) + 2f(x_{i-1}))}{h^2} + O(h^3)$$

Central

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))}{h^2} + O(h^2)$$

$x_0 \rightarrow$ use forward
 $x_n \rightarrow$ use backward
 in between $\rightarrow$ use central

2) Unequally spaced points:

1st derivative:

$$f_2(x) = \sum_{i=0}^2 L_i(x) \cdot f(x_i)$$

$$f_2(x) = \left(\frac{x-x_1}{x_0-x_1} \right) \left(\frac{x-x_2}{x_0-x_2} \right) \cdot f(x_0) + \dots \rightarrow$$

$$\rightarrow \dots + \left(\frac{x-x_0}{x_1-x_0} \right) \left(\frac{x-x_2}{x_1-x_2} \right) \cdot f(x_1) + \left(\frac{x-x_0}{x_2-x_0} \right) \left(\frac{x-x_1}{x_2-x_1} \right) \cdot f(x_2)$$

$$\therefore f_0(x) = \frac{x^2 - x x_2 - x x_1 + x_1 x_2}{x_0^2 - x_0 x_2 - x_0 x_1 + x_1 x_2} \cdot f(x_0)$$

$$+ \frac{x^2 - x x_2 - x x_0 + x_0 x_2}{x_1^2 - x_1 x_2 - x_0 x_1 + x_0 x_2} \cdot f(x_1)$$

$$y_0 < x < x_2$$

Ex: Use Lagrange 1st order and 2nd order to evaluate $\ln(2)$ on the basis of the given data:

$$P_1(x) = \left(\frac{x-x_1}{x_0-x_1} \right) \cdot f(x_0) + \left(\frac{x-x_0}{x_1-x_0} \right) \cdot f(x_1)$$

x_0	x_i	$f(x_i) = \ln(x_i)$
1	2	$\ln(1) = 0$
4	4	$\ln(4) = 1.386$
6	6	$\ln(6) = 1.7917$

$$= \frac{2-4}{1-4} * 0 + \frac{2-1}{4-1} * 1.386 = 0.462098133$$

$$P_2(x) = \frac{x-x_1}{x_0-x_1} * \frac{x-x_2}{x_0-x_2} \cdot f(x_0) + \frac{x-x_0}{x_1-x_0} * \frac{x-x_2}{x_1-x_2} \cdot f(x_1) + \frac{x-x_0}{x_2-x_0} * \frac{x-x_1}{x_2-x_1} \cdot f(x_2) = 0.56584437$$

[True Answer: $\ln(2) = 0.693$]

* Using Newton's:

$$f_1(x) = f(x_0) + \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] (x - x_0)$$

$$= 0.462098133$$

$$f_2(x) = 0 + \left[\frac{1.3862944 - 0}{4 - 1} \right] (2 - 1) + \left[\left[\frac{1.791759 - 1.386}{6 - 4} \right] - \left[\frac{1.386 - 0}{4 - 1} \right] \right] (2 - 1)(2 - 4)$$

81

$$= 0.5657443$$

* Ex: Find the 4th order polynomial to fit:

$$f_4(x) = \left(\frac{x-0}{-2-0} \right) \left(\frac{x+1}{-2+1} \right)$$

$$\left(\frac{x-1}{-2-1} \right) \left(\frac{x-2}{-2-2} \right)$$

(4)

$$+ \left(\frac{x+2}{0+2} \right) \left(\frac{x+1}{0+1} \right)$$

$$\left(\frac{x-1}{0-1} \right) \left(\frac{x-2}{0-2} \right)$$

x_i	y
$x_0 \leftarrow -2$	4
$x_1 \leftarrow 0$	2
$x_2 \leftarrow -1$	-1
$x_3 \leftarrow 1$	1
$x_4 \leftarrow 2$	8

(2)

$$+ \left(\frac{x+2}{-1+2} \right) \left(\frac{x-0}{-1-0} \right)$$

$$\left(\frac{x-1}{-1-1} \right) (-1)$$

$$\left(\frac{x-2}{-1-2} \right)$$

$$+ \left(\frac{x+2}{1+2} \right) \left(\frac{x-0}{1-0} \right) \left(\frac{x+1}{1+1} \right)$$

$$\left(\frac{x-2}{1-2} \right) (1)$$

$$+ \left(\frac{x+2}{2+2} \right) \left(\frac{x-0}{2-0} \right) \left(\frac{x+1}{2+1} \right)$$

$$\left(\frac{x-1}{2-1} \right) (8)$$

$$\Rightarrow f_4(x) = x^4 - 3x^2 + x + 2$$

* Ex: Consider the following data:

$$\begin{aligned}
 f(x) &= \left(\frac{x+1}{2+1} \right) \left(\frac{x-0}{-2-0} \right) \left(\frac{x-1}{-2-1} \right) (-15) \\
 &\quad + \left(\frac{x-2}{-2-2} \right) \left(\frac{x-3}{-2-3} \right) \left(\frac{x-4}{-2-4} \right) (-15) + \dots \\
 &\quad + \left(\frac{x+2}{4+2} \right) \left(\frac{x+1}{4+1} \right) \left(\frac{x-0}{4-0} \right) (15) \\
 &\quad + \left(\frac{x-1}{4-1} \right) \left(\frac{x-2}{4-2} \right) \left(\frac{x-3}{4-3} \right) (15) = \dots
 \end{aligned}$$

* Ex: Estimate $\ln(2)$ using linear interpolation:

x_i	$f(x_i)$	1 st D.D.C
$x_0 \leftarrow 1$	$f(x_0) \leftarrow 0$	q_0
$x_1 \leftarrow 6$	$f(x_1) \leftarrow 1.7917595$	q_1

$x^* \leftarrow 2$
 $f(x^*) \leftarrow 0.693147$

$$\begin{aligned}
 * f(x) &= f(x_0) + \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] (x - x_0) \\
 &= 0 + 0.3862949 (x - 1) \\
 f(x) &\leftarrow 0.3862949 (x - 1) + 0
 \end{aligned}$$

1st D.D.C

x_i	$f(x_i)$	1 st D.D.C	2 nd D.D.C
	$0 \rightarrow a_0$	$1.3862944 \rightarrow a_1$	
1	1.3862944	$\frac{1.7917595 - 1.3862944}{6-4}$	---
4	1.7917595	---	---
6		---	---

$$\left[\frac{(1.7917595 - 1.3862944)}{6-4} \right]$$

$$- \left[\frac{(1.3862944 - 0)}{4-1} \right] / (6-1)$$

$$\therefore f_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$

$$\downarrow \quad \downarrow$$

$$f(x_0) \quad \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

x_i	$f(x_i)$	1 st D.D.C	2 nd D.D.C
$x^0 \rightarrow 1$	$0 \rightarrow a_0$	$0.46209813 \rightarrow a_1$	$0.051873116 \rightarrow a_2$
$x^+ \rightarrow 4$	1.3862944	0.20273255	---
$x^- \rightarrow 5$	a_0	$\rightarrow a_1$	---
$x' \rightarrow 6$	1.7917595	---	---

$$f_1(5) = 1.3862944 + 0.20273255(5-4)$$

$$f_2(2) = 0 + 0.46209813(5-1) + 0.051873116(5-1)(5-4)$$

$$\ln f_1(2) = 0.46209813$$

$$\ln f_2(2) = 0.56584437$$

* Divided Difference Table:

x_i	$f(x_i)$	1 st D.D.C	2 nd D.D.C
x_0	$f(x_0)$	$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$	$\frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$
x_1	$f(x_1)$	$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$	$\frac{\frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_1}$
x_2	$f(x_2)$	$\frac{f(x_3) - f(x_2)}{x_3 - x_2}$	$\frac{\frac{f(x_4) - f(x_3)}{x_4 - x_3} - \frac{f(x_3) - f(x_2)}{x_3 - x_2}}{x_4 - x_2}$
x_3	$f(x_3)$	$\frac{f(x_4) - f(x_3)}{x_4 - x_3}$	$\dots$
x_4	$f(x_4)$	$\dots$	$\dots$

(5) $\frac{\frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_1}$

(4) $\frac{\frac{f(x_4) - f(x_3)}{x_4 - x_3} - \frac{f(x_3) - f(x_2)}{x_3 - x_2}}{x_4 - x_2}$

(3) $\dots$

$$\therefore f_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

* Differentiation:

$$x_i = \frac{\pi}{4}, \Delta h = \frac{\pi}{12}$$

$$x_{i+1} = x_i + \Delta h$$

$$\Rightarrow x_{i+1} = \frac{\pi}{3}$$

* Ex: Forward $O(h)$, $O(h^2)$
 Backward $O(h)$, $O(h^2)$
 Central $O(h^2)$, $O(h^4)$
 $f'(\frac{\pi}{4})$ for $y = \sin(x)$?

$$y' = \cos(x) \Rightarrow y'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \underline{0.7071}$$

True answer

$$\rightarrow \text{Forward: } f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2(h)} + O(h^2)$$

$$\Delta h = \frac{\pi}{12}$$

x_i	$f(x_i)$	$\frac{\pi}{4}$	$\frac{\pi}{4} + \frac{\pi}{12}$	$\frac{\pi}{4} + 2\frac{\pi}{12}$
$\frac{\pi}{4} = x_0$	$\sin(\frac{\pi}{4}) = 0.7071$	x_i	x_{i+1}	x_{i+2}
$\frac{\pi}{3} = x_1$	$\sin(\frac{\pi}{3}) = 0.866$			
$\frac{5\pi}{12} = x_2$	$\sin(\frac{5\pi}{12}) = 0.9659$			

$$f'(\frac{\pi}{4}) = \frac{0.866 - 0.7071}{\pi/12} = 0.607 \xrightarrow{\text{error}} 0.7071 + O(h)$$

$$f'(\frac{\pi}{4}) = \frac{-0.9659 + 4(0.866) - 3(0.7071)}{2(\pi/12)} + O(h^2)$$

$$= 0.7196 \rightarrow 0.7071$$

→ Backward: $f'(x_i) = \frac{f(x_i) - f(x_{i-1}))}{h} + o(h)$

$$f'\left(\frac{\pi}{4}\right) = \frac{f\left(\frac{\pi}{4}\right) - f\left(\frac{\pi}{6}\right)}{h}$$

x_i	$f(x_i)$	$\frac{\pi}{4} - \frac{\pi}{6}$	$\frac{\pi}{4} - \frac{\pi}{12}$	$\frac{\pi}{4}$
		x_{i-2}	x_{i-1}	x_i
$\frac{\pi}{6}$	0.5			
$\frac{\pi}{12}$	0.259			

$$f'\left(\frac{\pi}{4}\right) = \frac{0.7071 - 0.5}{\pi/12} + o(h) = 0.791$$

$$f'\left(\frac{\pi}{4}\right) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2}))}{2h} + o(h^2)$$

$$= \frac{3(0.7071) - 4(0.5) + 0.259}{2(\pi/12)} + o(h^2)$$

$$= 0.72632 \text{ compare } 0.7071$$

→ Central: $\frac{\pi}{12} \quad \frac{\pi}{6} \quad \frac{\pi}{4} \quad \frac{\pi}{3} \quad \frac{5\pi}{12}$
 $x_{i-2} \quad x_{i-1} \quad x_i \quad x_{i+1} \quad x_{i+2}$

$$f'\left(\frac{\pi}{4}\right) = \frac{f(x_{i+1}) - f(x_{i-1}))}{2h} + o(h^2)$$

$$= \frac{0.866 - 0.5}{2 \cdot \frac{\pi}{12}} = 0.699 + o(h^2)$$

87

$$\begin{aligned}
 f'(\pi/4) &= \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_i) + f(x_{i-1}))}{12h} + O(h^4) \\
 &= \frac{-[-0.9659] + 8(0.866) - 8(0.5) + 0.259}{12(\pi/4)} \\
 &= 0.70699 + O(h^4) \rightarrow 0.7071
 \end{aligned}$$

* $O(h)$, $O(h^2)$, $O(h^4)$, ... :

* if the error is $O(h)$; halving the step size will halve the error.

← (ah) step size → خطأ 1/2
(خطأ 1/2 → error)

* if the error is $O(h^2)$, halving the step size will quarter the error.
(error 1/4 → 0.25 error)

* Richardson Extrapolation:

$$D \approx D(h_2) + \frac{1}{(h_1/h_2)^2 - 1} \left[\begin{array}{c} D(h_2) \\ \downarrow \text{more accurate} \end{array} - \begin{array}{c} D(h_1) \\ \downarrow \text{less accurate} \end{array} \right]$$

$h_2 = 1/2 h_1$

* Forward: $f'(\frac{\pi}{4}) = \frac{f(\frac{\pi}{2}) - f(\frac{\pi}{4})}{\frac{\pi}{12}} + o(h)$

$\downarrow \frac{\pi}{12}$
 $\rightarrow$ less accurate

$$f'(\frac{\pi}{4}) = \frac{f(\frac{7\pi}{24}) - f(\frac{\pi}{4})}{\frac{\pi}{24}} + o(h)$$

(Δh) stepsize (خطوة) $\frac{\pi}{24}$
 (المسافة) more accurate

x_i
$\frac{\pi}{4}$
$\frac{\pi}{3}$
$\frac{5\pi}{12}$

$$\left(\frac{\pi}{4} + \frac{\pi}{24} = \frac{7\pi}{24} \right)$$

$$D_{best} = D|_{h_2} + \frac{1}{\left[\left(\frac{h_1}{h_2}\right)^2 - 1\right]} [D|_{h_2} - D|_{h_1}]$$

$\rightarrow \left(\frac{h_1}{\frac{1}{2}h_1}\right)^2 - 1 = 3$

$$\therefore D_{best} = D|_{h_2} + \frac{1}{3} [D|_{h_2} - D|_{h_1}] + o(h^2)$$

h $D_{correct} (h^2)$

$$D|_{h_1} \quad D|_{h_2} + \frac{1}{3} [D|_{h_2} - D|_{h_1}] = D|_{h_1} / h_2$$

$$D|_{h_2} \quad D|_{h_3} + \frac{1}{3} [D|_{h_3} - D|_{h_2}] = D|_{h_2} / h_3$$

$D|_{h_3}$

$\dots$ $D_{correct} (h^4)$

$$D|_{h_2} / 3 + \frac{1}{15} [D|_{h_2} / h_2 - D|_{h_1} / h_2]$$

*Ex: first derivative of $y = \ln(x)$ at $x=4$ using $h_1=2$ and $h_2=1=\frac{1}{2}h_1$, employ central differences of $O(h^2)$?

	x_i	$f(x_i)$	x_{i-2}	x_i	x_{i+2}
			+	+	+
x_{i-h_1}	2	0.693	$\ln(2)$	$\ln(4)$	$\ln(6)$
x_i	4	1.3863			
x_{i+h_1}	6	1.79176			

$$D|_{h=1}(4) = \frac{1.79176 - 0.693}{2 \times 2} = 0.2747$$

$$D|_{h=2} = \frac{1.6094 - 1.099}{2(1)} + O(h^2) = 0.255$$

x_i	$f(x_i)$
3	1.099
5	1.6094

h	$O(h^2)$
h_1	$D _{h_2} + \frac{1}{3} [D _{h_2} - D _{h_1}]^2$
h_2	$0.255 + \frac{1}{3} [0.255 - 0.2747]^2 + O(h^4)$

$$\Rightarrow 0.24843$$

*

$$h_n = \frac{1}{2} h_{n-1}$$

$$\rightarrow (h_3 = \frac{1}{2} h_2 \Rightarrow h_2 = \frac{1}{2} h_1)$$

*Ex: $f'(0) \rightarrow O(h^2)$?

Fit the data using Lagrange:

x	C
0	0.1 y_{i-1}
1	0.4 y_i
3	0.9 y_{i+1}

$$f'(x) = \frac{2x - x_{i+1} - x_{i-1}}{(x_{i-1})(x_i)(x_{i+1} - x_{i-1})} + \frac{2x - x_{i+1} - x_{i-1}}{(x_i - x_{i-1})(x_i - x_{i+1})} + \frac{2x - x_i - x_{i-1}}{(x_{i+1} - x_{i-1})(x_{i+1} - x_i)}$$

$y_{i-1} \leftarrow$ $y_i \leftarrow$ $y_{i+1} \leftarrow$

Valid for $0 \leq x \leq 3$

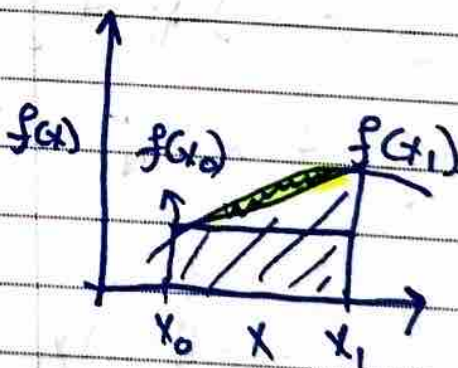
$$f'(0) = \frac{2(0) - 3 - 1}{(0-1)(0-3)} (0.1) + \frac{2(0) - 3 - 0}{(1-0)(1-3)} (0.4) + \frac{(2(0) - 1 - 0)(0.9)}{(3-0)(3-1)} = 0.31667$$

*Numerical Integration:

Equation $\rightarrow$ Table

Equally

Un-equally



x_i	$f(x_i)$
x_0	$f(x_0)$
x_1	$f(x_1)$

$$f_1(x) = q_0 + q_1(x - x_0)$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0)$$

$$\int_{x_0}^{x_1} f_1(x) dx = \int_{x_0}^{x_1} \left[f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) \right] dx$$

$$= f(x_0)(x_1 - x_0) + \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] \left[\frac{1}{2} (x_1^2 - x_0^2) - x_0(x_1 - x_0) \right]$$

$$f_1(x) = (x_1 - x_0) \left[\frac{f(x_1) + f(x_0)}{2} \right] - \left(\frac{1}{12} f''(\xi) \right) h^3$$

$$= \frac{h}{2} [f(x_1) + f(x_0)]$$

$$\text{Total } A = A_B + A_T = \frac{[f(x_1) - f(x_0)](x_1 - x_0)}{2} + f(x_0)(x_1 - x_0) + \frac{1}{2}$$

$$= f(x_0)x_1 - f(x_0)x_0 + \frac{1}{2} f(x_1)x_1 - \frac{1}{2} f(x_1)x_0 - \frac{1}{2} f(x_0)x_1 + \frac{1}{2} f(x_0)x_0$$

$$= -\frac{1}{2} f(x_0)x_0 + \frac{1}{2} f(x_0)x_1 + \frac{1}{2} f(x_1)x_1 - \frac{1}{2} f(x_1)x_0$$

$$= \frac{1}{2} \left[-f(x_0)x_0 + f(x_0)x_1 + -f(x_1)x_1 + f(x_1)x_0 \right]$$

$$= \frac{1}{2} [f(x_0) [x_1 - x_0] + f(x_1) [x_1 - x_0]]$$

$$= \frac{1}{2} (x_1 - x_0) [f(x_1) + f(x_0)] \quad \left\{ \begin{array}{l} \text{Trapezoidal} \\ \text{Rule} \end{array} \right.$$

$$\int_{x_0}^{x_4} f_1(x) dx = \int_{x_0}^{x_1} f_1(x) dx + \int_{x_1}^{x_2} f_1(x) dx + \int_{x_2}^{x_3} f_1(x) dx + \int_{x_3}^{x_4} f_1(x) dx$$

$$= \frac{h_1}{2} [f(x_1) + f(x_0)] + \frac{h_1}{2} [f(x_2) + f(x_1)] + \frac{h_1}{2} [f(x_3) + f(x_2)] + \frac{h_1}{2} [f(x_4) + f(x_3)]$$

$$= \frac{h_1}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

	x_i	$f(x_i)$
h_1	x_0	$f(x_0)$
$\neq h_2$	x_1	$f(x_1)$
$\neq h_3$	x_2	$f(x_2)$
	x_3	$f(x_3)$

$$f_2(x) = q_0 + q_1(x - x_0) + q_2(x - x_0)(x - x_1)$$

$$\int_{x_0}^{x_2} f_2(x) dx = \int_{x_0}^{x_2} f(x) dx + \frac{f(x_1) - f(x_0)}{x_1 - x_0} (x - x_0) + \frac{[f(x_2) - f(x_1)](x_2 - x_1) - [f(x_1) - f(x_0)](x_2 - x_0)}{(x_1 - x_0)(x_2 - x_0)} (x - x_0)(x - x_1)$$

$$= \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)] + o(h^5)$$

$\frac{1}{3}$ Simpson Rule

x_3 $\int_{x_0}^{x_3} f_3(x) dx =$

$$\frac{3}{8} h \left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right]$$

x_i	$f(x_i)$
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$

$\frac{3}{8}$ S.R. $\rightarrow$

$$* \text{Boole's} = \frac{b-a}{90} \left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4) \right]$$

* Romberg Integration:

for integration $\left\{ \begin{array}{l} D_{\text{best}} = D_{\text{more}} + \frac{1}{3} \left[D_{\text{more}} - D_{\text{less}} \right], \quad (h_2 = \frac{1}{2} h_1) \\ I_{\text{best}} = I_{\text{more}} + \frac{1}{3} \left[I_{\text{more}} - I_{\text{less}} \right] \end{array} \right.$

$$\text{Ex 1: } H = C_{eh} \int_0^t q A dt = C_{eh} A \int_0^t q dt$$

$$C_{eh} = 0.45, A = 150000 \text{ cm}^2$$

Common step size, $\Delta x = 1$

① Trapezoidal :

$$\int_0^{14} q dt = \frac{\Delta x}{2} \left[q(0) + 2 \sum_{i=1}^{13} q_i + q(14) \right]$$

step size

$$= \frac{1}{2} \left[0.1 + 2 \{ 1.62 + 5.32 + \dots + 1 \} + 0.2 \right] = 78$$

$$\therefore H = 0.45 \times 150000 \times 78 = 5265000$$

Time (h)	Heat Flux (q)
0	0.1
1	1.62
2	5.32
3	6.29
4	
5	
6	
7	
8	
9	
10	
11	
12	3.54
13	1
14	0.2

② Using 1/3 S.R (Simpson's 1/3 Rule)

$$\int_0^{14} q dt \Big|_{1/3 \text{ S.R}} = \frac{\text{step size}}{3} \left[q(0) + 2 \{ q(2) + q(4) + q(6) + q(8) + q(10) + q(12) \} + 4 \{ q(1) + q(3) + q(5) + q(7) + q(9) + q(11) + q(13) \} + q(14) \right] = 77.92667$$

95

Extreme points	* 1
Joint points	* 2
mid points	* 4

$$\therefore H = 0.45 * 150000 * 77.9267 = 526050$$

③ $\frac{3}{8}$ Simpson Rule (بؤنؤ كؤ ذقؤؤ)

$$\int_0^{12} \frac{3}{8} = \frac{3}{8} \left[q(0) + 2 \{ q(3) + q(6) + q(9) \} + 3 \{ q(1) + q(2) + q(4) + q(5) + q(7) + q(8) + q(10) + q(11) \} + q(12) \right] + \frac{1}{3} [q(12) + 4 q(13) + q(14)]$$

$$= 78.42 \quad \left(\int_0^{14} \frac{1}{3} S.R \right) \leftarrow \frac{1}{3} S.R$$

ذقؤؤ ذقؤؤ = بؤؤ

Extreme points	* 1
Joint points	* 2
mid points	* 4

$$\therefore H = 0.45 * 150000 * 78.42 = 5293350$$

(الطريقه اللاديه)

$$\text{Ex 2: } M = \int_{t_1}^{t_2} Q_c dt = Q \int_{t_1}^{t_2} c dt$$

$$Q = 4 \text{ m}^3/\text{min}$$

① Trapezoidal:

$$\Delta t = 5$$

$$\int_0^{50} T.R = \frac{5}{2} [10 + 2 \{ 22 + \dots + 32 \} + 34]$$

$$= 2000$$

$$\therefore M = 4 \int_0^{50} c dt = 4 \times 2000 = 8000$$

t	c (mg/m ³)
0	10
5	22
10	35
15	47
20	55
25	58
30	52
35	46
40	37
45	32
50	34

② $\frac{1}{3}$ S.R :

$$\int_0^{50} \frac{1}{3} \text{ S.R} = \frac{1}{3} (5) [10 + 2 \{ 35 + 55 + 52 + 37 \} + 4 \{ 22 + 47 + 58 + 40 + 32 \} + 34]$$

$$= 1996.667$$

$$\therefore M = 4 \times 1996.667 = 7986.668$$

$$\textcircled{3} \quad 3/8 \text{ S.R.}$$

$$\int_0^{50} = \int_0^{45} 3/8 + \int_{45}^{50} \text{T.R}$$

$$= \frac{3}{8} (5) [10 + 2 \{ 47 + 52 \} + 3 \{ 22 + 35 + 55 + 58 + 40 + 37 \} + 22]$$

$$= 1839.375$$

$$\int_{45}^{50} \text{T.R} = \frac{1}{2} (5) [32 + 34] = 165$$

$$\therefore M = 8017.5$$

$$\text{Ex: } Q=12, M = Q \int_0^{20} c \, dt$$

$$\textcircled{1} \int_0^{20} \text{T.R} + \int_0^8 \text{T.R}$$

$$+ \int_8^{20} \text{T.R} \quad \Delta t=2 \quad \Delta t=2$$

$$\therefore \int_0^8 \text{T.R} \quad \Delta t=2$$

	c
0	10
2	20
4	30
6	40
8	50
12	72
16	70
20	50

$$= \frac{1}{2} (2) [10 + 2 \{ 20 + 30 + 40 \} + 60]$$

$$= 250$$

$$\int_0^{20} T.R. \quad \Delta t = 4 = \frac{1}{2}(4) [60 + 2\{72 + 70\} + 50] = 788$$

$$\therefore \int_0^{20} = 250 + 788 = 1038$$

$$\therefore M = 12 \cdot 1038 = 12456$$

[2] Simpson Rules:

$$\int_0^{20} = \int_0^2 \frac{1}{3} S.R. + \int_2^{20} \frac{3}{8} S.R.$$

$$\therefore \int_0^2 \frac{1}{3} S.R. = \frac{1}{3}(2) [10 + 2(30) + 4\{20 + 40\} + 60] = 246.666$$

$$\int_2^{20} \frac{3}{8} S.R. = \frac{3}{8}(4) [60 + 50 + 3\{72 + 70\}] = 804$$

$$\therefore \int_0^{20} = 246.666 + 804 = 1050.666$$

$$\therefore M = 12 \cdot 1050.666 = 126055$$

(most accurate)

Ex 4: $\int_0^1 \frac{1+x}{1+x^3} dx$

(= 1.1666) (5)

① Trapezoidal with 2 subintervals

$$\int_0^1 = \int_0^{0.5} + \int_{0.5}^1$$

$$\int_0^{0.5}$$

$$= \frac{1}{2} (0.5) [1 + 2(1.333) + 1]$$

$$= 1.1666$$

x	f(x)
0	$\frac{1+0}{1+0^3}$
0.5	$\frac{1+0.5}{1+(0.5)^3}$
1	$\frac{1+1}{1+1^3}$

② Trapezoidal with 10 subintervals.

x	f(x)
0	1
0.1	1.098901
0.2	,
0.3	,
0.4	,
0.5	1.3333
0.6	,
0.7	,
0.8	,
0.9	1.190476
1	1.098901

x	f(x)
0	1
0.5	1.3332
1	1



$$\int_0^1 T.R = \frac{0.1}{2} \left[1 + 2 \{ 1.098901 + \dots + 1.190476 + 1.098901 \} + 1 \right]$$

$$= 1.20753124$$

③ Simpson Rule ⁽¹⁾/₃ with 2 subintervals.

$$\int_0^1 \frac{1}{3} S.R = \frac{1}{3} (0.5) \left[1 + 4(1.333) + 1 \right]$$

x	f(x)
0	1
0.5	1.333
1	1

(2 subintervals) 8.1 جملہ 8.1

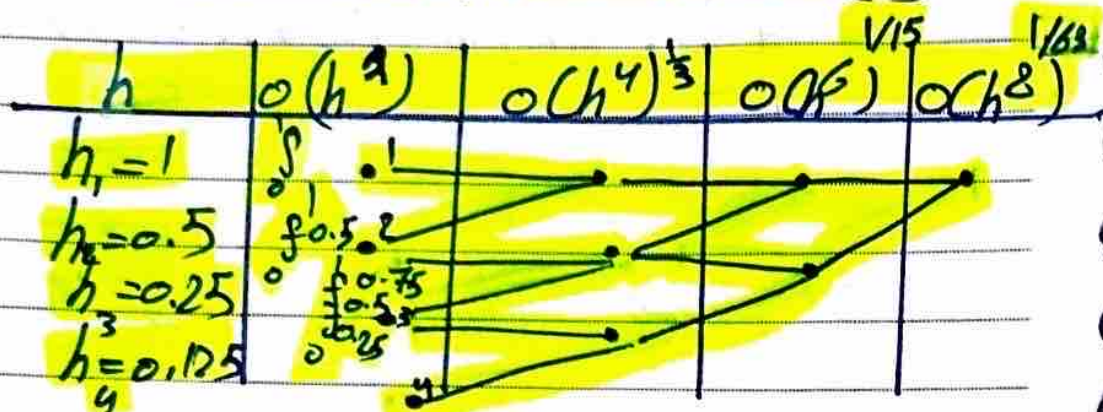
$$= 1.2222$$

④ $\frac{1}{3} S.R \Rightarrow$ with 10 subintervals

$$\int_0^1 \frac{1}{3} S.R = \frac{1}{3} (0.1) \left[1 + 2 \{ 1.190476 + 1.315789 + 1.190476 \} + 4 \{ 1.098901 + 1.265823 + 1.315785 + 1.3333 + 1.265823 + 1.098901 \} + 1 \right]$$

$$= 1.20926235$$

5] Using Trapezoidal Romberg integration to achieve an error of $O(h^8)$



x	f(x)
0	1
0.125	1.122807
0.25	1.230769
0.375	1.306122
0.5	1.3333
0.625	1.306122
0.75	1.230769
0.875	1.122807
1	1

$$\begin{aligned}
 1 &= \int_0^1 \frac{1+x}{1+x^3} dx = \frac{1}{2} (1) [1+1] = 1 \\
 2 &= \int_0^1 f(x) dx = \int_0^{0.5} f(x) dx + \int_{0.5}^1 f(x) dx \\
 &= \frac{1}{2} (0.5) [1 + 2(1.333) + 1] \\
 &= 1.1\overline{11}
 \end{aligned}$$

$$\cdot 3 = \int_0^1 f(x) dx = \int_0^{0.25} f(x) dx + \int_{0.25}^{0.5} f(x) dx + \int_{0.5}^{0.75} f(x) dx + \int_{0.75}^1 f(x) dx$$

$$= \frac{1}{2}(0.25) [1 + 2 \{1.230769 + 1.333 + 1.230769\} + 1]$$

$$\cdot 4 = \int_0^1 f(x) dx = \int_0^{0.125} f(x) dx + \int_{0.125}^{0.25} f(x) dx + \int_{0.25}^{0.375} f(x) dx + \int_{0.375}^{0.5} f(x) dx + \int_{0.5}^{0.625} f(x) dx + \int_{0.625}^{0.75} f(x) dx + \int_{0.75}^{0.875} f(x) dx + \int_{0.875}^1 f(x) dx$$

$$= \frac{1}{2}(0.125) [1 + 2 \{1.122807 + 1.230769 + 1.306122 + 1.3333 + 1.306122 + 1.230769 + 1.122807\} + 1] = 1.20659134$$

$$\therefore O(h^4)^{\frac{1}{3}}$$

$$\begin{aligned} & \rightarrow 1.1666 + \frac{1}{3} [1.666 - 1] = 1.222222 \\ & \rightarrow 1.9871795 + \frac{1}{3} [1.1987 - 1.166] = 1.20940139 \\ & \rightarrow 1.20659 + \frac{1}{3} [1.20659 - 1.1987] = 1.209215805 \end{aligned}$$

103

$(1/15)^{No.}$

$O(h^6)$

$$\rightarrow 1.209401709 + \frac{1}{15} [1.2094 - 1.222] \\ = 1.208547$$

$$\rightarrow 1.209215 + \frac{1}{15} [1.20921 - 1.209] \\ = 1.209203$$

$O(1/3)(1/3)$

$$\rightarrow 1.209203 + \frac{1}{63} [1.209203 \\ - 1.208547] \\ = 1.209214$$

104



Ordinary differential equation → Boundary value

initial value

1st O.D.E higher O.D.E

(2nd, 3rd, ...)

Euler's

Runge-Kutta

system of 1st O.D.E

1) Heun's method
2) polygon method

Modification of Euler's

* 1st order ordinary differential equation

① Euler Method:

This is a derivative

$$y_{i+1} = y_i + f(x_i, y_i) h$$

$\begin{matrix} y_i & y_{i+1} \\ | & | \\ x_i & x_{i+1} \end{matrix}$

* Ex: $\frac{dy}{dx} = 4e^{0.8x} - 0.5y$, $y(0) = 2$

$$y' + 0.5y = 4e^{0.8x}$$

* Mathematically: $y = e^{-\int P(x) dx} \left[\int e^{\int P(x) dx} \cdot Q(x) dx + C \right]$

$$= e^{-0.5x} \left[\int e^{0.5x} \cdot 4e^{0.8x} dx + C \right]$$

105

ask :: believe & recieve

$$= e^{-0.5x} \left[4 \int e^{1.3x} dx + C \right]$$

$$= e^{-0.5x} \left[4/1.3 e^{1.3x} + C \right]$$

$$y = 4/1.3 e^{0.8x} + C e^{-0.3x}$$

$$2 = 4/1.3 (1) + C(1) \quad y(0) = 2$$

* Numerically:

① Euler's method:

• at $x=1$:

$$\begin{array}{cccccc} y(0)=2 & y(1) & y(2) & y(3) & y(4) \\ \hline x=0 & x=1 & x=2 & x=3 & x=4 \end{array}$$

$$y(1) = \underbrace{y(0)}_{=2} + f'(0, 2)(1)$$

$$(f'(0, 2) = 4e^{+0.8(0)} - 0.5(2) = 3)$$

$$\therefore y(1) = 2 + 3(1) = 5$$

• at $x=2$:

$$y(2) = y(1) + f(1, 5)(1)$$

$$= 5 + 6.402163714 = 11.40216371$$

$$y'(1.5) = 4e^{0.8(1.5)} - 0.5(5) = 6.40216371$$

• at $x = 3$:

$$\begin{aligned}
 y(3) &= y(2) + y'(2, 11.40216371)(1) \\
 &= 11.40216371 + 14.11104784 = 25.51321155 \\
 y' &= 4e^{0.8(2)} - 0.5(11.40216) \leftarrow \\
 &= 14.11104784
 \end{aligned}$$

• at $x = 4$:

$$y(4) = y(3) + y'(3, 25.513...) = 56.84931$$

② Heun's Method:

$$y_{i+1} = y_i + \left| \frac{y'(x_i, y_i) + y'(x_{i+1}, y_{i+1})}{2} \right| h$$

[Use Euler's method to calculate $y(x_{i+1})$]

$$y(x_{i+1}) = y(x_i) + y'(x_i, y(x_i)) h$$

$$\therefore y(1) = y(0) + \left| \frac{y'(0, 2) + y'(1, y(1))}{2} \right| (1)$$

But:

$$\begin{aligned}
 y'(0, 2) &= 4e^{0.8(0)} - 0.5(2) = 3 \\
 y(1) &= y(0) + y'(0, 2)h = 2 + 3(1) = 5 \\
 y'(1, 5) &= 4e^{0.8(1)} - 0.5(5) = 6.40216 \\
 \therefore y(1) &= 2 + \frac{3 + 6.40216}{2} (1) = 6.70108
 \end{aligned}$$

* O.D.E

1st O.D.E

□ Euler's Method : $y(x_{i+1}) = y(x_i) + y'(x_i - y(x_i))h$

□ Heun's Method :

$$y(x_{i+1}) = y(x_i) + \left[\frac{y'(x_i, y(x_i)) + y'(x_i + (y_{i+1})h)}{2} \right] h$$

□ Polygon Method:

$$y(x_{i+1}) = y(x_i) + y'(x_i + \frac{1}{2}, y(x_i + \frac{1}{2}))h$$

$$y'_{i+\frac{1}{2}} = y'(x_i + \frac{1}{2}, y_{i+\frac{1}{2}})$$

$$y(x_i + \frac{1}{2}) = y(x_i) + y'(x_i, y(x_i)) \frac{h}{2}$$

* Ex: $y' = 4e^{0.8x} - 0.5y$, $y(0) = 2, h=1$

• $x=1$: $y(1) = y(0) + y'(0.5, y(0.5))$

$$y(0.5) = y(0) + y'(0.25, y(0.25)) \times \frac{1}{2}$$

$$= (2 + 4e^{0.8(0)} - 0.5(2)) \times \frac{1}{2}$$

$$= 3.5$$

$$y'(0.5, 3.5) = 4e^{0.8(0.5)} - 0.5(3.5)$$

$$= 4.217298$$

$$\therefore y(1) = 2 + (4.217298)(1) = 6.217298$$

$$\begin{aligned}
 \bullet \quad x=2 : y(2) &= y(1) + y'(1.5, y(1.5)) (1) \\
 y(1.5) &= y(1) + y'(1.6, 817298) (1/2) \\
 &= [6.817298 + 4e^{0.8(1)} - 0.5(6.817298)] \\
 &= 9.114055357
 \end{aligned}$$

$$\begin{aligned}
 \therefore y'(1.5, 9.114055357) &= \left\{ 4e^{0.8(1.5)} - 0.5 \right. \\
 &\quad \left. (9.114055357) \right\} \\
 &= 8.723440091
 \end{aligned}$$

$$\begin{aligned}
 \therefore y'(1.5, 9.114055357) &= 4e^{0.8(1.5)} - 0.5(9.114055337) \\
 &= 8.723440091
 \end{aligned}$$

$$\begin{aligned}
 \therefore y(2) &= 6.217292 \\
 &\quad + 8.723440091 (1) \\
 &= \underline{14.941042}
 \end{aligned}$$

* Runge-Kutta Method:

$$y(x_{i+1}) = y(x_i) + \phi(x_i, y_i, h)$$

$$\phi = a_1 k_1 + a_2 k_2 + a_3 k_3 + \dots + a_n k_n$$

↳ if 1st order Runge-Kutta Methods:

$$\phi = a_1 k_1$$

if 2nd order Runge-Kutta Methods:

$$\phi = a_1 k_1 + a_2 k_2 \quad [109]$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_1 k_1 h)$$

* 1st order Range (خطات لا) - 1st order Range-kutta
 * 2nd order Range-kutta (خطات لا) - 2nd order Range-kutta
 * 3rd order Range-kutta (خطات لا) - 3rd order Range-kutta

* Ex: $y' = 4e^{0.8x} - 0.5y$, $y(0) = 2$, $q_2 = 1/2$
 2nd order R-K, $h = 1$

$$y(x_{i+1}) = y(x_i) + \left[\frac{1}{2} k_1 + \frac{1}{2} k_2 \right] h$$

$$y(1) = 2 + \frac{1}{2} \times 1 \left[3 + 6.4021637 \right] =$$

but: $k_1 = f(x_i, y_i) = f(0, 2) = 4e^{0.8} - 0.5 \times 2$
 $= 3$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

$$= f(1, 5) = 4e^{0.8} - 0.5 \times 5 = 6.4021637$$

$$\therefore y(1) = 6.7010818$$

$\Rightarrow$ Assume $q_2 = 1$: $y(1) = y(0) + k_2 \times 1$

but $k_2 = f(1/2, 3.5) = 4.21729879$

$$k_1 = f(0, 2) = 3$$

$$\therefore y(1) = 2 + k_2 = 6.21729879$$

110

* Ex: $y' = 4e^{0.8x} - 0.5y$, $h=1$, 4th order

at $x=1$: $y(1) = y(0) + \left[\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \right] h$ $\rightarrow =1$

But: $k_1 = f(0, 2) = 3$

$k_2 = f(0.5, 3.5) = 4.217298$

$k_3 = f(0.5, 4.108649) = 3.9129743$

$k_4 = f(1, 5.9129743) = 5.945676564$

$\therefore y(1) = 2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 6.201034$

* System of 1st O.D.E:

$y_1' = f_1(x, y_1, y_2, y_3)$

$y_2' = f_2(x, y_1, y_2, y_3)$

$y_3' = f_3(x, y_1, y_2, y_3)$

integrate from 0 to 1 using $h=0.5$

$y_1(0.5) = ??$

$y_2(0.5) = ??$

$y_3(0.5) = ??$

$y_1(1) = ?$

$y_2(1) = ?$

$y_3(1) = ?$

بیشتر
[y_1, y_2, y_3]

* Ex: $y_1' = -0.5y_1$, ($f_1(x, y_1, y_2)$)

$y_2' = 4 - 0.3y_2 - 0.1y_1$, ($f_2(x, y_1, y_2)$)

$y_1(0) = 4, y_2(0) = 6$, use $h=0.5$ to calculate
 $y_1(1) = y_2(1) = ?$
 (using 4th order Runge-Kutta)

$$y(x_i + 1) = y(x_i) + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4) h$$

K_1, K_2, K_3, K_4 (قوانين موجودة بالشئ)

$$\therefore \text{at } x = 0.5 :$$

$$y_1(0.5) = y_1(0) + \frac{1}{6} (K_1^{(1)} + 2K_2^{(1)} + 2K_3^{(1)} + K_4^{(1)}) h$$

$$K_1^{(1)} = y_1'(0, 4) = -0.5 y_1 = -0.5(4) = -2$$

$$K_2^{(1)} = y_1'(0 + \frac{1}{2}(0.5), 4 + \frac{1}{2}(0.5)(-2)) = y_1'(0.25, 3.5) =$$

$$K_3^{(1)} = y_1'(0 + \frac{1}{2}(0.5), y_1 + \frac{1}{2}(0.5)(-1.75)) = -1.75$$

$$= y_1'(0.25, 3.5625) = -1.78125$$

$$K_4^{(1)} = y_1'(0 + 0.5, 4 + (0.5)(-1.78125)) = y_1'(0.5, 3.109375) = -1.5546875$$

$$\therefore y_1(0.5) = 3.11523$$

$$y_2(0.5) = y_2(0) + \frac{1}{6} [K_1^{(2)} + 2K_2^{(2)} + 2K_3^{(2)} + K_4^{(2)}] h$$

$$K_1^{(2)} = y_2'(0, 6) = 4 - 0.3 y_2 - 0.1 y_1 = 4 - 0.3(6) - 0.1(4) = 1.8$$

$$K_2^{(2)} = y_2'(0 + \frac{1}{2}(0.5), \frac{1}{2}(0.5) K_1^{(2)}) = y_2'(0.25, 6.45) = 1.715$$

$$K_3^{(2)} = y_2'(0.25, 6.42875) = 1.715125$$

$$K_4^{(2)} = y_2'(0.5, 6.8575625) = 4 - 0.3(6.85756) - 0.1(3.10937) = 1.63179375$$

112

$$\therefore y_2(0.5) = 6 + \frac{0.5}{6} [1.8 + 2(1.715 + 1.715125) + 1.6317] \\ = 6.857670313$$

for $x=1$:

$$y_1(1) = y_1(0.5) + \frac{0.5}{6} [k_1^{(1)} + 2k_2^{(1)} + 2k_3^{(1)} + k_4^{(1)}]$$

$$k_1^{(1)} = y_1'(0.5, 3.115234375, 6.857670213) \\ = -0.5(3.115234375) = -1.5576$$

$$k_2^{(1)} = -1.362915 = y_1'(0.75, 2.725830078)$$

$$k_3^{(1)} = -1.21080398 = y_1'(0.75, 2.774505615)$$

$$k_4^{(1)} = y_1'(1, 2.421607971) = -1.210804$$

$$\therefore y_1(1) = 2.926171308$$

$$y_2(1) = y_2(0.5) + \frac{0.5}{6} [k_1^{(2)} + 2k_2^{(2)} + 2k_3^{(2)} + k_4^{(2)}]$$

$$k_1^{(2)} = y_2'(0.5, 3.115234375, 6.8576703) = 1.631175469$$

$$k_2^{(2)} = y_2'(0.75, 2.72583, 7.26546418) = 1.54777$$

$$k_3^{(2)} = y_2'(0.75, 2.774505, 7.2446177) = 1.549165$$

$$k_4^{(2)} = y_2'(1, 2.421607971, 7.63225282) = 1.4686$$

$$\therefore y_2(1) = 7.632105674 \quad \#$$

* Higher O.O.D.E :

$y'' = g(x, y, y')$ Convert to 2 1st O.O.D.E

$y'' \Rightarrow$ Convert to 3 1st O.O.D.E

$y''' \Rightarrow$ " " 4 " " " " " " " " " " " "

Let $u = y \xrightarrow{\text{derive}} u' = y' = v$
 $v = y' \xrightarrow{\text{derive}} v' = y'' = g(x, u, v)$
 $u(0) = v, v(0) = v$

$y'' = g(x, y', y'', y''')$

$u = y \rightarrow u' = y' = r$

$v = y' \rightarrow v' = y'' = w$

$w = y'' \rightarrow w' = y''' = g(x, u, v, w)$

$y(x_0), y'(x_0), y''(x_0)$

$u(x_0)$

$v(x_0)$

$w(x_0)$

* Ex: $y'' + 0.5y' + 5y = 0$

$y(0) = 4, y'(0) = 0$, Solve from $x=0$ to $x=1$ with $h=0.5$ using 4th O. B. K?

$$u = y \rightarrow u' = y' = v$$

$$v = y' \rightarrow v' = y'' = -0.5v - 5u$$

$$y(0) = 4 \Rightarrow u(0) = 4$$

$$y'(0) = 0 \Rightarrow v(0) = 0$$

at $x = 0.5$:

$$u(0.5) = u(0) + \frac{0.5}{6} [K_1^u + 2K_2^u + 2K_3^u + K_4^u]$$

$$K_1^u = u'(0, 4, 0) = 0, K_1^v = v'(0, 4, 0) = -0.5(0) - 5(4) = -20$$

$$K_2^u = u'(0.25, 4, 5) = -5, K_2^v = v'(0.25, 4, 5) = -17.5$$

$$K_3^u = u'(0.25, 2.75, -4.375) = -4.375, K_3^v = v'(0.25, 2.75, -4.375) = -11.5625$$

$$K_4^u = u'(0.5, 1.8125, -5.78125) = -5.78125, K_4^v = v'(0.5, 1.8125, -5.78125) = -6.171875$$

$$\therefore u(0.5) = 1.955729167$$

$$v(0.5) = -7.024739583$$

$$u(1) = u(0.5) + \frac{0.5}{6} [K_1^u + 2K_2^u + 2K_3^u + K_4^u]$$

$$v(1) = v(0.5) + \frac{0.5}{6} [K_1^v + 2K_2^v + 2K_3^v + K_4^v]$$

$$K_1^u = -7.204739503, K_1^v = -6.176276$$

$$\begin{aligned}K_2^u &= -8.74880259, K_2^v = 3.6068299 \\K_3^u &= -6.124318848, K_3^v = 4.219524329 \\K_4^u &= -4.91497742, K_4^v = 7.98983999\end{aligned}$$

$$\therefore y(1,0) = -15.331010829$$

