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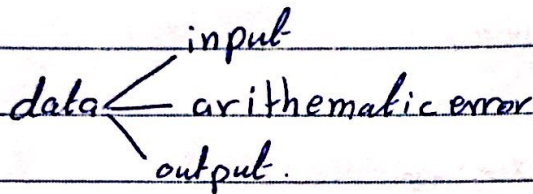
Numerical

Dr. Naim Al_faqeer



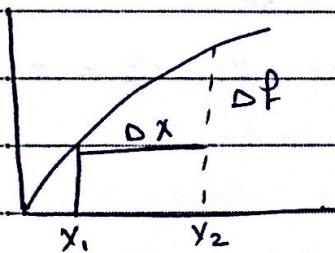
• Type of errors:-

- ① inherent errors (model)
- ② error of method (Truncation error) (numerical) = approximation
- ③ computer: Rounding errors.



• True error:-

True error = true value - Approximate value



$$\text{Slope} = \frac{\Delta f}{\Delta x} = \lim_{\Delta x \rightarrow 0} = f'$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$h = x_2 - x_1$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Example:-

$$f(x) = 7e^{0.5x}, \quad h = 0.3$$

• Find approximate value $f'(2)$

• True value of $f'(e)$

• True error for first part

$$P'(x) = \frac{P(x+h) - P(x)}{h}$$

$$P'(x) = 7 \times 0.5 e^{0.3x}$$

$$P'(x) = 3.5 e^{0.5x}$$

$$x = 2$$

$$P'(2) = 3.5 e^{0.5 \times 2} = 9.5140$$

$$\begin{aligned} \text{given } x_0 = 2 \quad \& \quad x_2 = x_0 + h \\ h = 0.3 \quad \& \quad = 2 + 0.3 \end{aligned}$$

$$\begin{aligned} P'(2) \approx \frac{P(2+0.3) - P(2)}{0.3} &= \frac{7 e^{0.5(2+0.3)} - 7 e^{0.5 \times 2}}{0.3} \\ &= \frac{22.107 - 19.028}{0.3} = 10.263 \end{aligned}$$

$$E_L = 9.5140 - 10.263 = -0.7490$$

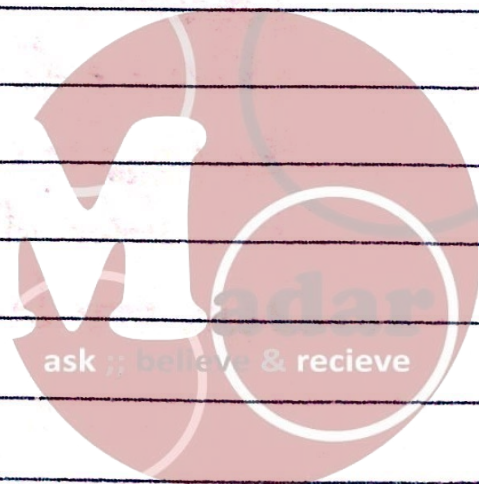
*Relative true error:-

$$E_L = \frac{\text{True error}}{\text{True value}}$$

$$= \frac{-0.7490}{9.5140} = -0.0787$$

As a percentage

$$E_L = -0.0787 \times 100 \% = -7.87 \%$$



* Approximate error = present approximate - previous approximate
(E_a).

~~prev~~ In previous example:-

a) $h = 0.3$, $f'(2)$ we find it.

b) $h = 0.15$, $f'(2)$

$$f'(2) \approx f(0.15 + 2) - f(2)$$

$$\approx f(2.15) - f(2)$$

$$= 7e^{2.15} - 7e^{2}$$

$$= 20.50 - 19.028 = 9.8800$$

$$E_a = 9.880 - 10.263 = -0.3830$$

ans

Relative approximate error = $\frac{\text{Approx. error}}{\text{present approxi.}}$

12/2/2020

• Non linear equation

$$f(x) = 0 \quad (1 - \text{var})$$

Find root α of $f(x)$, it will make $f(x) = 0$

quadrating Eqn.

$$ax^2 + bx + c = 0$$

Polynomial of degree 2 ($p_2(x)$).

$$\text{Solution } x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \leftarrow \text{Analytical Solution}$$

- computer Solution

$$x_1 = \frac{-b - (\text{sign } b) \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b + (\text{sign } b) \sqrt{b^2 - 4ac}}{2a}$$

Vieta's formula

$$x_1 = \frac{w}{2a}, \quad x_2 = \frac{2c}{w} \leftarrow \text{Real solution}$$

$$w = -b - \text{sign}(b) \sqrt{b^2 - 4ac}$$

$$\begin{aligned} x_1 + x_2 &= -\frac{b}{a} \\ x_1 x_2 &= \frac{c}{a} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} a, b, c \text{ \# order of magnitude}$$

$$\text{sign } b = \begin{cases} 1 & \text{if } b \geq 0 \\ -1 & \text{if } b < 0 \end{cases} \quad \begin{aligned} \text{sign}(3) &= 1 \\ \text{sign}(-3) &= -1 \end{aligned}$$

$$ax^2 + bx + c = a(x - x_1)(x - x_2) = 0$$

Example:-

$$a = \frac{1}{9999}, \quad b = 16, \quad c = -10^{-8}$$

$$\begin{aligned} w &= -b - \text{sign}(b) \cdot \sqrt{b^2 - 4ac} \\ &= -16 - 1 \times \sqrt{16^2 - 4 \times \frac{1}{9999} \times -10^{-8}} \\ &= -32 \end{aligned}$$

$$x_1 = \frac{w}{2a} = \frac{-32}{2 \times \frac{1}{9999}} = -15984.00$$

$$x_2 = \frac{2c}{w} = \frac{-2 \times 10^{-8}}{-32} = 6.2500 \times 10^{-10}$$

- Methods:-

① method without derivatives

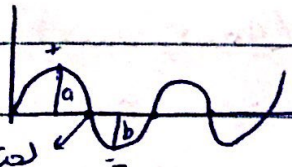
② method with derivatives

- Remark:-

Linear: $ax + b = 0 \Rightarrow 1 \text{ root}$

non linear: $ax^2 + bx + c = 0 \Rightarrow 2 \text{ roots}$

$$\sin x = 0$$



فترة متتوية متغير الإشارة

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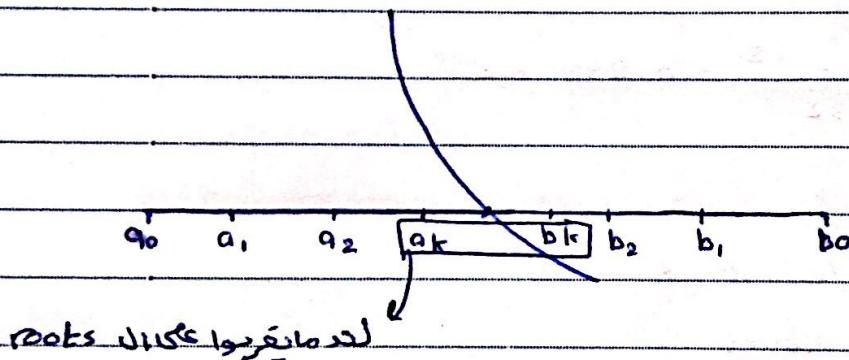
(change of sign).

First we plot $f(x)$, $x \in (a, b)$.

- if function $f(x)$ doesn't change sign between two points
~~if~~ there may not be any roots for the equation between the two points.

* Search method without derivative

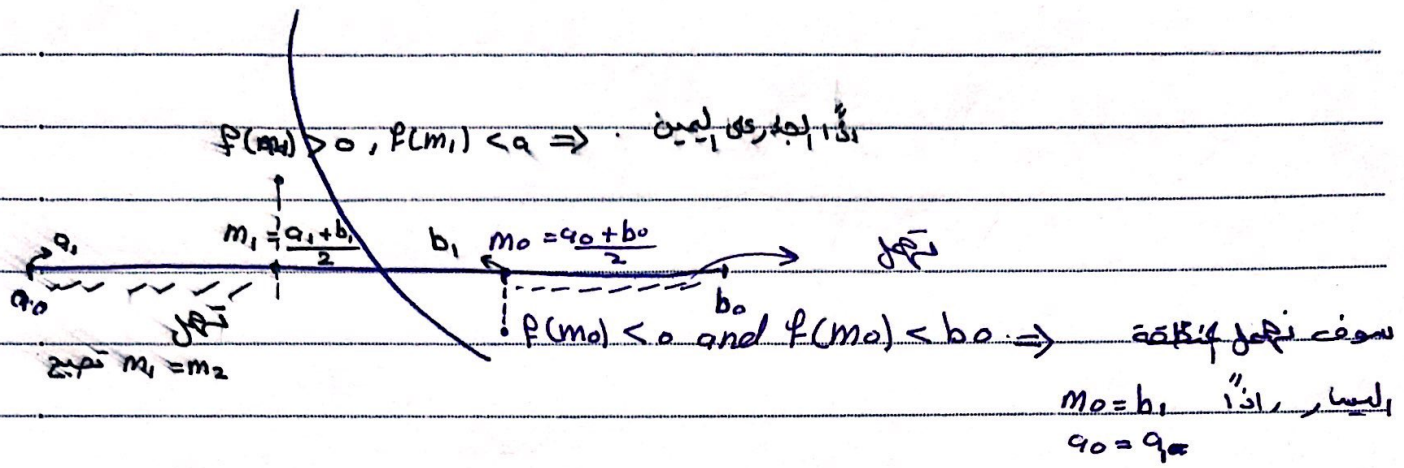
- choose an interval (a_0, b_0) in which a root lies
- find a series of successively smaller intervals $(a_1, b_1), (a_2, b_2)$ which contain the roots
- when $a_k = b_k$ to enough decimal places, stop and declare root $f(x) = a_k = b_k$



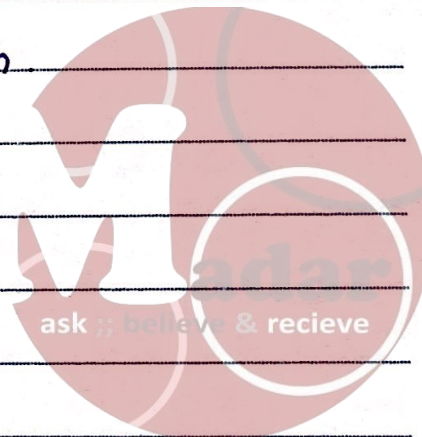
$$f(x) = a_k = b_k$$

* How we determine the intervals

- pick $m_k = (a_k + b_k) / 2$ (midpoint of interval)
- if $f(m_k) \cdot f(b_k) < 0$ let $(a_{k+1}, b_{k+1}) = (m_k, b_k)$
- if $f(m_k) \cdot f(a_k) < 0$ let $(a_{k+1}, b_{k+1}) = (a_k, m_k)$
- Repeat the above until $a_k = b_k$ to enough decimal places
 Stop and declare root $f(x) = b_k = m_k$.



$f(a_1) * f(b_1) < 0 \Rightarrow$ There's change in sign



* Example:-

$$f(x) = x^2 - 3$$

$$\text{Roots: } x_1 = 1.7321, x_2 = -1.7321$$

$$a_0 = -1 \Rightarrow f(a_0) = -2$$

$$b_0 = 3 \Rightarrow f(b_0) = 6$$

$$m_0 = \frac{-1+3}{2} = 1 \Rightarrow f(m_0) = -2$$

} eliminate interval $(-1, 1)$

new interval $(1, 3)$

$$a_1 = 1 \Rightarrow f(1) = -2$$

$$b_1 = 3 \Rightarrow f(3) = 6$$

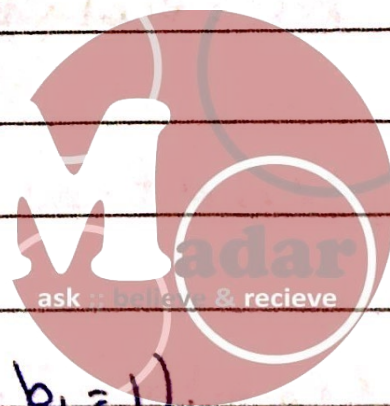
$$m_1 = 1 \Rightarrow f(1) = -2$$

⋮

we stop when $a_k = b_k = m_k$

$$E = 1.7325 - 1.7315 = 0.001$$

To get $x_2 = -1.7321$ choose $(a_0 = -2, b_1 = 1)$



* (a_0, b_0)

new interval = $(a_1, b_1) = \frac{b_0 - a_0}{2}$
 يكونه في القوسه

$$(a_2, b_2) = \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2}$$

$$(a_3, b_3) = \frac{b_2 - a_2}{2} = \frac{b_0 - a_0}{2^3}$$

$$* (b_n - a_n) = \frac{b_0 - a_0}{2^n} \quad n^{\text{th}} \text{ iteration}$$

$$a_n \approx b_n \quad | \quad b_n - a_n = 0.05 \leftarrow \text{error}$$

Eg. $a_0 = 1$ & $b_n - a_n = 0.05$
 $b_0 = 3$

$$0.05 = \frac{3 - 1}{2^n}$$

$$2^n = \frac{4}{0.05} \Rightarrow n$$

* Newton-Raphson Method (with deriv.)

$$f(x), \alpha \Rightarrow f(\alpha) = 0$$

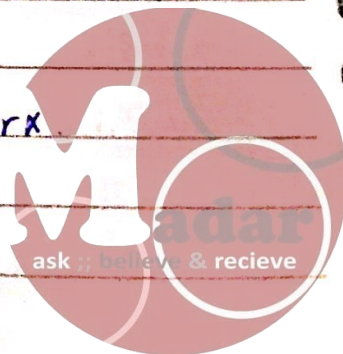
Given x_0

Expand $f(x)$ and x_0 using Taylor Series

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \dots$$

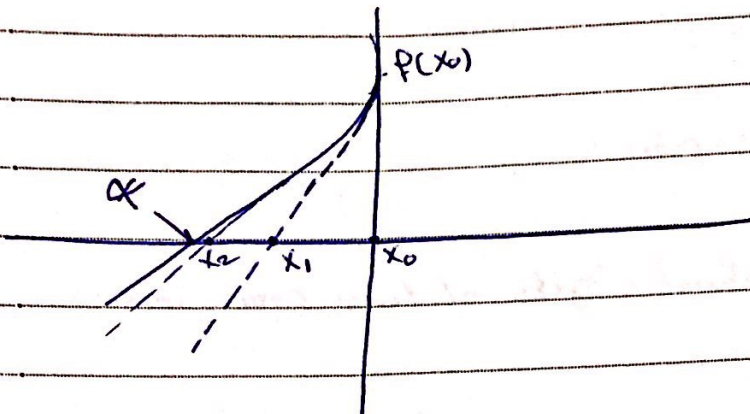
$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) \stackrel{\text{Linear}}{=} 0 \quad \leftarrow \text{Solve for } x$$

$$(x - x_0) = -\frac{f(x_0)}{f'(x_0)} \Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$



$$\bullet X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)} \quad n = 0, 1, 2, \dots$$

$$x_0, x_1, x_2, \dots, x_n \rightarrow \alpha$$



$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$

→ خط مماس
في x_0

* اننا نقول ان $f'(x_0) = 0$ في هذه الحالة لا يمكننا ان نستخدم هذه الطريقة

~~* Example~~

Example:-

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

polynomial with 3 degree
it has at least one real root.

$$f'(x) = 3x^2 - 0.33x$$

$$\bullet n = 0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.05 - \frac{(0.05)^3 - 0.165 \times (0.05)^2}{3 \times (0.05)^2 - 0.33 \times 0.05} = 0.06242$$

$$\frac{x_1 - x_0}{x_1} = \frac{0.06242 - 0.05}{0.06242} = \dots < \epsilon_{TOL}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.06242 - \dots$$

$$= 0.06238$$

$$\frac{x_2 - x_1}{x_1} \times 100\% = 0.716\%$$

The number significant digits at least correct in the answer 2

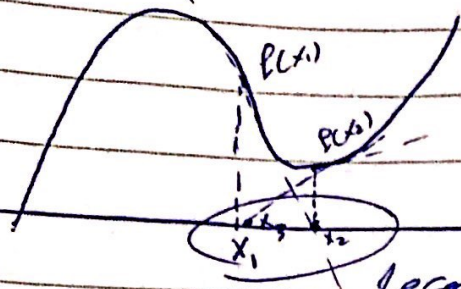
x_2

⋮

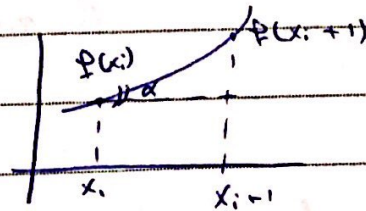
$$E_a = x_3$$

$$= 0\%$$

The number significant digits at least correct is 4, ask only 4
Significant digits are carried through all the calculation.



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



$$\tan \alpha = \frac{\Delta f}{\Delta x}$$

$$P'(x) = \frac{P'(x_{i+1}) - P'(x_i)}{x_{i+1} - x_i}$$

- ~~Iterative~~ Iterative Formula

$$X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)} \quad n = 1, 2, 3,$$

Sequence $\{x_n\} \rightarrow \alpha$ (root)

Given X_0 .

Example:-

$$P(x) = (x-1)^3 + 0.512$$

$$f'(x) = 3(x-1)^2$$

$$X_0 = 1$$

$P'(1) = 0 \Rightarrow$ inflection point (maybe can be max., min.)
• overflow.

[illegible]

$$P(x) = x^3 - 0.03x^2 + 2.4 \times 10^{-1} = 0.$$

$$P'(x) = 3x^2 - 0.03 \times 2x = 0.$$

$$\text{Root} = 0.$$

$$x(3x - 0.03 \times 2) = 0.$$

$$\left\{ \begin{array}{l} x = 0 \end{array} \right.$$

$$x = \frac{0.03 \times 2}{3} = \frac{0.06}{3} = 0.02$$

؟
 overflow.
 overflow.
 overflow.

- t-digit computer

$$X = \pm (0.d_1 d_2 d_3 \dots d_t) \times 10^{\textcircled{9}}$$

؟
 overflow.

$$X = 123.1 = 0.1231 \times 10^3$$

؟
 overflow.

$$d_1 \neq 0 \Rightarrow \text{overflow}$$

$$\text{if } t = 4$$

$$0.d_1 d_2 d_3 d_4$$

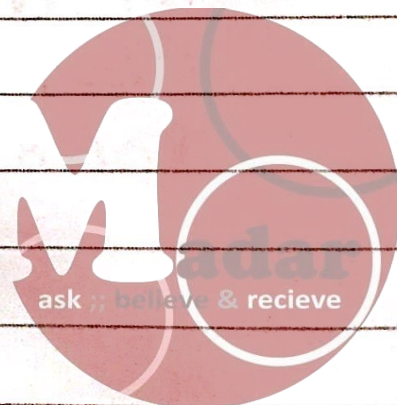
$$\text{max. fraction} \rightarrow 0.9999$$

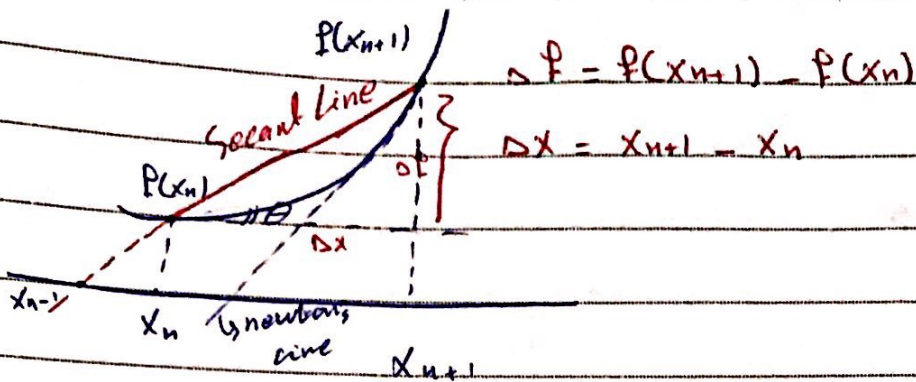
$$\text{min. fraction} \rightarrow 0.1000$$

- Secant method (when I can't get derivative)

$$P'(x_n) ?$$

approximate function.





$$\tan \theta = \frac{\Delta f}{\Delta x}$$

$$\text{Slope} = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}$$

$$\lim_{x_{n+1} \rightarrow x_n} \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = f'(x_n)$$

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}$$

Given x_0, x_1

$$n_1 = 1$$

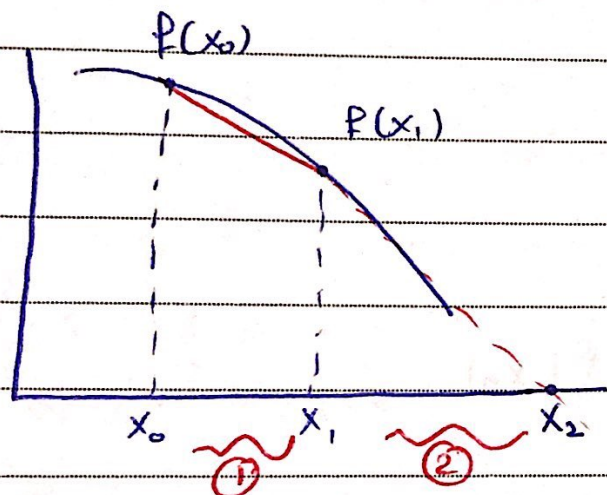
$$x_2 = x_1 - \frac{(x_1 - x_0) f(x_1)}{f(x_1) - f(x_0)}$$

$$E_n = \frac{|x_{n+1} - x_n|}{x_{n+1}} \times 100\%$$



$$x_1 = \frac{f(x_0)(x_2 - x_0)}{f(x_0) - f(x_1)}$$

$$x_1 = x_0 - \frac{x_0 - x_1}{f(x_0) - f(x_1)} f(x_0)$$



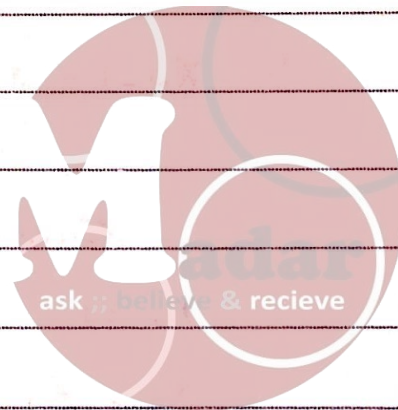
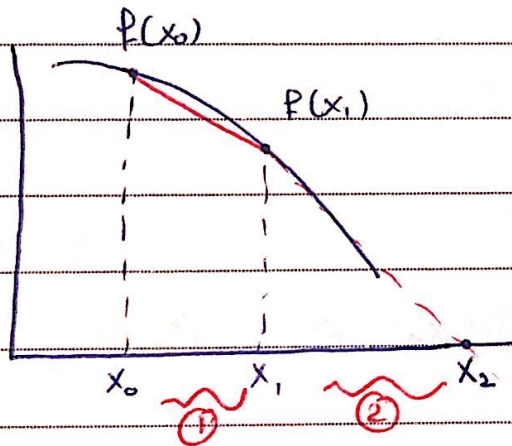
Example

$$P(x) = x^3 - 0.165x^2 +$$

$$x - 1 = 0.02$$

$$x_0 = 0.05$$

$$x_1 = \frac{P(x_0)(x_0 - x_1)}{P(x_0) - P(x_1)} \quad x_1 = \frac{x_0 - x_1}{\frac{P(x_0)}{P(x_0) - P(x_1)}}$$



• Newton's

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}, \text{ given } X_0$$

• Secant

$$X_{n+1} = X_n - \left(\frac{X_n - X_{n-1}}{f(X_n) - f(X_{n-1})} \right) f(X_n), \text{ given } X_0, X_1$$

Example:-

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

Iteration 1:-

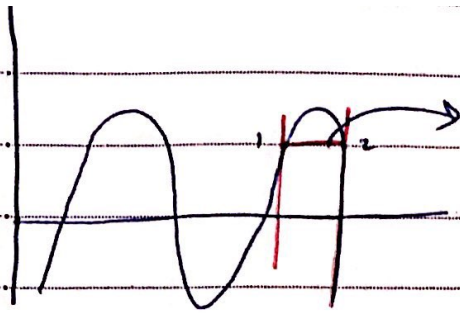
$$X_0 = 0.02$$

$$X_1 = 0.05$$

$$X_1 = X_0 - \frac{f(X_0)(X_0 - X_{-1})}{f(X_0) - f(X_{-1})} = 0.05 - \frac{(0.05^3 - 0.165(0.05)^2 + 3.993 \times 10^{-4}) \times (0.05 - 0.02)}{(0.05^3 - 0.165(0.05)^2 + 3.993 \times 10^{-4}) - (0.02^3 - 0.165(0.02)^2 + 3.993 \times 10^{-4})}$$

Iteration 2:-

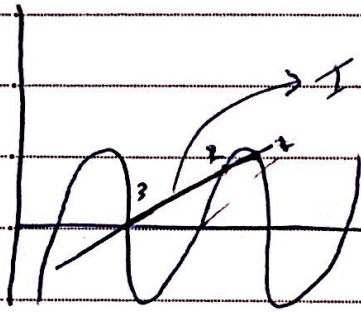
$$X_2 = X_1 - \frac{f(X_1)(X_1 - X_0)}{f(X_1) - f(X_0)}$$



نقطہ کی تلاش

$$f(x_1) = f(x_2)$$

$$f(x_1) - f(x_2) = 0 \Rightarrow \text{overflow}$$



I can't have converge

• Fixed point iteration

(one point iteration) (Successive substitution method (SS))

given

$$f(x) = 0 \quad \text{find } x = \alpha$$

$$f(x) = 0 \quad (1)$$

$$x = x + f(x) = g(x)$$

if α root.

$$f(\alpha) = 0$$

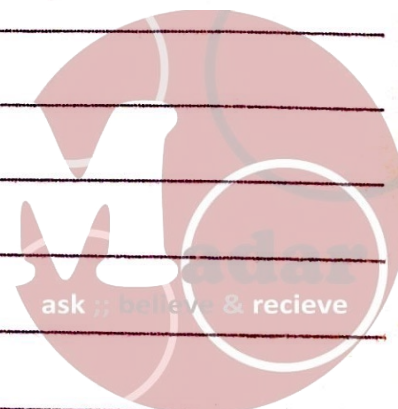
$$\alpha + f(\alpha) = g(\alpha)$$

"zero"

$$x = \alpha = g(x)$$

$$x_{n+1} = g(x_n)$$

$$\{x_n\} \rightarrow \alpha \quad n = 0, 1, 2, 3, \dots$$

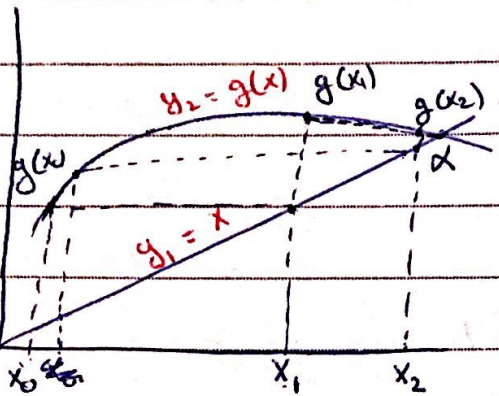


$$x = g(x)$$

Let $y_1 = x \rightarrow$ Line (origin)

$$y_2 = g(x)$$

$$y_1 = y_2 \Rightarrow x = \alpha$$



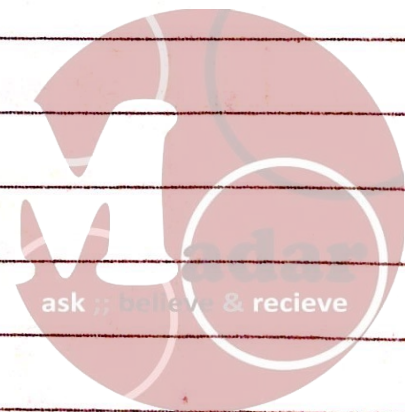
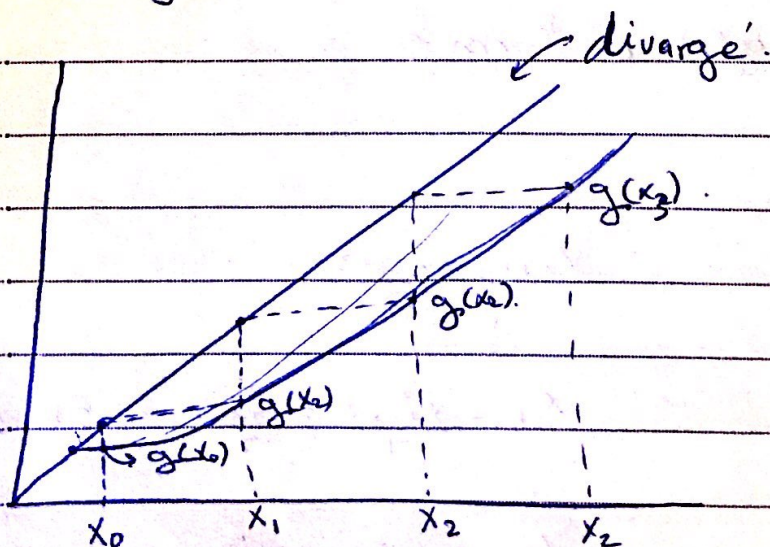
$$n = 0$$

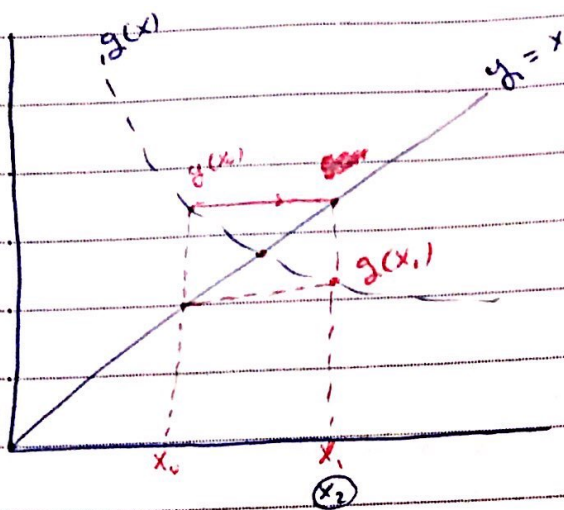
$$x_1 = g(x_0)$$

$$n = 1$$

$$x_2 = g(x_1)$$

$$y = g(x)$$





منه مخرج (جذر)
 * change the point.

$$\frac{|x_{n+1} - x_n|}{x_{n+1}} \times 100\% \leq \text{Tol}$$

Example:-

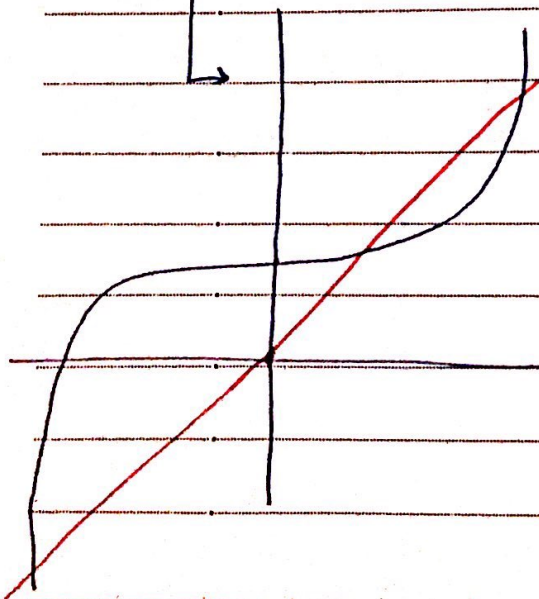
$$f(x) = x^3 - 7x + 3 = 0$$

$$x^3 - 7x + 3 = 0$$

$$x = \frac{x^3 + 3}{7} = g(x) \quad \leftarrow \text{Form 1}$$

or

$$x = \sqrt[3]{7x - 3} = g(x) \quad \leftarrow \text{Form 2}$$



$$x_{n+1} = \frac{x_n^3 + 3}{7}$$

$$x_1 = \frac{x_0^3 + 3}{7} = \frac{2^3 + 3}{7} = 1.57143$$

$$x_2 = \frac{x_1^3 + 3}{7} = \frac{1.57143^3 + 3}{7} = 0.98292$$

$$\alpha = 0.4408 \approx 0.441$$

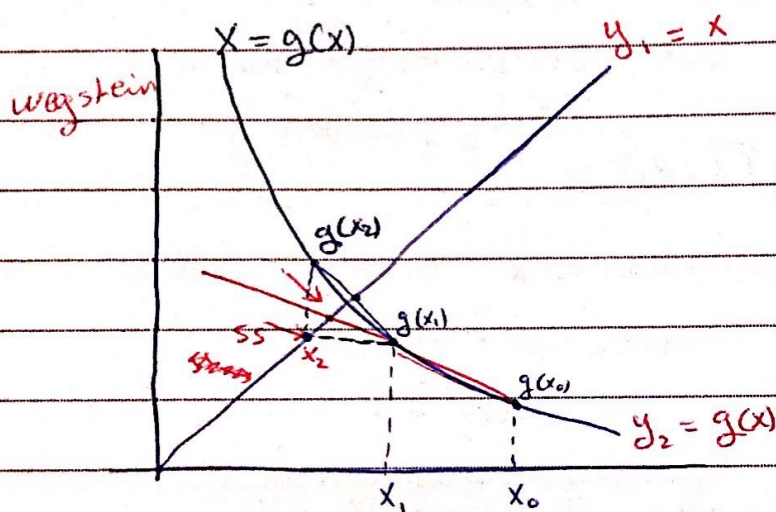
• Wegstein method

$$X = g(x), \text{ given } x_0$$

Expand $g(x)$ and x_0

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \dots$$

$$g'(x_0) = \frac{g(x_1) - g(x_0)}{x_1 - x_0}$$



Bounded Wegstein method:

$$\text{Given } x_0, x_1$$

$$\downarrow \quad \downarrow$$

$$g(x_0) \quad g(x_1)$$

Fit a line (secant)

between $(x_0, g(x_0))$ and $(x_1, g(x_1))$ extend the line to $\cap$ with $y_1 = x$

$$y_1 = x$$

$$y_2 = g(x) \text{ expand } g'(x)$$

using Taylor series:

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \dots$$

Linear Function

$$y = g(x) = g(x_0) + g'(x_0)(x - x_0)$$

$$g'(x_0) = \frac{g(x_1) - g(x_0)}{x_1 - x_0} = \text{Slope} = 1 - \frac{1}{\theta} \quad \text{--- (7)}$$

$$g(x) = g(x_1) + g'(x_1)(x - x_1)$$

Solve for θ :

$$\text{Slope} = \frac{\theta - 1}{\theta}$$

$$x = g(x) = g(x_1) + \left(\frac{\theta - 1}{\theta}\right)(x - x_1)$$

Solve for x

$$X = x_1(1 - \theta) + \theta g(x_0)$$

$$X_{n+1} = x_n(1 - \theta) + \theta g(x_n) \leftarrow \text{general formula.}$$

$$n = 1, 2, 3, \dots$$

$$\{x_n\} \rightarrow x$$

• Given x_0 and x_1

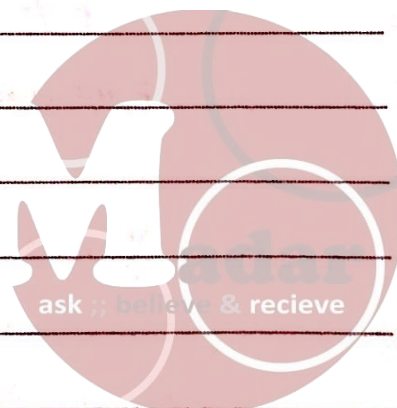
compute $g(x_0)$ and $g(x_1)$

calculate

$$\text{Slope} = \frac{g(x_1) - g(x_0)}{x_1 - x_0}$$

$$x_1 - x_0$$

$$\theta = \frac{1}{1 - \text{Slope}}$$



if $\theta > 5$ set $\theta = +5$

if $\theta < -10$ set $\theta = -10$

$$x_{n+1} = x_n (1 - \theta) + \theta g(x_n)$$

we stop when $\frac{|x_{n+1} - x_n|}{x_{n+1}} \leq \text{Tot}$

$$n = 1, \quad x_2 = x_1 (1 - \theta) + \theta g(x_1)$$

$$\rightarrow \text{slope} = \frac{g(x_1) - g(x_0)}{x_1 - x_0}, \quad \theta_1 = 1 - \frac{1}{\text{slope}}$$

$$n = 2, \quad x_3 = x_2 (1 - \theta) + \theta g(x_2)$$

$$\text{slope} = \frac{g(x_2) - g(x_1)}{x_2 - x_1}, \quad \theta = 1 - \frac{1}{\text{slope}}$$

$$\text{Example: } g(x) = \frac{x^3 + 3}{7}$$

$$\text{given } x_0 = 1$$

$$x_1 = 1.5$$

• Linear system of equation:-

variables = # Eqs

$$\text{DoF} = 0$$

deter. of coefficient $\neq 0 \Rightarrow$ unique solution.

$$2x_1 - 3x_2 + 2x_3 + 5x_4 = 3$$

$$x_1 - x_2 + x_3 + 2x_4 = 1$$

$$3x_1 + 2x_2 + 2x_3 + x_4 = 0$$

$$x_1 + x_2 - 3x_3 - x_4 = 0$$

Solution: $x_1 = 1$

$$x_2 = -2$$

$$x_3 = 0$$

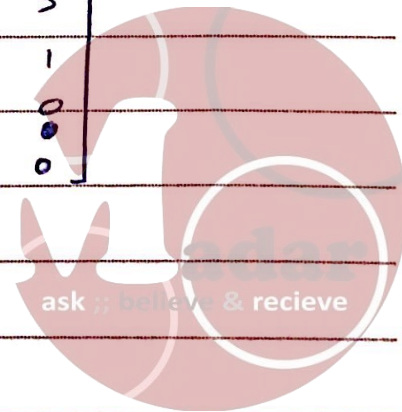
$$x_4 = -1$$

in matrix form.

$$\begin{bmatrix} 2 & -3 & 2 & 5 \\ 1 & -1 & 1 & 2 \\ 3 & 2 & 2 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$A \quad X \quad = \quad b$

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



• Eliminate $X_1 = 1.5 + 1.5X_2 - X_3 - 2.5X_4$

$$0.5X_2 + 0X_3 - 0.5X_4 = -0.5$$

$$6.5X_2 - X_3 - 8.5X_4 = -4.5$$

$$2.5X_2 - 4X_3 - 3.9X_4 = -1.5$$

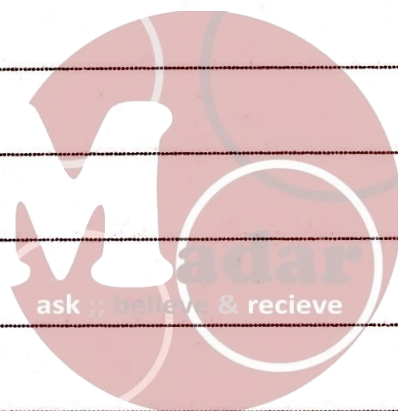
eliminate $X_2 = -1 + X_4$

$$-X_3 - 2X_4 = 2$$

$$-4X_3 - X_4 = 1$$

eliminate $X_3 = -2X_4 - 2$

$$7X_4 = -7 \Rightarrow X_4 = -1$$



Gaussian elimination method.

$$* 2x_1 - 3x_2 + 2x_3 + 5x_4 = 3 \quad \text{--- (1)}$$

$$x_1 - x_2 + x_3 + 2x_4 = 1 \quad \text{--- (2)}$$

$$3x_1 + 2x_2 + 2x_3 - x_4 = 0 \quad \text{--- (3)}$$

$$x_1 + x_2 + 3x_3 - x_4 = 0 \quad \text{--- (4)}$$

حل النظام أولاً سوف نقوم بحذف x_1 من المعادلات 2, 3, 4
 وحذف x_1 من المعادلة 2 بفرج معادلة 1
 (نحذف x_1 من المعادلة 2 بفرج معادلة 1)
 نضرب المعادلة 1 بـ $\frac{1}{2}$ ونحذف x_1 من المعادلة 2

multiply eq #1 by $-\frac{1}{2}$

$$-x_1 + \frac{3}{2}x_2 - x_3 - \frac{5}{2}x_4 = -\frac{3}{2}$$

$$x_1 - x_2 + x_3 + 2x_4 = 1$$

$$0.5x_2 + 0x_3 - 0.5x_4 = -0.5 \quad \text{--- *}$$

multiply eq #1 by $-\frac{3}{2}$ to eliminate x_1 from eq 3

$$-3x_1 + \frac{9}{2}x_2 - \frac{6}{2}x_3 - \frac{15}{2}x_4 = -\frac{9}{2}$$

$$3x_1 - x_2 + x_3 + 2x_4 = 1$$

$$6.5x_1 - x_3 - 8.5x_4 = -4.5 \quad \text{--- **}$$

multiply eq #1 by $-\frac{1}{2}$ to eliminate x_1 from eq 4

$$-x_1 + \frac{3}{2}x_2 - x_3 - \frac{5}{2}x_4 = -\frac{3}{2}$$

$$x_1 + x_2 + 3x_3 - x_4 = 0$$

$$2.5x_2 - 4x_3 - 3.9x_4 = -1.5 \quad \text{--- ***}$$

The system will be:-

$$0.5x_2 + 0x_3 - 0.5x_4 = -0.5 \quad \text{--- (1')}$$

$$6.5x_2 - x_3 - 8.5x_4 = -4.5 \quad \text{--- (2')}$$

$$2.5x_2 - 4x_3 - 3.9x_4 = -1.5 \quad \text{--- (3')}$$

multiply eq # 0' by $-\frac{6.5}{0.5}$ to eliminate x_2 from eq 2'

$$-6.5/x_2 + 6.5x_4 = 6.5$$

$$6.5/x_2 - x_3 - 8.5x_4 = -4.5$$

$$-x_3 - 2x_4 = 2 \quad *$$

multiply eq # 1' by $-\frac{2.5}{0.5}$ to eliminate x_2 from eq 3'

$$-2.5/x_2 + 0x_3 + 2.5x_4 = 2.5$$

$$2.5/x_2 - 4x_3 - 3.9x_4 = -1.5$$

$$-4x_3 - x_4 = 1 \quad **$$

$$-x_3 - 2x_4 = 2 \quad 1''$$

$$-4x_3 - x_4 = 1 \quad 2''$$

multiply eq 1'' by $+4$ to eliminate x_3 from eq 2''

$$+4x_3 + 8x_4 = -8$$

$$-4x_3 - x_4 = 1$$

$$7x_4 = -7 \quad 1'''$$

$x_4 = -1$ we will go back to find x_1, x_2, x_3 .

L					U				
1	0	0	0	2	-3	2.5	3		
-1/2	1	0	0	0	0.5	0	-0.5		
-3/2	-6.5	1	0	0	0	-1	2		
-1/2	-2.5	-4	1	0	0	0	7		

Lower Triangular

Upper Triangular

$$[L][A] = [U]$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -3/2 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & 1 \end{bmatrix} \quad \boxed{A}$$

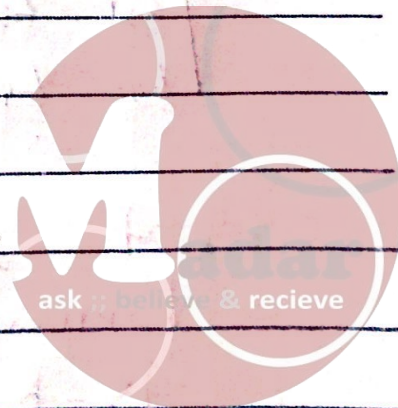
$$= \boxed{A_1} \rightarrow \text{المعادلة بعد حذف } x_1$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -6.5/0.5 & 1 & 0 \\ 0 & -2.5/0.5 & 0 & 1 \end{bmatrix} \quad \boxed{A_1}$$

$$= \boxed{A_2} \rightarrow \text{المعادلة بعد حذف } x_2$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix} \quad \boxed{A_2}$$

$$= \boxed{A_3} \rightarrow \text{المعادلة بعد حذف } x_3$$



$$A_3 = U$$

$$E_3 E_2 E_1 A = U$$

$$E_3 E_2 E_1 b = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 7 \end{bmatrix}$$

~~Elimination~~

ل₂₁ ← -1/2
ل₃₁ ← -3/2
ل₄₁ ← -1/2

$$\begin{bmatrix} 1 & \phi & \phi & \phi \\ -1/2 & 1 & 0 & 0 \\ -3/2 & -6.5 & 1 & 0 \\ -1/2 & -2.5 & -4 & 1 \end{bmatrix}$$

$$E = E_3 E_2 E_1$$

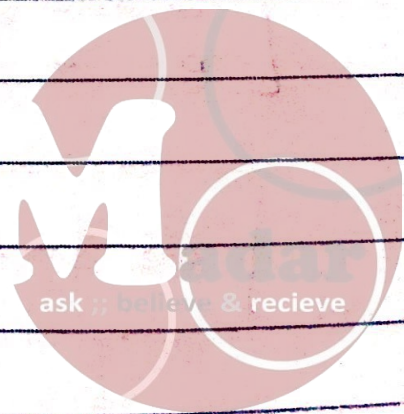
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix}$$

$$A \rightarrow U$$

Forward elimination.

$$Ax = b \Rightarrow UX = b'$$

They have the same solution.



$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\
 &\vdots \\
 a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n &= b_n
 \end{aligned}$$

(n-1) Steps forward elimination.

Matrix $A_{n \times n}$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ i & a_{i1} & a_{i2} & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

$$\left. \begin{matrix} i=1, n \\ i \end{matrix} \right\} \begin{bmatrix} & & j \\ & & \vdots \\ & & a_{ij} \\ & & \vdots \\ & & \end{bmatrix} \left. \begin{matrix} j=1, n \end{matrix} \right\}$$

$$\{a_{ij}\} \quad i=1, \dots, n; \quad j=1, \dots, n$$

* when we say i we mean equation i

" " " j " " variable j

Diagonal matrix

$$D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ & d_{ii} & & \\ 0 & & & \\ & & & d_{nn} \end{bmatrix} \quad \begin{matrix} \{d_{ii}\} \\ i=1, n \\ j=1, n \\ \text{coefficient.} \end{matrix}$$

$$\det(D) = d_{11} \times \dots \times d_{ii} \times \dots \times d_{nn}$$

$$D^{-1} = \frac{1}{d_{11}} \times \dots \times \frac{1}{d_{ii}} \times \dots \times \frac{1}{d_{nn}}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$[B][A] = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 44 \end{bmatrix}$$

$$[A][B] = \begin{bmatrix} 19 & 28 \\ 22 & 31 \end{bmatrix}$$

$$BA \neq AB \leftarrow \text{in general}$$

Diagram illustrating a system of linear equations:

Matrix A (size $m \times n$) is multiplied by vector X (size $n \times 1$) to equal vector b (size $m \times 1$).

Matrix A is labeled with m equations and n variables. The element a_{ij} is shown. The vector X contains variables $x_1, x_2, \dots, x_n$. The vector b contains constants $b_1, \dots, b_m$.

$$m > n$$

$$\text{Dof} = n - m = \ominus \rightarrow \text{over specified}$$

ممكن ان يكون عدد معادلات اعمد الجاهل وبنجد اذا كانا واحد
ففي بعض الحالات يكون واحد مع

Diagram illustrating a system of linear equations:

Matrix A (size $m \times n$) is multiplied by vector X (size $n \times 1$) to equal vector b (size $m \times 1$).

Matrix A is labeled with m equations and n variables. The vector X contains variables $x_1, x_2, \dots, x_n$. The vector b contains constants $b_1, \dots, b_m$.

$$\text{Dof} = n - m > 0$$

$$3x_1 + 3x_2 = 7$$

$$m = 1$$

$$n = 2$$

$$x_1 = 7 - 3x_2$$

$$5$$

* infinite # of solution

Identity matrix
3x3

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$I_{1 \times 1} = 1$$

$$1 \times b = b$$

$$b \times 1 = b$$

$$I_{2 \times 2} \quad A_{2 \times 2}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

↳ The same

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2 \times 1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

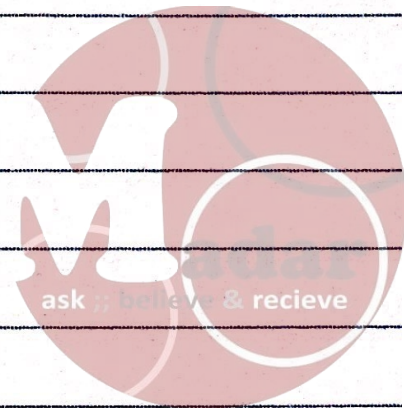
A matrix $n \times 2$

A^{-1} inverse.

$$A_{1 \times 1} = 5$$

$$A^{-1}_{1 \times 1} = \frac{1}{5}$$

$$A^{-1} A = I = I$$



$$AX = b$$

$$E_1 AX = E_1 b$$

$$E_1 = \begin{bmatrix} 1 & & & \\ -2_{21} & 1 & & \\ -2_{31} & 0 & 1 & \\ \vdots & & & \\ -2_{n1} & 0 & 0 & 1 \end{bmatrix}$$

Elimination step k

$$\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ 0 & \dots & 1 & & \\ & & & \ddots & \\ 0 & & -2_{k,k+1} & & 1 \\ & & & \ddots & \\ 0 & & -2_{k,n} & 0 & 1 \end{bmatrix}$$

$$E_{n-1}, E_{n-2}, \dots, E_3, E_2, E_1 = U$$

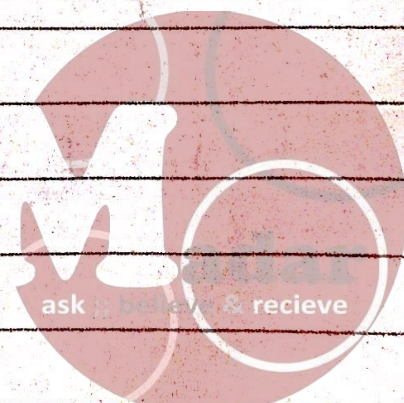
upper triangular matrix

$$E_n, E_3, E_2, E_1, AX = E_n \dots E_3, E_2, E_1$$

$$UX = b'$$

Solve for X.

215 multipliers



$$Ax = b$$

$$F_1 F_2 F_3 \leftarrow UX = b'$$

←

$$Ax = b$$

$$F_1 = \begin{bmatrix} 1 & & & \\ L_{21} & 1 & & \\ L_{31} & 0 & 1 & \\ L_{41} & 0 & 0 & 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & L_{32} & 1 & \\ 0 & L_{42} & 0 & 1 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & L_{43} & 1 \end{bmatrix}$$

$$F_1 F_2 F_3 = \begin{bmatrix} 1 & & & \\ L_{21} & 1 & & \\ L_{31} & L_{32} & 1 & \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix}$$

⇐ Lower triangular matrix

ask :: believe & recieve

$$\underbrace{F_1 F_2 F_3}_L U = A$$

$$A = LU$$

Factorization of matrix A

$$[A] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ L & & 1 \end{bmatrix} * \begin{bmatrix} \diagup & & 0 \\ & \ddots & \\ 0 & & \diagdown \end{bmatrix}$$

$$A = L \cdot U$$

Solve $AX = b$

$$\begin{matrix} [A] & = & \begin{bmatrix} 1 & & 0 \\ x & \ddots & \\ x & & 1 \end{bmatrix} & \begin{bmatrix} \diagup & x & x \\ & \ddots & \\ 0 & & \diagdown \end{bmatrix} \\ A & & L & U \end{matrix}$$

$$A = LU$$

How to use LU Factorization in Solving Linear system.

$$AX = b \Rightarrow L \cdot U \cdot X = b$$

$$\text{Let } UX = c$$

$$\begin{bmatrix} \diagup & \\ & \ddots \\ 0 & & \diagdown \end{bmatrix} [X] = [c]$$

$$\Rightarrow^* LC = b \text{ solve for } c$$

$$\text{solve } UX = c \text{ for } x$$



Gaussian elimination

$$Ax = b \rightarrow UX = b'$$

$$\boxed{A} \boxed{x} = \boxed{b}$$

} equivalent (Same solution)

pivots $\leftarrow$

$$\boxed{\begin{array}{c|c} u & \\ \hline 0 & \end{array}} \boxed{x} = \boxed{b'}$$

~~Algorithm (logical procedure to solve problem)~~

$$\begin{array}{l|l} Ax = b & A^{-1} \text{ exist if } \det(A) \neq 0. \\ \hline A^{-1}Ax = A^{-1}b & \text{Unique solution} \\ \hline IX = X = A^{-1}b & A \text{ singular.} \end{array}$$

Example:-

$$\begin{array}{l} +\frac{3}{2} \quad 2x_1 - 4x_2 = -8 \\ -3x_1 + 6x_2 = 7 \end{array} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{inconsistent.}$$

$$2x_1 - 4x_2 = -8$$

$$0x_1 + 0x_2 = -5$$

$$\left[\begin{array}{cc|c} 2 & -4 & -8 \\ 0 & 0 & -5 \end{array} \right]$$

$$0x_2 = -5 \Rightarrow \text{inconsistent.}$$

ask, believe & receive

$$13/2 \quad 2x_1 - 4x_2 = -6$$

$$-3x_1 + 6x_2 = 9$$

$$2x_1 - 4x_2 = -6$$

$$0x_1 + 0x_2 = 0$$

$$\begin{bmatrix} 1 & -4 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = 0$. trivial solution

~~XXXXXXXXXXXX~~

$$-3 \quad 2x_1 - 4x_2 = -6$$

$$2 \quad -3x_1 + 6x_2 = 9$$

$$-6x_1 + 12x_2 = +18$$

$$-6x_1 + 12x_2 = 98$$

no solution

Graphical method:-

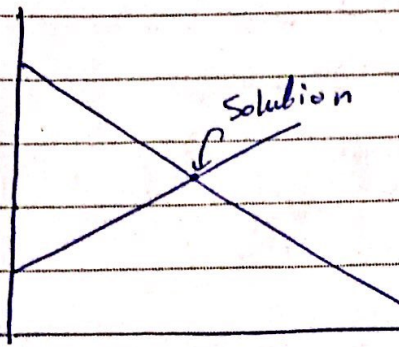
$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

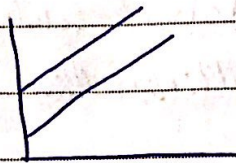
Solve both eq. for x_2 .

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \Rightarrow x_2 = (\text{slope})x_1 + \text{intercept}$$

$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$



There is no solution $\Leftarrow$ different intercept $\Rightarrow$ (inconsistent).
(inconsistent).



multiplier

$$L_{21} = \frac{a_{21}}{a_{11}} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

zero pivot (0)

zero L_{21} (0)

non zero element (non zero pivot)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1' \\ b_2' \\ b_3' \end{bmatrix}$$

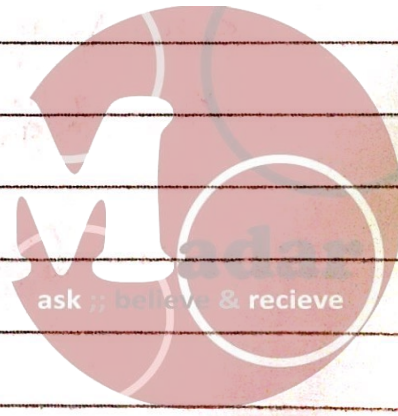
non zero.

pivots (unique solution).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1' \\ b_2' \\ b_3' \end{bmatrix}$$

inconsistent.

$$0x_3 = b_3' \neq$$



* اذا كانت عندى ان coefficient بينها فرق كبير ينفذ ال max coefficient ويتكون معادله

رقم 1 x_1

0.003

$5 \times 10^3 \Rightarrow \text{max. coeff.}$

- system where small changes in coefficient result in large changes in the solution. it happen when two or more equations are nearly identical

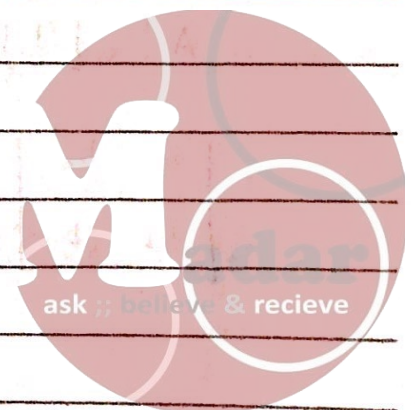
• ~~Summary~~ Summary of gaussian elimination

$$\begin{bmatrix} A \end{bmatrix}_{n \times n} \begin{bmatrix} X \end{bmatrix}_{n \times 1} = \begin{bmatrix} b \end{bmatrix}_{n \times 1}$$

$\Downarrow$

$$\begin{bmatrix} \diagup & & \\ & u & \\ & 0 & \end{bmatrix} \begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} b' \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & & \\ L_{21} & 1 & & \\ L_{31} & L_{32} & 1 & \\ \vdots & \vdots & \vdots & \ddots \\ L_{n1} & L_{n2} & \dots & L_{nn} & 1 \end{bmatrix}$$



LU Factorization

$$A = \begin{bmatrix} \times & 0 \\ \times \times & \times \end{bmatrix} = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \begin{bmatrix} \times & \times \\ 0 & \times \end{bmatrix}$$

$L \quad U$

$$A \cdot x = b$$

$$LU \cdot x = b$$

$$\text{Let } Ux = c$$

$$\Rightarrow Lc = b$$

Solve for c

$$Ux = c$$

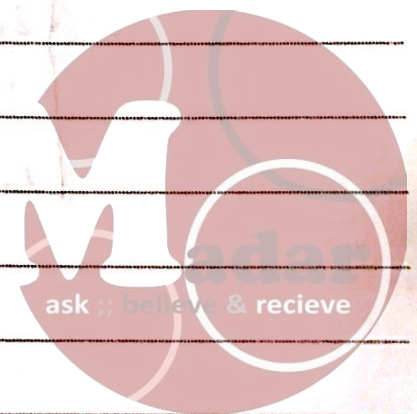
Solve for x

• How to find inverse?

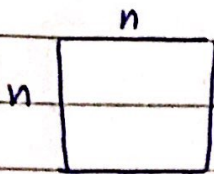
$$A = [5] \quad \left\{ \begin{array}{l} A^{-1} A = 1 \\ A^{-1} = \left[\frac{1}{5} \right] \end{array} \right.$$

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Identity matrix.



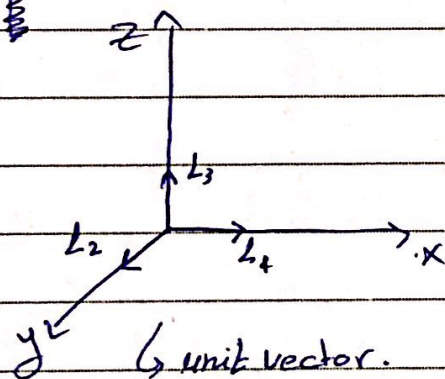
For square matrix.



if $A \rightarrow \det A \neq 0$
 A^{-1} exist.

$$A^{-1} A = A A^{-1} = I$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Find matrix B such that:-

$$BA = AB = I$$

$$\begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

دہاں لائنوں کے لیے