



NUMERICAL

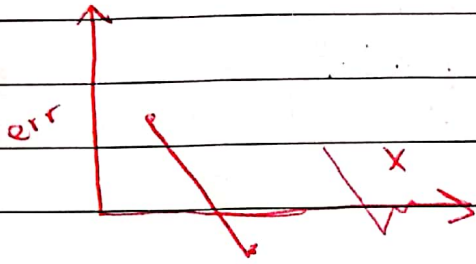
Dr. Reyad Al-Shawabkeh



Solution of non-linear equation

Sol of single equation

$$x^2 + e^x = 3 \rightarrow x = \pm \sqrt{3 - e^x}$$
$$x^2 e^x + \ln x \quad x^3 = \dots \rightarrow \dots$$



- (a) fixed point iterative method
- (b) Bisection method
- (c) Newton-Rapison method
- (d) Newton - 2nd order method

Secant Method :

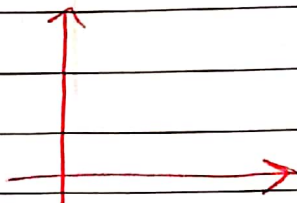
(f) Huler method

Fixed point method ?

$$f(x) = 0 \rightarrow x = ? \quad f(x) = 0 \rightarrow \text{are the Roots}$$

ex) : $f(x) = e^{-x}$ $\rightarrow$ what is the value of x where $f(x) = 0$!?

x	e^{-x}
0	1
-1	$e = 2.7$
1	$\frac{1}{2.7}$



Fixed point iterated Method!

Ex: $f(x) = x^3 + x^2 - 3x - 3$

P.S: conv $\rightarrow$ Roots exist

div $\rightarrow$ Root do not exist

$$\begin{array}{c} 4 \\ 0 \quad -12 \quad 60 \end{array}$$

$$x = g(x)$$

$$x \pm g(x) = f(x) = 0 \Rightarrow x = g(x)$$

① $x^3 + x^2 - 3x - 3 = 0$

$$x = \frac{x^3 + x^2 - 3}{3}$$

x_{old}

x_{new}

1

$-\frac{1}{3}$

$-\frac{1}{3}$

-0.975

-0.975

-0.992 $\rightarrow$ Reaching one Root

② $x = \sqrt[3]{3x + 3 - x^2} = (3x + 3 - x^2)^{1/3}$

x_{old}

x_{new}

1

1.708

1.708

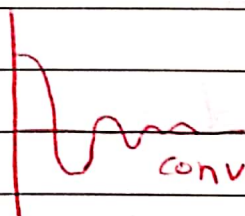
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$\rightarrow$ err = 0.4
 $\rightarrow$ er = 0.2

1.73

1.732

$\rightarrow$ er = 0.1



③ $x^2 = 3x + 3 - x^3$

$$x = \sqrt[2]{3x + 3 - x^3} = (3x + 3 - x^3)^{1/2}$$

x_{old}

x_{new}

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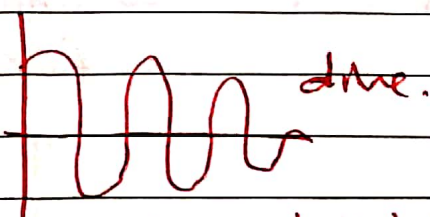
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imaginary Root



conv use old as initial value $\rightarrow$ List 1

⊗ to find the proper root

proper initial guess

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for fixed method iterate key point ; high order in side and the other in another side

highest order = other orders f_{xn}

⊗ [highest = other → initial guess] → can the Right

Newton's Method

Root, its speed ⊗
more powerfull ⊗

$$f(x) = f(x_0) + f'(x_0) \frac{\Delta x}{1!} + \frac{f''(x_0) (\Delta x)^2}{2!} + \dots$$

1 ← err →

Jap uplo uplo

$$= 0$$

$$f(x_0) + f'(x_0) \Delta x = 0$$

$$\Delta x = - \frac{f(x_0)}{f'(x_0)}$$

$$x_{0 \text{ New}} - x_{0 \text{ old}} = - \frac{f(x_0)}{f'(x_0)}$$

$$x_{0 \text{ New}} = x_{0 \text{ old}} - \frac{f(x_0)}{f'(x_0)}$$

(Ex) $f(x) = x^3 + x^2 - 3x - 3$
 $f'(x) = 3x^2 + 2x - 3$

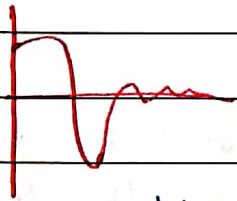
$$x_{i+1} = x_i - \frac{x_i^3 + x_i^2 - 3x_i - 3}{3x_i^2 + 2x_i - 3}$$

x_i	x_{i+1}	x_i	x_{i+1}	x_i	x_{i+1}
0.1	-1.08	2	1.769	-2	-1.8
-1.08	-0.989	1.769	1.732	-1.8	-1.738
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-0.999	-1			-1.72	-1.72
-1	-1 → conv				

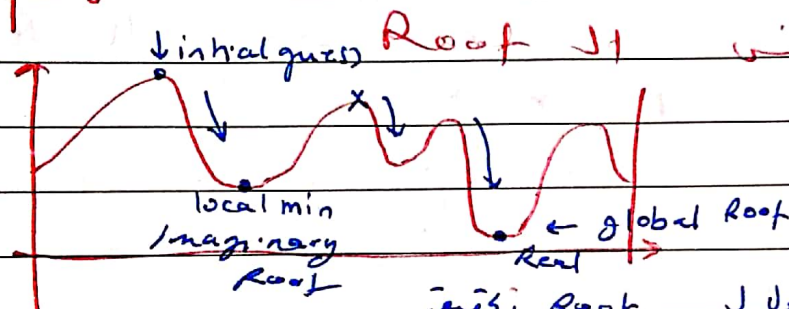
first Root

sec Root

Third Root



highly -ve & highly +ve
Roots are 2, -2, 2



I wanna the global minimum!!

Roots are 2, -2, 2

Initial guess + proper writing of equation

Initial guess + proper writing of equation

Newton - 2nd order Method.

$$f(x) = f(x_0) + f'(x_0) \frac{\Delta x}{1!} + f''(x_0) \frac{(\Delta x)^2}{2!} + O(\text{err})$$

$$= 0$$

$$f(x_0) + f'(x_0) \frac{\Delta x}{1} + f''(x_0) \frac{(\Delta x)^2}{2} = 0$$

From Newton 1st order $\Delta x = -\frac{f(x_0)}{f'(x_0)}$

$$f(x_0) + \Delta x \left[f'(x_0) + f''(x_0) \frac{\Delta x}{2} \right] = 0$$

$$f(x_0) + \Delta x \left[f'(x_0) + \frac{f''(x_0)}{2} \left(-\frac{f(x_0)}{f'(x_0)} \right) \right] = 0$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\left[f'(x_i) - \frac{f(x_i) f''(x_i)}{2 f'(x_i)} \right]}$$

(Ex) $f(x) = 2x^3 - 3x^2 + x - 6$

$$f'(x) = 6x^2 - 6x + 1$$

$$f''(x) = 12x - 6$$

x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$
1	-6	1	6
1.3	-5.3	3.5	9.7
1.8	-2.3	9.6	15.6
1.99	-0.04	12.9	17.9
2	0	13	18

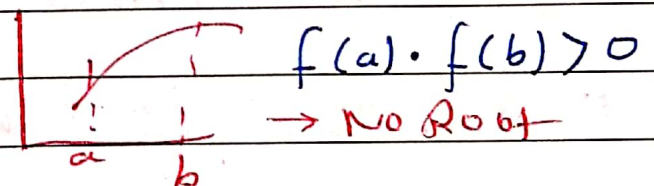
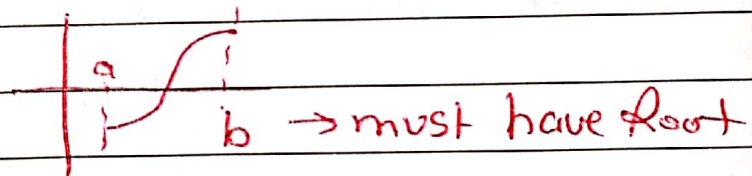
4) Bisection Method.

نقطة، نقطة وقوة

$$a \leq x \leq b$$

$$f(a) \cdot f(b) < 0$$

⊖ و ⊕



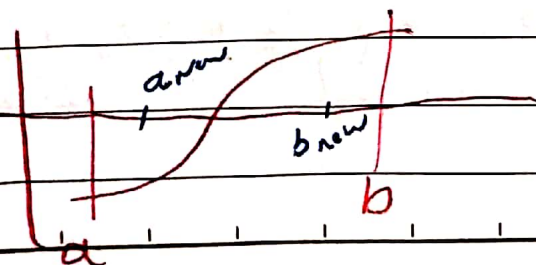
$$x_n = \frac{a+b}{2}, f(x_n) ??$$

if $f(a) \cdot f(x_n) > 0$

$$a_{\text{new}} = x_n$$

if $f(b) \cdot f(x_n) > 0$

$$b = x_n$$

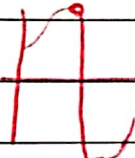


error ↑
high guess

ex) $f(x) = x^2 + x - 5$

في افترقا $\left[\begin{array}{l} a=0 \\ b=3 \end{array} \right] \left[\begin{array}{l} f(a) = f(0) = -5 \\ f(b) = f(3) = 7 \end{array} \right] \rightarrow f(a) * f(b) < 0$
the assumption is valid

a	b	x_n	$f(a)$	$f(b)$	$f(x_n)$	chang.
0	3	1.5	-5	7	-1.25	$a = x_n$
1.5	3	2.25	-1.25	7	2.3	$b = x_n$
1.5	2.25	1.87	-1.25	2.3	0.39	$b = x_n$
1.5	1.87	1.68	-1.25	0.39	-0.46	$a = x_n$
1.68	1.87	1.79	-0.46	0.39	-0.04	$a = x_n$

stiffness  sluggish

5) Secant method

$$f'(x) = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}} \quad \text{↑ error ↑}$$

From Newton-R Method

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \rightarrow x_n = x_{n-1} - f(x_{n-1}) \left[\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} \right]$$

في افترقا $\rightarrow$ two initial guesses
في افترقا $\rightarrow$ two initial guesses

ex) $f(x) = 2x^3 - 3x^2 + x - 6$

$x_{n-1} = 3$

$x_{n-2} = 4$

في افترقا $\rightarrow$ Root
+ $\int$ bisection Method.

$$b = \frac{f_1 - f_0 - a h_1^2}{h_1}$$

$$\delta = \frac{h_2}{h_1}$$

$$\Delta x = -2c$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$X = X_0 - \frac{2f_0}{\dots}$$

$$b \pm \sqrt{b^2 - 4ac}$$

اکوفا عکین

صبر الخير اصغر ما عكس
صبر الامام

+ve side is $\leftarrow +ve \leftarrow b | a$

-ve side is $\leftarrow -ve \leftarrow b | a$

$$f(x) = \sin x - \frac{x}{2} \quad \text{--- (Rad)}$$

(Root = 1.89)

$$h = h_1 = h_2 = 0.2$$

for the first time we assume they are equal for simplicity but it is not obligated to assume that

$$x_0 = 2$$

$$x_1 = x_0 + h_1 = 2 + 0.2 = \boxed{2.2}$$

$$x_2 = x_0 - h = 2 - 0.2 = 1.8$$

three initial guesses.

~~$$f(x_0) = f_0 = \sin(2) - \frac{2}{2} = -0.0907$$~~

$$f(x_4) = f_1 = \sin(2.2) - \frac{2.2}{2} = -0.991$$

$$f(x_2) = f_2 = \sin(1.8) - \frac{1.8^2}{2} = -0.0738$$

$\uparrow x$ $\uparrow h$ $\textcircled{*}$

$\delta = 1$ for First trial

$$a = \frac{(1)(-0.991) - (-0.0907)(1+1) + 0.07}{(1)(0.2)^2(1+1)} = -0.453$$

$$b = \frac{(-0.991) - (-0.6907)(-0.453)}{0.2} (0.2)^2 = -0.913$$

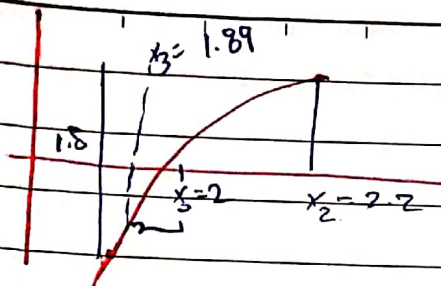
$$c = f(x_0) = f_0 = -0.907$$

the root

$$X = 2 - 2(-0.0907)$$

$$= 1.89526$$

$$-0.918 \pm \sqrt{(-0.913)^2 - 4(10.453)(-10907)}$$



$$x_0 = 1.89526$$

$$f_0 = \sin(1.89526) - \frac{1.8926}{2}$$

$$= 1.918 \times 10^{-4}$$

$$x_1 = 2$$

$$h_1 = x - x_0 = 2 - 1.89526$$

$$x_2 = 1.8$$

$$h_2 = 0.10474$$

$$x_2 = 1.8$$

$$h_2 = x_0 - x_2 = 0.09526$$

$$\therefore x = \frac{h_2}{h_1} = \frac{0.09526}{0.0104} = 0.9094$$

$$c = f(x_0) = 1.918 \times 10^{-4}$$

$$f_1 = f(x_1) = f(2) = -0.0907$$

$$f_2 = f(x_2) = f(1.8) = 0.0738$$

$$a = -0.4728$$

$$b = -0.81820$$

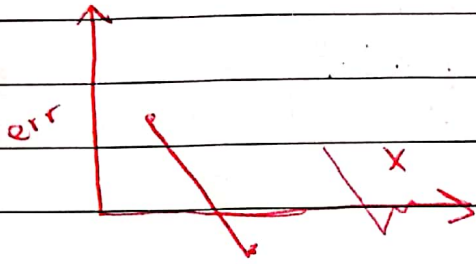
$$x \approx 0 \text{ (Red noting value)}$$

$$x = 1.8954 \text{ exactly the root}$$

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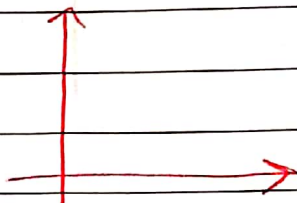
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Fixed point iterated Method!

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-0.992 $\rightarrow$ Reaching one Root

② $x = \sqrt[3]{3x + 3 - x^2} = (3x + 3 - x^2)^{1/3}$

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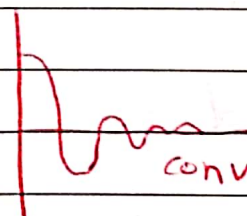
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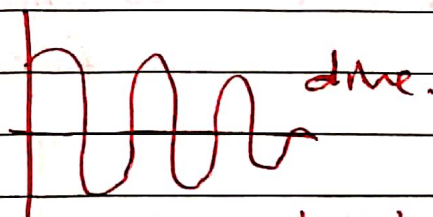
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Jawab upo upo

$$f(x_0) + f'(x_0) \Delta x = 0$$

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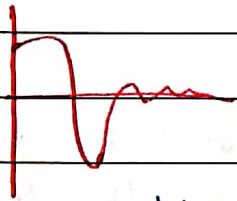
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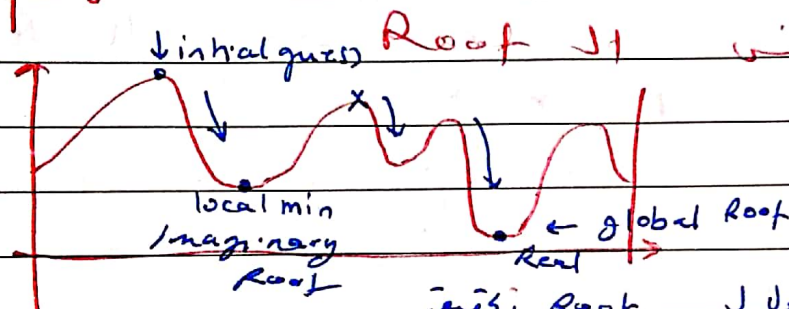
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minamum!!

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Initial guess + proper writing of equation

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(Ex) $f(x) = 2x^3 - 3x^2 + x - 6$

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$$f''(x) = 12x - 6$$

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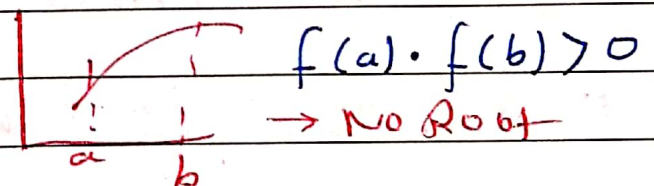
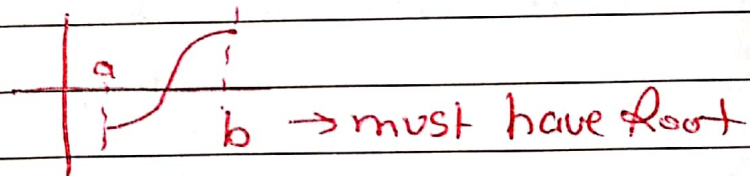
④ Bisection Method.

نقطة، نقطة وقوة

$$a \leq x \leq b$$

$$f(a) \cdot f(b) < 0$$

⊖ و ⊕



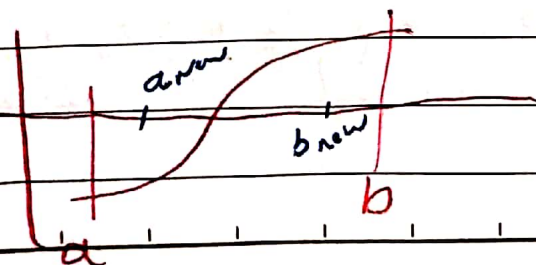
$$x_n = \frac{a+b}{2}, f(x_n) ??$$

if $f(a) \cdot f(x_n) > 0$

$$a_{\text{new}} = x_n$$

if $f(b) \cdot f(x_n) > 0$

$$b = x_n$$

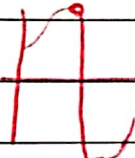


error ↑
high guess

ex) $f(x) = x^2 + x - 5$

في افترقا $\left[\begin{array}{l} a=0 \\ b=3 \end{array} \right. \left. \begin{array}{l} f(a) = f(0) = -5 \\ f(b) = f(3) = 7 \end{array} \right] \rightarrow f(a) * f(b) < 0$
the assumption is valid

a	b	x_n	$f(a)$	$f(b)$	$f(x_n)$	chang.
0	3	1.5	-5	7	-1.25	$a = x_n$
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1.68	1.87	1.79	-0.46	0.39	-0.04	$a = x_n$

stiffness  sluggish

⑤ Secant method

$$f'(x) = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}} \quad \text{↑ error ↑}$$

From Newton-R Method

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \rightarrow x_n = x_{n-1} - f(x_{n-1}) \left[\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} \right]$$

④ لا يكون عندنا أكثر من معادلة ← عندنا نقطتين فقط، لكننا نحتاج إلى نقطتين
two initial guesses

ex) $f(x) = 2x^3 - 3x^2 + x - 6$

$x_{n-1} = 3$

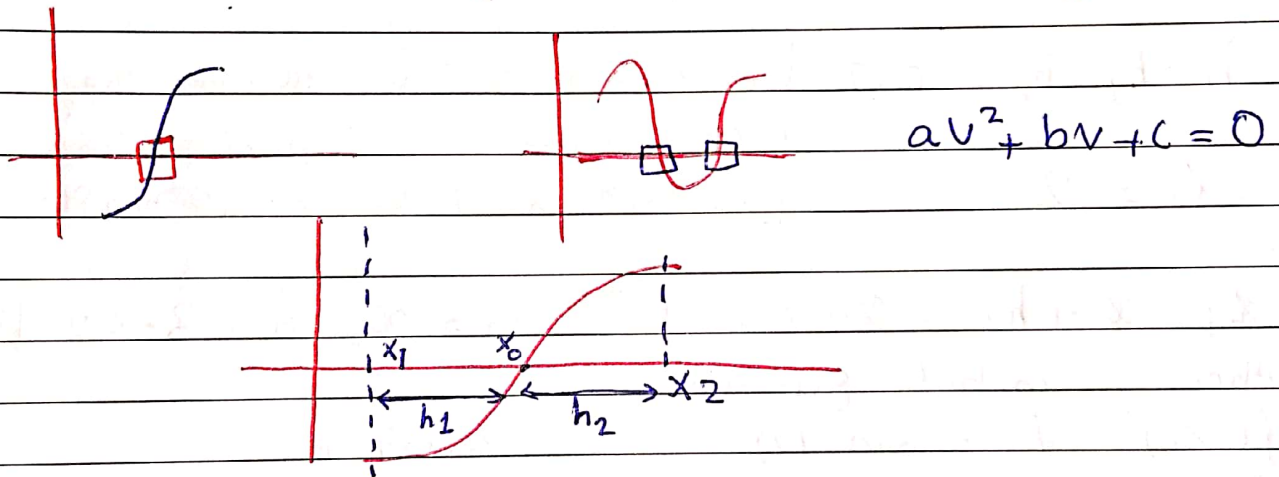
$x_{n-2} = 4$

④ في هذه الحالة يكون Root بين 3 و 4
+ ∫ bisection Method.

x_{n-2}	x_{n-1}	$f(x_{n-2})$	$f(x_{n-1})$	x_n	$f(x_n)$
4	3	78	24	2.6	10.3
← $\frac{f(x_{n-1})}{f'(x_{n-1})}$ ←					
3	2.6	24	10.3	2.2	3.3
2.6	2.2	10.3	3.3	2.1	0.8
2.2	2.1	3.3	0.8	2.0001	0.1
2.1	2.0001	0.8	0.1	2.0	0
← $\frac{f(x_{n-1})}{f'(x_{n-1})}$ ←					

← bisection / Newton

⑥ Müller method (يفضل) Find the root right away



$$h_1 = x_1 - x_0$$

$$h_2 = x_0 - x_2$$

$$av^2 + bv + c = 0 = f(v) \quad ; \text{ substitute by } x_0, x_1, x_2$$

$$x_1 = h_1 + x_0$$

$$x_2 = x_0 - h_2$$

$$x_0 = x_0$$

$$a h_1^2 + b h_1 + c = 0 = f(h_1)$$

$$a h_2^2 + b h_2 + c = 0 = f(h_2)$$

$$c = 0 = f(x_0)$$

$$a = \frac{\delta f_1 - f_0(1 + \delta) + f_2}{\delta h_1^2 (1 + \delta)}$$

من ههنا مطلوب

$$b = \frac{f_1 - f_0 - a h_1^2}{h_1}$$

$$\delta = \frac{h_2}{h_1}$$

$$\Delta x = -2c$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$X = X_0 - \frac{2f_0}{\dots}$$

$$b \pm \sqrt{b^2 - 4ac}$$

اکوفا عکین

صبر الحزير اصغر ما عكس
من الامام

+ve side is $\leftarrow +ve \leftarrow b | a$

-ve side is $\leftarrow -ve \leftarrow b | a$

$$f(x) = \sin x - \frac{x}{2} \quad \text{--- (Rad)}$$

(Root = 1.89)

$$h = h_1 = h_2 = 0.2$$

for the first time we assume they are equal for simplicity but it's not obligated to assume that

$$x_0 = 2$$

$$x_1 = x_0 + h_1 = 2 + 0.2 = 2.2 \quad x_2 = x_0 - h = 2 - 0.2 = 1.8$$

three initial guesses.

~~$$f(x_0) = f_0 = \sin(2) - \frac{2}{2} = -0.0907$$~~

$$f(x_4) = f_1 = \sin(2.2) - \frac{2.2}{2} = -0.991$$

$$f(x_2) = f_2 = \sin(1.8) - \frac{1.8^2}{2} = -0.0738$$

$\uparrow x$ $\uparrow h$ $\odot *$

$\delta = 1$ for First trial

$$a = \frac{(1)(-0.991) - (-0.0907)(1+1) + 0.07}{(1)(0.2)^2(1+1)} = -0.453$$

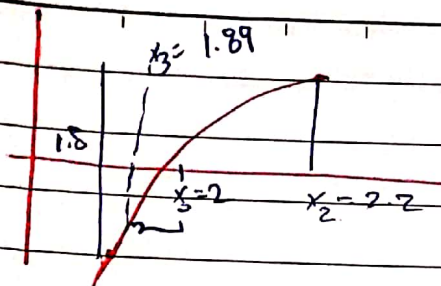
$$b = \frac{(-0.991) - (-0.6907) - (-0.453)}{0.2} (0.2)^2 = -0.913$$

$$c = f(x_0) = f_0 = -0.907$$

the root

~~$$X = 2 - 2(-0.0907) = 1.89526$$~~

$$-0.918 \pm \sqrt{(-0.913)^2 - 4(10.453)(-10907)}$$



$$x_0 = 1.89526$$

$$f_0 = \sin(1.89526) - \frac{1.8926}{2}$$

$$= 1.918 \times 10^{-4}$$

$$x_1 = 2$$

$$h_1 = x - x_0 = 2 - 1.89526$$

$$x_2 = 1.8$$

$$h_2 = 0.10474$$

$$x_2 = 1.8$$

$$h_2 = x_0 - x_2 = 0.09526$$

$$\therefore x = \frac{h_2}{h_1} = \frac{0.09526}{0.0104} = 0.9094$$

$$c = f(x_0) = 1.918 \times 10^{-4}$$

$$f_1 = f(x_1) = f(2) = -0.0907$$

$$f_2 = f(x_2) = f(1.8) = 0.0738$$

$$a = -0.4728$$

$$b = -0.81820$$

$$x \approx 0 \text{ (Red not in box)}$$

$$x = 1.8954 \text{ exactly the root}$$

Solution of linear equation

New chapter
23th Jun - 2019

Solution linear equation

- 1) Gaussian elimination Method
- 2) Gaussian Jordan Method
- 3) Jacobi iterated Method
- 4) Gauss - sidel Method
- 5) L V Method

#1 Gaussian elimination Method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots = b_2$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots = b_n$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$\vec{X} = A^{-1}b$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

ex

$$2x_1 - x_2 + 3x_3 = 2$$

$$x_1 + 2x_2 - x_3 = 3$$

$$3x_1 - 2x_2 + x_3 = -1$$

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -2 & 1 & -1 \end{bmatrix}$$

diagonal ← استاذنا

اصرف بالرقم الى يمينه وظهره على اول

الرقم diagonal

$$-\frac{1}{2} R_1 + R_2 = R_2$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ \rightarrow 3 & -2 & 1 & -1 \end{array}$$

$$-\frac{3}{2} + R_1 + R_3 = R_3$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ 0 & -0.5 & -3.5 & -4 \end{array}$$

$$\frac{0.5}{2.5} R_2 + R_3 = R_3$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ 0 & 0 & -4 & -3.6 \end{array}$$

OR $-\frac{0.5}{2.5} R_2 - R_3 = R_3$

Algorithm:

For $K=1:N$

For $J=1:N+1 \rightarrow$ Right hand column.

Read a_{KJ}

END

END

For $k=1:N-1$

For $i=k+1:N$

For $j=k+1:N+1$

$$a_{ij} = a_{ij} - \frac{a_{ik} a_{kj}}{a_{kk}}$$

Now solve:

$$-4x_3 = -3.6$$

$$x_3 = 0.9$$

$$2.5x_2 - 2.5x_3 = 2$$

$$2.5x_2 - 2.5(0.9) = 2$$

$$x_2 = 1.7$$

$$2x_1 - x_2 + x_3 = 2$$

$$2x_1 - \frac{4}{5} = 2$$

$$+\frac{4}{5} \quad +\frac{4}{5}$$

$$x_1 = 1/2$$

$$X_n = \frac{a_{n, n+1}}{a_{nn}}$$

for $k = n-1 : 1$

$$\text{sum} = 0$$

for $j = k+1 : N$

$$\text{sum} = \text{sum} + a_{kj} \cdot X_j$$

END

$$X_k = \frac{1}{a_{kk}} (a_{k, n+1} - \text{sum})$$

END k.

ON previous example :

$$a_{21} = -\frac{1}{2} \times 2 + 1 = 0$$

$$a_{31} = -\frac{3}{2} \times 2 + 3 = 0$$

$$a_{23} = -\frac{1}{2} \times 3 + -1 = -2.5$$

$$a_{32} = -\frac{3}{2} \times -1 - 2 = -0.5$$

$$a_{22} = -\frac{1}{2} \times -1 + 2 = 2.5$$

$$a_{33} = -\frac{3}{2} \times 3 + 1 = -3.5$$

$$b_1 = 2$$

$$b_2 = -\frac{3}{2} \times 2 - 1 = -4$$

$$a_{32} = \frac{0.5}{2.5} \times 2.5 - 0.5 = 0$$

$$a_{33} = \frac{0.5}{2.5} \times -2.5 + (-3.5) = -4$$

$$b_3 = \frac{0.5}{2.5} \times 2 - 4 = -3.6$$

2) Gauss-Jordan Method.

For $k=1 : N-1$

$i = k+1 : N$

$j = N+1 : k_1-1$

$$a_{ij} = a_{ij} - \frac{a_{ik} a_{kj}}{a_{kk}}$$

END,

$$\begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

not necessary to be 0's
just to equal (1)

(ex) from previous example bring from it the eq's.

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -4 \\ 0 & 0 & 40 & 36 \end{array} \right]$$

Ⓢ ملاحظه:

$$\left[\begin{array}{l} 36 \\ 40 \end{array} \right] \leftarrow \begin{array}{l} 3.6 \\ 4 \end{array}$$

نہیں امیڑ ہوا 10 دکان عینہ کوزہ
سب او عوجہ جب عبادتی ای اختیرا لیسیر

① set diagonal = 1 (دکتر کتا میں ضروری اصول 1=1)

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1 & 4/5 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$E_3 + E_2 = E_2$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1 & 4/5 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$\frac{1}{2} E_2 + E_1 = E_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 74/40 \\ 0 & 1 & 0 & 68/40 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$-\frac{3}{2} E_3 + E_1 = E_1$$

Then solve

$$\begin{bmatrix} 1 & 0 & 0 & | & 29/40 \\ 0 & 1 & 0 & | & 68/40 \\ 0 & 0 & 1 & | & 36/40 \end{bmatrix}$$

$$x_1 = 29/40 = 1/2$$

$$x_2 = 68/40 = 17/10$$

$$x_3 = 36/40 = 9/10$$

APPLY THE CODE:

$$N = 3$$

$$K = 1 \rightarrow 2$$

$$i = 2 \rightarrow 3$$

$$J = 4 \rightarrow K$$

$$K = 1$$

$$i = 2$$

$$J = 4, 3, 2, 1$$

→ looping

$$a_{24} = a_{24} - (a_{21}/a_{11})(a_{14})$$

$$a_{23} = a_{23} - \left(\frac{a_{21}}{a_{11}}\right)(a_{13})$$

#3) Jacobi - iterative Method

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots = b_2$$

$$a_{n1} x_1 + a_{n2} x_2 + a_{n3} x_3 + \dots = b_n$$

④ ملاحظہ کریں کہ coeff درجہ اولیٰ و اولیٰ درجہ اولیٰ coeff درجہ اولیٰ و اولیٰ درجہ اولیٰ
by mistake fixed point
div

$$\prod_{i=1}^n |a_{ij}| = \max$$

$$\begin{vmatrix} a_{11} & a_{22} & a_{33} \end{vmatrix} = \max \text{imum}$$

Ex

$$2x_1 + 7x_2 - 8x_3 = 10$$

$$5x_1 - 2x_2 + 6x_3 = 6$$

$$3x_1 + 5x_2 - 7x_3 = 3$$

x_i which largest value!!

re arrange

$$5x_1 - 2x_2 + 6x_3 = 6$$

$$2x_1 + 7x_2 - 8x_3 = 10$$

$$3x_1 + 5x_2 - 7x_3 = 3$$

$$5 * 7 * 7 = 245$$

$$5 * 5 + 8 = 200 \quad \text{NOT THE Max}$$

*) جربنا ترتيب اخر هين باح بعضيني Max الجرب

$$x_1 = \frac{6 + 2x_2 - 6x_3}{5}$$

$$x_2 = \frac{10 - 2x_1 + 8x_3}{7}$$

$$x_3 = \frac{3 - 3x_1 - 5x_2}{7}$$

	#1	#2	#3	#4	...	#40
x_1	0	6/5	2.28	0.11		-0.305
x_2	0	10/7	0.545	2.04		4.76
x_3	0	-3/7	1.106	0.98		2.283 → The Root

4 : Gauss's Sidel Method

	#1	#2	#3	#22
x_1	0	6/5	0.6	0.306
x_2	0	$10 - 2(6/5) + 0 = 1.08$	2.24	4.76
x_3	0	$(3 - 3(6/5) - 5(1.08)) / -7 = 0.86$	1.42	2.84

لما انا احسها بوضوح $(x_1 = 6/5)$ مشي
(6) يعني بوضوح الآلة الحاسبة

~~Development / off / find / matrix /~~

(5) lower - Upper limit Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots a_{2n} \\ \vdots & \vdots & \vdots \dots \vdots \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & & & \\ a_{21} & 1 & & \\ a_{n1} & & 1 & \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

L

U

$$\vec{x} = \vec{b}$$

$$L^{-1} / \checkmark \quad \vec{y} = L^{-1} \vec{b} \Rightarrow \vec{y} = L^{-1} \vec{b} \rightarrow U\vec{x} = \vec{y}$$

$$\vec{x} = U^{-1} \vec{y}$$

Development of L & U Matrix

Elements of lower Matrix !!

$$l_{ij} = \frac{a_{ij}}{a_{ii}}$$

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & & 1 \\ \vdots & & & \ddots & \vdots \end{bmatrix}$$

$$U_{ij\text{new}} = U_{ij\text{old}} = l_{ij} a_{ii}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ & u_{22} & \dots & u_{2n} \\ & & \ddots & \\ & & & u_{nn} \end{bmatrix}$$

(Ex)

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$\Rightarrow LU \Rightarrow$

$$L = \begin{bmatrix} 1 & & \\ p_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$p_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$a_{21} = a_{21} - p_{21} a_{11} = 64 - (2.56)(25) = 0$$

$$p_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

$$a_{22} = a_{22} - p_{21} a_{12} = 8 - (2.56)(5) = -4.8$$

$$l_{32} = \frac{a_{32}}{a_{22}} = ??$$

$$a_{23} = a_{23} - p_{21} a_{13} = 1 - (2.56)(1) = -1.56$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{bmatrix}$$

upper !!

$$p_{31} = 5.76 \text{ so } a_{31} = a_{31} - p_{31} a_{11}$$

$$a_{31} = 144 - (5.76)(25) = 0$$

$$a_{32} = 12 - (5.76)(5) = 16.8$$

$$a_{33} = 1 - (5.76)(1) = -4.76$$

new one !!

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.75 & -1.56 \\ 0 & 16.8 & -4.76 \end{bmatrix}$$

$\Rightarrow$

$$p_{32} = \frac{a_{32}}{a_{22}} = \frac{16.8}{-4.75} = 3.53$$

$$a_{32} = a_{32} - p_{32} a_{22} = (16.8) - (3.53)(-4.75) = 0$$

$$a_{33} = a_{33} - p_{32} a_{23} = -4.76 - (3.53)(-1.56) = 0.7$$

$$-4.76 - (3.53)(-1.56) = 0.7$$

Five Apple

$$\begin{matrix} & U & & L & & = & A \end{matrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.75 & 1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.53 & 1 \end{bmatrix}$$

$$Ax = b \rightarrow L U x = b \rightarrow \boxed{L y = b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.53 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

straight forward!!

$$y_1 = 1$$

$$2.56 y_1 + y_2 = 2 \Rightarrow y_2 = -0.56$$

$$5.76 y_1 + 3.53 y_2 + y_3 = 3 \Rightarrow y_3 = -0.78$$

$$\boxed{L U x = b} \rightarrow \boxed{L y = b} \rightarrow \boxed{U x = y}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.75 & 1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.56 \\ -0.78 \end{bmatrix}$$

backward calculation

$$0.7 * x_3 = -0.78 \rightarrow x_3 = -1.11$$

$$x_2 = -0.48$$

$$x_1 = -0.0116$$

$$X = \begin{bmatrix} -1.11 \\ 0.48 \\ -0.6116 \end{bmatrix}$$

Algorithm !!

Read A, b

set $U = A$

$L = I \rightarrow [1, 0]$

for $k = 1, n-1$

for $j = k+1, n$

$$l_{jk} = \frac{U_{jk}}{U_{kk}}$$

$$U_{j,k:n} = U_{j,k:n} - l_{jk} U_{k,k:n}$$

$U_{k,k:n}$ no. of variable

END

ماتريك صفر، الدكتور

$$A = \begin{bmatrix} 5 & 5 & 1 & 4 & 0 & 1 & 4 & 1 & 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$[n, n] = \text{size}(A)$$

$$L = \text{eye}(n)$$

$$U = A$$

For $k = 1:n-1$

For $j = k+1:n$

$$L(j, k) = U(j, k) / U(k, k)$$

$$U(j, :) = U(j, :) - L(j, k) * U(k, :)$$

END

END

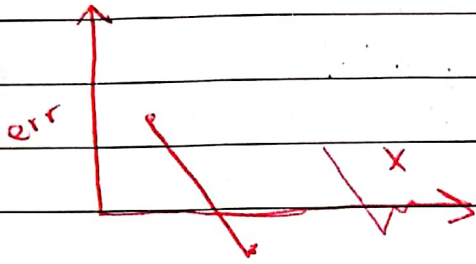
$$y = \text{Inv}(L) * b$$

$$x = \text{Inv}(U) * y$$

Solution of non-linear equation

Sol of single equation

$$x^2 + e^x = 3 \rightarrow x = \pm \sqrt{3 - e^x}$$
$$x^2 e^x + \ln x \quad x^3 = \dots \rightarrow \dots$$



- (a) fixed point iterative method
- (b) Bisection method
- (c) Newton-Rapison method
- (d) Newton - 2nd order method

Secant Method :

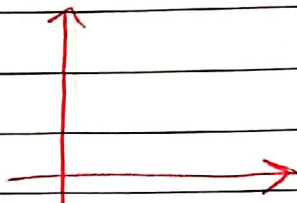
(f) Huler method

Fixed point method ?

$$f(x) = 0 \rightarrow x = ? \quad f(x) = 0 \rightarrow \text{are the Roots}$$

ex) : $f(x) = e^{-x}$ $\rightarrow$ what is the value of x where $f(x) = 0$!?

x	e^{-x}
0	1
-1	$e = 2.7$
1	$\frac{1}{2.7}$



Fixed point iterated Method!

Ex: $f(x) = x^3 + x^2 - 3x - 3$

P.S: conv $\rightarrow$ Roots exist

div $\rightarrow$ Root do not exist

$$\begin{array}{c} 4 \\ 0 \quad -12 \quad 0 \end{array}$$

$$x = g(x)$$

$$x \pm g(x) = f(x) = 0 \Rightarrow x = g(x)$$

① $x^3 + x^2 - 3x - 3 = 0$

$$x = \frac{x^3 + x^2 - 3}{3}$$

x_{old}

x_{new}

1

$-\frac{1}{3}$

$-\frac{1}{3}$

-0.975

-0.975

-0.992 $\rightarrow$ Reaching one Root

② $x = \sqrt[3]{3x + 3 - x^2} = (3x + 3 - x^2)^{1/3}$

x_{old}

x_{new}

1

1.708

1.708

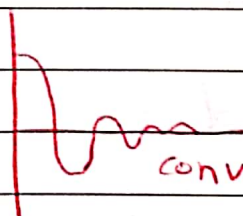
1.73

$\rightarrow$ err = 0.4
 $\rightarrow$ er = 0.2

1.73

1.732

$\rightarrow$ er = 0.1



③ $x^2 = 3x + 3 - x^3$

$$x = \sqrt{3x + 3 - x^3} = (3x + 3 - x^3)^{1/2}$$

x_{old}

x_{new}

1

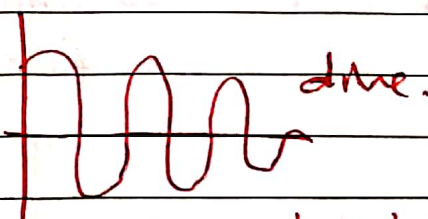
2.23

2.23

6.32

6.32

imaginary Root



conv use old as initial value $\rightarrow$ List 1

⊗ to find the proper root

proper initial guess

!!

for fixed method iterate key point ; high order in side and the other in another side

highest order = other orders f_{xn}

⊗ [highest = other → initial guess] → can the Right

Newton's Method

Root, its speed ⊗
more powerfull ⊗

$$f(x) = f(x_0) + f'(x_0) \frac{\Delta x}{1!} + f''(x_0) \frac{(\Delta x)^2}{2!} + \dots$$

← err →
↓
Jap uplo uplo

$$f(x_0) + f'(x_0) \Delta x = 0$$

$$\Delta x = - \frac{f(x_0)}{f'(x_0)}$$

$$x_{0 \text{ New}} - x_{0 \text{ old}} = - \frac{f(x_0)}{f'(x_0)}$$

$$x_{0 \text{ New}} = x_{0 \text{ old}} - \frac{f(x_0)}{f'(x_0)}$$

⊗ (Ex) $f(x) = x^3 + x^2 - 3x - 3$
 $f'(x) = 3x^2 + 2x - 3$

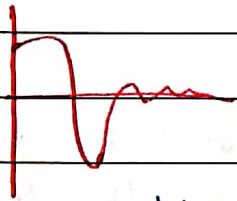
$$x_{i+1} = x_i - \frac{x_i^3 + x_i^2 - 3x_i - 3}{3x_i^2 + 2x_i - 3}$$

x_i	x_{i+1}	x_i	x_{i+1}	x_i	x_{i+1}
0.1	-1.08	2	1.769	-2	-1.8
-1.08	-0.989	1.769	1.732	-1.8	-1.738
-0.989	-0.999	1.732	1.732	-1.738	-1.72
-0.999	-1			-1.72	-1.72
-1	-1 → conv				

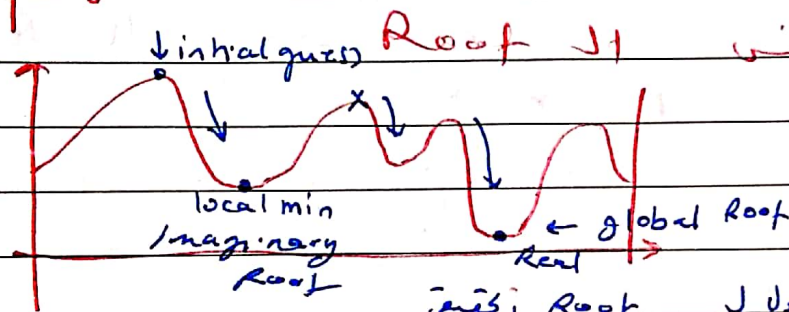
first Root

sec Root

Third Root



highly -ve & highly +ve
root find -2 > 2



Root 11

I wanna the global minimum!!

Roots are 10

initial guess is 2

Initial guess + proper writing of equation

Newton - 2nd order Method.

$$f(x) = f(x_0) + f'(x_0) \frac{\Delta x}{1!} + f''(x) \frac{(\Delta x)^2}{2!} + O(\text{err})$$

$$= 0$$

$$f(x_0) + f'(x_0) \frac{\Delta x}{1} + f''(x_0) \frac{(\Delta x)^2}{2} = 0$$

From Newton 1st order $\Delta x = -\frac{f(x_0)}{f'(x_0)}$

$$f(x_0) + \Delta x \left[f'(x_0) + f''(x_0) \frac{\Delta x}{2} \right] = 0$$

$$f(x_0) + \Delta x \left[f'(x_0) + \frac{f''(x_0)}{2} \left(-\frac{f(x_0)}{f'(x_0)} \right) \right] = 0$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\left[f'(x_i) - \frac{f(x_i) f''(x_i)}{2 f'(x_i)} \right]}$$

(Ex) $f(x) = 2x^3 - 3x^2 + x - 6$

$$f'(x) = 6x^2 - 6x + 1$$

$$f''(x) = 12x - 6$$

x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$
1	-6	1	6
1.3	-5.3	3.5	9.7
1.8	-2.3	9.6	15.6
1.99	-0.04	12.9	17.9
2	0	13	18

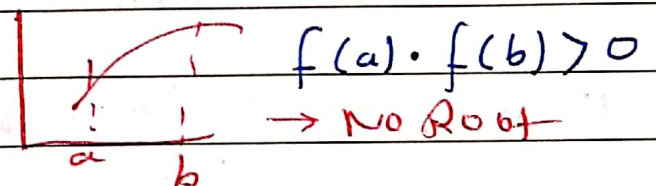
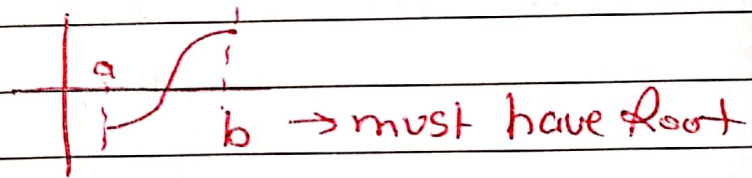
4) Bisection Method.

نقطة، نقطة وقف

$$a \leq x \leq b$$

$$f(a) \cdot f(b) < 0$$

⊖ و ⊕



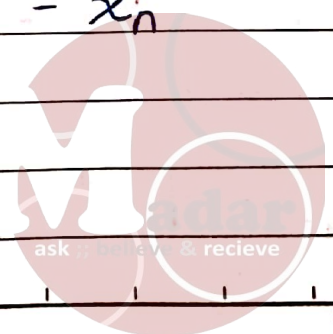
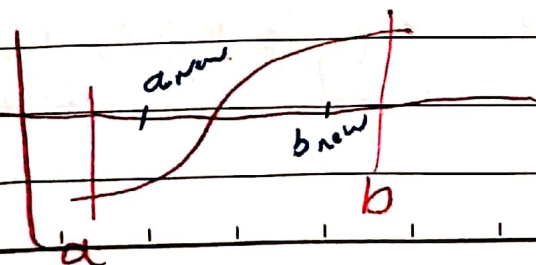
$$x_n = \frac{a+b}{2}, f(x_n) ??$$

if $f(a) \cdot f(x_n) > 0$

$$a_{\text{new}} = x_n$$

if $f(b) \cdot f(x_n) > 0$

$$b = x_n$$

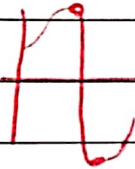


error ↑
high guess

ex) $f(x) = x^2 + x - 5$

في افترضا $\left[\begin{array}{l} a=0 \\ b=3 \end{array} \right] \left[\begin{array}{l} f(a) = f(0) = -5 \\ f(b) = f(3) = 7 \end{array} \right] \rightarrow f(a) * f(b) < 0$
the assumption is valid

a	b	x_n	$f(a)$	$f(b)$	$f(x_n)$	chang.
0	3	1.5	-5	7	-1.25	$a = x_n$
1.5	3	2.25	-1.25	7	2.3	$b = x_n$
1.5	2.25	1.87	-1.25	2.3	0.39	$b = x_n$
1.5	1.87	1.68	-1.25	0.39	-0.46	$a = x_n$
1.68	1.87	1.79	-0.46	0.39	-0.04	$a = x_n$

stiffness  sluggish

5) Secant method

$$f'(x) = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}$$
 ↑ error ↑

From Newton-R Method

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \rightarrow x_n = x_{n-1} - f(x_{n-1}) \left[\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} \right]$$

⊗ لا يكون عندك أكثر من معادلة عندك نقطتان فقط، لا تكفي، السبب اني
two initial guesses

ex) $f(x) = 2x^3 - 3x^2 + x - 6$

$x_{n-1} = 3$

$x_{n-2} = 4$

⊗ في البداية يكون Root بين 3 و 4
+ ∫ bisection Method.

$$b = \frac{f_1 - f_0 - a h_1^2}{h_1}$$

$$\delta = \frac{h_2}{h_1}$$

$$\Delta x = -2c$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$X = X_0 - 2f_0$$

$$b \pm \sqrt{b^2 - 4ac}$$

اکرم ما عکرم

صبر له خير اصغر ما عكس
صبر الامام

$+ve$ origin $\rightarrow +ve \leftarrow b \text{ is } \odot$
 $-ve$ origin $\rightarrow -ve \leftarrow b \text{ is } |$

$$f(x) = \sin x - \frac{x}{2} \quad \text{--- (Rad)}$$

(Root = 1.89)

$$h = h_1 = h_2 = 0.2$$

for the first time we assume they are equal for simplicity but it's not obligated to assume that

$$x_0 = 2$$

$$x_1 = x_0 + h_1 = 2 + 0.2 = 2.2 \quad x_2 = x_0 - h = 2 - 0.2 = 1.8$$

three initial guesses.

~~$$f(x_0) = f_0 = \sin(2) - \frac{2}{2} = -0.0907$$~~

$$f(x_4) = f_1 = \sin(2.2) - \frac{2.2}{2} = -0.991$$

$$f(x_2) = f_2 = \sin(1.8) - \frac{1.8^2}{2} = -0.0738$$

$\uparrow x$ $\uparrow h$ $\odot *$

$\delta = 1$ for First trial

$$a = \frac{(1)(-0.991) - (-0.0907)(1+1) + 0.07}{(1)(0.2)^2(1+1)} = -0.453$$

$$b = \frac{(-0.991) - (-0.6907) - (-0.453)}{0.2} (0.2)^2 = -0.913$$

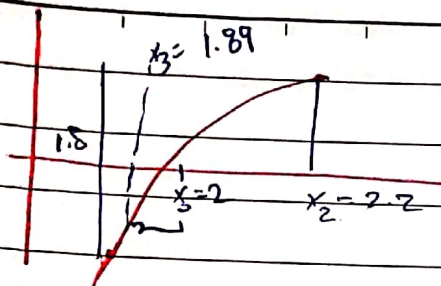
$$c = f(x_0) = f_0 = -0.907$$

the foot

$$X = 2 - 2(-0.0907)$$

$$= 1.89526$$

$$-0.918 \pm \sqrt{(-0.913)^2 - 4(10.453)(-10907)}$$



$$x_0 = 1.89526$$

$$f_0 = \sin(1.89526) - \frac{1.8927}{2}$$

$$= 1.918 \times 10^{-4}$$

$$x_1 = 2$$

$$h_1 = x - x_0 = 2 - 1.89526$$

$$x_2 = 1.8$$

$$h_2 = 0.10474$$

$$x_2 = 1.8$$

$$h_2 = x_0 - x_2 = 0.09526$$

$$\therefore x = \frac{h_2}{h_1} = \frac{0.09526}{0.0104} = 0.9094$$

$$c = f(x_0) = 1.918 \times 10^{-4}$$

$$f_1 = f(x_1) = f(2) = -0.0907$$

$$f_2 = f(x_2) = f(1.8) = 0.0738$$

$$a = -0.4728$$

$$b = -0.81820$$

$$x \approx 0 \text{ (Red noting lines)}$$

$$x = 1.8954 \text{ exactly the root}$$



Solution of linear equation

New chapter
23th Jun - 2019

Solution linear equation

- 1) Gaussian elimination Method
- 2) Gaussian Jordan Method
- 3) Jacobi iterated Method
- 4) Gauss - sidel Method
- 5) L V Method

#1 Gaussian elimination Method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots = b_2$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots = b_n$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$\vec{X} = A^{-1}b$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

ex

$$2x_1 - x_2 + 3x_3 = 2$$

$$x_1 + 2x_2 - x_3 = 3$$

$$3x_1 - 2x_2 + x_3 = -1$$

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -2 & 1 & -1 \end{bmatrix}$$

diagonal ← استعملنا

اصرف بالرقم الي بدله في الصفه على اول

الرقم diagonal

$$-\frac{1}{2} R_1 + R_2 = R_2$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ \rightarrow 3 & -2 & 1 & -1 \end{array}$$

$$-\frac{3}{2} + R_1 + R_3 = R_3$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ 0 & -0.5 & -3.5 & -4 \end{array}$$

$$\frac{0.5}{2.5} R_2 + R_3 = R_3$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ 0 & 0 & -4 & -3.6 \end{array}$$

OR $-\frac{0.5}{2.5} R_2 - R_3 = R_3$

Algorithm:

For $K=1:N$

For $J=1:N+1 \rightarrow$ Right hand column.

Read a_{KJ}

END

END

For $k=1:N-1$

For $i=k+1:N$

For $j=k+1:N+1$

$$a_{ij} = a_{ij} - \frac{a_{ik} a_{kj}}{a_{kk}}$$

Now solve:

$$-4x_3 = -3.6$$

$$x_3 = 0.9$$

$$2.5x_2 - 2.5x_3 = 2$$

$$2.5x_2 - 2.5(0.9) = 2$$

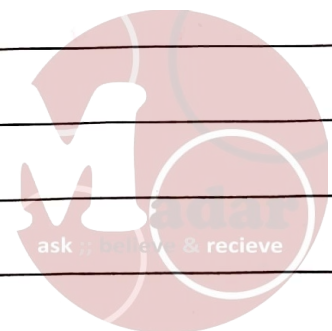
$$x_2 = 1.7$$

$$2x_1 - x_2 + x_3 = 2$$

$$2x_1 - \frac{4}{5} = 2$$

$$+\frac{4}{5} \quad +\frac{4}{5}$$

$$x_1 = 1/2$$



$$X_n = \frac{a_{n, n+1}}{a_{nn}}$$

for $k = n-1 : 1$

$$\text{sum} = 0$$

for $j = k+1 : N$

$$\text{sum} = \text{sum} + a_{kj} \cdot X_j$$

END

$$X_k = \frac{1}{a_{kk}} (a_{k, n+1} - \text{sum})$$

END k.

ON previous example :

$$a_{21} = -\frac{1}{2} \times 2 + 1 = 0$$

$$a_{31} = -\frac{3}{2} \times 2 + 3 = 0$$

$$a_{23} = -\frac{1}{2} \times 3 + -1 = -2.5$$

$$a_{32} = -\frac{3}{2} \times -1 - 2 = -0.5$$

$$a_{22} = -\frac{1}{2} \times -1 + 2 = 2.5$$

$$a_{33} = -\frac{3}{2} \times 3 + 1 = -3.5$$

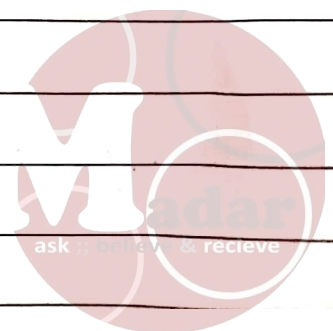
$$b_1 = 2$$

$$b_2 = -\frac{3}{2} \times 2 - 1 = -4$$

$$a_{32} = \frac{0.5}{2.5} \times 2.5 - 0.5 = 0$$

$$a_{33} = \frac{0.5}{2.5} \times -2.5 + (-3.5) = -4$$

$$b_3 = \frac{0.5}{2.5} \times 2 - 4 = -3.6$$



2) Gauss-Jordan Method.

For $k=1 : N-1$

$i = k+1 : N$

$j = N+1 : k_1-1$

$$a_{ij} = a_{ij} - \frac{a_{ik} a_{kj}}{a_{kk}}$$

END,

$$\begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

NOT necessary to be 0's
just to equal (1)

(ex) from previous example bring from it the eq's.

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -4 \\ 0 & 0 & 40 & 36 \end{array} \right]$$

Ⓢ ملاحظه:

$$\left[\begin{array}{c} 36 \\ 40 \end{array} \right] \leftarrow \begin{array}{c} 3.6 \\ 4 \end{array}$$

نہیں امیڑ ہوا 10 دکان عینہ کوڑا
سب او عوجہ جب عادی ای اختیار کرنا

① set diagonal = 1 (دکان کے ضروری اصول = 1)

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1 & 4/5 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$E_3 + E_2 = E_2$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1 & 4/5 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$\frac{1}{2} E_2 + E_1 = E_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 74/40 \\ 0 & 1 & 0 & 68/40 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$-\frac{3}{2} E_3 + E_1 = E_1$$

Then solve

$$\begin{bmatrix} 1 & 0 & 0 & | & 29/40 \\ 0 & 1 & 0 & | & 68/40 \\ 0 & 0 & 1 & | & 36/40 \end{bmatrix}$$

$$x_1 = 29/40 = 1/2$$

$$x_2 = 68/40 = 17/10$$

$$x_3 = 36/40 = 9/10$$

APPLY THE CODE:

$$N=3$$

$$K=1 \rightarrow 2$$

$$i=2 \rightarrow 3$$

$$J=4 \rightarrow K$$

$$K=1$$

$$i=2$$

$$J=4, 3, 2, 1$$

→ looping

$$a_{24} = a_{24} - (a_{21}/a_{11})(a_{14})$$

$$a_{23} = a_{23} - \left(\frac{a_{21}}{a_{11}}\right)(a_{13})$$

#3) Jacobi - iterative Method

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots = b_2$$

$$a_{n1} x_1 + a_{n2} x_2 + a_{n3} x_3 + \dots = b_n$$

④ ملاحظه اني اكر coeff درجة ٠ او اعلى منه بـ coeff لدرجة اعلى منه بنوعه
 fixed point by mistake اخطا الى كبر درجة الـ coeff (مفاد)
 ولاحظ ان معادلات بروج

$$\prod_{i=1}^n |a_{ij}| = \max$$

$$\begin{vmatrix} a_{11} & a_{22} & a_{33} \end{vmatrix} = \max \text{imum}$$



Ex

$$2x_1 + 7x_2 - 8x_3 = 10$$

$$5x_1 - 2x_2 + 6x_3 = 6$$

$$3x_1 + 5x_2 - 7x_3 = 3$$

x_1 which largest value!!

re arrange

$$5x_1 - 2x_2 + 6x_3 = 6$$

$$2x_1 + 7x_2 - 8x_3 = 10$$

$$3x_1 + 5x_2 - 7x_3 = 3$$

$$5 * 7 * 7 = 245$$

$$5 * 5 + 8 = 200 \quad \text{NOT The Max}$$

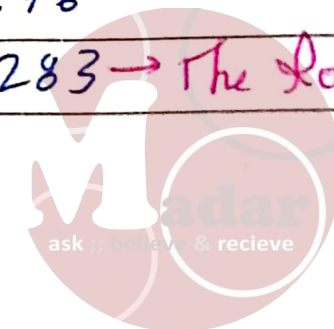
ل جربنا ترتيب اخر هين باح بعضيني Max بالجزء

$$x_1 = \frac{6 + 2x_2 - 6x_3}{5}$$

$$x_2 = \frac{10 - 2x_1 + 8x_3}{7}$$

$$x_3 = \frac{3 - 3x_1 - 5x_2}{7}$$

	#1	#2	#3	#4	...	#40
x_1	0	6/5	2.28	0.11		-0.305
x_2	0	10/7	0.545	2.04		4.76
x_3	0	-3/7	1.106	0.98		2.283 → The Root



4 : Gauss's Sidel Method

	#1	#2	#3	#22
x_1	0	6/5	0.6	6.306
x_2	0	$\frac{10 - 2(6/5) + 0}{-7} = 1.08$	2.24	4.76
x_3	0	$(3 - 3(6/5) - 5(1.08)) / -7 = 0.86$	1.42	2.84

لها آتي احدها بوزن $(\frac{6}{5} = x_1)$ مشحون
(5) يعني بوزن الملكية الجديدة

~~Development of kind & matrix~~

(5) lower - Upper limit McPhaul

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & & & \\ a_{21} & 1 & & \\ & & \ddots & \\ a_{n1} & & & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \vec{x} = \vec{b}$$

L

U

8

(1)

75

L^{-1} / L

75

$$= l' b$$

⇒

5 =

6

U

1511

ask ~~x~~ $\approx$ receive u

Development of L & U Matrix

Elements of lower Matrix !!

$$l_{ij} = \frac{a_{ij}}{a_{ii}}$$

$$L = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & & 1 & \\ \vdots & & & \ddots \end{bmatrix}$$

$$U_{ij\text{new}} = U_{ij\text{old}} - l_{ij} a_{ii}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ & u_{22} & \dots & u_{2n} \\ & & \ddots & \\ & & & u_{nn} \end{bmatrix}$$

Ex

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$\Rightarrow LU \Rightarrow$

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$l_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$a_{21} = a_{21} - l_{21} a_{11} = 64 - (2.56)(25) = 0$$

$$l_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

$$a_{22} = a_{22} - l_{21} a_{12} = 8 - (2.56)(5) = -4.8$$

$$l_{32} = \frac{a_{32}}{a_{22}} = ??$$

$$a_{23} = a_{23} - l_{21} a_{13} = 1 - (2.56)(1) = -1.56$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{bmatrix}$$

upper !!

$$l_{31} = 5.76 \text{ so } a_{31} = a_{31} - l_{31} a_{11}$$

$$a_{31} = 144 - (5.76)(25) = 0$$

$$a_{32} = 12 - (5.76)(5) = 16.8$$

New one !!

$$a_{33} = 1 - (5.76)(1) = -4.76$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.75 & -1.56 \\ 0 & 16.8 & -4.76 \end{bmatrix}$$

$\Rightarrow$

$$l_{32} = \frac{a_{32}}{a_{22}} = \frac{16.8}{-4.75} = 3.53$$

$$a_{32} = a_{32} - l_{32} a_{22} = (16.8) - (3.53)(-4.75) = 0$$

$$a_{33} = a_{33} - l_{32} a_{23} = -4.76 - (3.53)(-1.56) = 0.7$$

$$-4.76 - (3.53)(-1.56) = 0.7$$

$$\begin{matrix} & U & & L & & = & A \end{matrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.75 & 1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.53 & 1 \end{bmatrix}$$

$$Ax = b \rightarrow L U X = b \rightarrow \boxed{L y = b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.53 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

straight forward!!

$$y_1 = 1$$

$$2.56 y_1 + y_2 = 2 \Rightarrow y_2 = -0.56$$

$$5.76 y_1 + 3.53 y_2 + y_3 = 3 \Rightarrow y_3 = -0.78$$

$$\boxed{L U X = b} \rightarrow \boxed{L y = b} \rightarrow \boxed{U X = y}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.75 & 1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.56 \\ -0.78 \end{bmatrix}$$

backward calculation

$$0.7 * x_3 = -0.78 \rightarrow x_3 = -1.11$$

$$x_2 = -0.48$$

$$x_1 = -0.0116$$

$$X = \begin{bmatrix} -1.11 \\ 0.48 \\ -0.6116 \end{bmatrix}$$

Algorithm !!

Read A, b

set $U = A$

$L = I \rightarrow [1, 0]$

for $k = 1, n-1$

for $j = k+1, n$

$$l_{jk} = \frac{U_{jk}}{U_{kk}}$$

$$U_{j,k:n} = U_{j,k:n} - l_{jk} U_{k,k:n}$$

$U_{k,k:n}$ no. of variable

END

مثال من الدكتور

$$A = \begin{bmatrix} 5 & 5 & 1 & 4 & 0 & 1 & 4 & 1 & 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$[n, n] = \text{size}(A)$$

$$L = \text{eye}(n)$$

$$U = A$$

For $k = 1:n-1$

For $j = k+1:n$

$$L(j, k) = U(j, k) / U(k, k)$$

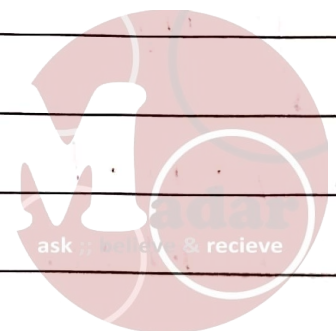
$$U(j, :) = U(j, :) - L(j, k) * U(k, :)$$

END

END

$$y = \text{Inv}(L) * b$$

$$x = \text{Inv}(U) * y$$



Numerical solution of system of non-linear eq.

1. Iterative method
2. Newton's Raphson method.

① Iterative Method

Equation $\rightarrow$ Rearrangement

(ex)

$$2x_1^2 + x_2 = 3 \rightarrow x_1 = \left(\frac{3-x_2}{2}\right)^{1/2}$$

$$x_1 + x_2^3 = 2 \rightarrow x_2 = (2-x_1)^{1/3}$$

x_1	x_2
0	1.26
0.93	1.02
0.994	1.001
0.999	1.0001

First trail: $x_2 = 1.26$ when $x_1 = 0 \dots$ so on.

(ex): ~~$x_1^3 + x_2 = 2$~~

$$2x_1^2 + x_2 = 3$$

$$x_1^3 + x_2 = 2$$

← have higher order degree x_1 ←

Ⓐ $x_2 = 3 - 2x_1^2$

$$x_1 = (2 - x_2)^{1/3}$$

conv $\rightarrow$ odd $(\frac{1}{3})$

x_1	x_2
0	3
-1	1
1	1

x_2 مبدئي x_1 ← initial guess
 x_1 مبدئي x_2 ←

Ⓐ x_1 مبدئي x_2 ←
 Root 1

Ⓑ $x_1 = \left(\frac{3-x_2}{2}\right)^{1/2}$

$$x_2 = 2 - x_1^3$$

(ex) Solve: $f_1 = 3x_1 - \cos(x_2 x_3) - \frac{1}{2}$
 $f_2 = x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06$
 $f_3 = e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3}$

by Newton Raphson Method

start with $x_1 = 0.1$ $x_2 = 0.1$ $x_3 = -0.1$

$$\begin{bmatrix} f_{1x_1} & f_{1x_2} & f_{1x_3} \\ f_{2x_1} & f_{2x_2} & f_{2x_3} \\ f_{3x_1} & f_{3x_2} & f_{3x_3} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2 \\ -f_3 \end{bmatrix}$$

$$f_{1x_1} = 3 \quad f_{1x_2} = x_3 \sin(x_2 x_3) \quad f_{1x_3} = x_2 \sin(x_2 x_3)$$

$$f_{2x_1} = 2x_1 \quad f_{2x_2} = -162(x_2 + 0.1) \quad f_{2x_3} = \cos(x_3)$$

$$f_{3x_1} = -x_2 e^{-x_1 x_2} \quad f_{3x_2} = -x_1 e^{-x_1 x_2} \quad f_{3x_3} = 20$$

$$\begin{bmatrix} 3 & 0.001 & -0.001 \\ 0.2 & -32.4 & 0.995 \\ -0.099 & -0.1 & 20 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} -1.2 \\ -2.27 \\ 8.46 \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -0.08 \\ -0.42 \end{bmatrix}$$

2nd iteration

$x_{1, \text{new}}$	$x_{1, \text{old}}$	Δx_1	0.1	0.5
$x_{2, \text{new}}$	$x_{2, \text{old}}$	Δx_2	0.1	$+0.02$
$x_{3, \text{new}}$	$x_{3, \text{old}}$	Δx_3	-0.42	-0.52

$$\epsilon = 0.001$$

i	x_1	x_2	x_3	ϵ
0	0.1	0.1	-0.1	—
1	0.5	0.02	-0.52	0.9
2	0.5	0.0015	-0.523	0.018
3	0.5	0.000012	-0.523	0.0015
4	0.5	0	-0.523	1.2×10^{-5}

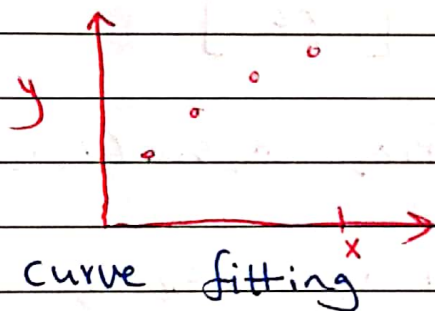
مقدار، کوچک شود
 RMS (error)
 خطا کم، به جهت
 $\epsilon = 0.001$
 جدول 8 و 10 و 11
 RMS می باشد

$$err = \sum_{i=1}^n (x_{i\text{ new}} - x_{i\text{ old}})^2$$

$$err = (x_{1N} - x_{1old})^2 + (x_{2N} - x_{2old})^2 + (x_{3N} - x_{3old})^2$$

Curve fitting & interpolation

ساختن



curve fitting

1- Linear Regression $\Rightarrow$ fitting

$$y = a + bx$$

2- Multi linear Regression

$$y = a + bx_1 + cx_2$$

3- Non-linear Regression

$$y = a + bx + cx^2$$

4- Multi Non-linear Regression

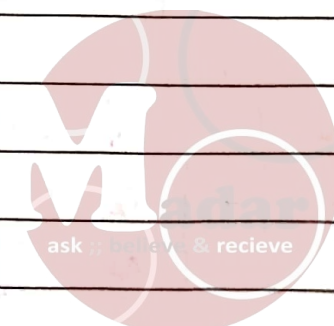
$$y = a + bx_1 + cx_1^2 + \dots + dx_2 + ex_2^2 + \dots$$

More general Regression It's including all the above Regression It's so powerfull

Linear Regression

X y_{exp} $y_{calculated}$ err

$$err = \min$$



$$\text{err} = \sum_{i=1}^n (y_{\text{exp}} - y_{\text{calc}})^2 = \sum (y_{\text{exp}} - (a + bx))^2$$

here, the best values for a, b should be; when err is the minimum

$$\frac{\partial \text{err}}{\partial a} = 0 \rightarrow 2 \sum (y_{\text{exp}} - (a + bx)) (-1) = -2 \left[\sum y_{\text{exp}} - (a + bx) \right]$$

$$\frac{\partial \text{err}}{\partial b} = 0 \rightarrow 2 \sum (y_{\text{exp}} - a - bx) (-x) = -2 \sum (x y_{\text{exp}} - a x - b x^2) = 0$$

$$\sum y_{\text{exp}} = \sum_{i=1}^n a + \sum b x$$

$$\sum x y_{\text{exp}} = \sum a x + \sum b x^2$$

$$\sum y_{\text{exp}} = na + b \sum x$$

$$\sum x y_{\text{exp}} = a \sum x + b \sum x^2$$

$$\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_{\text{exp}} \\ \sum x y_{\text{exp}} \end{bmatrix} \Rightarrow \text{simple linear eq. with two known}$$

$$a = \frac{\sum y_{\text{exp}} - b \sum x}{n}$$

$$b = \frac{n \sum y_i x_i - \sum y_i \sum x_i}{n \sum x_i^2 - (\sum x_i)^2}$$

Ex

n	x_i	y_{exp}	x_i^2	xy	y_{cal}	err^2 $(y - y_{\text{cal}})^2$	R^2 & L.V. $(y_i - y_{\text{avg}})^2$
1	1	4	1	4	3.3	$(4 - 3.3)^2 = 0.49$	$(4 - 6.75) = 7.5$
2	2	6	4	12	5.6	0.16	0.56
3	3	5	9	15	7.9	8.41	27.56
4	4	12	16	48	10.2	3.24	38.74
$\sum$	10	27	30	79		12.3	

Minimum value of err

↓ err ↑ fit → not indicated for the fit

$$b = \frac{4(79) - (27)(10)}{4(30) - (10)2} = 2.3 \quad a = \frac{27 - (2.3)(10)}{4} = 1$$

$$y_{\text{calc}} = a + bx \rightarrow y_{\text{calc}} = 1 + 2.3x$$

Regression coefficient
coefficient of determination

$$R^2 = \frac{\sum (y_{\text{exp}} - y_{\text{avg}})^2 - \sum (y_{\text{exp}} - y_{\text{cal}})^2}{\sum (y_{\text{exp}} - y_{\text{avg}})^2}$$

$0 < R^2 < 1 \rightarrow [0 - 1] \rightarrow$ best fit

المعيار $\rightarrow 0.97$ 0.95 $R^2 > 0.99$

ave value $\rightarrow y \rightarrow y_{avg} = \frac{\sum y_{exp}}{n} = \frac{27}{4} = 6.75$

ف. عاصود في ركبول والباقي تابع لها.

$$R^2 = \frac{38.74 - 12.3}{38.74} = 0.68 \rightarrow \text{how accurate our fitting}$$

Multi linear Regression

$$y = a + bx + cx_2 + dx_3 + \dots$$

Previously for a single Regression linear we do have this

$$\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \end{bmatrix}$$

$$y = a_0 + a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

$\sum x_1$	$\sum x_2$	$\dots$	$\sum x_n$	a_0	$\sum y$
$\sum x_1^2$	$\sum x_1 x_2$	$\dots$	$\sum x_1 x_n$	a_1	$\sum x_1 y$
$\sum x_2$	$\sum x_1 x_2$	$\sum x_2^2$	$\sum x_2 x_n$	a_2	$\sum x_2 y$
$\sum x_n$	$\sum x_1 x_n$	$\sum x_2 x_n$	$\sum x_n^2$	a_3	$\sum x_n y$

Ex Given That

n	x_1	x_2	y	x_1^2	$x_1 x_2$	x_2^2	$x_1 y$	$x_2 y$
1	0	0	3	0	0	0	0	0
2	1	3	1	1	3	9	1	3
3	2	2	7	4	4	4	14	14
4	3	5	5	9	15	25	15	25
5	4	7	5	16	28	49	20	35
6	5	1	12	25	5	1	105	21
	15	18	42	55	55	88	155	98

↑ values to fit ↑ ind var

$$\begin{bmatrix} 6 & 15 & 18 \\ 15 & 55 & 55 \\ 18 & 55 & 88 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 42 \\ 155 \\ 98 \end{bmatrix} \rightarrow \text{system of the linear eq.}$$

$$y = 3 + 4x_1 - 2x_2$$

Non-linear regression

1. $y = ax^b \rightarrow \text{take a log} \rightarrow \ln y = \ln a + b \ln x$

$\underbrace{\quad}_D \quad \underbrace{\quad}_Z \quad \underbrace{\quad}_Z \quad \underbrace{\quad}_C \quad \underbrace{\quad}_D$

x	y	$\ln x$	$\ln y$
---	---	---------	---------

$\Rightarrow b \checkmark$

$\Rightarrow c = \ln a \rightarrow a = e^c$

2. $y = a e^{bx} \rightarrow \text{Take } \ln \rightarrow \ln y = \ln a + bx$

x	y	$\ln y$
$\vdots$	$\vdots$	$\vdots$

$$\ln a = \checkmark \quad b = \checkmark \quad a = e^{\text{value}}$$

3. $y = a x^b e^{cx} \rightarrow \text{take a log}$

$$\ln y = \ln a + b \ln x + cx = \ln a$$

Polynomial Regression !!

$$y = a_1 + a_2 x + a_3 x^2 + \dots$$

$$\text{err} = \sum (y - y_e)^2$$

$$\frac{\partial \text{err}}{\partial a_1} = \frac{\partial}{\partial a_1} \sum (a_1 + a_2 x + a_3 x^2 - y_e)^2$$

$$\frac{\partial \text{err}}{\partial a_2} = \frac{\partial}{\partial a_2} \sum ($$

المصنوفة الهاتية الى 2 - انقل الى

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

n	$\sum x$	$\sum x^2$	$\sum x^3$	$\dots \sum x^n$	a_0		$\sum y$
$\sum x$	$\sum x^2$	$\sum x^3$	$\sum x^4$	$\sum x^{n+1}$	a_1		$\sum xy$
$\sum x^2$	$\sum x^3$	$\sum x^4$	$\sum x^5$	$\sum x^{n+2}$	a_2	=	$\sum x^2 y$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\sum x^n$	$\sum x^{n+1}$	$\sum x^{n+2}$	$\sum x^{n+3}$	$\sum x^{2n}$	a_n		$\sum x^n y$

n	$\sum x$	$\sum x^2$	x	y	x^2	x^3	x^4	xy	$x^2 y$
$\sum x$	$\sum x^2$	$\sum x^3$							
$\sum x^2$	$\sum x^3$	$\sum x^4$							
			$\sum$	$\sum$	$\sum$	$\sum$	$\sum$	$\sum$	$\sum$

$$y = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + b_1 x_2 + b_2 x_2^2 + b_3 x_2^3 +$$

$$c_1 x_1 x_2 + c_2 x_1 x_2^2 + c_3 x_1^2 x_2$$

$$y_1 = f(x_1, x_2)$$

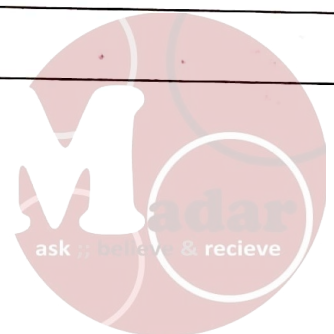
	element										قبلی
n	$\sum x_1$	$\sum x_1^2$	$\sum x_1^3$	$\sum x_2$	$\sum x_2^2$	$\sum x_2^3$	$\sum x_1 x_2$	$\sum x_1 x_2^2$	$\sum x_1^2 x_2$		$\sum y$
$\sum x_1$	$\sum x_1^2$	$\sum x_1^3$	$\sum x_1^4$	$\sum x_1 x_2$	$\sum x_1 x_2^2$	$\sum x_1^2 x_2$	$\sum x_1^2 x_2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2$		$\sum x_1 y$
$\sum x_1^2$	$\sum x_1^3$	$\sum x_1^4$	$\sum x_1^5$	$\sum x_1^2 x_2$	$\sum x_1^2 x_2^2$	$\sum x_1^2 x_2^3$	$\sum x_1^3 x_2$	$\sum x_1^3 x_2^2$	$\sum x_1^4 x_2$		$\sum x_1^2 y$
$\sum x_1^3$	$\sum x_1^4$	$\sum x_1^5 x_2^2$	$\sum x_1^6$	$\sum x_1^3 x_2$	$\sum x_1^3 x_2^2$	$\sum x_1^3 x_2^3$	$\sum x_1^4 x_2$	$\sum x_1^4 x_2^2$	$\sum x_1^5 x_2$		$\sum x_1^3 y$
$\sum x_2$	$\sum x_1 x_2$	$\sum x_1^2 x_2$	$\sum x_1^3 x_2$	$\sum x_2^2$	$\sum x_2^3$	$\sum x_2^4$	$\sum x_1 x_2^2$	$\sum x_1 x_2^3$	$\sum x_1^2 x_2^2$		$\sum x_2 y$
$\sum x_2^2$	$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2^2$	$\sum x_2^3$	$\sum x_2^4$	$\sum x_2^5$	$\sum x_1 x_2^3$	$\sum x_1 x_2^4$	$\sum x_1^2 x_2^3$		$\sum x_2^2 y$
$\sum x_2^3$	$\sum x_1 x_2^3$	$\sum x_1^2 x_2^3$	$\sum x_1^3 x_2^3$	$\sum x_2^4$	$\sum x_2^5$	$\sum x_2^6$	$\sum x_1 x_2^4$	$\sum x_1 x_2^5$	$\sum x_1^2 x_2^4$		$\sum x_2^3 y$
$\sum x_1 x_2$	$\sum x_1^2 x_2$	$\sum x_1^3 x_2$	$\sum x_1^4 x_2$	$\sum x_1 x_2^2$	$\sum x_1 x_2^3$	$\sum x_1 x_2^4$	$\sum x_1^2 x_2^2$	$\sum x_1^2 x_2^3$	$\sum x_1^3 x_2^2$		$\sum x_1 x_2 y$
$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2^2$	$\sum x_1^4 x_2^2$	$\sum x_1 x_2^3$	$\sum x_1 x_2^4$	$\sum x_1 x_2^5$	$\sum x_1^2 x_2^3$	$\sum x_1^2 x_2^4$	$\sum x_1^3 x_2^3$		$\sum x_1 x_2^2 y$
$\sum x_1^2 x_2$	$\sum x_1^3 x_2$	$\sum x_1^4 x_2$	$\sum x_1^5 x_2$	$\sum x_1^2 x_2^2$	$\sum x_1^2 x_2^3$	$\sum x_1^2 x_2^4$	$\sum x_1^3 x_2^2$	$\sum x_1^3 x_2^3$	$\sum x_1^4 x_2^2$		$\sum x_1^2 x_2 y$

المعاجيب

$$y = 1 + 2x_1 + 3x_1^2 + 4x_1^3 + 5x_2 + 6x_2^2 + 7x_2^3 + 8x_1x_2 + 9x_1x_2^2 + 10x_1^2x_2$$

x_1	x_2	y	$x_1^2 x_2$
0	0		
↓	↓		

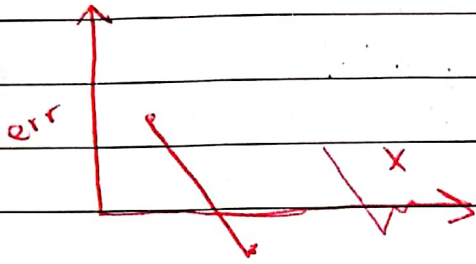
0	10
↓	↓
1	10



Solution of non-linear equation

Sol of single equation

$$x^2 + e^x = 3 \rightarrow x = \pm \sqrt{3 - e^x}$$
$$x^2 e^x + \ln x \cdot x^3 = \dots \rightarrow \dots$$



- (a) fixed point iterative method
- (b) Bisection method
- (c) Newton-Rapison method
- (d) Newton - 2nd order method

Secant Method :

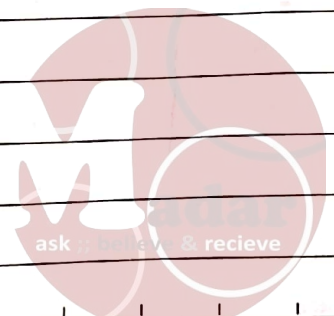
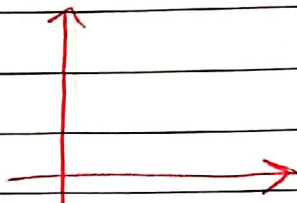
(f) Huler method

Fixed point method ?

$f(x) = 0 \rightarrow x = ?$ $f(x) = 0$ are the Roots

ex - $f(x) = e^{-x}$ $\rightarrow$ what is the value of x where $f(x) = 0$!?

x	e^{-x}
0	1
-1	$e = 2.7$
1	$\frac{1}{2.7}$

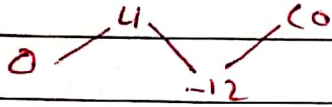


Fixed point iterated Method!

Ex: $f(x) = x^3 + x^2 - 3x - 3$

P.S: conv $\rightarrow$ Roots exist

div $\rightarrow$ Root do not exist



$$x = g(x)$$

$$x \pm g(x) = f(x) = 0 \Rightarrow x = g(x)$$

① $x^3 + x^2 - 3x - 3 = 0$

$$x = \frac{x^3 + x^2 - 3}{3}$$

x_{old}

x_{new}

1

$-\frac{1}{3}$

$-\frac{1}{3}$

-0.975

-0.975

-0.992 $\rightarrow$ Reaching one Root

② $x = \sqrt[3]{3x + 3 - x^2} = (3x + 3 - x^2)^{1/3}$

x_{old}

x_{new}

1

1.708

1.708

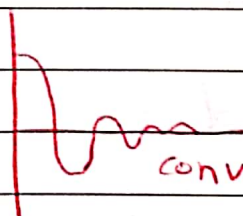
1.73

$\rightarrow$ err = 0.4
 $\rightarrow$ er = 0.2

1.73

1.732

$\rightarrow$ er = 0.1



③ $x^2 = 3x + 3 - x^3$

$$x = \sqrt[2]{3x + 3 - x^3} = (3x + 3 - x^3)^{1/2}$$

x_{old}

x_{new}

1

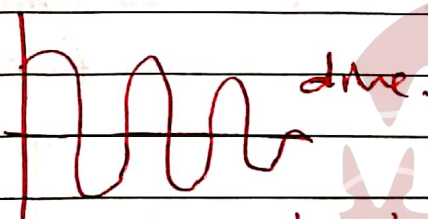
2.23

2.23

6.32

6.32

imaginary Root



conv use old as initial

div use old as initial

⊛ to find the proper root

proper initial guess

!!

for fixed method iterate key point ; high order in side and the other in another side

highest order = other orders f_{xn}

⊛ [highest = other → initial guess] → can the Right

Newton's Method

Root, its speed ⊛
more powerfull ⊛

$$f(x) = f(x_0) + f'(x_0) \frac{\Delta x}{1!} + \frac{f''(x_0) (\Delta x)^2}{2!} + \dots$$

1 ← err →
Jap uplo uplo

$$= 0$$

$$f(x_0) + f'(x_0) \Delta x = 0$$

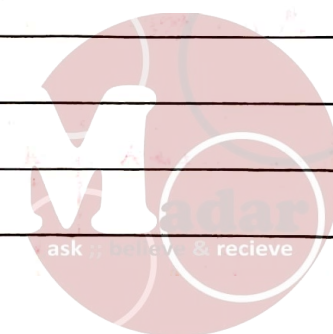
$$\Delta x = - \frac{f(x_0)}{f'(x_0)}$$

$$x_{0 \text{ New}} - x_{0 \text{ old}} = - \frac{f(x_0)}{f'(x_0)}$$

$$x_{0 \text{ New}} = x_{0 \text{ old}} - \frac{f(x_0)}{f'(x_0)}$$

(Ex) $f(x) = x^3 + x^2 - 3x - 3$
 $f'(x) = 3x^2 + 2x - 3$

$$x_{i+1} = x_i - \frac{x_i^3 + x_i^2 - 3x_i - 3}{3x_i^2 + 2x_i - 3}$$

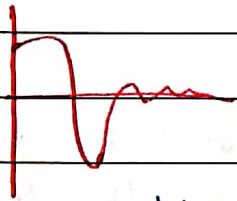


x_i	x_{i+1}	x_i	x_{i+1}	x_i	x_{i+1}
0.1	-1.08	2	1.769	-2	-1.8
-1.08	-0.989	1.769	1.732	-1.8	-1.738
-0.989	-0.999	1.732	1.732	-1.738	-1.72
-0.999	-1			-1.72	-1.72
-1	-1 $\rightarrow$ conv				

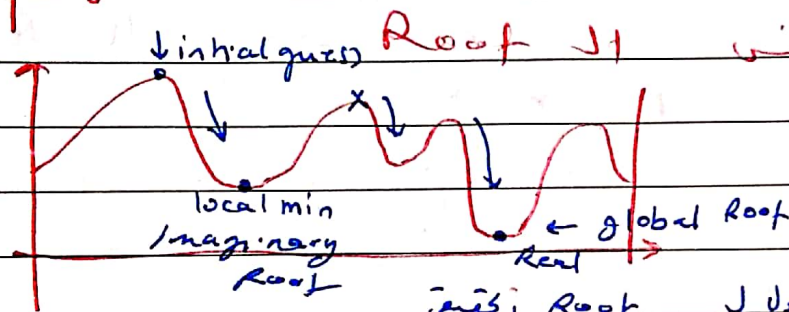
first Root

sec Root

Third Root



highly -ve & highly +ve
root find -2 > 2



Root 11

I wanna the global minimum!!

Roots are 10

initial guess is 2

Initial guess + proper writing of equation

Newton - 2nd order Method.

$$f(x) = f(x_0) + f'(x_0) \frac{\Delta x}{1!} + f''(x) \frac{(\Delta x)^2}{2!} + O(\text{err})$$

$$= 0$$

$$f(x_0) + f'(x_0) \frac{\Delta x}{1} + f''(x_0) \frac{(\Delta x)^2}{2} = 0$$

From Newton 1st order $\Delta x = \frac{-f(x_0)}{f'(x_0)}$

$$f(x_0) + \Delta x \left[f'(x_0) + f''(x_0) \frac{\Delta x}{2} \right] = 0$$

$$f(x_0) + \Delta x \left[f'(x_0) + \frac{f''(x_0)}{2} \left(-\frac{f(x_0)}{f'(x_0)} \right) \right] = 0$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\left[f'(x_i) - \frac{f(x_i) f''(x_i)}{2 f'(x_i)} \right]}$$

(Ex) $f(x) = 2x^3 - 3x^2 + x - 6$

$$f'(x) = 6x^2 - 6x + 1$$

$$f''(x) = 12x - 6$$

x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$
1	-6	1	6
1.3	-5.3	3.5	9.7
1.8	-2.3	9.6	15.6
1.99	-0.04	12.9	17.9
2	0	13	18

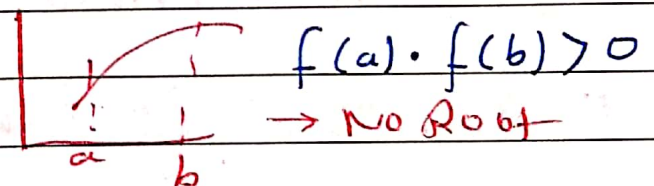
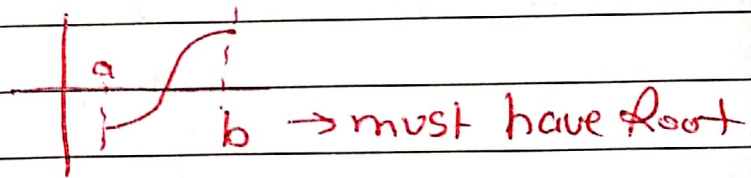
4) Bisection Method.

نقطة، نقطة وقوة

$$a \leq x \leq b$$

$$f(a) \cdot f(b) < 0$$

⊖ و ⊕



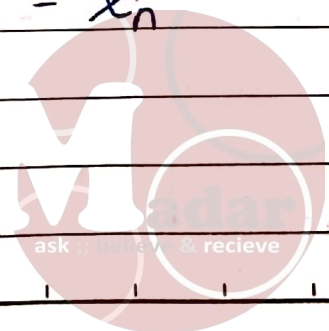
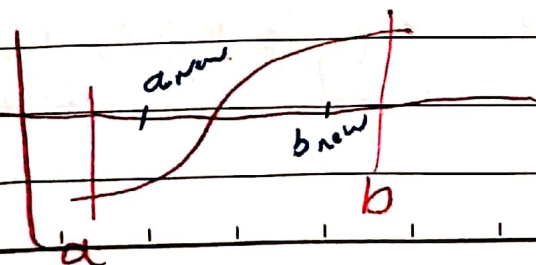
$$x_n = \frac{a+b}{2}, f(x_n) ??$$

if $f(a) \cdot f(x_n) > 0$

$$a_{\text{new}} = x_n$$

if $f(b) \cdot f(x_n) > 0$

$$b = x_n$$



error ↑
high gues

ex) $f(x) = x^2 + x - 5$

في افترقا $\left[\begin{array}{l} a=0 \\ b=3 \end{array} \right] \left[\begin{array}{l} f(a) = f(0) = -5 \\ f(b) = f(3) = 7 \end{array} \right] \rightarrow f(a) * f(b) < 0$
the assumption is valid

a	b	x_n	$f(a)$	$f(b)$	$f(x_n)$	chang.
0	3	1.5	-5	7	-1.25	$a = x_n$
1.5	3	2.25	-1.25	7	2.3	$b = x_n$
1.5	2.25	1.87	-1.25	2.3	0.39	$b = x_n$
1.5	1.87	1.68	-1.25	0.39	-0.46	$a = x_n$
1.68	1.87	1.79	-0.46	0.39	-0.04	$a = x_n$

stiffness  sluggish

5) Secant method

$$f'(x) = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}$$
 ↑ error ↑

From Newton-R Method

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \rightarrow x_n = x_{n-1} - f(x_{n-1}) \left[\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} \right]$$

⊗ لا يكون عندك أكثر من معادلة عندك نقطتان فقط، لكنك تحتاج إلى اثنين
two initial guesses

ex) $f(x) = 2x^3 - 3x^2 + x - 6$

$x_{n-1} = 3$

$x_{n-2} = 4$

⊗ في البداية يكون Root في بين 3 و 4
+ ∫ bisection Method.

$$b = \frac{f_1 - f_0 - a h_1^2}{h_1}$$

$$\delta = \frac{h_2}{h_1}$$

$$\Delta x = -2c$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$X = X_0 - 2f_0$$

$$b \pm \sqrt{b^2 - 4ac}$$

اکرم ما عکرم

صبر له خير اصغر ما عكس
صبر الامام

$+ve$ origin $\rightarrow +ve \leftarrow b \text{ is } \infty$
 $-ve$ origin $\rightarrow -ve \leftarrow b \text{ is } 1$

$$f(x) = \sin x - \frac{x}{2} \quad \text{--- (Rad)}$$

(Root = 1.89)

$$h = h_1 = h_2 = 0.2$$

for the first time we assume they are equal for simplicity but it's not obligated to assume that

$$x_0 = 2$$

$$x_1 = x_0 + h_1 = 2 + 0.2 = \boxed{2.2}$$

$$x_2 = x_0 - h = 2 - 0.2 = 1.8$$

three initial guesses.

~~$$f(x_0) = f_0 = \sin(2) - \frac{2}{2} = -0.0907$$~~

$$f(x_4) = f_1 = \sin(2.2) - \frac{2.2}{2} = -0.991$$

$$f(x_2) = f_2 = \sin(1.8) - \frac{1.8^2}{2} = 0.0738$$

$\uparrow x$ $\uparrow h$ $\odot *$

$\delta = 1$ for First trial

$$a = \frac{(1)(-0.991) - (-0.0907)(1+1) + 0.07}{(1)(0.2)^2(1+1)} = -0.453$$

$$b = \frac{(-0.991) - (-0.6907)(-0.453)}{0.2} (0.2)^2 = -0.913$$

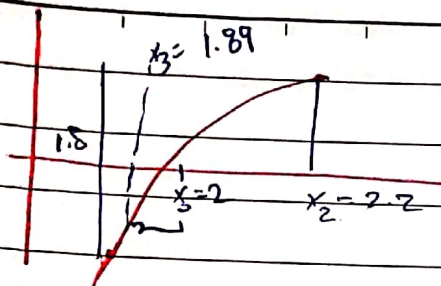
$$c = f(x_0) = f_0 = -0.907$$

the foot

$$X = 2 - 2(-0.0907)$$

$$= 1.89526$$

$$-0.918 \pm \sqrt{(-0.913)^2 - 4(10.453)(-10907)}$$



$$x_0 = 1.89526$$

$$f_0 = \sin(1.89526) - \frac{1.8927}{2}$$

$$= 1.918 \times 10^{-4}$$

$$x_1 = 2$$

$$h_1 = x - x_0 = 2 - 1.89526$$

$$x_2 = 1.8$$

$$h_2 = 0.10474$$

$$x_2 = 1.8$$

$$h_2 = x_0 - x_2 = 0.09526$$

$$\therefore \delta = \frac{h_2}{h_1} = \frac{0.09526}{0.0104} = 0.9094$$

$$c = f(x_0) = 1.918 \times 10^{-4}$$

$$f_1 = f(x_1) = f(2) = -0.0907$$

$$f_2 = f(x_2) = f(1.8) = 0.0738$$

$$a = -0.4728$$

$$b = -0.81820$$

$$X \approx 0 \text{ (Red noting value)}$$

$$X = 1.8954 \text{ exactly the root}$$



Solution of linear equation

New chapter
23th Jun - 2019

Solution linear equation

- 1) Gaussian elimination Method
- 2) Gaussian Jordan Method
- 3) Jacobi iterated Method
- 4) Gauss - sidel Method
- 5) L V Method

#1 Gaussian elimination Method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots = b_2$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots = b_n$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$\vec{X} = A^{-1}b$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

ex

$$2x_1 - x_2 + 3x_3 = 2$$

$$x_1 + 2x_2 - x_3 = 3$$

$$3x_1 - 2x_2 + x_3 = -1$$

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -2 & 1 & -1 \end{bmatrix}$$

diagonal ← استعملنا

اصرف بالرقم الي بدله تصفوه على اول

diagonal رقم

$$-\frac{1}{2} R_1 + R_2 = R_2$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ \rightarrow 3 & -2 & 1 & -1 \end{array}$$

$$-\frac{3}{2} + R_1 + R_3 = R_3$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ 0 & -0.5 & -3.5 & -4 \end{array}$$

$$\frac{0.5}{2.5} R_2 + R_3 = R_3$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ 0 & 0 & -4 & -3.6 \end{array}$$

OR $-\frac{0.5}{2.5} R_2 - R_3 = R_3$

Algorithm:

For $K=1:N$

For $J=1:N+1 \rightarrow$ Right hand column.

Read a_{kj}

END

END

For $k=1:N-1$

For $i=k+1:N$

For $j=k+1:N+1$

$$a_{ij} = a_{ij} - \frac{a_{ik} a_{kj}}{a_{kk}}$$

Now solve:

$$-4x_3 = -3.6$$

$$x_3 = 0.9$$

$$2.5x_2 - 2.5x_3 = 2$$

$$2.5x_2 - 2.5(0.9) = 2$$

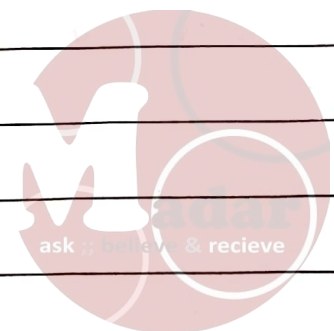
$$x_2 = 1.7$$

$$2x_1 - x_2 + x_3 = 2$$

$$2x_1 - \frac{4}{5} = 2$$

$$+\frac{4}{5} \quad +\frac{4}{5}$$

$$x_1 = 1/2$$



$$X_n = \frac{a_{n, n+1}}{a_{nn}}$$

for $k = n-1 : 1$

$$\text{sum} = 0$$

for $j = k+1 : N$

$$\text{sum} = \text{sum} + a_{kj} \cdot X_j$$

END

$$X_k = \frac{1}{a_{kk}} (a_{k, n+1} - \text{sum})$$

END k.

ON previous example :

$$a_{21} = -\frac{1}{2} \times 2 + 1 = 0$$

$$a_{31} = -\frac{3}{2} \times 2 + 3 = 0$$

$$a_{23} = -\frac{1}{2} \times 3 + -1 = -2.5$$

$$a_{32} = -\frac{3}{2} \times -1 - 2 = -0.5$$

$$a_{22} = -\frac{1}{2} \times -1 + 2 = 2.5$$

$$a_{33} = -\frac{3}{2} \times 3 + 1 = -3.5$$

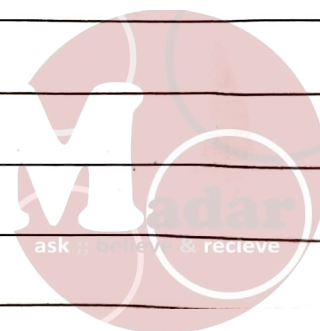
$$b_1 = 2$$

$$b_2 = -\frac{3}{2} \times 2 - 1 = -4$$

$$a_{32} = \frac{0.5}{2.5} \times 2.5 - 0.5 = 0$$

$$a_{33} = \frac{0.5}{2.5} \times -2.5 + (-3.5) = -4$$

$$b_3 = \frac{0.5}{2.5} \times 2 - 4 = -3.6$$



2) Gauss-Jordan Method.

For $k=1 : N-1$

$i = k+1 : N$

$j = N+1 : k_1-1$

$$a_{ij} = a_{ij} - \frac{a_{ik} a_{kj}}{a_{kk}}$$

END,

$$\begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

not necessary to be 0's
just to equal (1)

(ex) from previous example bring from it the eq's.

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -4 \\ 0 & 0 & 40 & 36 \end{array} \right]$$

Ⓢ ملاحظه:

$$\left[\begin{array}{l} 36 \\ 40 \end{array} \right] \leftarrow \begin{array}{l} 3.6 \\ 4 \end{array}$$

نہیں امیڑ ہوا 10 دکان عینہ کوڑا
سب او عوجہ جب عادی ای اختیار کرنا

① set diagonal = 1 (دکان کے ضروری اصول = 1)

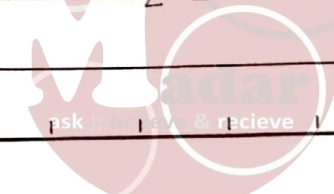
$$\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1 & 4/5 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$E_3 + E_2 = E_2$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1 & 4/5 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$\frac{1}{2} E_2 + E_1 = E_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 74/40 \\ 0 & 1 & 0 & 68/40 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$



$$-\frac{3}{2} E_3 + E_1 = E_1$$

Then solve

$$\begin{bmatrix} 1 & 0 & 0 & | & 29/40 \\ 0 & 1 & 0 & | & 68/40 \\ 0 & 0 & 1 & | & 36/40 \end{bmatrix}$$

$$x_1 = 29/40 = 1/2$$

$$x_2 = 68/40 = 17/10$$

$$x_3 = 36/40 = 9/10$$

APPLY THE CODE:

$$N = 3$$

$$K = 1 \rightarrow 2$$

$$i = 2 \rightarrow 3$$

$$J = 4 \rightarrow K$$

$$K = 1$$

$$i = 2$$

$$J = 4, 3, 2, 1$$

→ looping

$$a_{24} = a_{24} - (a_{21}/a_{11})(a_{14})$$

$$a_{23} = a_{23} - \left(\frac{a_{21}}{a_{11}}\right)(a_{13})$$

#3) Jacobi - iterative Method

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots = b_2$$

$$a_{n1} x_1 + a_{n2} x_2 + a_{n3} x_3 + \dots = b_n$$

④ ملاحظہ کریں کہ coeff درجہ اولیٰ و اولیٰ درجہ اولیٰ coeff درجہ اولیٰ و اولیٰ درجہ اولیٰ
by mistake fixed print
div

$$\prod_{i=1}^n |a_{ij}| = \max$$

$$\begin{vmatrix} a_{11} & a_{22} & a_{33} \end{vmatrix} = \max \text{imum}$$



Ex

$$2x_1 + 7x_2 - 8x_3 = 10$$

$$5x_1 - 2x_2 + 6x_3 = 6$$

$$3x_1 + 5x_2 - 7x_3 = 3$$

x_1 which largest value!!

re arrange

$$5x_1 - 2x_2 + 6x_3 = 6$$

$$2x_1 + 7x_2 - 8x_3 = 10$$

$$3x_1 + 5x_2 - 7x_3 = 3$$

$$5 * 7 * 7 = 245$$

$$- 5 * 5 + 8 = 200 \quad \text{NOT The Max}$$

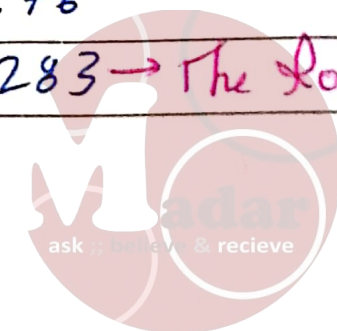
ل جربا ترتيب عز هين باح بعدين Max الجواب

$$x_1 = \frac{6 + 2x_2 - 6x_3}{5}$$

$$x_2 = \frac{10 - 2x_1 + 8x_3}{7}$$

$$x_3 = \frac{3 - 3x_1 - 5x_2}{7}$$

	#1	#2	#3	#4	...	#40
x_1	0	6/5	2.28	0.11		-0.305
x_2	0	10/7	0.545	2.04		4.76
x_3	0	-3/7	1.106	0.98		2.283 → The Root



4 : Gauss's Sidel Method

	#1	#2	#3	#22
x_1	0	6/5	0.6	6.306
x_2	0	$10 - 2(6/5) + 0 = 1.08$	2.24	4.76
x_3	0	$(3 - 3(6/5) - 5(1.08)) / -7 = 0.86$	1.42	2.84

لما أتت أحباؤها بوخذ (6/5 = x_1) مش
(5) دعي بوخذ القيمة الجديدة

~~Develop ment / off / kind / V / nothing /~~

(5) lower - Upper limit Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \dots & \cancel{a_{1n}} \\ a_{21} & \cancel{a_{22}} & \dots & \cancel{a_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & \dots & \cancel{a_{nn}} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \cancel{a_{22}} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \dots & \cancel{a_{nn}} \end{bmatrix} \vec{x} = b$$

diagonal = 1

$$U \quad \vec{\alpha} = \vec{b}$$

$$L^{-1} / \checkmark \quad \vec{y} = L^{-1} \vec{b} \Rightarrow \vec{y} = L^{-1} \vec{b} \rightarrow U \vec{x} = \vec{y} \\ \vec{x} = U^{-1} \vec{y}$$

Development of L & U Matrix

Elements of lower Matrix !!

$$l_{ij} = \frac{a_{ij}}{a_{ii}}$$

$$L = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & & 1 & \\ \vdots & & & \ddots \end{bmatrix}$$

$$U_{ij\text{new}} = U_{ij\text{old}} - l_{ij} a_{ii}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ & u_{22} & \dots & u_{2n} \\ & & \ddots & \\ & & & u_{nn} \end{bmatrix}$$

(Ex)

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$\Rightarrow LU \Rightarrow$

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$l_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$a_{21} = a_{21} - l_{21} a_{11} = 64 - (2.56)(25) = 0$$

$$l_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

$$a_{22} = a_{22} - l_{21} a_{12} = 8 - (2.56)(5) = -4.8$$

$$l_{32} = \frac{a_{32}}{a_{22}} = ??$$

$$a_{23} = a_{23} - l_{21} a_{13} = 1 - (2.56)(1) = -1.56$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{bmatrix}$$

upper !!

$$l_{31} = 5.76 \text{ so } a_{31} = a_{31} - l_{31} a_{11}$$

$$a_{31} = 144 - (5.76)(25) = 0$$

$$a_{32} = 12 - (5.76)(5) = 16.8$$

$$a_{33} = 1 - (5.76)(1) = -4.76$$

new one !!

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.75 & -1.56 \\ 0 & 16.8 & -4.76 \end{bmatrix}$$

$\Rightarrow$

$$l_{32} = \frac{a_{32}}{a_{22}} = \frac{16.8}{-4.75} = 3.53$$

$$a_{32} = a_{32} - l_{32} a_{22} = (16.8) - (3.53)(-4.75) = 0$$

$$a_{33} = a_{33} - l_{32} a_{23} = -4.76 - (3.53)(-1.56) = 0.7$$

$$-4.76 - (3.53)(-1.56) = 0.7$$

$$\begin{array}{c}
 U \qquad \qquad \qquad L \qquad \qquad \qquad = A \\
 \left[\begin{array}{ccc} 25 & 5 & 1 \\ 0 & -4.75 & 1.56 \\ 0 & 0 & 0.7 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.53 & 1 \end{array} \right]
 \end{array}$$

$$Ax = b \rightarrow L U X = b \rightarrow \boxed{L y = b}$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.53 & 1 \end{array} \right] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

straight forward!!

$$y_1 = 1$$

$$2.56 y_1 + y_2 = 2 \Rightarrow y_2 = -0.56$$

$$5.76 y_1 + 3.53 y_2 + y_3 = 3 \Rightarrow y_3 = -0.78$$

$$\boxed{L U X = b} \rightarrow \boxed{L y = b} \rightarrow \boxed{U X = y}$$

$$\left[\begin{array}{ccc} 25 & 5 & 1 \\ 0 & -4.75 & 1.56 \\ 0 & 0 & 0.7 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.56 \\ -0.78 \end{bmatrix}$$

backward calculation

$$0.7 * x_3 = -0.78 \rightarrow x_3 = -1.11$$

$$x_2 = -0.48$$

$$x_1 = -0.0116$$

$$X = \begin{bmatrix} -1.11 \\ 0.48 \\ -0.6116 \end{bmatrix}$$

Algorithm !!

Read A, b

set $U = A$

$L = I \rightarrow [n, n]$

for $k = 1, n-1$

for $j = k+1, n$

$$l_{jk} = \frac{U_{jk}}{U_{kk}}$$

$$U_{j,k:n} = U_{j,k:n} - l_{jk} U_{k,k:n}$$

$U_{k,k:n}$ no. of variable

END

مثال من الدكتور

$$A = \begin{bmatrix} 5 & 5 & 1 & 4 & 0 & 1 & 4 & 1 & 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$[n, n] = \text{size}(A)$$

$$L = \text{eye}(n)$$

$$U = A$$

For $k = 1:n-1$

For $j = k+1:n$

$$L(j, k) = U(j, k) / U(k, k)$$

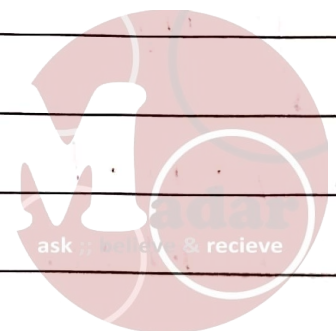
$$U(j, :) = U(j, :) - L(j, k) * U(k, :)$$

END

END

$$y = \text{Inv}(L) * b$$

$$x = \text{Inv}(U) * y$$



Numerical solution of system of non-linear eq.

1. Iterative method
2. Newton's Raphson method.

① Iterative Method

Equation $\rightarrow$ Rearrangement

(ex)

$$2x_1^2 + x_2 = 3 \rightarrow x_1 = \left(\frac{3-x_2}{2}\right)^{1/2}$$

$$x_1 + x_2^3 = 2 \rightarrow x_2 = (2-x_1)^{1/3}$$

x_1	x_2
0	1.26
0.93	1.02
0.994	1.001
0.999	1.0001

First trail: $x_2 = 1.26$ when $x_1 = 0 \dots$ so on.

(ex): ~~$x_1^3 + x_2 = 2$~~

$$2x_1^2 + x_2 = 3$$

$$x_1^3 + x_2 = 2$$

← have higher order degree x_1 ←

Ⓐ $x_2 = 3 - 2x_1^2$

$$x_1 = (2 - x_2)^{1/3}$$

conv $\rightarrow$ odd $(\frac{1}{3})$

x_1	x_2
0	3
-1	1
1	1

x_2 مبدئي x_1 ← initial guess
 x_1 مبدئي x_2 ←

Ⓐ x_1 مبدئي x_2 ←
 Root 1

Ⓑ $x_1 = \left(\frac{3-x_2}{2}\right)^{1/2}$

$$x_2 = 2 - x_1^3$$

x_1	x_2
0	2
0.7	1.65
0.82	1.445
0.88	1.32
0.91	1.23

$\text{div} \leftarrow (-) + (\frac{1}{2})^{\text{initial guess}}$ (*)
 jier, $\rightarrow$ imaginary
 add. o'ed
 (*) انبء لا initial
 guess
 Zero ال عرال $\leftarrow$ (0) $\leftarrow$ Zero
 img. al div $\leftarrow$

odd, $\rightarrow$ imaginary id

① انیسو لہ initial guess

مثلاً اتي عن ال zero ← مثلاً ← (0) (نظروا
عنا div لا div. imag.

Newton-Raphson Method

$$f(x, y) = 0 = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \frac{\Delta x}{1!} + \frac{\partial f}{\partial y}(x_0, y_0) \frac{\Delta y}{1!} +$$

$$g(x, y) = 0 = g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0) \frac{\Delta x}{I} + \frac{\partial g}{\partial y}(x_0, y_0) \frac{\Delta y}{II}$$

let $f_x = \frac{\partial f}{\partial x}$ $f_y = \frac{\partial f}{\partial y}$

$$g_x = \frac{\partial g}{\partial x} \quad g_y = \frac{\partial g}{\partial y}$$

$$\begin{aligned} f(x, y) &= f_0 + f_x \Delta x + f_y \Delta y = 0 \\ g(x, y) &= g_0 + g_x \Delta x + g_y \Delta y = 0 \end{aligned}$$

unknown!

$$f_x \Delta x + f_y \Delta y = -f_0$$

~~$$g_x \Delta x + g_y \Delta y = -J_0$$~~

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -f_0 \\ -g_0 \end{bmatrix}$$

$$\Delta x = x - x_0 \longrightarrow x = x_0 + \Delta x$$

$$\Delta y = y - y_0 \rightarrow y = y_0 + \Delta y$$

(ex) Solve: $f_1 = 3x_1 - \cos(x_2 x_3) - \frac{1}{2}$
 $f_2 = x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06$
 $f_3 = e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3}$

by Newton Raphson Method

start with $x_1 = 0.1$ $x_2 = 0.1$ $x_3 = -0.1$

$$\begin{bmatrix} f_{1x_1} & f_{1x_2} & f_{1x_3} \\ f_{2x_1} & f_{2x_2} & f_{2x_3} \\ f_{3x_1} & f_{3x_2} & f_{3x_3} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2 \\ -f_3 \end{bmatrix}$$

$$f_{1x_1} = 3 \quad f_{1x_2} = x_3 \sin(x_2 x_3) \quad f_{1x_3} = x_2 \sin(x_2 x_3)$$

$$f_{2x_1} = 2x_1 \quad f_{2x_2} = -162(x_2 + 0.1) \quad f_{2x_3} = \cos(x_3)$$

$$f_{3x_1} = -x_2 e^{-x_1 x_2} \quad f_{3x_2} = -x_1 e^{-x_1 x_2} \quad f_{3x_3} = 20$$

$$\begin{bmatrix} 3 & 0.001 & -0.001 \\ 0.2 & -32.4 & 0.995 \\ -0.099 & -0.1 & 20 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} -1.2 \\ -2.27 \\ 8.46 \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -0.08 \\ -0.42 \end{bmatrix}$$

2nd iteration

$x_{1, \text{new}}$	$x_{1, \text{old}}$	Δx_1	0.1	0.5
$x_{2, \text{new}}$	$x_{2, \text{old}}$	Δx_2	0.1	$+0.02$
$x_{3, \text{new}}$	$x_{3, \text{old}}$	Δx_3	-0.42	-0.52

$$\epsilon = 0.001$$

i	x_1	x_2	x_3	ϵ
0	0.1	0.1	-0.1	—
1	0.5	0.02	-0.52	0.9
2	0.5	0.0015	-0.523	0.018
3	0.5	0.000012	-0.523	0.0015
4	0.5	0	-0.523	1.2×10^{-5}

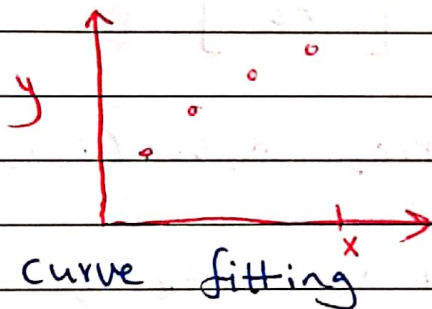
مقدار، کوچک (error)
 خطا کم، نه زیاد
 $\epsilon = 0.001$
 خطا 8 و 10 و 15
 خطا کم

$$err = \sum_{i=1}^n (x_{i\text{ new}} - x_{i\text{ old}})^2$$

$$err = (x_{1N} - x_{1old})^2 + (x_{2N} - x_{2old})^2 + (x_{3N} - x_{3old})^2$$

Curve fitting & interpolation

ساختن



curve fitting

1- Linear Regression $\Rightarrow$ fitting

$$y = a + bx$$

2- Multi linear Regression

$$y = a + bx_1 + cx_2$$

3- Non-linear Regression

$$y = a + bx + cx^2$$

4- Multi Non-linear Regression

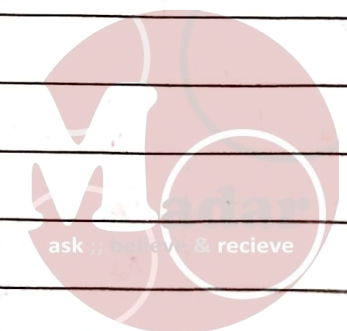
$$y = a + bx_1 + cx_1^2 + \dots + dx_2 + ex_2^2 + \dots$$

More general Regression It's including all the above Regression It's so powerfull

Linear Regression

X y_{exp} $y_{calculated}$ err

$$err = \min$$



$$\text{err} = \sum_{i=1}^n (y_{\text{exp}} - y_{\text{calc}})^2 = \sum (y_{\text{exp}} - (a + bx))^2$$

here, the best values for a, b should be; when err is the minimum

$$\frac{\partial \text{err}}{\partial a} = 0 \rightarrow 2 \sum (y_{\text{exp}} - (a + bx)) (-1) = -2 \left[\sum y_{\text{exp}} - (a + bx) \right]$$

$$\frac{\partial \text{err}}{\partial b} = 0 \rightarrow 2 \sum (y_{\text{exp}} - a - bx) (-x) = -2 \sum (x y_{\text{exp}} - a x - b x^2) = 0$$

$$\sum y_{\text{exp}} = \sum_{i=1}^n a + \sum b x$$

$$\sum x y_{\text{exp}} = \sum a x + \sum b x^2$$

$$\sum y_{\text{exp}} = na + b \sum x$$

$$\sum x y_{\text{exp}} = a \sum x + b \sum x^2$$

$$\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_{\text{exp}} \\ \sum x y_{\text{exp}} \end{bmatrix} \Rightarrow \text{simple linear eq. with two known}$$

$$a = \frac{\sum y_{\text{exp}} - b \sum x}{n}$$

$$b = \frac{n \sum y_i x_i - \sum y_i \sum x_i}{n \sum x_i^2 - (\sum x_i)^2}$$

Ex

n	x_i	y_{exp}	x_i^2	xy	y_{cal}	err^2 $(y - y_{\text{cal}})^2$	R^2 & L.V. $(y_i - y_{\text{avg}})^2$
1	1	4	1	4	3.3	$(4 - 3.3)^2 = 0.49$	$(4 - 6.75) = 7.5$
2	2	6	4	12	5.6	0.16	0.56
3	3	5	9	15	7.9	8.41	27.56
4	4	12	16	48	10.2	3.24	38.74
$\sum$	10	27	30	79		12.3	

Minimum value of err

↓ err ↑ fit → not indicated for the fit

$$b = \frac{4(79) - (27)(10)}{4(30) - (10)^2} = 2.3 \quad a = \frac{27 - (2.3)(10)}{4} = 1$$

$$y_{\text{calc}} = a + bx \rightarrow y_{\text{calc}} = 1 + 2.3x$$

Regression coefficient
coefficient of determination

$$R^2 = \frac{\sum (y_{\text{exp}} - y_{\text{avg}})^2 - \sum (y_{\text{exp}} - y_{\text{calc}})^2}{\sum (y_{\text{exp}} - y_{\text{avg}})^2}$$

$$0 < R^2 < 1 \rightarrow [0 - 1] \text{ best fit}$$

$$R^2 > 0.99 \quad 0.95 \quad 0.97 \rightarrow \text{accuracy}$$

$$\text{ave value} \rightarrow y \rightarrow y_{\text{avg}} = \frac{\sum y_{\text{exp}}}{n} = \frac{27}{4} = 6.75$$

یہاں پر y_{avg} کا استعمال ہے

$$R^2 = \frac{38.74 - 12.3}{38.74} = 0.68 \rightarrow \text{how accurate our fitting}$$

Multi linear Regression

$$y = a + bx + cx_2 + dx_3 + \dots$$

previously for a single Regression linear we do have this

$$\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \end{bmatrix}$$

$$y = a_0 + a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

$$\begin{bmatrix} n & \sum x_1 & \sum x_2 & \dots & \sum x_n \\ \sum x_1 & \sum x_1^2 & \sum x_1 x_2 & \dots & \sum x_1 x_n \\ \sum x_2 & \sum x_1 x_2 & \sum x_2^2 & \dots & \sum x_2 x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum x_n & \sum x_1 x_n & \sum x_2 x_n & \dots & \sum x_n^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum x_1 y \\ \sum x_2 y \\ \vdots \\ \sum x_n y \end{bmatrix}$$

(ex) Given That

n	x_1	x_2	y	x_1^2	$x_1 x_2$	x_2^2	$x_1 y$	$x_2 y$
1	0	0	3	0	0	0	0	0
2	1	3	1	1	3	9	1	3
3	2	2	7	4	4	4	14	14
4	3	5	5	9	15	25	15	25
5	4	7	5	16	28	49	20	35
6	5	1	12	25	5	1	105	21
	15	18	42	55	55	88	155	98

↑ values to fit ↑ ind var

$$\begin{bmatrix} 6 & 15 & 18 \\ 15 & 55 & 55 \\ 18 & 55 & 88 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 42 \\ 155 \\ 98 \end{bmatrix} \rightarrow \text{system of the linear eq.}$$

$$y = 3 + 4x_1 - 2x_2$$

Non-linear regression

1. $y = ax^b \rightarrow \text{take a log} \rightarrow \ln y = \ln a + b \ln x$

$\underbrace{\quad}_D \quad \underbrace{\quad}_Z \quad \underbrace{\quad}_C \quad \underbrace{\quad}_D$

x	y	$\ln x$	$\ln y$
---	---	---------	---------

$\Rightarrow b \checkmark$

$\Rightarrow c = \ln a \rightarrow a = e^c$

2. $y = a e^{bx} \rightarrow \text{Take } \ln \rightarrow \ln y = \ln a + bx$

x	y	$\ln y$
$\vdots$	$\vdots$	$\vdots$

$$\ln a = \checkmark \quad b = \checkmark \quad a = e^{\text{value}}$$

3. $y = a x^b e^{cx} \rightarrow \text{take a log}$

$$\ln y = \ln a + b \ln x + cx = \ln a$$

Polynomial Regression !!

$$y = a_1 + a_2 x + a_3 x^2 + \dots$$

$$\text{err} = \sum (y - y_e)^2$$

$$\frac{\partial \text{err}}{\partial a_1} = \frac{\partial}{\partial a_1} \sum (a_1 + a_2 x + a_3 x^2 - y_e)^2$$

$$\frac{\partial \text{err}}{\partial a_2} = \frac{\partial}{\partial a_2} \sum ($$

المصنوفة الهاتية الى 2 - تنقل الى

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

n	$\sum x$	$\sum x^2$	$\sum x^3$	$\dots \sum x^n$	a_0		$\sum y$
$\sum x$	$\sum x^2$	$\sum x^3$	$\sum x^4$	$\sum x^{n+1}$	a_1		$\sum xy$
$\sum x^2$	$\sum x^3$	$\sum x^4$	$\sum x^5$	$\sum x^{n+2}$	a_2	=	$\sum x^2 y$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\sum x^n$	$\sum x^{n+1}$	$\sum x^{n+2}$	$\sum x^{n+3}$	$\sum x^{2n}$	a_n		$\sum x^n y$

n	$\sum x$	$\sum x^2$	x	y	x^2	x^3	x^4	xy	$x^2 y$
$\sum x$	$\sum x^2$	$\sum x^3$							
$\sum x^2$	$\sum x^3$	$\sum x^4$							
			$\sum$	$\sum$	$\sum$	$\sum$	$\sum$	$\sum$	$\sum$

$$y = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + b_1 x_2 + b_2 x_2^2 + b_3 x_2^3 +$$

$$c_1 x_1 x_2 + c_2 x_1 x_2^2 + c_3 x_1^2 x_2$$

$$y_1 = f(x_1, x_2)$$

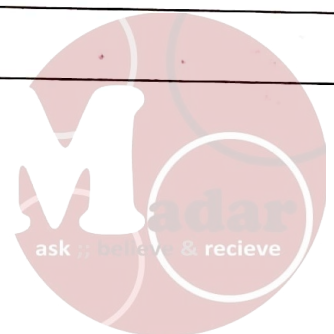
n	$\sum x_1$	$\sum x_1^2$	$\sum x_1^3$	$\sum x_2$	$\sum x_2^2$	$\sum x_2^3$	$\sum x_1 x_2$	$\sum x_1^2 x_2$	$\sum x_1 x_2^2$	element	قب
$\sum x_1$	$\sum x_1^2$	$\sum x_1^3$	$\sum x_1^4$	$\sum x_1 x_2$	$\sum x_1^2 x_2$	$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2$	$\sum x_1 x_2^3$	a_0	$\sum y$
$\sum x_1^2$	$\sum x_1^3$	$\sum x_1^4$	$\sum x_1^5$	$\sum x_1^2 x_2$	$\sum x_1^3 x_2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2^2$	$\sum x_1^4 x_2$	$\sum x_1^2 x_2^3$	a_1	$\sum x_1 y$
$\sum x_1^3$	$\sum x_1^4$	$\sum x_1^5 x_2$	$\sum x_1^6$	$\sum x_1^3 x_2$	$\sum x_1^4 x_2$	$\sum x_1^3 x_2^2$	$\sum x_1^4 x_2^2$	$\sum x_1^5 x_2$	$\sum x_1^3 x_2^3$	a_2	$\sum x_1^2 y$
$\sum x_2$	$\sum x_1 x_2$	$\sum x_1^2 x_2$	$\sum x_1^3 x_2$	$\sum x_2^2$	$\sum x_2^3$	$\sum x_2^4$	$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^2 x_2^3$	a_3	$\sum x_1^3 y$
$\sum x_2^2$	$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2^2$	$\sum x_2^3$	$\sum x_2^4$	$\sum x_2^5$	$\sum x_1 x_2^3$	$\sum x_1^2 x_2^3$	$\sum x_1^2 x_2^4$	b_1	$\sum x_2 y$
$\sum x_2^3$	$\sum x_1 x_2^3$	$\sum x_1^2 x_2^3$	$\sum x_1^3 x_2^3$	$\sum x_2^4$	$\sum x_2^5$	$\sum x_2^6$	$\sum x_1 x_2^4$	$\sum x_1^2 x_2^4$	$\sum x_1^2 x_2^5$	b_2	$\sum x_2^2 y$
$\sum x_1 x_2$	$\sum x_1^2 x_2$	$\sum x_1^3 x_2$	$\sum x_1^4 x_2$	$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2^2$	$\sum x_1^2 x_2^3$	$\sum x_1^3 x_2^3$	$\sum x_1^3 x_2^4$	b_3	$\sum x_2^3 y$
$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2^2$	$\sum x_1^4 x_2^2$	$\sum x_1^2 x_2^3$	$\sum x_1^3 x_2^3$	$\sum x_1^4 x_2^3$	$\sum x_1^2 x_2^4$	$\sum x_1^3 x_2^4$	$\sum x_1^3 x_2^5$	c_1	$\sum x_1 x_2 y$
$\sum x_1^2 x_2$	$\sum x_1^3 x_2$	$\sum x_1^4 x_2$	$\sum x_1^5 x_2$	$\sum x_1^2 x_2^4$	$\sum x_1^3 x_2^4$	$\sum x_1^4 x_2^4$	$\sum x_1^2 x_2^5$	$\sum x_1^3 x_2^5$	$\sum x_1^3 x_2^6$	c_2	$\sum x_1 x_2^2 y$
										c_3	$\sum x_1^2 x_2 y$

المعاجيب

$$y = 1 + 2x_1 + 3x_1^2 + 4x_1^3 + 5x_2 + 6x_2^2 + 7x_2^3 + 8x_1x_2 + 9x_1x_2^2 + 10x_1^2x_2$$

x_1	x_2	y	$x_1^2 x_2$
0	0		
↓	↓		

0	10
1	6
↓	↓
1	
	10



Interpolation

→ one variable

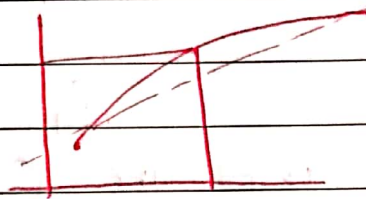
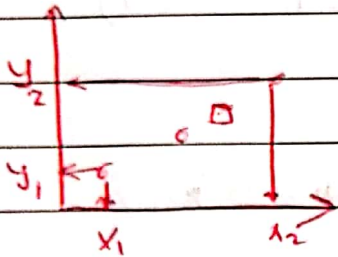
→ two variable

lagrange interpolation

Newton's interpolation

spline

langrage interpolation :



$$F(x) = a_1 + a_2 x + a_3 x^2 + \dots + a_n x^n$$

$$F_p(x) = \sum_{j=0}^n L_j(x) f(x_j)$$

$$L_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^n \frac{x - x_k}{x_j - x_k}$$

← j=0 ←

the order of the degree excluded → because the condition

Ex

$$f(x) = a + bx$$

$$f_1(x) = \sum_{j=0}^1 L_j(x) f(x_j)$$

$$= L_0(x) f(x_0) + L_1 f(x_1)$$

What is L_0 ?

$$L_0(x) = \prod_{\substack{k=0 \\ k \neq j}}^1 \frac{x - x_k}{x_j - x_k} = \frac{x - x_1}{x_0 - x_1}$$

the order of the degree excluded → because the condition

$$L_1(x) = \prod_{\substack{k=0 \\ k \neq j}}^1 \frac{x - x_k}{x_j - x_k} = \frac{x - x_0}{x_1 - x_0}$$

↓
= L

$$f_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

Ex Find $f_1(3)$, find the value $f(3)$ based on the first order langrage interpolation

x	y
$x_0 \leftarrow 2$	1.5 $\rightarrow f_0(x) = y_0$
$x_1 \leftarrow 5$	4 $\rightarrow f_1(3)$
8	6.5
11	9

$$f_1(x) = \frac{x-5}{2-5} f(2) + \frac{x-2}{5-2} f(5)$$

$$f_1(x) = \frac{x-5}{-3} (1.5) + \frac{x-2}{3} (4)$$

$$f_1(3) = \frac{3-5}{-3} (1.5) + \frac{3-2}{3} (4)$$

$$f_1(3) = 2.33$$

To check $\rightarrow$ use the (التحقق من الإجابة)

$$\frac{f(3) - 1.5}{4 - 1.5} = \frac{x - 2}{3 - 2} \quad (\text{check}) \quad \text{so;}$$

$$f(3) = \frac{3-2}{3} (4-1.5) + 1.5 = 2.33$$

we notice it's the same answer coz; it's linear

Quadratic Interpolation second order!!

$$f_2(x) = \sum_{j=0}^2 L_j(x) f(x_j) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

First $\rightarrow$ term of relation coeff const
 $\rightarrow$ term of n coffe of x

second $\rightarrow$ term of x^2

x	y
x_0	"
x_1	"
x_2	"

} interpolation

معدل (2) و معدل (2)
 قيمتين !!

order 2

$$L_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^2 \frac{x - x_k}{x_j - x_k} = \left(\frac{x - x_0}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right) = L_0$$

$$L_1(x) = \prod_{\substack{k=0 \\ k \neq 1}}^2 \frac{x - x_k}{x_1 - x_k} = \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right)$$

(excluded)

$$L_2(x) = \prod_{\substack{k=0 \\ k \neq 2}}^2 \frac{x - x_k}{x_2 - x_k} = \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right)$$

(c) Find $f_2(2.5)$

x	y		$L_0(x) = \left(\frac{x-3}{2-3} \right) \left(\frac{x-4}{2-4} \right)$
1	0		
2	1	$\rightarrow x_0$	$L_1(x) = \left(\frac{x-2}{3-2} \right) \left(\frac{x-4}{3-4} \right)$
3	2	$\rightarrow x_1$	$L_2(x) = \left(\frac{x-2}{4-2} \right) \left(\frac{x-3}{4-3} \right)$
4	3	$\rightarrow x_2$	
5	5		
6	7		

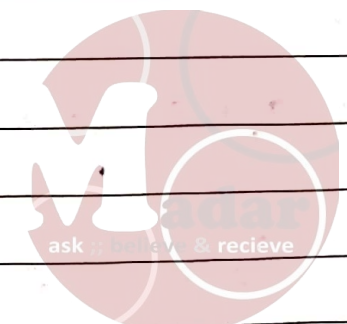
Substitute !! $L_0 = \frac{1}{2} (x-3)(x-4)$

$$L_1 = - (x-2)(x-4)$$

$$L_2 = \frac{1}{2} (x-2)(x-3)$$

$$F_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$F_2(2.5) = \frac{1}{2} (2.5-3)(2.5-4)(1) + - (2.5-2)(2.5-4)(2) + \frac{1}{2} (2.5-2)(2.5-3)(3)$$



Numerical Interpolation

2. Newton's interpolation

1st order

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) \\ &= f(x_0) + f'(x_0)(x - x_0) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \end{aligned}$$

x	f(x)
-	-
-	- $\rightarrow x_0$
-	- $\rightarrow x$
-	- $\rightarrow x_1$

2nd order $\rightarrow$ 2nd derivative

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &= f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)(x - x_1) \\ &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] (x - x_0)(x - x_1) \end{aligned}$$

$$\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] (x - x_0)(x - x_1)$$

Higher order:

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \\ &= a_0 + \sum_{i=1}^n \prod_{j=0}^{i-1} (x - x_j) a_{i+1} \end{aligned}$$

$\rightarrow$ simple case $\leftarrow$

(Ex)

i	x	y
0	0	1
1	1	2.25
2	2	3.75
3	3	4.25
4	4	5.81

Find $f(2.4)$ using 3rd order method !!

$$F_3(2.4) = \sum_{k=0}^3 a_k \prod_{j=0}^{k-1} (x - x_j)$$

$$= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$= f(x_0) + \overset{f'(x_0)}{f(x_0, x_1)}(x - x_0) + f(x_0, x_1, x_2)(x - x_0)(x - x_1) + f(x_0, x_1, x_2, x_3)(x - x_0)(x - x_1)(x - x_2)$$

$$a_0 = f(x_0) = 1$$

$$a_1 = f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{2.25 - 1}{2 - 0} = 1.5$$

$$a_2 = f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{4.25 - 2.25}{3 - 1} = 1$$

$$f(x_0, x_1, x_2) = \frac{1 - 1.5}{3 - 0} = -0.5$$

$$a_3 = f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{5.81 - 4.25}{4 - 3} = 1.56$$

$$f(x_1, x_2, x_3) = \frac{1.56 - 1}{4 - 2} = 0.53$$

$$a_3 = \frac{0.53 - (-0.5)}{4-1} = 0.343$$

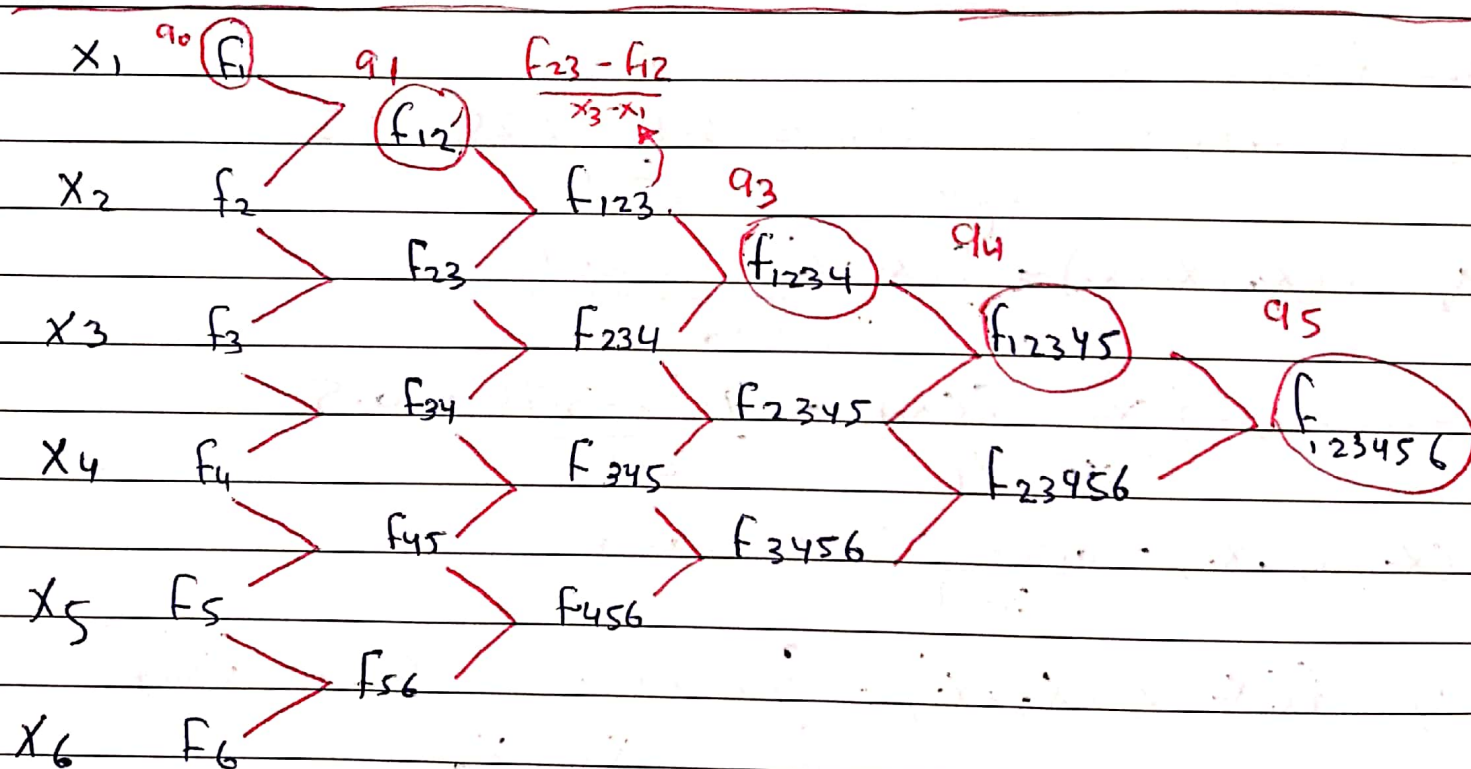
$$a_2 = -0.5 \quad a_1 = 1.5 \quad a_0 = 2.25$$

$$\begin{aligned} f(x) &= 2.25 + 1.5(x-x_0)(x-x_1) + 0.34(x-x_0)(x-x_1)(x-x_2) \\ &= 2.25 + 1.5(2.4-1) - 0.5(2.4-1)(2.4-2) + 0.34(2.4-1) \\ &\quad (2.4-2)(2.4-3) = 3.95 \end{aligned}$$

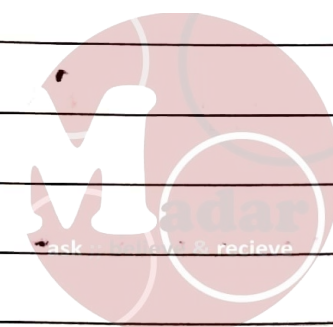
Newton Divided Difference

Table

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(5)}(x)$
		d_1	d_2	d_3	d_4	d_5



↓
no. of data = no. of order



Newton's Divided Difference

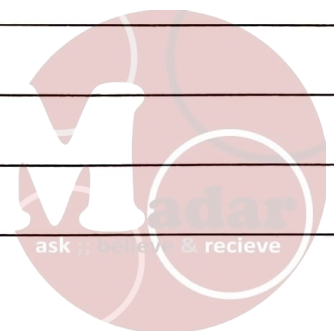
(EX)

X	Y	d ₁	d ₂	d ₃	d ₄
0	1	1.25	0.125	-0.208	0.137
1	2.25	1.5	-0.5	0.31	
2	3.75	0.55	0.33		
3	4.25	1.56			
4	5.81				

$$f(2.4) = 1 + 1.25(2.4-0) + 0.125(2.4-0)(2.4-1) - 0.208(2.4-0)(2.4-1)(2.4-2) + 0.137(2.4-0)(2.4-1)(2.4-2)(2.4-3) = 4.03$$

→ more accurate

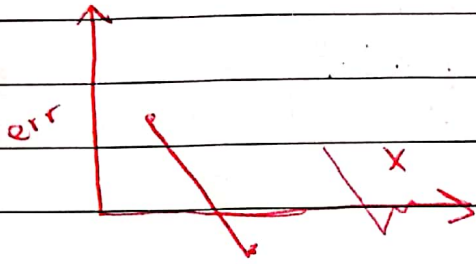
بہتر اور زیادہ دقیق نتیجہ دیتا ہے *



Solution of non-linear equation

Sol of single equation

$$x^2 + e^x = 3 \rightarrow x = \pm \sqrt{3 - e^x}$$
$$x^2 e^x + \ln x \cdot x^3 = \dots \rightarrow \dots$$



- (a) fixed point iterative method
- (b) Bisection method
- (c) Newton-Rapison method
- (d) Newton - 2nd order method

Secant Method :

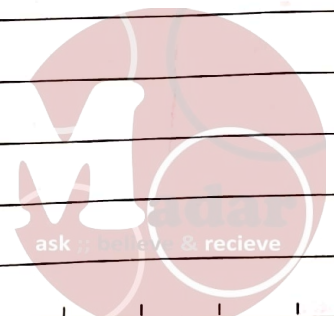
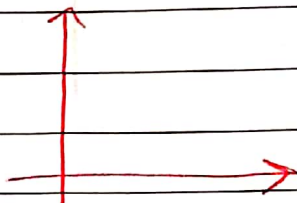
(f) Huler method

Fixed point method ?

$f(x) = 0 \rightarrow x = ?$ $f(x) = 0$ are the Roots

ex - $f(x) = e^{-x}$ $\rightarrow$ what is the value of x where $f(x) = 0$!?

x	e^{-x}
0	1
-1	$e = 2.7$
1	$\frac{1}{2.7}$

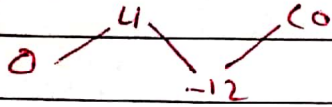


Fixed point iterated Method!

Ex: $f(x) = x^3 + x^2 - 3x - 3$

P.S: conv $\rightarrow$ Roots exist

div $\rightarrow$ Root do not exist



$$x = g(x)$$

$$x \pm g(x) = f(x) = 0 \Rightarrow x = g(x)$$

① $x^3 + x^2 - 3x - 3 = 0$

$$x = \frac{x^3 + x^2 - 3}{3}$$

x_{old}

x_{new}

1

$-\frac{1}{3}$

$-\frac{1}{3}$

-0.975

-0.975

-0.992 $\rightarrow$ Reaching one Root

② $x = \sqrt[3]{3x + 3 - x^2} = (3x + 3 - x^2)^{1/3}$

x_{old}

x_{new}

1

1.708

1.708

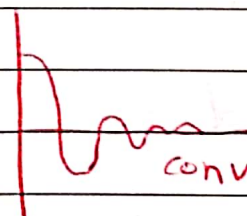
1.73

$\rightarrow$ err = 0.4
 $\rightarrow$ er = 0.2

1.73

1.732

$\rightarrow$ er = 0.1



③ $x^2 = 3x + 3 - x^3$

$$x = \sqrt[2]{3x + 3 - x^3} = (3x + 3 - x^3)^{1/2}$$

x_{old}

x_{new}

1

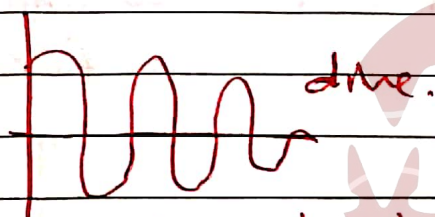
2.23

2.23

6.32

6.32

imaginary Root



conv use old as initial

list d @ recieve

⊛ to find the proper root

proper initial guess

!!

for fixed method iterate key point ; high order in side and the other in another side

highest order = other orders f_{xn}

⊛ [highest = other → initial guess] → can the Right

Newton's Method

Root, its speed ⊛
more powerfull ⊛

$$f(x) = f(x_0) + f'(x_0) \frac{\Delta x}{1!} + \frac{f''(x_0) (\Delta x)^2}{2!} + \dots$$

← err →
↓
Jap uplo uplo

$$f(x_0) + f'(x_0) \Delta x = 0$$

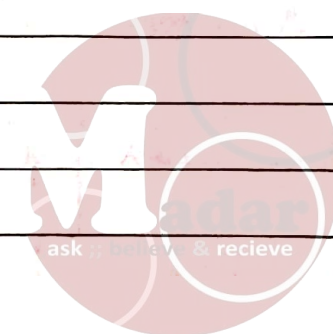
$$\Delta x = - \frac{f(x_0)}{f'(x_0)}$$

$$x_{0 \text{ New}} - x_{0 \text{ old}} = - \frac{f(x_0)}{f'(x_0)}$$

$$x_{0 \text{ New}} = x_{0 \text{ old}} - \frac{f(x_0)}{f'(x_0)}$$

⊛ (Ex) $f(x) = x^3 + x^2 - 3x - 3$
 $f'(x) = 3x^2 + 2x - 3$

$$x_{i+1} = x_i - \frac{x_i^3 + x_i^2 - 3x_i - 3}{3x_i^2 + 2x_i - 3}$$

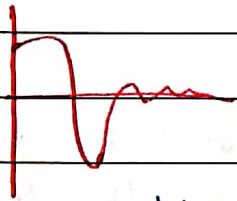


x_i	x_{i+1}	x_i	x_{i+1}	x_i	x_{i+1}
0.1	-1.08	2	1.769	-2	-1.8
-1.08	-0.989	1.769	1.732	-1.8	-1.738
-0.989	-0.999	1.732	1.732	-1.738	-1.72
-0.999	-1			-1.72	-1.72
-1	-1 → conv				

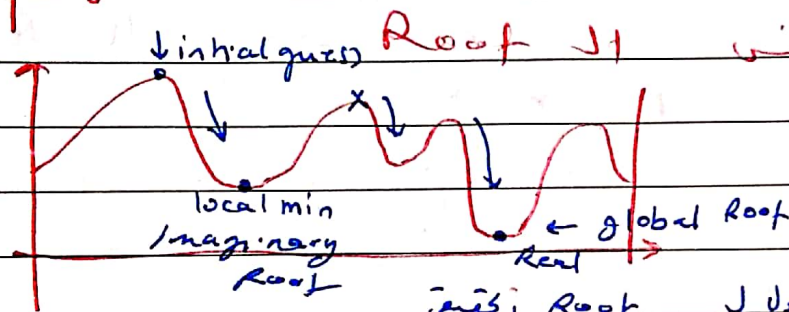
first Root

sec Root

Third Root



highly -ve & highly +ve
root find -2 > 2



Root 11

I wanna the global minimum!!

Roots are 10

initial guess is 2

Initial guess + proper writing of equation

Newton - 2nd order Method.

$$f(x) = f(x_0) + f'(x_0) \frac{\Delta x}{1!} + f''(x) \frac{(\Delta x)^2}{2!} + O(\text{err})$$

$$= 0$$

$$f(x_0) + f'(x_0) \frac{\Delta x}{1} + f''(x_0) \frac{(\Delta x)^2}{2} = 0$$

From Newton 1st order $\Delta x = \frac{-f(x_0)}{f'(x_0)}$

$$f(x_0) + \Delta x \left[f'(x_0) + f''(x_0) \frac{\Delta x}{2} \right] = 0$$

$$f(x_0) + \Delta x \left[f'(x_0) + \frac{f''(x_0)}{2} \left(-\frac{f(x_0)}{f'(x_0)} \right) \right] = 0$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\left[f'(x_i) - \frac{f(x_i) f''(x_i)}{2 f'(x_i)} \right]}$$

(Ex) $f(x) = 2x^3 - 3x^2 + x - 6$

$$f'(x) = 6x^2 - 6x + 1$$

$$f''(x) = 12x - 6$$

x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$
1	-6	1	6
1.3	-5.3	3.5	9.7
1.8	-2.3	9.6	15.6
1.99	-0.04	12.9	17.9
2	0	13	18

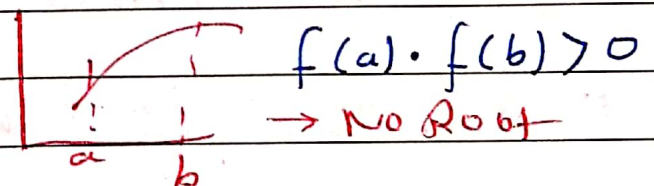
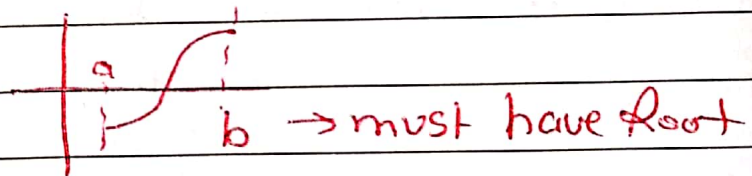
4) Bisection Method.

نقطة، نقطة وقف

$$a \leq x \leq b$$

$$f(a) \cdot f(b) < 0$$

⊖ و ⊕



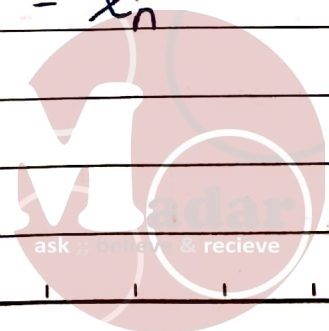
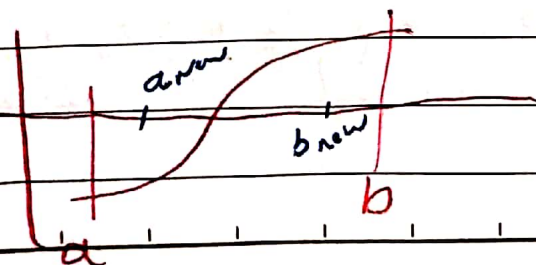
$$x_n = \frac{a+b}{2}, f(x_n) ??$$

if $f(a) \cdot f(x_n) > 0$

$$a_{\text{new}} = x_n$$

if $f(b) \cdot f(x_n) > 0$

$$b = x_n$$

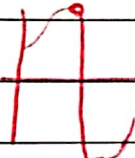


error ↑
high guess

ex) $f(x) = x^2 + x - 5$

في افترقا $\left[\begin{array}{l} a=0 \\ b=3 \end{array} \right] \left[\begin{array}{l} f(a) = f(0) = -5 \\ f(b) = f(3) = 7 \end{array} \right] \rightarrow f(a) * f(b) < 0$
the assumption is valid

a	b	x_n	$f(a)$	$f(b)$	$f(x_n)$	chang.
0	3	1.5	-5	7	-1.25	$a = x_n$
1.5	3	2.25	-1.25	7	2.3	$b = x_n$
1.5	2.25	1.87	-1.25	2.3	0.39	$b = x_n$
1.5	1.87	1.68	-1.25	0.39	-0.46	$a = x_n$
1.68	1.87	1.79	-0.46	0.39	-0.04	$a = x_n$

stiffness  sluggish

5) Secant method

$$f'(x) = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}$$
 ↑ error ↑

From Newton-R Method

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \rightarrow x_n = x_{n-1} - f(x_{n-1}) \left[\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} \right]$$

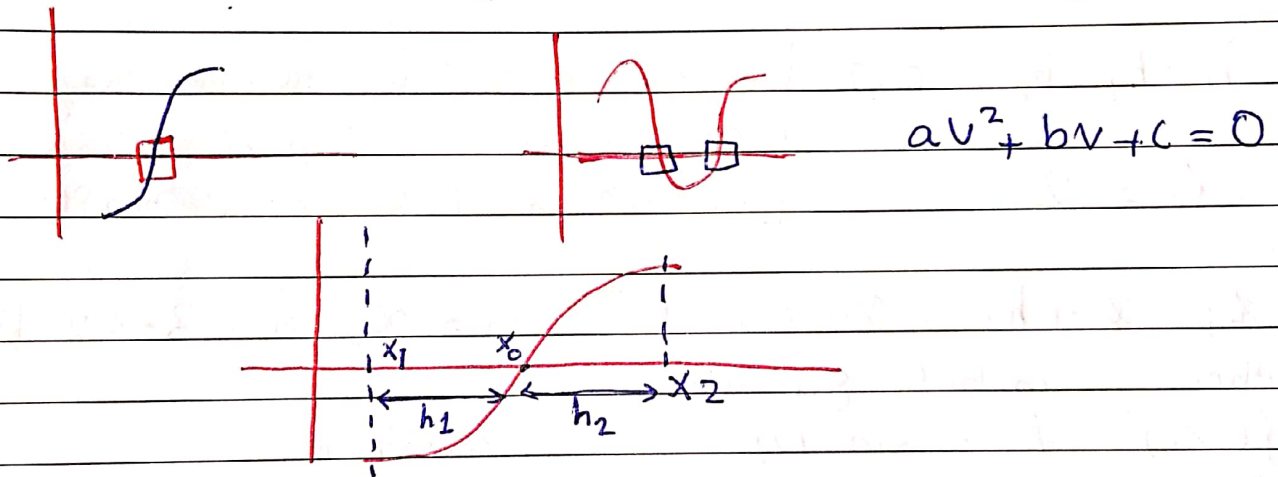
⊗ لا يكون عندك أكثر من معادلة عندك نقطتان فقط، لا تكفي، السبب اني
two initial guesses

ex) $f(x) = 2x^3 - 3x^2 + x - 6$

$x_{n-1} = 3$

$x_{n-2} = 4$

⊗ في البداية يكون Root في بين 3 و 4
+ ∫ bisection Method.



$$b = \frac{f_1 - f_0 - a h_1^2}{h_1}$$

$$\delta = \frac{h_2}{h_1}$$

$$\Delta x = -2c$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$X = X_0 - 2f_0$$

$$b \pm \sqrt{b^2 - 4ac}$$

اکرم ما عکرم

صبر له خير اصغر ما عكس
صبر الامام

$+ve$ origin $\rightarrow +ve \leftarrow b \text{ is } \infty$
 $-ve$ origin $\rightarrow -ve \leftarrow b \text{ is } 1$

$$f(x) = \sin x - \frac{x}{2} \quad \text{--- (Rad)}$$

(Root = 1.89)

$$h = h_1 = h_2 = 0.2$$

for the first time we assume they are equal for simplicity but it's not obligated to assume that

$$x_0 = 2$$

$$x_1 = x_0 + h_1 = 2 + 0.2 = 2.2 \quad x_2 = x_0 - h = 2 - 0.2 = 1.8$$

three initial guesses.

~~$$f(x_0) = f_0 = \sin(2) - \frac{2}{2} = -0.0907$$~~

$$f(x_4) = f_1 = \sin(2.2) - \frac{2.2}{2} = -0.991$$

$$f(x_2) = f_2 = \sin(1.8) - \frac{1.8^2}{2} = -0.0738$$

$\uparrow x$ $\uparrow h$ $\odot *$

$\delta = 1$ for First trial

$$a = \frac{(1)(-0.991) - (-0.0907)(1+1) + 0.07}{(1)(0.2)^2(1+1)} = -0.453$$

$$b = \frac{(-0.991) - (-0.6907)(-0.453)}{0.2} (0.2)^2 = -0.913$$

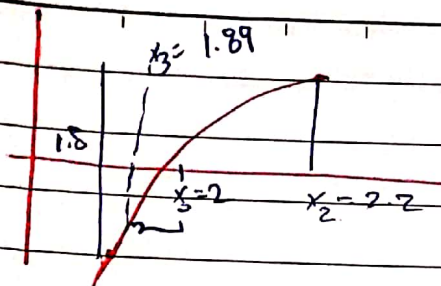
$$c = f(x_0) = f_0 = -0.907$$

the foot

$$X = 2 - 2(-0.0907)$$

$$= 1.89526$$

$$-0.918 \pm \sqrt{(-0.913)^2 - 4(10.453)(-10907)}$$



$$x_0 = 1.89526$$

$$f_0 = \sin(1.89526) - \frac{1.8927}{2}$$

$$= 1.918 \times 10^{-4}$$

$$x_1 = 2$$

$$h_1 = x - x_0 = 2 - 1.89526$$

$$x_2 = 1.8$$

$$h_2 = 0.10474$$

$$x_2 = 1.8$$

$$h_2 = x_0 - x_2 = 0.09526$$

$$\therefore x = \frac{h_2}{h_1} = \frac{0.09526}{0.0104} = 0.9094$$

$$c = f(x_0) = 1.918 \times 10^{-4}$$

$$f_1 = f(x_1) = f(2) = -0.0907$$

$$f_2 = f(x_2) = f(1.8) = 0.0738$$

$$a = -0.4728$$

$$b = -0.81820$$

$$x \approx 0 \text{ (Red not in line)}$$

$$x = 1.8954 \text{ exactly the root}$$



Solution of linear equation

New chapter
23th Jun - 2019

Solution linear equation

- 1) Gaussian elimination Method
- 2) Gaussian Jordan Method
- 3) Jacobi iterated Method
- 4) Gauss - sidel Method
- 5) L V Method

#1 Gaussian elimination Method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots = b_2$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots = b_n$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$\vec{X} = A^{-1}b$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

ex

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 3 \\ 3x_1 - 2x_2 + x_3 &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -2 & 1 & -1 \end{array} \right]$$

diagonal ← استاذنا

اصرف بالرقم الي بدله ديفعه على اول

diagonal رقم

$$-\frac{1}{2} R_1 + R_2 = R_2$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ \rightarrow 3 & -2 & 1 & -1 \end{array}$$

$$-\frac{3}{2} + R_1 + R_3 = R_3$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ 0 & -0.5 & -3.5 & -4 \end{array}$$

$$\frac{0.5}{2.5} R_2 + R_3 = R_3$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ 0 & 0 & -4 & -3.6 \end{array}$$

OR $-\frac{0.5}{2.5} R_2 - R_3 = R_3$

Algorithm:

For $K=1:N$

For $J=1:N+1 \rightarrow$ Right hand column.

Read a_{KJ}

END

END

For $k=1:N-1$

For $i=k+1:N$

For $j=k+1:N+1$

$$a_{ij} = a_{ij} - \frac{a_{ik} a_{kj}}{a_{kk}}$$

Now solve:

$$-4x_3 = -3.6$$

$$x_3 = 0.9$$

$$2.5x_2 - 2.5x_3 = 2$$

$$2.5x_2 - 2.5(0.9) = 2$$

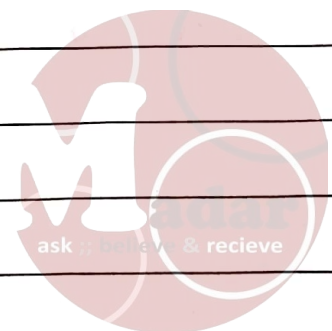
$$x_2 = 1.7$$

$$2x_1 - x_2 + x_3 = 2$$

$$2x_1 - \frac{4}{5} = 2$$

$$+\frac{4}{5} \quad +\frac{4}{5}$$

$$x_1 = 1/2$$



$$X_n = \frac{a_{n, n+1}}{a_{nn}}$$

for $k = n-1 : 1$

$$\text{sum} = 0$$

for $j = k+1 : N$

$$\text{sum} = \text{sum} + a_{kj} \cdot X_j$$

END

$$X_k = \frac{1}{a_{kk}} (a_{k, n+1} - \text{sum})$$

END k.

ON previous example :

$$a_{21} = -\frac{1}{2} \times 2 + 1 = 0$$

$$a_{31} = -\frac{3}{2} \times 2 + 3 = 0$$

$$a_{23} = -\frac{1}{2} \times 3 + -1 = -2.5$$

$$a_{32} = -\frac{3}{2} \times -1 - 2 = -0.5$$

$$a_{22} = -\frac{1}{2} \times -1 + 2 = 2.5$$

$$a_{33} = -\frac{3}{2} \times 3 + 1 = -3.5$$

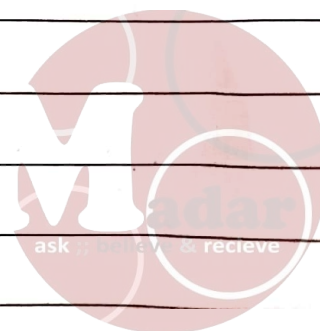
$$b_1 = 2$$

$$b_2 = -\frac{3}{2} \times 2 - 1 = -4$$

$$a_{32} = \frac{0.5}{2.5} \times 2.5 - 0.5 = 0$$

$$a_{33} = \frac{0.5}{2.5} \times -2.5 + (-3.5) = -4$$

$$b_3 = \frac{0.5}{2.5} \times 2 - 4 = -3.6$$



2) Gauss-Jordan Method.

For $k=1 : N-1$

$i = k+1 : N$

$j = N+1 : k_1-1$

$$a_{ij} = a_{ij} - \frac{a_{ik} a_{kj}}{a_{kk}}$$

END,

$$\begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

NOT necessary to be 0's
just to equal (1)

(ex) from previous example bring from it the eq's.

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -4 \\ 0 & 0 & 40 & 36 \end{array} \right]$$

Ⓢ ملاحظه:

$$\left[\begin{array}{l} 36 \\ 40 \end{array} \right] \leftarrow \begin{array}{l} 3.6 \\ 4 \end{array}$$

نہیں امیڑ ہوا 10 دکان عینہ کوڑا
سب او عوجہ جب عادی ای اختیار کرنا

① set diagonal = 1 (دکان کے ضروری اصول = 1)

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1 & 4/5 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$E_3 + E_2 = E_2$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1 & 4/5 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$\frac{1}{2} E_2 + E_1 = E_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 74/40 \\ 0 & 1 & 0 & 68/40 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$-\frac{3}{2} E_3 + E_1 = E_1$$

Then solve

$$\begin{bmatrix} 1 & 0 & 0 & | & 29/40 \\ 0 & 1 & 0 & | & 68/40 \\ 0 & 0 & 1 & | & 36/40 \end{bmatrix}$$

$$x_1 = 29/40 = 1/2$$

$$x_2 = 68/40 = 17/10$$

$$x_3 = 36/40 = 9/10$$

APPLY THE CODE:

$$N = 3$$

$$K = 1 \rightarrow 2$$

$$i = 2 \rightarrow 3$$

$$J = 4 \rightarrow K$$

$$K = 1$$

$$i = 2$$

$$J = 4, 3, 2, 1$$

→ looping

$$a_{24} = a_{24} - (a_{21}/a_{11})(a_{14})$$

$$a_{23} = a_{23} - \left(\frac{a_{21}}{a_{11}}\right)(a_{13})$$

#3) Jacobi - iterative Method

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots = b_2$$

$$a_{n1} x_1 + a_{n2} x_2 + a_{n3} x_3 + \dots = b_n$$

④ ملاحظہ کریں کہ coeff درجہ اولیٰ و اولیٰ درجہ اولیٰ coeff درجہ اولیٰ و اولیٰ درجہ اولیٰ
by mistake fixed print
div

$$\prod_{i=1}^n |a_{ij}| = \max$$

$$\begin{vmatrix} a_{11} & a_{22} & a_{33} \end{vmatrix} = \max \text{imum}$$



Ex

$$2x_1 + 7x_2 - 8x_3 = 10$$

$$5x_1 - 2x_2 + 6x_3 = 6$$

$$3x_1 + 5x_2 - 7x_3 = 3$$

x_1 which largest value!!

re arrange

$$5x_1 - 2x_2 + 6x_3 = 6$$

$$2x_1 + 7x_2 - 8x_3 = 10$$

$$3x_1 + 5x_2 - 7x_3 = 3$$

$$5 * 7 * 7 = 245$$

$$5 * 5 + 8 = 200 \quad \text{NOT THE Max}$$

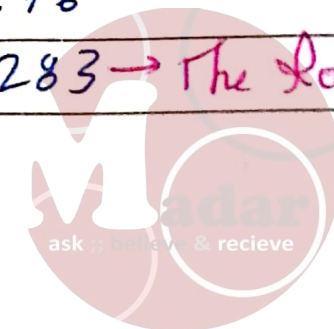
ل جربنا ترتيب اخر هين باح بعضيني Max بالترتيب

$$x_1 = \frac{6 + 2x_2 - 6x_3}{5}$$

$$x_2 = \frac{10 - 2x_1 + 8x_3}{7}$$

$$x_3 = \frac{3 - 3x_1 - 5x_2}{7}$$

	#1	#2	#3	#4	...	#40
x_1	0	6/5	2.28	0.11		-0.305
x_2	0	10/7	0.545	2.04		4.76
x_3	0	-3/7	1.106	0.98		2.283 → The Root



4 : Gauss's Sidel Method

	#1	#2	#3	#22
x_1	0	6/5	0.6	6.306
x_2	0	$10 - 2(6/5) + 0 = 1.08$	2.24	4.76
x_3	0	$(3 - 3(6/5) - 5(1.08)) / -7 = 0.86$	1.42	2.84

لما تم احسابها بوضوح $(x_1 = 6/5)$ مش
(6) بوضوح آلفه اربعة

~~Development of / find U matrix /~~

(5) lower - Upper limit Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots a_{2n} \\ \vdots & \vdots & \vdots \dots \vdots \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots a_{2n} \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

L diagonal = 1

$$U \vec{x} = \vec{b}$$

$$L^{-1} \vec{y} = L^{-1} \vec{b} \Rightarrow \vec{y} = L^{-1} \vec{b} \rightarrow U \vec{x} = \vec{y}$$

$$\vec{x} = U^{-1} \vec{y}$$

Development of L & U Matrix

Elements of lower Matrix !!

$$l_{ij} = \frac{a_{ij}}{a_{ii}}$$

$$L = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & & 1 & \\ \vdots & & & \ddots \end{bmatrix}$$

$$U_{ij\text{new}} = U_{ij\text{old}} - l_{ij} a_{ii}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ & u_{22} & \dots & u_{2n} \\ & & \ddots & \\ & & & u_{nn} \end{bmatrix}$$

Ex

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$\Rightarrow LU \Rightarrow$

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$l_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$a_{21} = a_{21} - l_{21} a_{11} = 64 - (2.56)(25) = 0$$

$$l_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

$$a_{22} = a_{22} - l_{21} a_{12} = 8 - (2.56)(5) = -4.8$$

$$l_{32} = \frac{a_{32}}{a_{22}} = ??$$

$$a_{23} = a_{23} - l_{21} a_{13} = 1 - (2.56)(1) = -1.56$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{bmatrix}$$

upper !!

$$l_{31} = 5.76 \text{ so } a_{31} = a_{31} - l_{31} a_{11}$$

$$a_{31} = 144 - (5.76)(25) = 0$$

$$a_{32} = 12 - (5.76)(5) = 16.8$$

New one !!

$$a_{33} = 1 - (5.76)(1) = -4.76$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.75 & -1.56 \\ 0 & 16.8 & -4.76 \end{bmatrix}$$

$\Rightarrow$

$$l_{32} = \frac{a_{32}}{a_{22}} = \frac{16.8}{-4.75} = 3.53$$

$$a_{32} = a_{32} - l_{32} a_{22} = (16.8) - (3.53)(-4.75) = 0$$

$$a_{33} = a_{33} - l_{32} a_{23} = -4.76 - (3.53)(-1.56) = 0.7$$

$$-4.76 - (3.53)(-1.56) = 0.7$$

$$\begin{matrix} & U & & L & & = & A \end{matrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.75 & 1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.53 & 1 \end{bmatrix}$$

$$Ax = b \rightarrow L U X = b \rightarrow \boxed{L y = b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.53 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

straight forward!!

$$y_1 = 1$$

$$2.56 y_1 + y_2 = 2 \Rightarrow y_2 = -0.56$$

$$5.76 y_1 + 3.53 y_2 + y_3 = 3 \Rightarrow y_3 = -0.78$$

$$\boxed{L U X = b} \rightarrow \boxed{L y = b} \rightarrow \boxed{U X = y}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.75 & 1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.56 \\ -0.78 \end{bmatrix}$$

backward calculation

$$0.7 * x_3 = -0.78 \rightarrow x_3 = -1.11$$

$$x_2 = -0.48$$

$$x_1 = -0.0116$$

$$X = \begin{bmatrix} -1.11 \\ 0.48 \\ -0.6116 \end{bmatrix}$$

Algorithm !!

Read A, b

set $U = A$

$L = I \rightarrow [1, 0]$

for $k = 1, n-1$

for $j = k+1, n$

$$l_{jk} = \frac{U_{jk}}{U_{kk}}$$

$$U_{j,k:n} = U_{j,k:n} - l_{jk} U_{k,k:n}$$

$U_{k,k:n}$ no. of variable

END

مثال من الدكتور

$$A = \begin{bmatrix} 5 & 5 & 1 & 4 & 0 & 1 & 4 & 1 & 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$[n, n] = \text{size}(A)$$

$$L = \text{eye}(n)$$

$$U = A$$

For $k = 1:n-1$

For $j = k+1:n$

$$L(j, k) = U(j, k) / U(k, k)$$

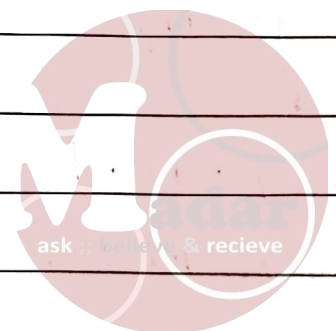
$$U(j, :) = U(j, :) - L(j, k) * U(k, :)$$

END

END

$$y = \text{Inv}(L) * b$$

$$x = \text{Inv}(U) * y$$



Numerical solution of system of non-linear eq.

1. Iterative method
2. Newton's Raphson method.

① Iterative Method

Equation $\rightarrow$ Rearrangement

(ex)

$$2x_1^2 + x_2 = 3 \rightarrow x_1 = \left(\frac{3-x_2}{2}\right)^{1/2}$$

$$x_1 + x_2^3 = 2 \rightarrow x_2 = (2-x_1)^{1/3}$$

x_1	x_2
0	1.26
0.93	1.02
0.994	1.001
0.999	1.0001

First trail: $x_2 = 1.26$ when $x_1 = 0 \dots$ so on.

(ex): ~~$x_1^3 + x_2 = 2$~~

$$2x_1^2 + x_2 = 3$$

$$x_1^3 + x_2 = 2$$

← have higher order degree x_1 ←

Ⓐ $x_2 = 3 - 2x_1^2$

$$x_1 = (2 - x_2)^{1/3}$$

conv $\rightarrow$ odd $(\frac{1}{3})$

x_1	x_2
0	3
-1	1
1	1

x_2 مبدئي x_1 ← initial guess
 x_1 مبدئي x_2 ←

Ⓐ x_1 مبدئي x_2 ←
 Root 1

Ⓑ $x_1 = \left(\frac{3-x_2}{2}\right)^{1/2}$

$$x_2 = 2 - x_1^3$$

x_1	x_2
0	2
0.7	1.65
0.82	1.445
0.88	1.32
0.91	1.23

* $\text{div} \leftarrow (-) + (\frac{1}{2})^{\text{اگر } 0 \text{ کی}} \quad (*)$
 $\text{if } x_i \rightarrow y_j \leftarrow \text{imaginary}$
 add. zero
 * انہی کے initial guess
 $\text{Zero} \leftarrow \text{عز ال صلا} \leftarrow (0) \text{ بطور}$
 img. di div

For, $\frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = \text{imaginary value}$

④ انیسو لہ initial guess لاؤ اختیار

مثلاً اتي عن ال zero ← مثلاً ← (0) (بطول
عننا div و div. imag.

Newton-Raphson Method

$$g(x, y) - 0 = g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0) \frac{\Delta x}{I} + \frac{\partial g}{\partial y}(x_0, y_0) \frac{\Delta y}{II}$$

$$\text{let } f_x = \frac{\partial f}{\partial x} \quad f_y = \frac{\partial f}{\partial y}$$

$$g_x = \frac{\partial g}{\partial x} \quad g_y = \frac{\partial g}{\partial y}$$

$$\begin{aligned} f(x, y) &= f_0 + f_x \Delta x + f_y \Delta y = 0 \\ g(x, y) &= g_0 + g_x \Delta x + g_y \Delta y = 0 \end{aligned}$$

unknown!!

~~$$f_x \Delta x + f_y \Delta y = -f_0$$~~

$$\cancel{g_x \Delta x + g_y \Delta y} = -g_0$$

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -f_0 \\ -g_0 \end{bmatrix}$$

$$\Delta x = x - x_0 \quad \rightarrow \quad x = x_0 + \Delta x$$

$$\Delta y = y - y_0 \rightarrow y = y_0 + \Delta y$$

(ex) Solve: $f_1 = 3x_1 - \cos(x_2 x_3) - \frac{1}{2}$
 $f_2 = x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06$
 $f_3 = e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3}$

by Newton Raphson Method

start with $x_1 = 0.1$ $x_2 = 0.1$ $x_3 = -0.1$

$$\begin{bmatrix} f_{1x_1} & f_{1x_2} & f_{1x_3} \\ f_{2x_1} & f_{2x_2} & f_{2x_3} \\ f_{3x_1} & f_{3x_2} & f_{3x_3} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2 \\ -f_3 \end{bmatrix}$$

$$f_{1x_1} = 3 \quad f_{1x_2} = x_3 \sin(x_2 x_3) \quad f_{1x_3} = x_2 \sin(x_2 x_3)$$

$$f_{2x_1} = 2x_1 \quad f_{2x_2} = -162(x_2 + 0.1) \quad f_{2x_3} = \cos(x_3)$$

$$f_{3x_1} = -x_2 e^{-x_1 x_2} \quad f_{3x_2} = -x_1 e^{-x_1 x_2} \quad f_{3x_3} = 20$$

$$\begin{bmatrix} 3 & 0.001 & -0.001 \\ 0.2 & -32.4 & 0.995 \\ -0.099 & -0.1 & 20 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} -1.2 \\ -2.27 \\ 8.46 \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -0.08 \\ -0.42 \end{bmatrix}$$

2nd iteration

$x_{1, \text{new}}$	$x_{1, \text{old}}$	Δx_1	0.1	0.5
$x_{2, \text{new}}$	$x_{2, \text{old}}$	Δx_2	0.1	$+0.02$
$x_{3, \text{new}}$	$x_{3, \text{old}}$	Δx_3	-0.42	-0.52

$$\epsilon = 0.001$$

i	x_1	x_2	x_3	ϵ
0	0.1	0.1	-0.1	—
1	0.5	0.02	-0.52	0.9
2	0.5	0.0015	-0.523	0.018
3	0.5	0.000012	-0.523	0.0015
4	0.5	0	-0.523	1.2×10^{-5}

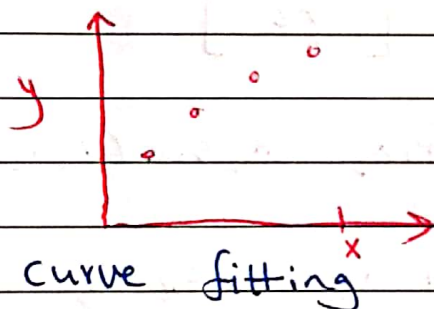
مقدار، کوچک شود
 RMS (error)
 خطا کم، نو جی
 $= x_{old} (0.001)$
 جوی، 8 و 0.01
 RMS mb

$$err = \sum_{i=1}^n (x_{i\text{ new}} - x_{i\text{ old}})^2$$

$$err = (x_{1N} - x_{1old})^2 + (x_{2N} - x_{2old})^2 + (x_{3N} - x_{3old})^2$$

Curve fitting & interpolation

ساختن



curve fitting

1- Linear Regression $\Rightarrow$ fitting

$$y = a + bx$$

2- Multi linear Regression

$$y = a + bx_1 + cx_2$$

3- Non-linear Regression

$$y = a + bx + cx^2$$

4- Multi Non-linear Regression

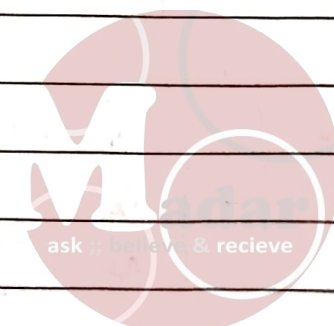
$$y = a + bx_1 + cx_1^2 + \dots + dx_2 + ex_2^2 + \dots$$

More general Regression It's including all the above Regression It's so powerfull

Linear Regression

X y_{exp} $y_{calculated}$ err

$$err = \min$$



$$\text{err} = \sum_{i=1}^n (y_{\text{exp}} - y_{\text{calc}})^2 = \sum (y_{\text{exp}} - (a + bx))^2$$

here, the best values for a, b should be; when err is the minimum

$$\frac{\partial \text{err}}{\partial a} = 0 \rightarrow 2 \sum (y_{\text{exp}} - (a + bx)) (-1) = -2 \left[\sum y_{\text{exp}} - (a + bx) \right]$$

$$\frac{\partial \text{err}}{\partial b} = 0 \rightarrow 2 \sum (y_{\text{exp}} - a - bx) (-x) = -2 \sum (x y_{\text{exp}} - a x - b x^2) = 0$$

$$\sum y_{\text{exp}} = \sum_{i=1}^n a + \sum b x$$

$$\sum x y_{\text{exp}} = \sum a x + \sum b x^2$$

$$\sum y_{\text{exp}} = na + b \sum x$$

$$\sum x y_{\text{exp}} = a \sum x + b \sum x^2$$

$$\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_{\text{exp}} \\ \sum x y_{\text{exp}} \end{bmatrix} \Rightarrow \text{simple linear eq. with two known}$$

$$a = \frac{\sum y_{\text{exp}} - b \sum x_i}{n}$$

$$b = \frac{n \sum y_i x_i - \sum y_i \sum x_i}{n \sum x_i^2 - (\sum x_i)^2}$$

Ex

n	x_i	y_{exp}	x_i^2	xy	y_{cal}	err^2 $(y - y_{\text{cal}})^2$	R^2 & L.V. $(y_i - y_{\text{avg}})^2$
1	1	4	1	4	3.3	$(4 - 3.3)^2 = 0.49$	$(4 - 6.75) = 7.5$
2	2	6	4	12	5.6	0.16	6.56
3	3	5	9	15	7.9	8.41	27.56
4	4	12	16	48	10.2	3.24	38.74
$\sum$	10	27	30	79		12.3	

Minimum value of err

↓ err ↑ fit → not indicated for the fit

$$b = \frac{4(79) - (27)(10)}{4(30) - (10)^2} = 2.3 \quad a = \frac{27 - (2.3)(10)}{4} = 1$$

$$y_{\text{calc}} = a + bx \rightarrow y_{\text{calc}} = 1 + 2.3x$$

Regression coefficient
coefficient of determination

$$R^2 = \frac{\sum (y_{\text{exp}} - y_{\text{avg}})^2 - \sum (y_{\text{exp}} - y_{\text{calc}})^2}{\sum (y_{\text{exp}} - y_{\text{avg}})^2}$$

$$0 < R^2 < 1 \rightarrow [0 - 1] \rightarrow \text{best fit}$$

$$R^2 > 0.99 \quad 0.95 \quad 0.97 \rightarrow \text{accuracy}$$

$$\text{ave value} \rightarrow y \rightarrow y_{\text{avg}} = \frac{\sum y_{\text{exp}}}{n} = \frac{27}{4} = 6.75$$

یہاں پر y_{avg} کا استعمال ہے

$$R^2 = \frac{38.74 - 12.3}{38.74} = 0.68 \rightarrow \text{how accurate our fitting}$$

Multi linear Regression

$$y = a + bx + cx_2 + dx_3 + \dots$$

previously for a single Regression linear we do have this

$$\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \end{bmatrix}$$

$$y = a_0 + a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

$$\begin{bmatrix} n & \sum x_1 & \sum x_2 & \dots & \sum x_n \\ \sum x_1 & \sum x_1^2 & \sum x_1 x_2 & \dots & \sum x_1 x_n \\ \sum x_2 & \sum x_1 x_2 & \sum x_2^2 & \dots & \sum x_2 x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum x_n & \sum x_1 x_n & \sum x_2 x_n & \dots & \sum x_n^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum x_1 y \\ \sum x_2 y \\ \vdots \\ \sum x_n y \end{bmatrix}$$

Ex Given That

n	x_1	x_2	y	x_1^2	$x_1 x_2$	x_2^2	$x_1 y$	$x_2 y$
1	0	0	3	0	0	0	0	0
2	1	3	1	1	3	9	1	3
3	2	2	7	4	4	4	14	14
4	3	5	5	9	15	25	15	25
5	4	7	5	16	28	49	20	35
6	5	1	12	25	5	1	105	21
	15	18	42	55	55	88	155	98

↑ values to fit ↑ ind var

$$\begin{bmatrix} 6 & 15 & 18 \\ 15 & 55 & 55 \\ 18 & 55 & 88 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 42 \\ 155 \\ 98 \end{bmatrix} \rightarrow \text{system of the linear eq.}$$

$$y = 3 + 4x_1 - 2x_2$$

Non-linear regression

1. $y = ax^b \rightarrow \text{take a log} \rightarrow \underbrace{\ln y}_Z = \underbrace{\ln a}_C + b \underbrace{\ln x}_D$

x	y	$\ln x$	$\ln y$
---	---	---------	---------

$\Rightarrow b \checkmark$

$\Rightarrow c = \ln a \rightarrow a = e^c$

2. $y = a e^{bx} \rightarrow \text{Take } \ln \rightarrow \ln y = \ln a + bx$

x	y	$\ln y$
⋮	⋮	⋮

$$\ln a = \checkmark \quad b = \checkmark \quad a = e^{\text{value}}$$

3. $y = a x^b e^{cx} \rightarrow \text{take a log}$

$$\ln y = \ln a + b \ln x + cx = \ln a$$

Polynomial Regression !!

$$y = a_1 + a_2 x + a_3 x^2 + \dots$$

$$\text{err} = \sum (y - y_e)^2$$

$$\frac{\partial \text{err}}{\partial a_1} = \frac{\partial}{\partial a_1} \sum (a_1 + a_2 x + a_3 x^2 - y_e)^2$$

$$\frac{\partial \text{err}}{\partial a_2} = \frac{\partial}{\partial a_2} \sum ($$

المصنوفة الهاتية الى 2 - تنقل الى

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

n	$\sum x$	$\sum x^2$	$\sum x^3$	$\dots \sum x^n$	a_0		$\sum y$
$\sum x$	$\sum x^2$	$\sum x^3$	$\sum x^4$	$\sum x^{n+1}$	a_1		$\sum xy$
$\sum x^2$	$\sum x^3$	$\sum x^4$	$\sum x^5$	$\sum x^{n+2}$	a_2	=	$\sum x^2 y$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\sum x^n$	$\sum x^{n+1}$	$\sum x^{n+2}$	$\sum x^{n+3}$	$\sum x^{2n}$	a_n		$\sum x^n y$

n	$\sum x$	$\sum x^2$	x	y	x^2	x^3	x^4	xy	$x^2 y$
$\sum x$	$\sum x^2$	$\sum x^3$							
$\sum x^2$	$\sum x^3$	$\sum x^4$							
			$\sum$	$\sum$	$\sum$	$\sum$	$\sum$	$\sum$	$\sum$

$$y = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + b_1 x_2 + b_2 x_2^2 + b_3 x_2^3 +$$

$$c_1 x_1 x_2 + c_2 x_1 x_2^2 + c_3 x_1^2 x_2$$

$$y_1 = f(x_1, x_2)$$

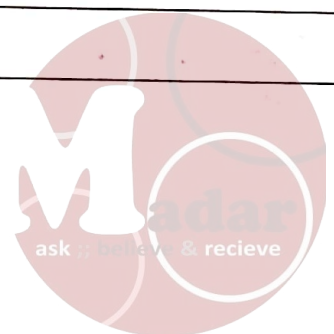
n	$\sum x_1$	$\sum x_1^2$	$\sum x_1^3$	$\sum x_2$	$\sum x_2^2$	$\sum x_2^3$	$\sum x_1 x_2$	$\sum x_1^2 x_2$	$\sum x_1 x_2^2$	element	قب
$\sum x_1$	$\sum x_1^2$	$\sum x_1^3$	$\sum x_1^4$	$\sum x_1 x_2$	$\sum x_1^2 x_2$	$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2$	$\sum x_1 x_2^3$	a_0	$\sum y$
$\sum x_1^2$	$\sum x_1^3$	$\sum x_1^4$	$\sum x_1^5$	$\sum x_1^2 x_2$	$\sum x_1^3 x_2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2^2$	$\sum x_1^4 x_2$	$\sum x_1^2 x_2^3$	a_1	$\sum x_1 y$
$\sum x_1^3$	$\sum x_1^4$	$\sum x_1^5 x_2$	$\sum x_1^6$	$\sum x_1^3 x_2$	$\sum x_1^4 x_2$	$\sum x_1^3 x_2^2$	$\sum x_1^4 x_2^2$	$\sum x_1^5 x_2$	$\sum x_1^3 x_2^3$	a_2	$\sum x_1^2 y$
$\sum x_2$	$\sum x_1 x_2$	$\sum x_1^2 x_2$	$\sum x_1^3 x_2$	$\sum x_2^2$	$\sum x_2^3$	$\sum x_2^4$	$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^2 x_2^3$	a_3	$\sum x_1^3 y$
$\sum x_2^2$	$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2^2$	$\sum x_2^3$	$\sum x_2^4$	$\sum x_2^5$	$\sum x_1 x_2^3$	$\sum x_1^2 x_2^3$	$\sum x_1^2 x_2^4$	b_1	$\sum x_2 y$
$\sum x_2^3$	$\sum x_1 x_2^3$	$\sum x_1^2 x_2^3$	$\sum x_1^3 x_2^3$	$\sum x_2^4$	$\sum x_2^5$	$\sum x_2^6$	$\sum x_1 x_2^4$	$\sum x_1^2 x_2^4$	$\sum x_1^2 x_2^5$	b_2	$\sum x_2^2 y$
$\sum x_1 x_2$	$\sum x_1^2 x_2$	$\sum x_1^3 x_2$	$\sum x_1^4 x_2$	$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2^2$	$\sum x_1^2 x_2^3$	$\sum x_1^3 x_2^3$	$\sum x_1^3 x_2^4$	b_3	$\sum x_2^3 y$
$\sum x_1 x_2^2$	$\sum x_1^2 x_2^2$	$\sum x_1^3 x_2^2$	$\sum x_1^4 x_2^2$	$\sum x_1^2 x_2^3$	$\sum x_1^3 x_2^3$	$\sum x_1^4 x_2^3$	$\sum x_1^2 x_2^4$	$\sum x_1^3 x_2^4$	$\sum x_1^3 x_2^5$	c_1	$\sum x_1 x_2 y$
$\sum x_1^2 x_2$	$\sum x_1^3 x_2$	$\sum x_1^4 x_2$	$\sum x_1^5 x_2$	$\sum x_1^2 x_2^4$	$\sum x_1^3 x_2^4$	$\sum x_1^4 x_2^4$	$\sum x_1^2 x_2^5$	$\sum x_1^3 x_2^5$	$\sum x_1^3 x_2^6$	c_2	$\sum x_1 x_2^2 y$
										c_3	$\sum x_1^2 x_2 y$

المعاجيب

$$y = 1 + 2x_1 + 3x_1^2 + 4x_1^3 + 5x_2 + 6x_2^2 + 7x_2^3 + 8x_1x_2 + 9x_1x_2^2 + 10x_1^2x_2$$

x_1	x_2	y	$x_1^2 x_2$
0	0		
↓	↓		

0	10
↓	↓
1	10



Interpolation

→ one variable

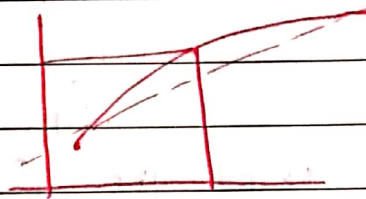
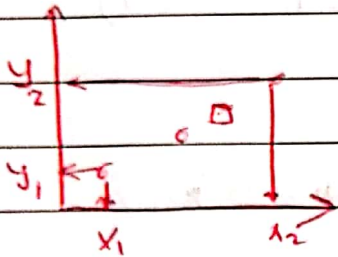
→ two variable

lagrange interpolation

Newton's interpolation

spline

langrage interpolation :



$$f(x) = a_1 + a_2 x + a_3 x^2 + \dots + a_n x^n$$

$$f_p(x) = \sum_{j=0}^n L_j(x) f(x_j)$$

$$L_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^n \frac{x - x_k}{x_j - x_k}$$

← j=0 ←

the order of the degree excluded → because the condition

Ex

$$f(x) = a + bx$$

$$f_1(x) = \sum_{j=0}^1 L_j(x) f(x_j)$$

$$= L_0(x) f(x_0) + L_1 f(x_1)$$

What is L_0 ?

$$L_0(x) = \prod_{\substack{k=0 \\ k \neq j}}^1 \frac{x - x_k}{x_j - x_k} = \frac{x - x_1}{x_0 - x_1}$$

the order of the degree excluded → because the condition

$$L_1(x) = \prod_{\substack{k=0 \\ k \neq j}}^1 \frac{x - x_k}{x_j - x_k} = \frac{x - x_0}{x_1 - x_0}$$

↓
= L

$$f_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

Ex Find $f_1(3)$, find the value $f(3)$ based on the first order langrage interpolation

x	y
$x_0 \leftarrow 2$	1.5 $\rightarrow f_0(x) = y_0$
$x_1 \leftarrow 5$	4 $\rightarrow f_1(3)$
8	6.5
11	9

$$f_1(x) = \frac{x-5}{2-5} f(2) + \frac{x-2}{5-2} f(5)$$

$$f_1(x) = \frac{x-5}{-3} (1.5) + \frac{x-2}{3} (4)$$

$$f_1(3) = \frac{3-5}{-3} (1.5) + \frac{3-2}{3} (4)$$

$$f_1(3) = 2.33$$

To check $\rightarrow$ use the (التحقق من الإجابة)

$$\frac{f(3) - 1.5}{4 - 1.5} = \frac{x - 2}{3 - 2} \quad \text{so;}$$

$$f(3) = \frac{3-2}{3} (4-1.5) + 1.5 = 2.33$$

we notice it's the same answer coz; it's linear

Quadratic Interpolation second order!!

$$f_2(x) = \sum_{j=0}^2 L_j(x) f(x_j) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

First $\rightarrow$ term of relation coeff const
 $\rightarrow$ term of n coffe of x

second $\rightarrow$ term of x^2

x	y
x_0	" } interpolation
x_1	"
x_2	"

سأذكرها في سؤال ٢
 فيماتين !!

order 2

$$L_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^2 \frac{x - x_k}{x_j - x_k} = \left(\frac{x - x_0}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right) = L_0$$

$$L_1(x) = \prod_{\substack{k=0 \\ k \neq 1}}^2 \frac{x - x_k}{x_1 - x_k} = \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right)$$

(excluded)

$$L_2(x) = \prod_{\substack{k=0 \\ k \neq 2}}^2 \frac{x - x_k}{x_2 - x_k} = \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right)$$

(c) Find $f_2(2.5)$

x	y		$L_0(x) = \left(\frac{x-3}{2-3} \right) \left(\frac{x-4}{2-4} \right)$
1	0		
2	1	$\rightarrow x_0$	$L_1(x) = \left(\frac{x-2}{3-2} \right) \left(\frac{x-4}{3-4} \right)$
3	2	$\rightarrow x_1$	$L_2(x) = \left(\frac{x-2}{4-2} \right) \left(\frac{x-3}{4-3} \right)$
4	3	$\rightarrow x_2$	
5	5		
6	7		

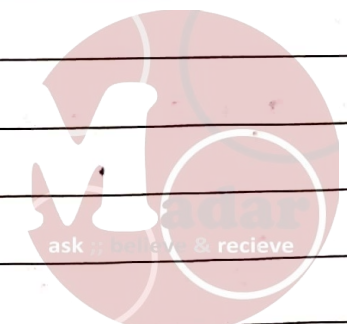
Substitute !! $L_0 = \frac{1}{2} (x-3)(x-4)$

$$L_1 = - (x-2)(x-4)$$

$$L_2 = \frac{1}{2} (x-2)(x-3)$$

$$F_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$F_2(2.5) = \frac{1}{2} (2.5-3)(2.5-4)(1) + - (2.5-2)(2.5-4)(2) + \frac{1}{2} (2.5-2)(2.5-3)(3)$$



Numerical Interpolation

2. Newton's interpolation

1st order

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) \\ &= f(x_0) + f'(x_0)(x - x_0) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \end{aligned}$$

x	f(x)
-	-
-	- $\rightarrow x_0$
-	- $\rightarrow x$
-	- $\rightarrow x_1$

2nd order $\rightarrow$ 2nd derivative

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &= f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)(x - x_1) \\ &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] (x - x_0)(x - x_1) \end{aligned}$$

$$* (x - x_0) \quad * (x - x_1)$$

Higher order:

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \\ &= a_0 + \sum_{i=0}^n \prod_{j=0}^{i-1} (x - x_j) a_{i+1} \end{aligned}$$

$\rightarrow$ simple case $\leftarrow$

(Ex)

i	x	y
0	0	1
1	1	2.25
2	2	3.75
3	3	4.25
4	4	5.81

Find $f(2.4)$ using 3rd order method !!

$$F_3(2.4) = \sum_{k=0}^3 a_k \prod_{j=0}^{k-1} (x - x_j)$$

$$= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$= f(x_0) + \overset{f'(x_0)}{f(x_0, x_1)}(x - x_0) + f(x_0, x_1, x_2)(x - x_0)(x - x_1) + f(x_0, x_1, x_2, x_3)(x - x_0)(x - x_1)(x - x_2)$$

$$a_0 = f(x_0) = 1$$

$$a_1 = f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{2.25 - 1}{2 - 0} = 1.5$$

$$a_2 = f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{4.25 - 2.25}{3 - 1} = 1$$

$$f(x_0, x_1, x_2) = \frac{1 - 1.5}{3 - 0} = -0.5$$

$$a_3 = f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{5.81 - 4.25}{4 - 3} = 1.56$$

$$f(x_1, x_2, x_3) = \frac{1.56 - 1}{4 - 2} = 0.53$$

$$a_3 = \frac{0.53 - (-0.5)}{4-1} = 0.343$$

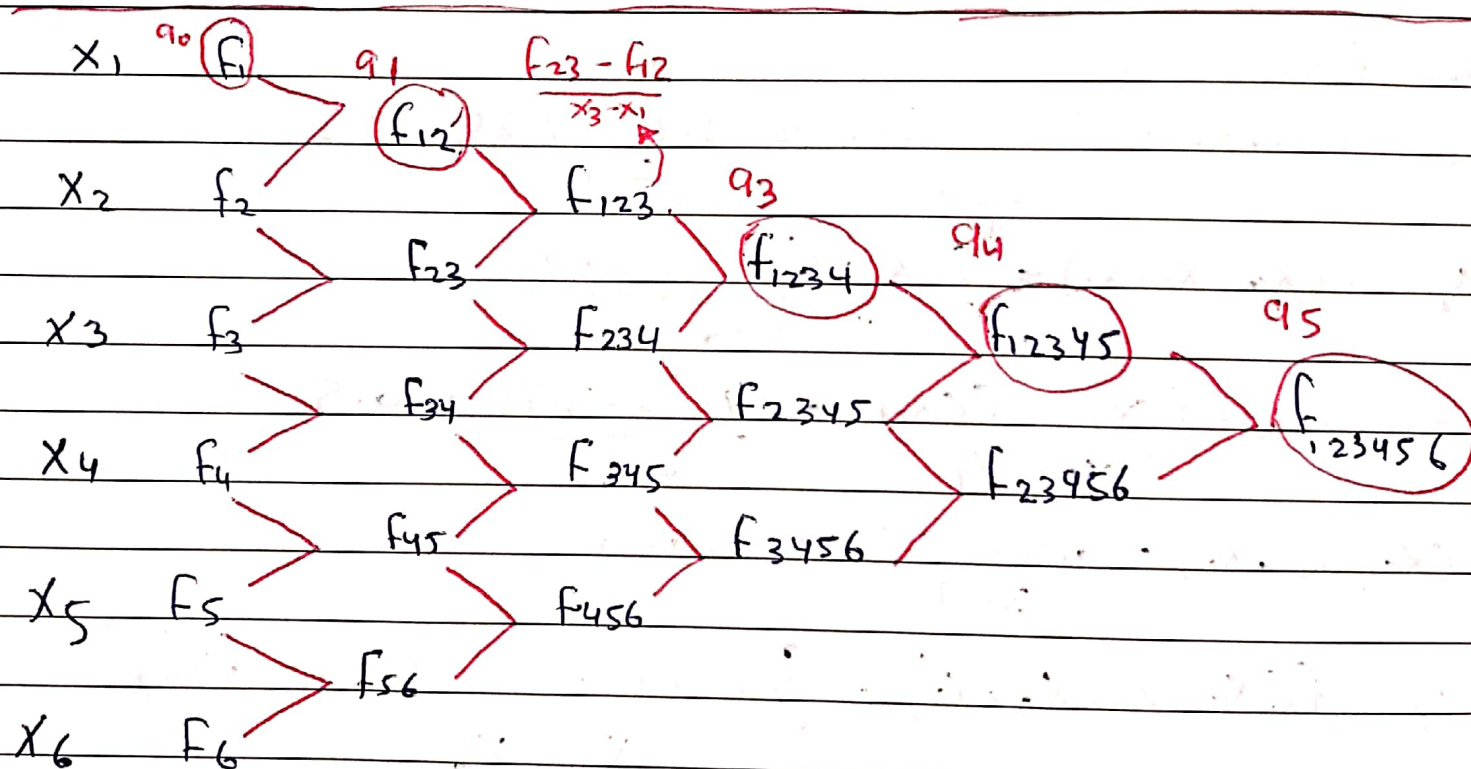
$$a_2 = -0.5 \quad a_1 = 1.5 \quad a_0 = 2.25$$

$$\begin{aligned} f(x) &= 2.25 + 1.5(x-x_0)(x-x_1) + 0.34(x-x_0)(x-x_1)(x-x_2) \\ &= 2.25 + 1.5(2.4-1) - 0.5(2.4-1)(2.4-2) + 0.34(2.4-1) \\ &\quad (2.4-2)(2.4-3) = 3.95 \end{aligned}$$

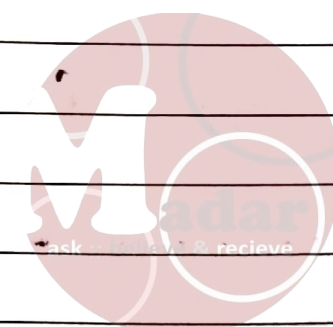
Newton Divided Difference

Table

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(5)}(x)$
		d_1	d_2	d_3	d_4	d_5



↓
no. of data = no. of order



Newton's Divided Difference

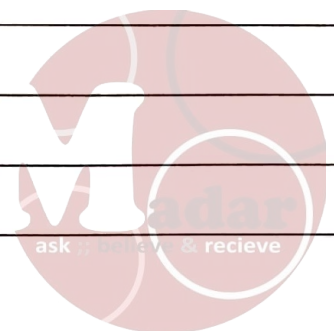
(EX)

X	Y	d_1	d_2	d_3	d_4
0	a_0 1	a_1 1.25	a_2 0.125	a_3 -0.208	a_4 0.137
1	2.25	1.5			
2	3.75	0.55	-0.5	0.34	
3	4.25	1.56	0.33		
4	5.81				

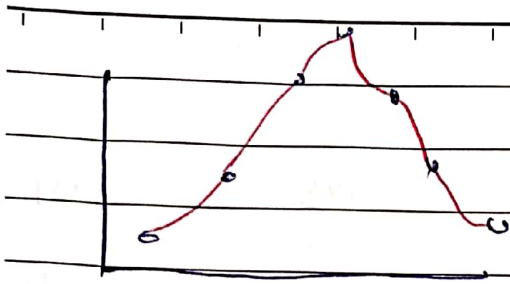
$$f(2.4) = 1 + 1.25(2.4-0) + 0.125(2.4-0)(2.4-1) - 0.208(2.4-0)(2.4-1)(2.4-2) + 0.137(2.4-0)(2.4-1)(2.4-2)(2.4-3) = 4.03$$

→ more accurate

بہت زیادہ درست جواب دے گا



Spline Method



$$y_1 = a_1 + b_1 x + c_1 x^2$$

$$y_2 = a_2 + b_2 x + c_2 x^2$$

$$\frac{dy_1}{dx} = \frac{dy_2}{dx} \text{ @ that given } x$$

eq's < plots.

Linear spline Method:

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

(*) (x_2, y_2)
(*) (x_1, y_1)
(*) (x_0, y_0)

$$y = \left(\frac{y_1 - y_0}{x_1 - x_0} \right) (x - x_0) + y_0 \quad x_0 \leq x \leq x_1$$

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x - x_1) + y_1 \quad x_1 \leq x \leq x_2$$

$$y = \left(\frac{y_3 - y_2}{x_3 - x_2} \right) (x - x_2) + y_2 \quad x_2 \leq x \leq x_3$$

(ex) given the following Data find $v(16)$

t	v	$v(16)?$
0	0	
10	227	
15	362	
20	517	
30	901	

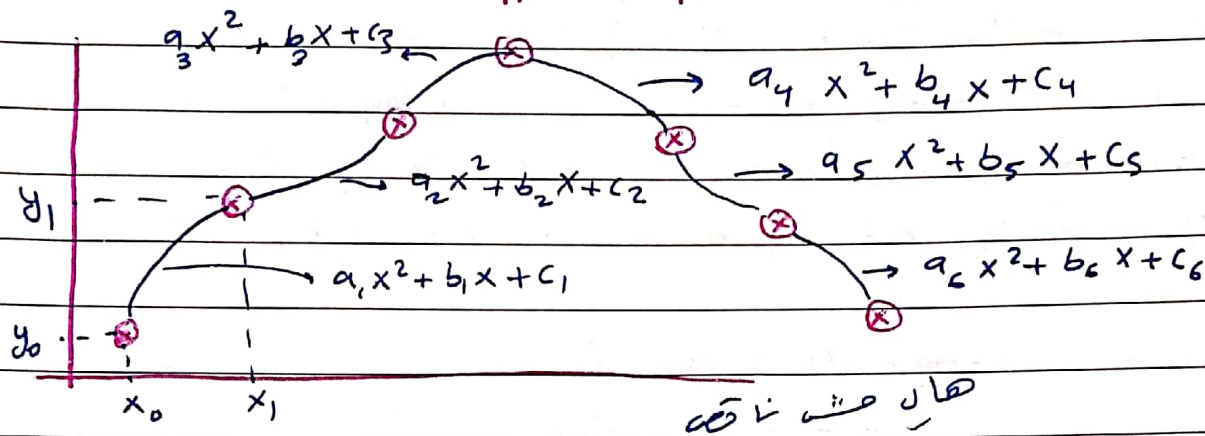
Sol: $x_0 = 15$ $y_0 = 362$
 $x_1 = 20$ $y_1 = 517$

$$\begin{aligned} v(16) &= \frac{(y_1 - y_0)}{x_1 - x_0} (t - t_0) + y_0 \\ &= \left(\frac{517 - 362}{20 - 15} \right) (16 - 15) + 362 \\ &= 393 \end{aligned}$$

Non-linear spline !!

* Assume all data passes a nonlinear function

* Assume $f(x) = a_1 x^2 + b_1 x + c_1$
 $= a_2 x^2 + b_2 x + c_2$
 $= a_n x^n + b_n x + c_n$



of basic equations $= (n-1 \text{ data}) - 1$

of unknown $3 + b = 18$ unknown
 we need $Bn - 3$ eq = 18

نقاط، 3، 18

$$(x_0, y_0) - (x_1, y_1) \begin{cases} a_1 x_0^2 + b_1 x_0 + c_1 = y_0 \\ a_1 x_1^2 + b_1 x_1 + c_1 = y_1 \end{cases}$$

$$(x_1, y_1) - (x_2, y_2) \begin{cases} a_2 x_1^2 + b_2 x_1 + c_2 = y_1 \\ a_2 x_2^2 + b_2 x_2 + c_2 = y_2 \end{cases}$$

$$(x_5, y_5) - (x_6, y_6) \begin{cases} a_6 x_5^2 + b_6 x_5 + c_6 = y_5 \\ a_6 x_6^2 + b_6 x_6 + c_6 = y_6 \end{cases}$$

⚡

12 equation

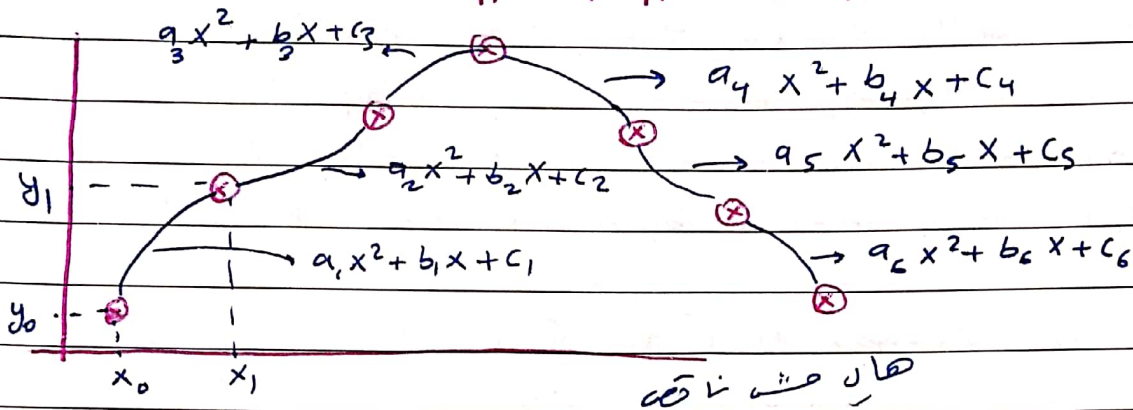
slope = slope

نقاط، 3، 18

non-linear spline !!

* Assume all data passes a nonlinear function.

* Assume $f(x) = a_1 x^2 + b_1 x + c_1$
 $= a_2 x^2 + b_2 x + c_2$
 $= a_n x^n + b_n x + c_n$



of basic equations = (n-1 data) - 1

of unknown $3 + 6 = 18$ unknown
 we need $18 - 3 = 15$ eq = 18

$(x_0, y_0) - (x_1, y_1) \begin{cases} a_1 x_0^2 + b_1 x_0 + c_1 = y_0 \\ a_1 x_1^2 + b_1 x_1 + c_1 = y_1 \end{cases}$

$(x_1, y_1) - (x_2, y_2) \begin{cases} a_2 x_1^2 + b_2 x_1 + c_2 = y_1 \\ a_2 x_2^2 + b_2 x_2 + c_2 = y_2 \end{cases}$

$(x_5, y_5) - (x_6, y_6) \begin{cases} a_6 x_5^2 + b_6 x_5 + c_6 = y_5 \\ a_6 x_6^2 + b_6 x_6 + c_6 = y_6 \end{cases}$

12 equation.

slope = slope

ask & relieve

$$\textcircled{a} \quad x \quad \frac{d}{dx} (a_1 x^2 + b_1 x + c_1) = \frac{d}{dx} (a_2 x^2 + b_2 x + c_2)$$

$$2a_1 x + b_1 = 2a_2 x + b_2$$

$$\textcircled{a} \quad (x_1, y_1) \quad 2a_1 x_1 + b_1 - 2a_2 x_1 - b_2 = 0$$

$$(x_2, y_2) \quad 2a_2 x_2 + b_2 - 2a_3 x_2 - b_3 = 0$$

$$(x_3, y_3) \quad 2a_3 x_3 + b_3 - 2a_4 x_3 - b_4 = 0$$

$$(x_4, y_4) \quad 2a_4 x_4 + b_4 - 2a_5 x_4 - b_5 = 0$$

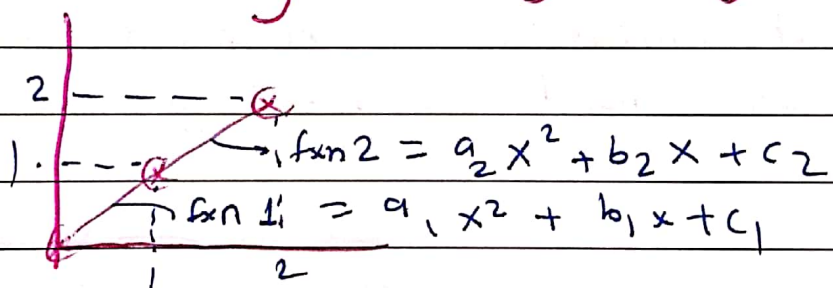
$$(x_5, y_5) \quad 2a_5 x_5 + b_5 - 2a_6 x_5 - b_6 = 0$$

$$12 \text{ eq} \rightarrow \text{either or } \boxed{a_1 = 0}$$

$$\boxed{a_7 = 0}$$

$\textcircled{\text{ex}}$ Given the following data found $y(1.5)$

x	y
0	0
1	1
2	2



$$n\text{-data} = 3$$

$$n\text{-eq} = n-1 = 3-1 = 2$$

$$n\text{-unknown} = 3(3) - 3 = 6$$

$$E_1 : a_1 x_0^2 + b_1 x_0 + c_1 = y_0$$

$$E_2 : a_1 x_1^2 + b_1 x_1 + c_1 = y_1$$

$$E_3 : a_2 x_1^2 + b_2 x_1 + c_2 = y_1$$

$$E_4 : a_2 x_2^2 + b_2 x_2 + c_2 = y_2$$

$$E_5 : 2a_1 x_1 + b_1 - 2a_2 x_1 - b_2 = 0$$

$$E_6 : a_1 = 0$$



non-linear w.r.s.t a

$$0 = a_1(0)^2 + b_1(0) + c_1$$

$$1 = a_1(1)^2 + b_1(1) + c_1$$

$$1 = a_2(1)^2 + b_2(1) + c_2$$

$$2 = a_2(2)^2 + b_2(2) + c_2$$

$$0 = 2a_1(1) + b_1 - 2a_2(1) - b_2$$

$$0 = a_1$$

$$0 = a_1 + b_1 + c_1$$

$$1 = a_1 + b_1 + c_1$$

$$1 = a_2 + b_2 + c_2$$

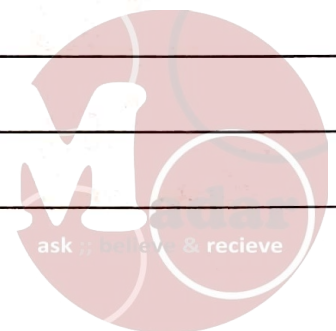
$$2 = 4a_2 + 2b_2 + c_2$$

$$0 = 2a_1 + b_1 - 2a_2 - b_2$$

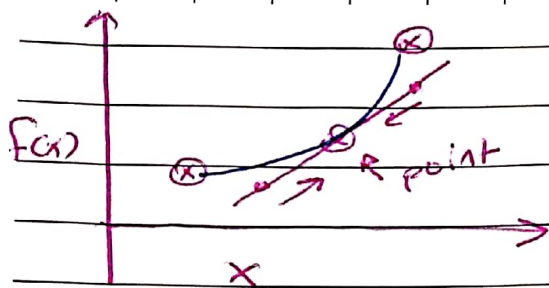
$$0 = a_1$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 2 & 1 \\ 2 & 1 & 0 & -2 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

straight for wand $y = (L, 5)$



Numerical Differentiation



$$\frac{dy}{dx} + y = 3$$

$$y = \ln x$$

$$\frac{dy}{dx} = ??$$

$$f'(x) = \frac{\Delta f(x)}{\Delta x}$$

Forward diff.

$$f'(x_i) = \frac{f(x_i + \Delta x) - f(x_i)}{x_i + \Delta x - x_i} =$$

$$= \boxed{\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}}$$

Backward Diff

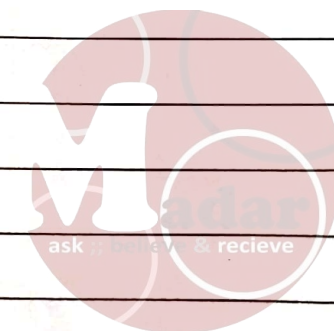
$$f'(x_i) = \frac{f(x_i) - f(x_i - \Delta x)}{x_i - (x_i - \Delta x)}$$

$$= \boxed{\frac{f(x_i) - f(x_i - \Delta x)}{\Delta x}}$$

Central diff

$$f'(x_i) = \frac{f(x_i + \Delta x) - f(x_i - \Delta x)}{(x_i + \Delta x) - (x_i - \Delta x)}$$

$$= \boxed{\frac{f(x_i + \Delta x) - f(x_i - \Delta x)}{2 \Delta x}}$$



Numerical differentiation

Ex) $f(x) = e^x$ Analytically? $f'(x) = e^x @ x = 1$
 $f'(1) = e^1 = 2.7182$

Forward! $f'(x) = \frac{f_{i+1} - f_i}{\Delta x} = \frac{e^{x+\Delta x} - e^x}{\Delta x}$

$\Delta x = 0.01 \Rightarrow \frac{f'(1)}{f} = \frac{e^{1.01} - e^1}{0.01} = 2.7319$

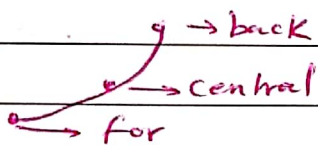
err = 0.5%

Backward: $f'(x) = \frac{f_i - f_{i-1}}{\Delta x}$; $f'(1) = \frac{e^1 - e^{0.99}}{0.01} = 2.7047$

err = 0.49%

central $f'(x) = \frac{f_{i+1} - f_{i-1}}{2\Delta x}$; $f'(1) = \frac{e^{1.01} - e^{0.99}}{2(0.01)} = 2.7183$

err = 0.001%



First derivative (3-points)

$$f'(x_i) = \frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2\Delta x} = \frac{-f(x_i + 2\Delta x) + 4f(x_i + \Delta x) - 3f(x_i)}{2\Delta x}$$

1st derivative !! (4-points)

$$f'(x_i) = \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x}$$

stiff Jacobian $\uparrow$ Terr diverges $\Delta x \uparrow$ (*)
 sharpness $\uparrow$ $\Delta x \downarrow$
 two factors $\Delta x \Rightarrow$ diverge & accuracy

Second derivative

$$f''_F(x_i) = (f_{i+2} - 2f_{i+1} + f_i) / (\Delta x)^2$$

$$f''_{B,C}(x_i) = (f_{i+1} - 2f_i + f_{i-1}) / (\Delta x)^2$$

ex $f''(1)$, $f(x) = e^x$ $x=1$ $\Delta x = 0.01$

$$f''_F(1) = (e^{1.02} - 2e^{1.01} + e^1) / (0.01)^2$$
$$= 2.745 \quad \text{err} = 1.33\%$$

$$f''_{B,C}(1) = (e^{1.01} - 2e^1 + e^{0.99}) / (0.01)^2$$
$$= 2.7183 \quad \text{err} = 0.02\%$$

Third derivative

$$f'''_F(x_i) = (f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i) / (\Delta x)^3$$

$$f'''_C(x_i) = (f_{i+2} - 2f_{i+1} + f_{i-1} - f_{i-2}) / 2(\Delta x)^3$$

ex $f(x) = x^3$ @ $x=1$ $\Delta x = 0.01$

$$f'(x) = 3x^2 \quad f''(x) = 6x \quad f'''(x) = 6$$

$$f'''_F(1) = \frac{(1.03)^3 - 3(1.02)^3 + 3(1.01)^3 - (1)^3}{(0.01)^3} = 6$$

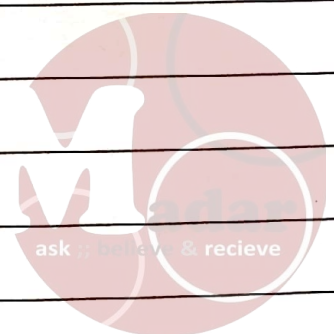
$$f'''_C(1) = \frac{(1.02)^3 - 2(1.01)^3 + 2(0.99)^3 - (0.98)^3}{2(0.01)^3}$$
$$= 6$$

4th derivative

$$f''''_f(x_i) = \frac{f_{i+4} - 4f_{i+3} + 6f_{i+2} - 4f_{i+1} + f_i}{(\Delta x)^4}$$

$$f''''_c(x_i) = \frac{f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2}}{(\Delta x)^4}$$

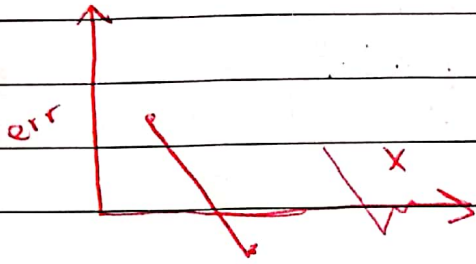
$$f(x) = \cos(x) e^{x^2} \quad \text{2nd order}$$



Solution of non-linear equation

Sol of single equation

$$x^2 + e^x = 3 \rightarrow x = \pm \sqrt{3 - e^x}$$
$$x^2 e^x + \ln x \cdot x^3 = \dots \rightarrow \dots$$



- (a) fixed point iterative method
- (b) Bisection method
- (c) Newton-Rapison method
- (d) Newton - 2nd order method

Secant Method :

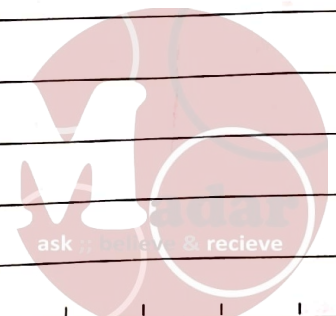
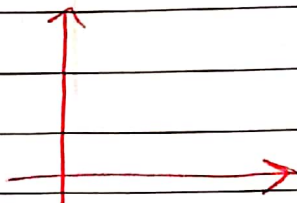
(f) Huler method

Fixed point method ?

$f(x) = 0 \rightarrow x = ?$ $f(x) = 0$ are the Roots

ex: $f(x) = e^{-x}$ $\rightarrow$ what is the value of x where $f(x) = 0$!?

x	e^{-x}
0	1
-1	$e = 2.7$
1	$\frac{1}{2.7}$

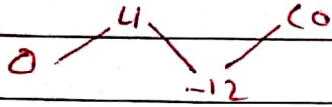


Fixed point iterated Method!

Ex: $f(x) = x^3 + x^2 - 3x - 3$

P.S: conv $\rightarrow$ Roots exist

div $\rightarrow$ Root do not exist



$$x = g(x)$$

$$x \pm g(x) = f(x) = 0 \Rightarrow x = g(x)$$

① $x^3 + x^2 - 3x - 3 = 0$

$$x = \frac{x^3 + x^2 - 3}{3}$$

x_{old}

x_{new}

1

$-\frac{1}{3}$

$-\frac{1}{3}$

-0.975

-0.975

-0.992 $\rightarrow$ Reaching one Root

② $x = \sqrt[3]{3x + 3 - x^2} = (3x + 3 - x^2)^{1/3}$

x_{old}

x_{new}

1

1.708

1.708

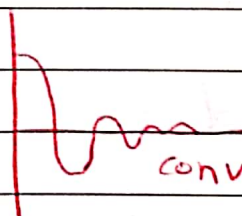
1.73

$\rightarrow$ err = 0.4
 $\rightarrow$ er = 0.2

1.73

1.732

$\rightarrow$ er = 0.1



③ $x^2 = 3x + 3 - x^3$

$$x = \sqrt[2]{3x + 3 - x^3} = (3x + 3 - x^3)^{1/2}$$

x_{old}

x_{new}

1

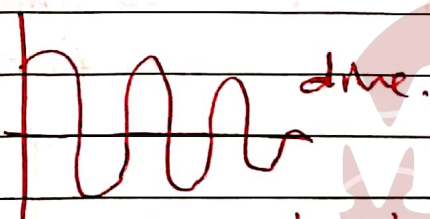
2.23

2.23

6.32

6.32

imaginary Root



conv use old as initial

list d @ recieve

⊗ to find the proper root

proper initial guess

!!

for fixed method iterate key point ; high order in side and the other in another side

highest order = other orders f_{xn}

⊗ [highest = other → initial guess] → can the Right

Newton's Method

Root, its speed ⊗
more powerfull ⊗

$$f(x) = f(x_0) + f'(x_0) \frac{\Delta x}{1!} + \frac{f''(x_0) (\Delta x)^2}{2!} + \dots$$

1 ← err →

Jap uplo uplo

$$f(x_0) + f'(x_0) \Delta x = 0$$

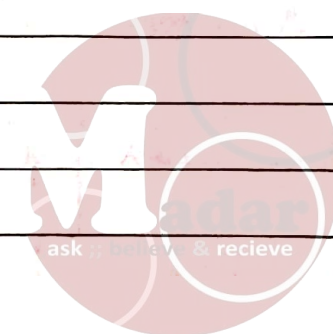
$$\Delta x = - \frac{f(x_0)}{f'(x_0)}$$

$$x_{0 \text{ New}} - x_{0 \text{ old}} = - \frac{f(x_0)}{f'(x_0)}$$

$$x_{0 \text{ New}} = x_{0 \text{ old}} - \frac{f(x_0)}{f'(x_0)}$$

(Ex) $f(x) = x^3 + x^2 - 3x - 3$
 $f'(x) = 3x^2 + 2x - 3$

$$x_{i+1} = x_i - \frac{x_i^3 + x_i^2 - 3x_i - 3}{3x_i^2 + 2x_i - 3}$$

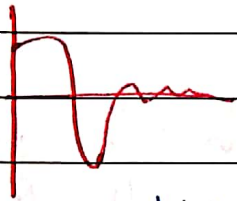


x_i	x_{i+1}	x_i	x_{i+1}	x_i	x_{i+1}
0.1	-1.08	2	1.769	-2	-1.8
-1.08	-0.989	1.769	1.732	-1.8	-1.738
-0.989	-0.999	1.732	1.732	-1.738	-1.72
-0.999	-1			-1.72	-1.72
-1	-1 → conv				

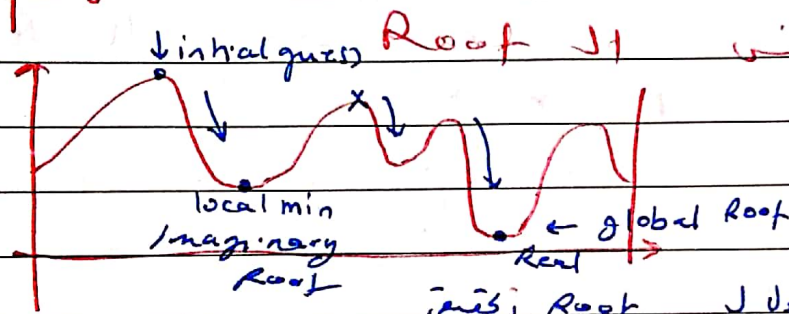
first Root

sec Root

Third Root



highly -ve & highly +ve
root find -2, 2



Root 11

I wanna the global minimum!!

Roots are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374, 375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387, 388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 402, 403, 404, 405, 406, 407, 408, 409, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440, 441, 442, 443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490, 491, 492, 493, 494, 495, 496, 497, 498, 499, 500, 501, 502, 503, 504, 505, 506, 507, 508, 509, 510, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 546, 547, 548, 549, 550, 551, 552, 553, 554, 555, 556, 557, 558, 559, 560, 561, 562, 563, 564, 565, 566, 567, 568, 569, 570, 571, 572, 573, 574, 575, 576, 577, 578, 579, 580, 581, 582, 583, 584, 585, 586, 587, 588, 589, 590, 591, 592, 593, 594, 595, 596, 597, 598, 599, 600, 601, 602, 603, 604, 605, 606, 607, 608, 609, 610, 611, 612, 613, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 625, 626, 627, 628, 629, 630, 631, 632, 633, 634, 635, 636, 637, 638, 639, 640, 641, 642, 643, 644, 645, 646, 647, 648, 649, 650, 651, 652, 653, 654, 655, 656, 657, 658, 659, 660, 661, 662, 663, 664, 665, 666, 667, 668, 669, 670, 671, 672, 673, 674, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690, 691, 692, 693, 694, 695, 696, 697, 698, 699, 700, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 721, 722, 723, 724, 725, 726, 727, 728, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 742, 743, 744, 745, 746, 747, 748, 749, 750, 751, 752, 753, 754, 755, 756, 757, 758, 759, 760, 761, 762, 763, 764, 765, 766, 767, 768, 769, 770, 771, 772, 773, 774, 775, 776, 777, 778, 779, 780, 781, 782, 783, 784, 785, 786, 787, 788, 789, 790, 791, 792, 793, 794, 795, 796, 797, 798, 799, 800, 801, 802, 803, 804, 805, 806, 807, 808, 809, 810, 811, 812, 813, 814, 815, 816, 817, 818, 819, 820, 821, 822, 823, 824, 825, 826, 827, 828, 829, 830, 831, 832, 833, 834, 835, 836, 837, 838, 839, 840, 841, 842, 843, 844, 845, 846, 847, 848, 849, 850, 851, 852, 853, 854, 855, 856, 857, 858, 859, 860, 861, 862, 863, 864, 865, 866, 867, 868, 869, 870, 871, 872, 873, 874, 875, 876, 877, 878, 879, 880, 881, 882, 883, 884, 885, 886, 887, 888, 889, 890, 891, 892, 893, 894, 895, 896, 897, 898, 899, 900, 901, 902, 903, 904, 905, 906, 907, 908, 909, 910, 911, 912, 913, 914, 915, 916, 917, 918, 919, 920, 921, 922, 923, 924, 925, 926, 927, 928, 929, 930, 931, 932, 933, 934, 935, 936, 937, 938, 939, 940, 941, 942, 943, 944, 945, 946, 947, 948, 949, 950, 951, 952, 953, 954, 955, 956, 957, 958, 959, 960, 961, 962, 963, 964, 965, 966, 967, 968, 969, 970, 971, 972, 973, 974, 975, 976, 977, 978, 979, 980, 981, 982, 983, 984, 985, 986, 987, 988, 989, 990, 991, 992, 993, 994, 995, 996, 997, 998, 999, 1000

Initial guess + proper writing of equation

Newton - 2nd order Method.

$$f(x) = f(x_0) + f'(x_0) \frac{\Delta x}{1!} + f''(x) \frac{(\Delta x)^2}{2!} + O(\text{err})$$

$$= 0$$

$$f(x_0) + f'(x_0) \frac{\Delta x}{1} + f''(x_0) \frac{(\Delta x)^2}{2} = 0$$

From Newton 1st order $\Delta x = -\frac{f(x_0)}{f'(x_0)}$

$$f(x_0) + \Delta x \left[f'(x_0) + f''(x_0) \frac{\Delta x}{2} \right] = 0$$

$$f(x_0) + \Delta x \left[f'(x_0) + \frac{f''(x_0)}{2} \left(-\frac{f(x_0)}{f'(x_0)} \right) \right] = 0$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\left[f'(x_i) - \frac{f(x_i) f''(x_i)}{2 f'(x_i)} \right]}$$

(Ex) $f(x) = 2x^3 - 3x^2 + x - 6$

$$f'(x) = 6x^2 - 6x + 1$$

$$f''(x) = 12x - 6$$

x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$
1	-6	1	6
1.3	-5.3	3.5	9.7
1.8	-2.3	9.6	15.6
1.99	-0.04	12.9	17.9
2	0	13	18

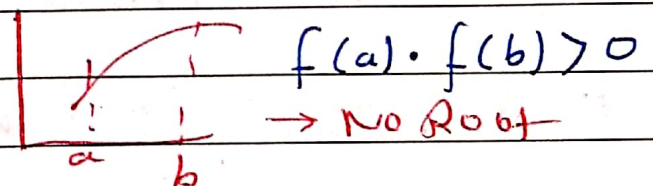
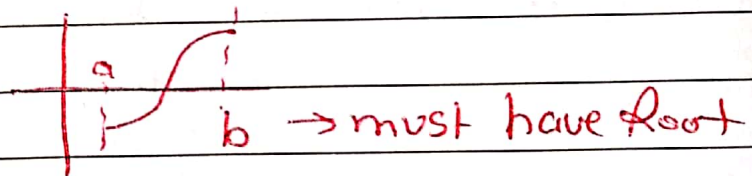
4) Bisection Method.

نقطة، نقطة وقف

$$a \leq x \leq b$$

$$f(a) \cdot f(b) < 0$$

⊖ و ⊕



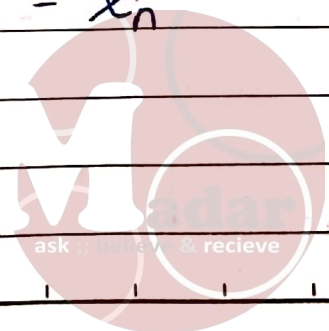
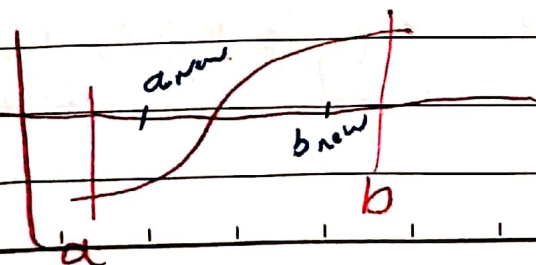
$$x_n = \frac{a+b}{2}, f(x_n) ??$$

if $f(a) \cdot f(x_n) > 0$

$$a_{\text{new}} = x_n$$

if $f(b) \cdot f(x_n) > 0$

$$b = x_n$$

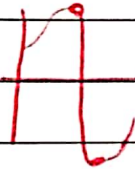


error ↑
high gues

ex) $f(x) = x^2 + x - 5$

في افترقا $\left[\begin{array}{l} a=0 \\ b=3 \end{array} \right] \left[\begin{array}{l} f(a) = f(0) = -5 \\ f(b) = f(3) = 7 \end{array} \right] \rightarrow f(a) * f(b) < 0$
the assumption is valid

a	b	x_n	$f(a)$	$f(b)$	$f(x_n)$	chang.
0	3	1.5	-5	7	-1.25	$a = x_n$
1.5	3	2.25	-1.25	7	2.3	$b = x_n$
1.5	2.25	1.87	-1.25	2.3	0.39	$b = x_n$
1.5	1.87	1.68	-1.25	0.39	-0.46	$a = x_n$
1.68	1.87	1.79	-0.46	0.39	-0.04	$a = x_n$

stiffness  sluggish

5) Secant method

$$f'(x) = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}$$
 ↑ error ↑

From Newton-R Method

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \rightarrow x_n = x_{n-1} - f(x_{n-1}) \left[\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} \right]$$

⊗ لا يكون عندك أكثر من معادلة عندك نقطتان فقط، لا تقدر انكسر الباش
two initial guesses

ex) $f(x) = 2x^3 - 3x^2 + x - 6$

$x_{n-1} = 3$

$x_{n-2} = 4$

⊗ في الباش يكون Root بين 3 و 4
+ ∫ bisection Method.

$$b = \frac{f_1 - f_0 - a h_1^2}{h_1}$$

$$\delta = \frac{h_2}{h_1}$$

$$\Delta x = -2c$$

$$\underline{b \pm \sqrt{b^2 - 4ac}}$$

$$X = X_0 - \frac{2f_0}{\dots}$$

$$b \pm \sqrt{b^2 - 4ac}$$

اکبر ما عکس

صبر الخير اصغر ما عكس
من الامام

$+ve$ 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100
 $-ve$ 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100

$$f(x) = \sin x - \frac{x}{2} \quad \text{--- (Rad)}$$

(Root = 1.89)

$$h = h_1 = h_2 = 0.2$$

for the first time we assume they are equal for simplicity but it's not obligated to assume that

$$x_0 = 2$$

$$x_1 = x_0 + h_1 = 2 + 0.2 = \boxed{2.2}$$

$$x_2 = x_0 - h = 2 - 0.2 = 1.8$$

three initial guesses.

~~$$f(x_0) = f_0 = \sin(2) - \frac{2}{2} = -0.0907$$~~

$$f(x_4) = f_1 = \sin(2.2) - \frac{2.2}{2} = -0.991$$

$$f(x_2) = f_2 = \sin(1.8) - \frac{1.8^2}{2} = -0.0738$$

$\uparrow x$ $\uparrow h$ $\odot$

$\delta = 1$ for First trial

$$a = \frac{(1)(-0.991) - (-0.0907)(1+1) + 0.07}{(1)(0.2)^2(1+1)} = -0.453$$

$$b = \frac{(-0.991) - (-0.6907) - (-0.453)}{0.2} (0.2)^2 = -0.913$$

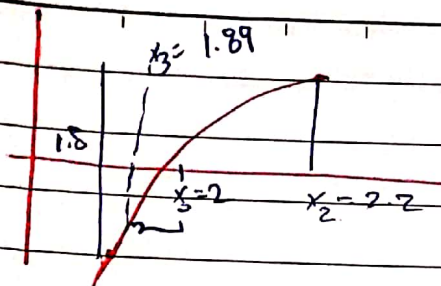
$$c = f(x_0) = f_0 = -0.907$$

the foot

$$X = 2 - 2(-0.0907)$$

$$= 1.89526$$

$$= 0.918 \pm \sqrt{(-0.913)^2 - 4(10.453)(-10907)}$$



$$x_0 = 1.89526$$

$$f_0 = \sin(1.89526) - \frac{1.8926}{2}$$

$$= 1.918 \times 10^{-4}$$

$$x_1 = 2$$

$$h_1 = x_1 - x_0 = 2 - 1.89526$$

$$x_2 = 1.8$$

$$h_2 = 0.10474$$

$$x_2 = 1.8$$

$$h_2 = x_0 - x_2 = 0.09526$$

$$\therefore \delta = \frac{h_2}{h_1} = \frac{0.09526}{0.10474} = 0.9094$$

$$c = f(x_0) = 1.918 \times 10^{-4}$$

$$f_1 = f(x_1) = f(2) = -0.0907$$

$$f_2 = f(x_2) = f(1.8) = 0.0738$$

$$a = -0.4728$$

$$b = -0.81820$$

$$X \approx 0 \text{ (Red noting value)}$$

$$X = 1.8954 \text{ exactly the root}$$



Solution of linear equation

New chapter
23th Jun - 2019

Solution linear equation

- 1) Gaussian elimination Method
- 2) Gaussian Jordan Method
- 3) Jacobi iterated Method
- 4) Gauss - sidel Method
- 5) L V Method

#1 Gaussian elimination Method

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots = b_2$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots = b_n$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$\vec{X} = A^{-1}b$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

ex

$$2x_1 - x_2 + 3x_3 = 2$$

$$x_1 + 2x_2 - x_3 = 3$$

$$3x_1 - 2x_2 + x_3 = -1$$

$$\begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -2 & 1 & -1 \end{bmatrix}$$

diagonal ← استاذنا

اصرف بالرقم الي به اول دهنه على اول

diagonal رقم

$$-\frac{1}{2} R_1 + R_2 = R_2$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ \rightarrow 3 & -2 & 1 & -1 \end{array}$$

$$-\frac{3}{2} + R_1 + R_3 = R_3$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ 0 & -0.5 & -3.5 & -4 \end{array}$$

$$\frac{0.5}{2.5} R_2 + R_3 = R_3$$

$$\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & 2.5 & -2.5 & 2 \\ 0 & 0 & -4 & -3.6 \end{array}$$

OR $-\frac{0.5}{2.5} R_2 - R_3 = R_3$

Algorithm:

For $K=1:N$

For $J=1:N+1 \rightarrow$ Right hand column.

Read a_{KJ}

END

END

For $k=1:N-1$

For $i=k+1:N$

For $j=k+1:N+1$

$$a_{ij} = a_{ij} - \frac{a_{ik} a_{kj}}{a_{kk}}$$

Now solve:

$$-4x_3 = -3.6$$

$$x_3 = 0.9$$

$$2.5x_2 - 2.5x_3 = 2$$

$$2.5x_2 - 2.5(0.9) = 2$$

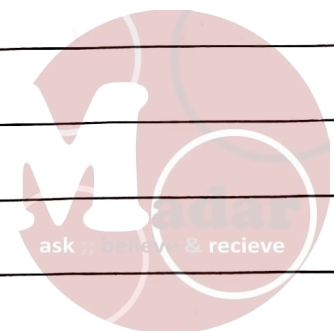
$$x_2 = 1.7$$

$$2x_1 - x_2 + x_3 = 2$$

$$2x_1 - \frac{4}{5} = 2$$

$$+\frac{4}{5} \quad +\frac{4}{5}$$

$$x_1 = 1/2$$



$$X_n = \frac{a_{n, n+1}}{a_{nn}}$$

for $k = n-1 : 1$

$$\text{sum} = 0$$

for $j = k+1 : N$

$$\text{sum} = \text{sum} + a_{kj} \cdot X_j$$

END

$$X_k = \frac{1}{a_{kk}} (a_{k, n+1} - \text{sum})$$

END k.

ON previous example :

$$a_{21} = -\frac{1}{2} \times 2 + 1 = 0$$

$$a_{31} = -\frac{3}{2} \times 2 + 3 = 0$$

$$a_{23} = -\frac{1}{2} \times 3 + -1 = -2.5$$

$$a_{32} = -\frac{3}{2} \times -1 - 2 = -0.5$$

$$a_{22} = -\frac{1}{2} \times -1 + 2 = 2.5$$

$$a_{33} = -\frac{3}{2} \times 3 + 1 = -3.5$$

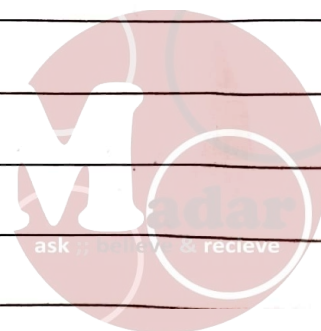
$$b_1 = 2$$

$$b_2 = -\frac{3}{2} \times 2 - 1 = -4$$

$$a_{32} = \frac{0.5}{2.5} \times 2.5 - 0.5 = 0$$

$$a_{33} = \frac{0.5}{2.5} \times -2.5 + (-3.5) = -4$$

$$b_3 = \frac{0.5}{2.5} \times 2 - 4 = -3.6$$



2) Gauss-Jordan Method.

For $k=1 : N-1$

$i = k+1 : N$

$j = N+1 : k_1-1$

$$a_{ij} = a_{ij} - \frac{a_{ik} a_{kj}}{a_{kk}}$$

END,

$$\begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

not necessary to be 0's
just to equal (1)

(ex) from previous example bring from it the eq's.

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -4 \\ 0 & 0 & 40 & 36 \end{array} \right]$$

Ⓢ ملاحظه:

$$\left[\begin{array}{c} 36 \\ 40 \end{array} \right] \leftarrow \begin{array}{c} 3.6 \\ 4 \end{array}$$

نہیں امیڑ ہوا 10 دکان عینہ کوڑا
سب او عوجہ جب عادی ای اختیار کرنا

① set diagonal = 1 (دکان کے ضروری اصول = 1)

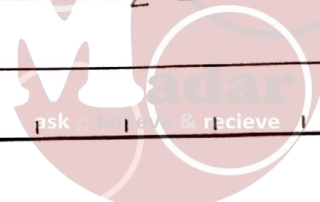
$$\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1 & 4/5 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$E_3 + E_2 = E_2$$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 3/2 & 1 \\ 0 & 1 & -1 & 4/5 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$

$$\frac{1}{2} E_2 + E_1 = E_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 74/40 \\ 0 & 1 & 0 & 68/40 \\ 0 & 0 & 1 & 36/40 \end{array} \right]$$



$$-\frac{3}{2} E_3 + E_1 = E_1$$

Then solve

$$\begin{bmatrix} 1 & 0 & 0 & | & 29/40 \\ 0 & 1 & 0 & | & 68/40 \\ 0 & 0 & 1 & | & 36/40 \end{bmatrix}$$

$$x_1 = 29/40 = 1/2$$

$$x_2 = 68/40 = 17/10$$

$$x_3 = 36/40 = 9/10$$

APPLY THE CODE:

$$N = 3$$

$$K = 1 \rightarrow 2$$

$$i = 2 \rightarrow 3$$

$$J = 4 \rightarrow K$$

$$K = 1$$

$$i = 2$$

$$J = 4, 3, 2, 1$$

→ looping

$$a_{24} = a_{24} - (a_{21}/a_{11})(a_{14})$$

$$a_{23} = a_{23} - \left(\frac{a_{21}}{a_{11}}\right)(a_{13})$$

#3) Jacobi - iterative Method

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots = b_2$$

$$a_{n1} x_1 + a_{n2} x_2 + a_{n3} x_3 + \dots = b_n$$

④ ملاحظہ کریں کہ coeff درجہ اولیٰ و اولیٰ درجہ اولیٰ coeff درجہ اولیٰ و اولیٰ درجہ اولیٰ
by mistake fixed print
div

$$\prod_{i=1}^n |a_{ij}| = \max$$

$$\begin{vmatrix} a_{11} & a_{22} & a_{33} \end{vmatrix} = \max \text{imum}$$



Ex

$$2x_1 + 7x_2 - 8x_3 = 10$$

$$5x_1 - 2x_2 + 6x_3 = 6$$

$$3x_1 + 5x_2 - 7x_3 = 3$$

x_1 which largest value!!

re arrange

$$5x_1 - 2x_2 + 6x_3 = 6$$

$$2x_1 + 7x_2 - 8x_3 = 10$$

$$3x_1 + 5x_2 - 7x_3 = 3$$

$$5 * 7 * 7 = 245$$

$$5 * 5 + 8 = 200 \quad \text{NOT THE Max}$$

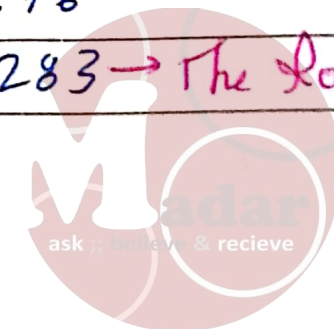
ل جربنا ترتيب اخر هين باح بعضيني Max بالجزء

$$x_1 = \frac{6 + 2x_2 - 6x_3}{5}$$

$$x_2 = \frac{10 - 2x_1 + 8x_3}{7}$$

$$x_3 = \frac{3 - 3x_1 - 5x_2}{7}$$

	#1	#2	#3	#4	...	#40
x_1	0	6/5	2.28	0.11		-0.305
x_2	0	10/7	0.545	2.04		4.76
x_3	0	-3/7	1.106	0.98		2.283 → The Root



4 : Gauss's Sidel Method

	#1	#2	#3	#22
x_1	0	6/5	0.6	6.306
x_2	0	$10 - 2(6/5) + 0 = 1.08$	2.24	4.76
x_3	0	$(3 - 3(6/5) - 5(1.08)) / -7 = 0.86$	1.42	2.84

لما تم احسابها بوضوح $(x_1 = 6/5)$ مش
(6) بوضوح آلفه اربعة

~~Development of / find U matrix /~~

(5) lower - Upper limit Method

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots a_{2n} \\ \vdots & \vdots & \vdots \dots \vdots \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots a_{2n} \\ a_{n1} & a_{n2} \dots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

L diagonal = 1

$$U \vec{x} = \vec{b}$$

$$L^{-1} \vec{y} = L^{-1} \vec{b} \Rightarrow \vec{y} = L^{-1} \vec{b} \rightarrow U \vec{x} = \vec{y}$$

$$\vec{x} = U^{-1} \vec{y}$$

Development of L & U Matrix

Elements of lower Matrix !!

$$l_{ij} = \frac{a_{ij}}{a_{ii}}$$

$$L = \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & & 1 & \\ \vdots & & & \ddots \end{bmatrix}$$

$$U_{ij\text{new}} = U_{ij\text{old}} - l_{ij} a_{ii}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ & u_{22} & \dots & u_{2n} \\ & & \ddots & \\ & & & u_{nn} \end{bmatrix}$$

(Ex)

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$\Rightarrow LU \Rightarrow$

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$l_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$a_{21} = a_{21} - l_{21} a_{11} = 64 - (2.56)(25) = 0$$

$$l_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

$$a_{22} = a_{22} - l_{21} a_{12} = 8 - (2.56)(5) = -4.8$$

$$l_{32} = \frac{a_{32}}{a_{22}} = ??$$

$$a_{23} = a_{23} - l_{21} a_{13} = 1 - (2.56)(1) = -1.56$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ & & u_{33} \end{bmatrix}$$

upper !!

$$l_{31} = 5.76 \text{ so } a_{31} = a_{31} - l_{31} a_{11}$$

$$a_{31} = 144 - (5.76)(25) = 0$$

$$a_{32} = 12 - (5.76)(5) = 16.8$$

$$a_{33} = 1 - (5.76)(1) = -4.76$$

New one !!

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.75 & -1.56 \\ 0 & 16.8 & -4.76 \end{bmatrix}$$

$\Rightarrow$

$$l_{32} = \frac{a_{32}}{a_{22}} = \frac{16.8}{-4.75} = 3.53$$

$$a_{32} = a_{32} - l_{32} a_{22} = (16.8) - (3.53)(-4.75) = 0$$

$$a_{33} = a_{33} - l_{32} a_{23} = -4.76 - (3.53)(-1.56) = 0.7$$

$$-4.76 - (3.53)(-1.56) = 0.7$$

$$\begin{array}{c} U \\ \left[\begin{array}{ccc|ccc} 25 & 5 & 1 & 1 & 0 & 0 \\ 0 & -4.75 & 1.56 & 2.56 & 1 & 0 \\ 0 & 0 & 0.7 & 5.76 & 3.53 & 1 \end{array} \right] \end{array} = A$$

$$Ax = b \rightarrow L U X = b \rightarrow \boxed{L y = b}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & y_1 \\ 2.56 & 1 & 0 & y_2 \\ 5.76 & 3.53 & 1 & y_3 \end{array} \right] = \left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & x_1 \\ 64 & 8 & 1 & x_2 \\ 144 & 12 & 1 & x_3 \end{array} \right] = \left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right]$$

straight forward!!

$$y_1 = 1$$

$$2.56 y_1 + y_2 = 2 \Rightarrow y_2 = -0.56$$

$$5.76 y_1 + 3.53 y_2 + y_3 = 3 \Rightarrow y_3 = -0.78$$

$$\boxed{L U X = b} \rightarrow \boxed{L y = b} \rightarrow \boxed{U X = y}$$

$$\left[\begin{array}{ccc|c} 25 & 5 & 1 & x_1 \\ 0 & -4.75 & -1.56 & x_2 \\ 0 & 0 & 0.7 & x_3 \end{array} \right] = \left[\begin{array}{c} 1 \\ -0.56 \\ -0.78 \end{array} \right]$$

backward calculation

$$0.7 * x_3 = -0.78 \rightarrow x_3 = -1.11$$

$$x_2 = -0.48$$

$$x_1 = -0.0116$$

$$X = \begin{bmatrix} -1.11 \\ 0.48 \\ -0.6116 \end{bmatrix}$$

Algorithm !!

Read A, b

set $U = A$

$L = I \rightarrow [1, 0]$

for $k = 1, n-1$

for $j = k+1, n$

$$l_{jk} = \frac{U_{jk}}{U_{kk}}$$

$$U_{j,k:n} = U_{j,k:n} - l_{jk} U_{k,k:n}$$

$U_{k,k:n}$ no. of variable

END

مثال من الدكتور

$$A = \begin{bmatrix} 5 & 5 & 1 & 4 & 0 & 1 & 4 & 1 & 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$[n, n] = \text{size}(A)$$

$$L = \text{eye}(n)$$

$$U = A$$

For $k = 1:n-1$

For $j = k+1:n$

$$L(j, k) = U(j, k) / U(k, k)$$

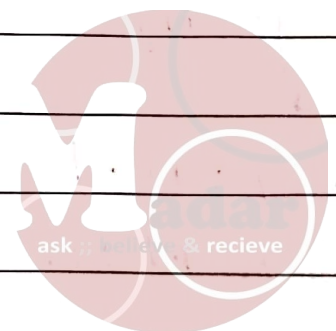
$$U(j, :) = U(j, :) - L(j, k) * U(k, :)$$

END

END

$$y = \text{Inv}(L) * b$$

$$x = \text{Inv}(U) * y$$



numerical solution of system of non-linear eq.

1. Iterative method
2. Newton's Raphson method.
- ① Iterative Method

Equation $\rightarrow$ Rearrangement

ex

$$2x_1^2 + x_2 = 3 \rightarrow x_1 = \left(\frac{3 - x_2}{2}\right)^{1/2}$$

$$x_1 + x_2^3 = 2 \rightarrow x_2 = (2 - x_1)^{1/3}$$

x_1	x_2
0	1.26
0.93	1.02
0.994	1.001
0.999	1.0001

First trial: $x_2 = 1.26$ when $x_1 = 0 \dots$ so on.

ex: ~~$\alpha \neq \beta \neq \gamma$~~

$$2x_1^2 + x_2 = 3$$

$$x_1^3 + x_2 = 2$$

دولت ← [have 'higher order degree' → x_1 'دولت']

⑦ $x_2 = 3 - 2x_1^2$

$$X_1 = (2 - X_2)^{1/3}$$

conv $\rightarrow$ odd $(\frac{1}{3})$ or 4

x_1	x_2
0	3
-1	1
1	1

$X_2 \leftarrow \text{نقطة في مساحة}$ $X_1 \leftarrow \text{initial guess}$ (*)
 $X_1 \leftarrow \text{نقطة في مساحة}$ $X_2 \leftarrow \text{نقطة في مساحة}$

④ روت نه جرسین... لاری... بلع صلیا
Root 1

(B) $x_1 = (3 - \frac{x_2}{2})^{1/2}$
 $x_2 = 2 - x_1^2$

(ex) Solve: $f_1 = 3x_1 - \cos(x_2 x_3) - \frac{1}{2}$
 $f_2 = x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06$
 $f_3 = e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3}$

by Newton Raphson Method

start with $x_1 = 0.1$ $x_2 = 0.1$ $x_3 = -0.1$

$$\begin{bmatrix} f_{1x_1} & f_{1x_2} & f_{1x_3} \\ f_{2x_1} & f_{2x_2} & f_{2x_3} \\ f_{3x_1} & f_{3x_2} & f_{3x_3} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2 \\ -f_3 \end{bmatrix}$$

$$f_{1x_1} = 3 \quad f_{1x_2} = x_3 \sin(x_2 x_3) \quad f_{1x_3} = x_2 \sin(x_2 x_3)$$

$$f_{2x_1} = 2x_1 \quad f_{2x_2} = -162(x_2 + 0.1) \quad f_{2x_3} = \cos(x_3)$$

$$f_{3x_1} = -x_2 e^{-x_1 x_2} \quad f_{3x_2} = -x_1 e^{-x_1 x_2} \quad f_{3x_3} = 20$$

$$\begin{bmatrix} 3 & 0.001 & -0.001 \\ 0.2 & -32.4 & 0.995 \\ -0.099 & -0.1 & 20 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} -1.2 \\ -2.27 \\ 8.46 \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -0.08 \\ -0.42 \end{bmatrix}$$

2nd iteration

$x_{1, \text{new}}$	$x_{1, \text{old}}$	Δx_1	0.1	0.5
$x_{2, \text{new}}$	$x_{2, \text{old}}$	Δx_2	0.1	$+0.02$
$x_{3, \text{new}}$	$x_{3, \text{old}}$	Δx_3	-0.42	-0.52

$$\epsilon = 0.001$$

i	x_1	x_2	x_3	ϵ
0	0.1	0.1	-0.1	—
1	0.5	0.02	-0.52	0.9
2	0.5	0.0015	-0.523	0.018
3	0.5	0.000012	-0.523	0.0015
4	0.5	0	-0.523	1.2×10^{-5}

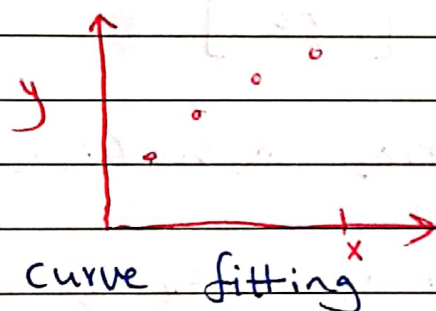
مقدار، کوچک (error)
 خطا کم، نه زیاد
 $\epsilon = 0.001$
 خطا 8 و 10 و 15
 خطا کم

$$err = \sum_{i=1}^n (x_{i\text{ new}} - x_{i\text{ old}})^2$$

$$err = (x_{1N} - x_{1old})^2 + (x_{2N} - x_{2old})^2 + (x_{3N} - x_{3old})^2$$

Curve fitting & interpolation

نمایش



curve fitting

1- Linear Regression $\Rightarrow$ fitting

$$y = a + bx$$

2- Multi linear Regression

$$y = a + bx_1 + cx_2$$

3- Non-linear Regression

$$y = a + bx + cx^2$$

4- Multi Non-linear Regression

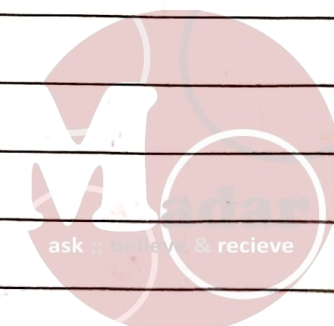
$$y = a + bx_1 + cx_1^2 + \dots + dx_2 + ex_2^2 + \dots$$

More general Regression It's including all the above Regression It's so powerfull

Linear Regression

X y_{exp} $y_{calculated}$ err

$$err = \min$$



$$\text{err} = \sum_{i=1}^n (y_{\text{exp}} - y_{\text{calc}})^2 = \sum (y_{\text{exp}} - (a + bx))^2$$

here, the best values for a, b should be; when err is the minimum

$$\frac{\partial \text{err}}{\partial a} = 0 \rightarrow 2 \sum (y_{\text{exp}} - (a + bx)) (-1) = -2 \left[\sum y_{\text{exp}} - (a + bx) \right]$$

$$\frac{\partial \text{err}}{\partial b} = 0 \rightarrow 2 \sum (y_{\text{exp}} - a - bx) (-x) = -2 \sum (x y_{\text{exp}} - a x - b x^2) = 0$$

$$\sum y_{\text{exp}} = \sum_{i=1}^n a + \sum b x$$

$$\sum x y_{\text{exp}} = \sum a x + \sum b x^2$$

$$\sum y_{\text{exp}} = na + b \sum x$$

$$\sum x y_{\text{exp}} = a \sum x + b \sum x^2$$

$$\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_{\text{exp}} \\ \sum x y_{\text{exp}} \end{bmatrix} \Rightarrow \text{simple linear eq. with two known}$$

$$a = \frac{\sum y_{\text{exp}} - b \sum x}{n}$$

$$b = \frac{n \sum y_i \text{exp} x - \sum y_i \sum x_i}{n \sum x_i^2 - (\sum x_i)^2}$$

Ex

n	x_i	y_{exp}	x_i^2	xy	y_{cal}	err^2 $(y - y_{\text{cal}})^2$	R^2 & L.V. $(y_i - y_{\text{avg}})^2$
1	1	4	1	4	3.3	$(4 - 3.3)^2 = 0.49$	$(4 - 6.75) = 7.5$
2	2	6	4	12	5.6	0.16	6.56
3	3	5	9	15	7.9	8.41	27.56
4	4	12	16	48	10.2	3.24	38.74
$\sum$	10	27	30	79		12.3	

Minimum value of err

↓ err ↑ fit → not indicated for the fit

$$b = \frac{4(79) - (27)(10)}{4(30) - (10)^2} = 2.3 \quad a = \frac{27 - (2.3)(10)}{4} = 1$$

$$y_{\text{calc}} = a + bx \rightarrow y_{\text{calc}} = 1 + 2.3x$$

Regression coefficient
coefficient of determination

$$R^2 = \frac{\sum (y_{\text{exp}} - y_{\text{avg}})^2 - \sum (y_{\text{exp}} - y_{\text{calc}})^2}{\sum (y_{\text{exp}} - y_{\text{avg}})^2}$$

$$0 < R^2 < 1 \rightarrow [0 - 1] \text{ best fit}$$

$$R^2 > 0.99 \quad 0.95 \quad 0.97 \rightarrow \text{accuracy}$$

$$\text{ave value} \rightarrow y \rightarrow y_{\text{avg}} = \frac{\sum y_{\text{exp}}}{n} = \frac{27}{4} = 6.75$$

یہاں پر y_{avg} کا استعمال ہے

$$R^2 = \frac{38.74 - 12.3}{38.74} = 0.68 \rightarrow \text{how accurate our fitting}$$

Multi linear Regression

$$y = a + bx + cx_2 + dx_3 + \dots$$

previously for a single Regression linear we do have this

$$\begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \end{bmatrix}$$

$$y = a_0 + a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

$$\begin{bmatrix} n & \sum x_1 & \sum x_2 & \dots & \sum x_n \\ \sum x_1 & \sum x_1^2 & \sum x_1 x_2 & \dots & \sum x_1 x_n \\ \sum x_2 & \sum x_1 x_2 & \sum x_2^2 & \dots & \sum x_2 x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum x_n & \sum x_1 x_n & \sum x_2 x_n & \dots & \sum x_n^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum x_1 y \\ \sum x_2 y \\ \vdots \\ \sum x_n y \end{bmatrix}$$

(ex) Given that

n	x_1	x_2	y	x_1^2	$x_1 x_2$	x_2^2	$x_1 y$	$x_2 y$
1	0	0	3	0	0	0	0	0
2	1	3	1	1	3	9	1	3
3	2	2	7	4	4	4	14	14
4	3	5	5	9	15	25	15	25
5	4	7	5	16	28	49	20	35
6	5	1	12	25	5	1	105	21
	15	18	42	55	55	88	155	98

↑ values to fit ↑ ind var

$$\begin{bmatrix} 6 & 15 & 18 \\ 15 & 55 & 55 \\ 18 & 55 & 88 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 42 \\ 155 \\ 98 \end{bmatrix} \rightarrow \text{system of the linear eq.}$$

$$y = 3 + 4x_1 - 2x_2$$

Non-linear regression

1. $y = ax^b \rightarrow \text{take a log} \rightarrow \underbrace{\ln y}_Z = \underbrace{\ln a}_C + b \underbrace{\ln x}_D$

x	y	$\ln x$	$\ln y$
---	---	---------	---------

$\Rightarrow b \checkmark$

$\Rightarrow c = \ln a \rightarrow a = e^c$

2. $y = a e^{bx} \rightarrow \text{Take } \ln \rightarrow \ln y = \ln a + bx$

x	y	$\ln y$
⋮	⋮	⋮

$$\ln a = \checkmark \quad b = \checkmark \quad a = e^{\text{value}}$$

3. $y = a x^b e^{cx} \rightarrow \text{take a log}$

$$\ln y = \ln a + b \ln x + cx = \ln a$$

Polynomial Regression !!

$$y = a_1 + a_2 x + a_3 x^2 + \dots$$

$$\text{err} = \sum (y - y_e)^2$$

$$\frac{\partial \text{err}}{\partial a_1} = \frac{\partial}{\partial a_1} \sum (a_1 + a_2 x + a_3 x^2 - y_e)^2$$

$$\frac{\partial \text{err}}{\partial a_2} = \frac{\partial}{\partial a_2} \sum ($$

المصنوفة الهاتية الى 2 - تنقل الى

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

n	$\sum x$	$\sum x^2$	$\sum x^3$	$\dots \sum x^n$	a_0		$\sum y$
$\sum x$	$\sum x^2$	$\sum x^3$	$\sum x^4$	$\sum x^{n+1}$	a_1		$\sum xy$
$\sum x^2$	$\sum x^3$	$\sum x^4$	$\sum x^5$	$\sum x^{n+2}$	a_2	=	$\sum x^2 y$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\sum x^n$	$\sum x^{n+1}$	$\sum x^{n+2}$	$\sum x^{n+3}$	$\sum x^{2n}$	a_n		$\sum x^n y$

n	$\sum x$	$\sum x^2$	x	y	x^2	x^3	x^4	xy	$x^2 y$
$\sum x$	$\sum x^2$	$\sum x^3$							
$\sum x^2$	$\sum x^3$	$\sum x^4$							
			$\sum$	$\sum$	$\sum$	$\sum$	$\sum$	$\sum$	$\sum$

$$y = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + b_1 x_2 + b_2 x_2^2 + b_3 x_2^3 +$$

$$c_1 x_1 x_2 + c_2 x_1 x_2^2 + c_3 x_1^2 x_2$$

$$y_1 = f(x_1, x_2)$$

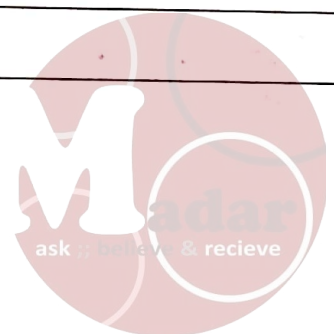
n	$\sum x_1$	$\sum x_1^2$	$\sum x_1^3$	$\sum x_1^4$	$\sum x_1^5$	$\sum x_1^6$	$\sum x_1^7$	$\sum x_1^8$	$\sum x_1^9$	$\sum x_1^{10}$	element	قالب
1	$\sum x_1$	$\sum x_1^2$	$\sum x_1^3$	$\sum x_1^4$	$\sum x_1^5$	$\sum x_1^6$	$\sum x_1^7$	$\sum x_1^8$	$\sum x_1^9$	$\sum x_1^{10}$	a_0	$\sum y$
2	$\sum x_1^2$	$\sum x_1^3$	$\sum x_1^4$	$\sum x_1^5$	$\sum x_1^6$	$\sum x_1^7$	$\sum x_1^8$	$\sum x_1^9$	$\sum x_1^{10}$	$\sum x_1^{11}$	a_1	$\sum x_1 y$
3	$\sum x_1^3$	$\sum x_1^4$	$\sum x_1^5$	$\sum x_1^6$	$\sum x_1^7$	$\sum x_1^8$	$\sum x_1^9$	$\sum x_1^{10}$	$\sum x_1^{11}$	$\sum x_1^{12}$	a_2	$\sum x_1^2 y$
4	$\sum x_1^4$	$\sum x_1^5$	$\sum x_1^6$	$\sum x_1^7$	$\sum x_1^8$	$\sum x_1^9$	$\sum x_1^{10}$	$\sum x_1^{11}$	$\sum x_1^{12}$	$\sum x_1^{13}$	a_3	$\sum x_1^3 y$
5	$\sum x_1^5$	$\sum x_1^6$	$\sum x_1^7$	$\sum x_1^8$	$\sum x_1^9$	$\sum x_1^{10}$	$\sum x_1^{11}$	$\sum x_1^{12}$	$\sum x_1^{13}$	$\sum x_1^{14}$	b_1	$\sum x_2 y$
6	$\sum x_1^6$	$\sum x_1^7$	$\sum x_1^8$	$\sum x_1^9$	$\sum x_1^{10}$	$\sum x_1^{11}$	$\sum x_1^{12}$	$\sum x_1^{13}$	$\sum x_1^{14}$	$\sum x_1^{15}$	b_2	$\sum x_2^2 y$
7	$\sum x_1^7$	$\sum x_1^8$	$\sum x_1^9$	$\sum x_1^{10}$	$\sum x_1^{11}$	$\sum x_1^{12}$	$\sum x_1^{13}$	$\sum x_1^{14}$	$\sum x_1^{15}$	$\sum x_1^{16}$	b_3	$\sum x_2^3 y$
8	$\sum x_1^8$	$\sum x_1^9$	$\sum x_1^{10}$	$\sum x_1^{11}$	$\sum x_1^{12}$	$\sum x_1^{13}$	$\sum x_1^{14}$	$\sum x_1^{15}$	$\sum x_1^{16}$	$\sum x_1^{17}$	c_1	$\sum x_1 x_2 y$
9	$\sum x_1^9$	$\sum x_1^{10}$	$\sum x_1^{11}$	$\sum x_1^{12}$	$\sum x_1^{13}$	$\sum x_1^{14}$	$\sum x_1^{15}$	$\sum x_1^{16}$	$\sum x_1^{17}$	$\sum x_1^{18}$	c_2	$\sum x_1 x_2^2 y$
10	$\sum x_1^{10}$	$\sum x_1^{11}$	$\sum x_1^{12}$	$\sum x_1^{13}$	$\sum x_1^{14}$	$\sum x_1^{15}$	$\sum x_1^{16}$	$\sum x_1^{17}$	$\sum x_1^{18}$	$\sum x_1^{19}$	c_3	$\sum x_1^2 x_2 y$

المعاجيب

$$y = 1 + 2x_1 + 3x_1^2 + 4x_1^3 + 5x_2 + 6x_2^2 + 7x_2^3 + 8x_1x_2 + 9x_1x_2^2 + 10x_1^2x_2$$

x_1	x_2	y	$x_1^2 x_2$
0	0		
↓	↓		

0	10
↓	↓
1	10



Interpolation

→ one variable

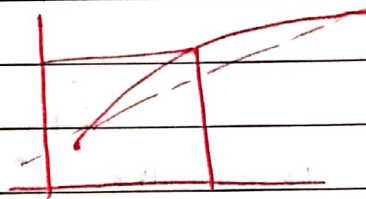
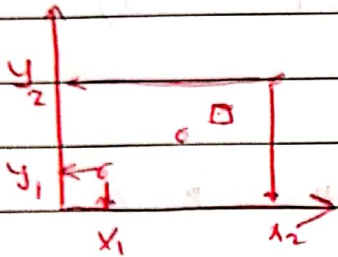
→ two variable

lagrange interpolation

Newton's interpolation

spline

langrage interpolation :



$$F(x) = a_1 + a_2 x + a_3 x^2 + \dots + a_n x^n$$

$$F_p(x) = \sum_{j=0}^n L_j(x) f(x_j)$$

$$L_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^n \frac{x - x_k}{x_j - x_k}$$

← j=0 ←

the order of the degree excluded → because the condition

Ex

$$f(x) = a + bx$$

$$f_1(x) = \sum_{j=0}^1 L_j(x) f(x_j)$$

$$= L_0(x) f(x_0) + L_1 f(x_1)$$

What is L_0 ?

$$L_0(x) = \prod_{\substack{k=0 \\ k \neq j}}^1 \frac{x - x_k}{x_j - x_k} = \frac{x - x_1}{x_0 - x_1}$$

the order of the degree excluded → because the condition

$$L_1(x) = \prod_{\substack{k=0 \\ k \neq j}}^1 \frac{x - x_k}{x_j - x_k} = \frac{x - x_0}{x_1 - x_0}$$

↓
= L

$$f_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

Ex Find $f_1(3)$, find the value $f(3)$ based on the first order langrage interpolation

x	y
$x_0 \leftarrow 2$	1.5 $\rightarrow f_0(x) = y_0$
$x_1 \leftarrow 5$	4 $\rightarrow f_1(3)$
8	6.5
11	9

$$f_1(x) = \frac{x-5}{2-5} f(2) + \frac{x-2}{5-2} f(5)$$

$$f_1(x) = \frac{x-5}{-3} (1.5) + \frac{x-2}{3} (4)$$

$$f_1(3) = \frac{3-5}{-3} (1.5) + \frac{3-2}{3} (4)$$

$$f_1(3) = 2.33$$

To check $\rightarrow$ use the (التحقق من الإجابة)

$$\frac{f(3) - 1.5}{4 - 1.5} = \frac{x - 2}{3 - 2} \quad (\text{check}) \quad \text{so;}$$

$$f(3) = \frac{3-2}{3} (4-1.5) + 1.5 = 2.33$$

we notice it's the same answer coz; it's linear

Quadratic Interpolation second order!!

$$f_2(x) = \sum_{j=0}^2 L_j(x) f(x_j) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

First $\rightarrow$ term of relation coeff const
 $\rightarrow$ term of n coffe of x

second $\rightarrow$ term of x^2

x	y
x_0	"
x_1	"
x_2	"

} interpolation

معدل (2) و معدل (2)
 قيمتين !!

order 2

$$L_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^2 \frac{x - x_k}{x_j - x_k} = \left(\frac{x - x_0}{x_0 - x_1} \right) \left(\frac{x - x_2}{x_0 - x_2} \right) = L_0$$

$$L_1(x) = \prod_{\substack{k=0 \\ k \neq 1}}^2 \frac{x - x_k}{x_1 - x_k} = \left(\frac{x - x_0}{x_1 - x_0} \right) \left(\frac{x - x_2}{x_1 - x_2} \right)$$

(excluded)

$$L_2(x) = \prod_{\substack{k=0 \\ k \neq 2}}^2 \frac{x - x_k}{x_2 - x_k} = \left(\frac{x - x_0}{x_2 - x_0} \right) \left(\frac{x - x_1}{x_2 - x_1} \right)$$

(c) Find $f_2(2.5)$

x	y		
1	0		
2	1	$\rightarrow x_0$	$L_0(x) = \left(\frac{x-3}{2-3} \right) \left(\frac{x-4}{2-4} \right)$
3	2	$\rightarrow 2.5$	$L_1(x) = \left(\frac{x-2}{3-2} \right) \left(\frac{x-4}{3-4} \right)$
4	3	$\rightarrow x_1$	$L_2(x) = \left(\frac{x-2}{4-2} \right) \left(\frac{x-3}{4-3} \right)$
5	5	$\rightarrow x_2$	
6	7		

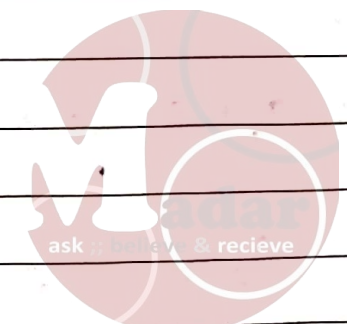
Substitute !! $L_0 = \frac{1}{2} (x-3)(x-4)$

$$L_1 = - (x-2)(x-4)$$

$$L_2 = \frac{1}{2} (x-2)(x-3)$$

$$F_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$F_2(2.5) = \frac{1}{2} (2.5-3)(2.5-4)(1) + - (2.5-2)(2.5-4)(2) + \frac{1}{2} (2.5-2)(2.5-3)(3)$$



Numerical Interpolation

2. Newton's interpolation

1st order

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) \\ &= f(x_0) + f'(x_0)(x - x_0) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \end{aligned}$$

x	f(x)
-	-
-	- $\rightarrow x_0$
-	- $\rightarrow x$
-	- $\rightarrow x_1$

2nd order $\rightarrow$ 2nd derivative

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &= f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)(x - x_1) \\ &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] (x - x_0)(x - x_1) \end{aligned}$$

$$\left[\begin{array}{l} * (x - x_0) \\ * (x - x_1) \end{array} \right]$$

Higher order:

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \\ &= a_0 + \sum_{i=1}^n \prod_{j=0}^{i-1} (x - x_j) a_{i+1} \end{aligned}$$

$\rightarrow$ simple case $\leftarrow$

(Ex)

i	x	y
0	0	1
1	1	2.25
2	2	3.75
3	3	4.25
4	4	5.81

Find $f(2.4)$ using 3rd order method !!

$$F_3(2.4) = \sum_{k=0}^3 a_k \prod_{j=0}^{k-1} (x - x_j)$$

$$= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$= f(x_0) + \overset{f'(x_0)}{f(x_0, x_1)}(x - x_0) + f(x_0, x_1, x_2)(x - x_0)(x - x_1) + f(x_0, x_1, x_2, x_3)(x - x_0)(x - x_1)(x - x_2)$$

$$a_0 = f(x_0) = 1$$

$$a_1 = f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{2.25 - 1}{2 - 0} = 1.5$$

$$a_2 = f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{4.25 - 2.25}{3 - 1} = 1$$

$$f(x_0, x_1, x_2) = \frac{1 - 1.5}{3 - 0} = -0.5$$

$$a_3 = f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{5.81 - 4.25}{4 - 3} = 1.56$$

$$f(x_1, x_2, x_3) = \frac{1.56 - 1}{4 - 2} = 0.53$$

$$a_3 = \frac{0.53 - (-0.5)}{4-1} = 0.343$$

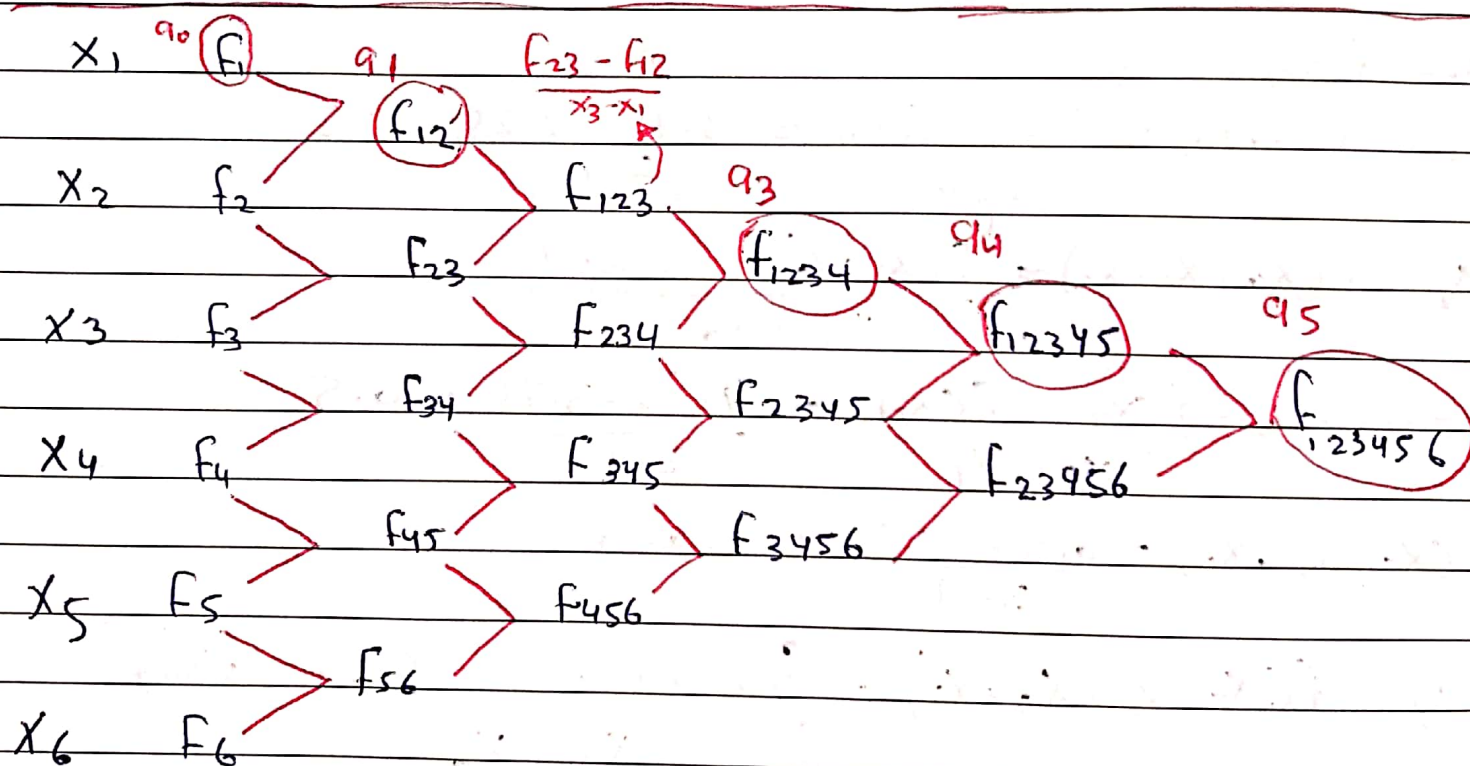
$$a_2 = -0.5 \quad a_1 = 1.5 \quad a_0 = 2.25$$

$$\begin{aligned} f(x) &= 2.25 + 1.5(x-x_0)(x-x_1) + 0.34(x-x_0)(x-x_1)(x-x_2) \\ &= 2.25 + 1.5(2.4-1) - 0.5(2.4-1)(2.4-2) + 0.34(2.4-1) \\ &\quad (2.4-2)(2.4-3) = 3.95 \end{aligned}$$

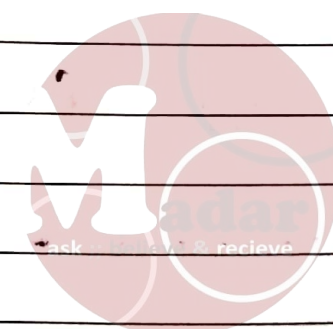
Newton Divided Difference

Table

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(5)}(x)$
		d_1	d_2	d_3	d_4	d_5



↓
no. of data = no. of order



Newton's Divided Difference

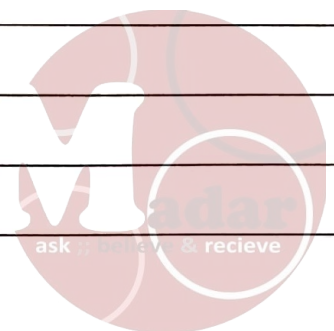
(EX)

X	Y	d_1	d_2	d_3	d_4
0	a_0 1	a_1 1.25	a_2 0.125	a_3 -0.208	a_4 0.137
1	2.25	1.5			
2	3.75	0.55	-0.5	0.34	
3	4.25	1.56	0.33		
4	5.81				

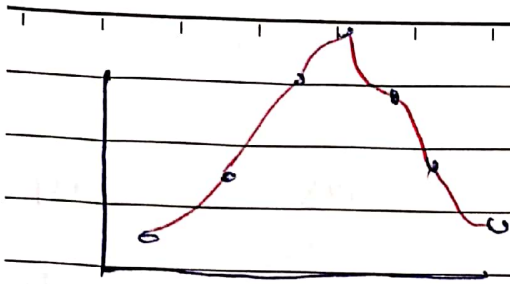
$$f(2.4) = 1 + 1.25(2.4-0) + 0.125(2.4-0)(2.4-1) - 0.208(2.4-0)(2.4-1)(2.4-2) + 0.137(2.4-0)(2.4-1)(2.4-2)(2.4-3) = 4.03$$

→ more accurate

بہتر اور زیادہ دقیق نتیجہ دیتا ہے *



Spline Method



$$y_1 = a_1 + b_1 x + c_1 x^2$$

$$y_2 = a_2 + b_2 x + c_2 x^2$$

$$\frac{dy_1}{dx} = \frac{dy_2}{dx} \text{ @ that given } x$$

eq's < plots.

Linear spline Method:

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

(*) (x_2, y_2)
(*) (x_1, y_1)
(*) (x_0, y_0)

$$y = \left(\frac{y_1 - y_0}{x_1 - x_0} \right) (x - x_0) + y_0 \quad x_0 \leq x \leq x_1$$

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x - x_1) + y_1 \quad x_1 \leq x \leq x_2$$

$$y = \left(\frac{y_3 - y_2}{x_3 - x_2} \right) (x - x_2) + y_2 \quad x_2 \leq x \leq x_3$$

(ex) given the following Data find $v(16)$

t	v	$v(16)?$
0	0	
10	227	
15	362	
20	517	
30	901	

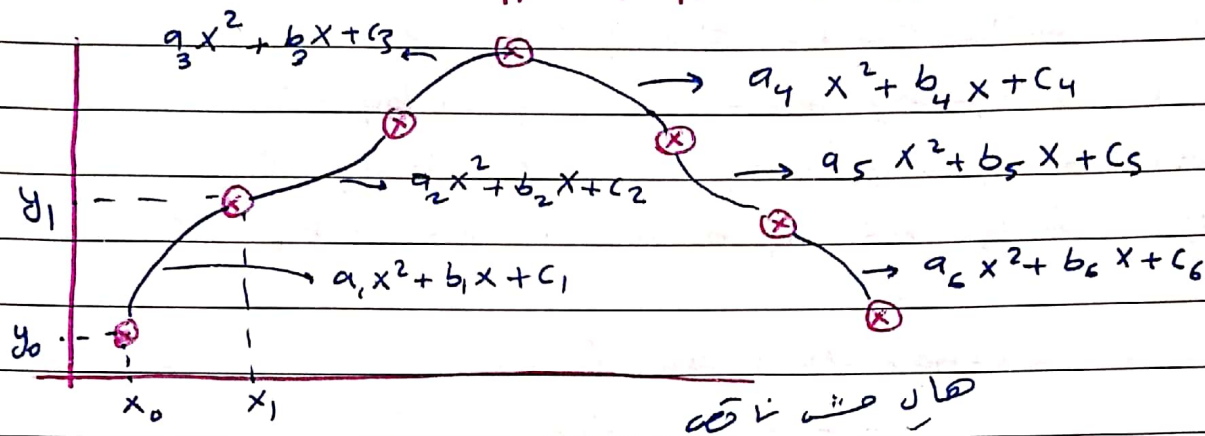
Sol: $x_0 = 15$ $y_0 = 362$
 $x_1 = 20$ $y_1 = 517$

$$\begin{aligned} v(16) &= \left(\frac{y_1 - y_0}{x_1 - x_0} \right) (t - t_0) + y_0 \\ &= \left(\frac{517 - 362}{20 - 15} \right) (16 - 15) + 362 \\ &= 393 \end{aligned}$$

Non-linear spline !!

* Assume all data passes a nonlinear function

* Assume $f(x) = a_1 x^2 + b_1 x + c_1$
 $= a_2 x^2 + b_2 x + c_2$
 $= a_n x^n + b_n x + c_n$



of basic equations $= (n-1 \text{ data}) - 1$

of unknown $3 + b = 18$ unknown
 we need $Bn - 3$ eq = 18

نقاط، 3، 18

$$(x_0, y_0) - (x_1, y_1) \begin{cases} a_1 x_0^2 + b_1 x_0 + c_1 = y_0 \\ a_1 x_1^2 + b_1 x_1 + c_1 = y_1 \end{cases}$$

$$(x_1, y_1) - (x_2, y_2) \begin{cases} a_2 x_1^2 + b_2 x_1 + c_2 = y_1 \\ a_2 x_2^2 + b_2 x_2 + c_2 = y_2 \end{cases}$$

$$(x_5, y_5) - (x_6, y_6) \begin{cases} a_6 x_5^2 + b_6 x_5 + c_6 = y_5 \\ a_6 x_6^2 + b_6 x_6 + c_6 = y_6 \end{cases}$$

⚡

12 equation

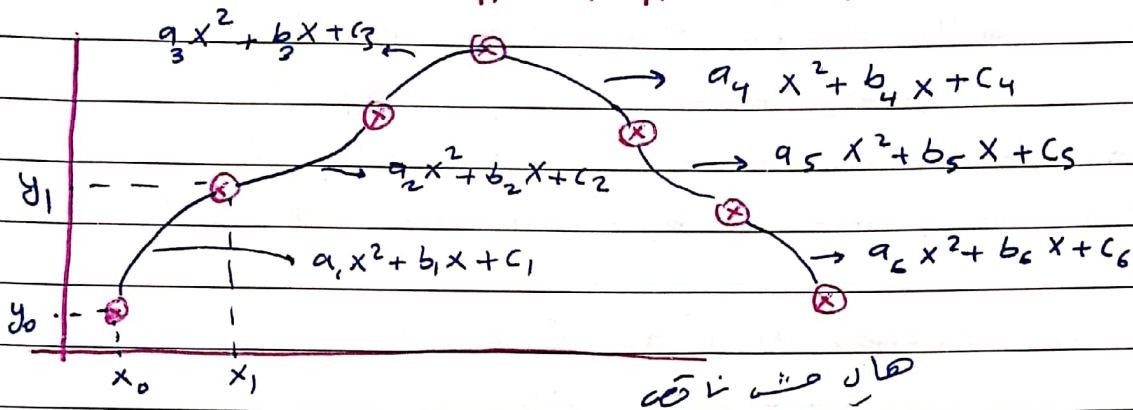
slope = slope

نقاط، 3، 18

Non-linear spline !!

* Assume all data passes a nonlinear function.

* Assume $f(x) = a_1 x^2 + b_1 x + c_1$
 $= a_2 x^2 + b_2 x + c_2$
 $= a_n x^n + b_n x + c_n$



of basic equations = (n-1 data) - 1

of unknown $3 + 6 = 18$ unknown
 we need $18 - 3 = 15$ eq = 18

= 18 معادلات، 3 نقاط *

$$(x_0, y_0) - (x_1, y_1) \begin{cases} a_1 x_0^2 + b_1 x_0 + c_1 = y_0 \\ a_1 x_1^2 + b_1 x_1 + c_1 = y_1 \end{cases}$$

$$(x_1, y_1) - (x_2, y_2) \begin{cases} a_2 x_1^2 + b_2 x_1 + c_2 = y_1 \\ a_2 x_2^2 + b_2 x_2 + c_2 = y_2 \end{cases}$$

$$(x_5, y_5) - (x_6, y_6) \begin{cases} a_6 x_5^2 + b_6 x_5 + c_6 = y_5 \\ a_6 x_6^2 + b_6 x_6 + c_6 = y_6 \end{cases}$$

⇓

12 equation.

slope = slope

→ 3 نقاط، 3 معادلات

$$\textcircled{a} \quad x \quad \frac{d}{dx} (a_1 x^2 + b_1 x + c_1) = \frac{d}{dx} (a_2 x^2 + b_2 x + c_2)$$

$$2a_1 x + b_1 = 2a_2 x + b_2$$

$$\textcircled{a} \quad (x_1, y_1) \quad 2a_1 x_1 + b_1 - 2a_2 x_1 - b_2 = 0$$

$$(x_2, y_2) \quad 2a_2 x_2 + b_2 - 2a_3 x_2 - b_3 = 0$$

$$(x_3, y_3) \quad 2a_3 x_3 + b_3 - 2a_4 x_3 - b_4 = 0$$

$$(x_4, y_4) \quad 2a_4 x_4 + b_4 - 2a_5 x_4 - b_5 = 0$$

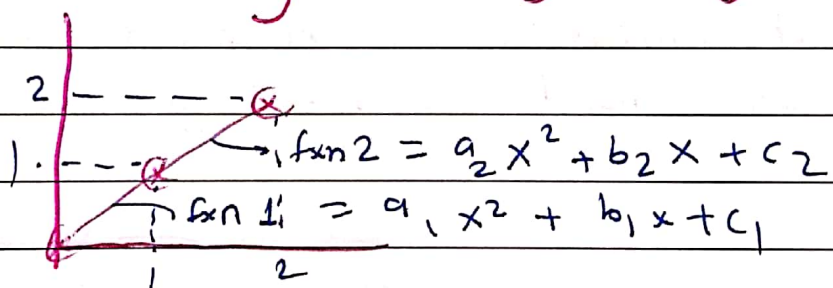
$$(x_5, y_5) \quad 2a_5 x_5 + b_5 - 2a_6 x_5 - b_6 = 0$$

$$12 \text{ eq} \rightarrow \text{either or } \boxed{a_1 = 0}$$

$$\boxed{a_7 = 0}$$

$\textcircled{\text{ex}}$ Given the following data found $y(1.5)$

x	y
0	0
1	1
2	2



$$n\text{-data} = 3$$

$$n\text{-eq} = n-1 = 3-1 = 2$$

$$n\text{-unknown} = 3(3) - 3 = 6$$

$$E_1 : a_1 x_0^2 + b_1 x_0 + c_1 = y_0$$

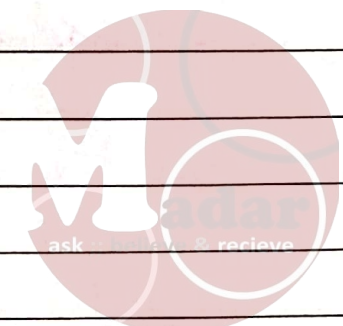
$$E_2 : a_1 x_1^2 + b_1 x_1 + c_1 = y_1$$

$$E_3 : a_2 x_1^2 + b_2 x_1 + c_2 = y_1$$

$$E_4 : a_2 x_2^2 + b_2 x_2 + c_2 = y_2$$

$$E_5 : 2a_1 x_1 + b_1 - 2a_2 x_1 - b_2 = 0$$

$$E_6 : a_1 = 0$$



non-linear w.r.s.t a

$$0 = a_1(0)^2 + b_1(0) + c_1$$

$$1 = a_1(1)^2 + b_1(1) + c_1$$

$$1 = a_2(1)^2 + b_2(1) + c_2$$

$$2 = a_2(2)^2 + b_2(2) + c_2$$

$$0 = 2a_1(1) + b_1 - 2a_2(1) - b_2$$

$$0 = a_1$$

$$0 = a_1 + b_1 + c_1$$

$$1 = a_1 + b_1 + c_1$$

$$1 = a_2 + b_2 + c_2$$

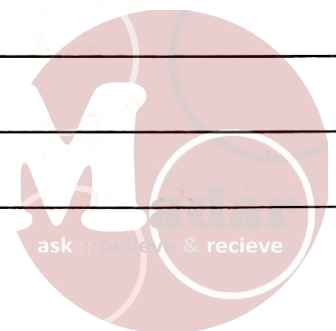
$$2 = 4a_2 + 2b_2 + c_2$$

$$0 = 2a_1 + b_1 - 2a_2 - b_2$$

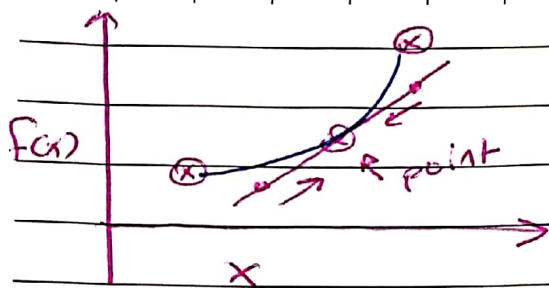
$$0 = a_1$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 2 & 1 \\ 2 & 1 & 0 & -2 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

straight for wand $y = (L, 5)$ $\leftarrow$



Numerical Differentiation



$$\frac{dy}{dx} + y = 3$$

$$y = \ln x$$

$$\frac{dy}{dx} = ??$$

$$f'(x) = \frac{\Delta f(x)}{\Delta x}$$

Forward diff.

$$f'(x_i) = \frac{f(x_i + \Delta x) - f(x_i)}{x_i + \Delta x - x_i} =$$

$$= \boxed{\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}}$$

Backward Diff

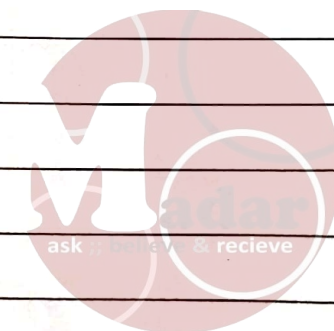
$$f'(x_i) = \frac{f(x_i) - f(x_i - \Delta x)}{x_i - (x_i - \Delta x)}$$

$$= \boxed{\frac{f(x_i) - f(x_i - \Delta x)}{\Delta x}}$$

Central diff

$$f'(x_i) = \frac{f(x_i + \Delta x) - f(x_i - \Delta x)}{(x_i + \Delta x) - (x_i - \Delta x)}$$

$$= \boxed{\frac{f(x_i + \Delta x) - f(x_i - \Delta x)}{2 \Delta x}}$$



Numerical differentiation

Ex) $f(x) = e^x$ Analytically? $f'(x) = e^x @ x = 1$
 $f'(1) = e^1 = 2.7182$

Forward! $f'(x) = \frac{f_{i+1} - f_i}{\Delta x} = \frac{e^{x+\Delta x} - e^x}{\Delta x}$

$\Delta x = 0.01 \Rightarrow \frac{f'(1)}{f} = \frac{e^{1.01} - e^1}{0.01} = 2.7319$

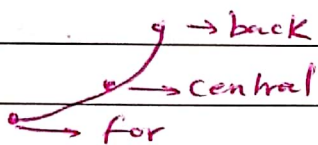
err = 0.5%

Backward: $f'(x) = \frac{f_i - f_{i-1}}{\Delta x}$; $f'(1) = \frac{e^1 - e^{0.99}}{0.01} = 2.7047$

err = 0.49%

central $f'(x) = \frac{f_{i+1} - f_{i-1}}{2\Delta x}$; $f'(1) = \frac{e^{1.01} - e^{0.99}}{2(0.01)} = 2.7183$

err = 0.001%



First derivative (3-points)

$$f'(x_i) = \frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2\Delta x} = \frac{-f(x_i + 2\Delta x) + 4f(x_i + \Delta x) - 3f(x_i)}{2\Delta x}$$

1st derivative !! (4-points)

$$f'(x_i) = \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x}$$

stiff Jacobian $\uparrow$ Terr diverges $\Delta x \uparrow$ (*)
 sharpness $\uparrow$ $\Delta x \downarrow$
 two factors $\Delta x \Rightarrow$ diverge & accuracy

Second derivative

$$f''_F(x_i) = (f_{i+2} - 2f_{i+1} + f_i) / (\Delta x)^2$$

$$f''_{B,C}(x_i) = (f_{i+1} - 2f_i + f_{i-1}) / (\Delta x)^2$$

ex $f''(1)$, $f(x) = e^x$ $x=1$ $\Delta x = 0.01$

$$f''_F(1) = (e^{1.02} - 2e^{1.01} + e^1) / (0.01)^2$$
$$= 2.745 \quad \text{err} = 1.33\%$$

$$f''_{B,C}(1) = (e^{1.01} - 2e^1 + e^{0.99}) / (0.01)^2$$
$$= 2.7183 \quad \text{err} = 0.02\%$$

Third derivative

$$f'''_F(x_i) = (f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i) / (\Delta x)^3$$

$$f'''_C(x_i) = (f_{i+2} - 2f_{i+1} + f_{i-1} - f_{i-2}) / 2(\Delta x)^3$$

ex $f(x) = x^3$ @ $x=1$ $\Delta x = 0.01$

$$f'(x) = 3x^2 \quad f''(x) = 6x \quad f'''(x) = 6$$

$$f'''_F(1) = \frac{(1.03)^3 - 3(1.02)^3 + 3(1.01)^3 - (1)^3}{(0.01)^3} = 6$$

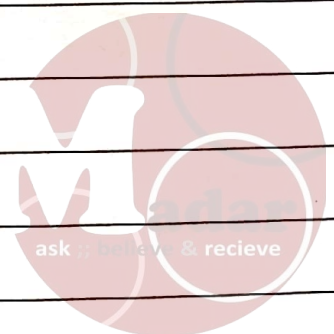
$$f'''_C(1) = \frac{(1.02)^3 - 2(1.01)^3 + 2(0.99)^3 - (0.98)^3}{2(0.01)^3}$$
$$= 6$$

4th derivative

$$f_f^{IV}(x_i) = \frac{f_{i+4} - 4f_{i+3} + 6f_{i+2} - 4f_{i+1} + f_i}{(\Delta x)^4}$$

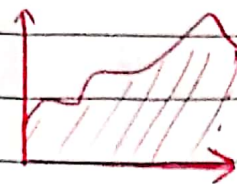
$$f_c^{IV}(x_i) = \frac{f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2}}{(\Delta x)^4}$$

$$f(x) = \cos(x) e^{x^2} \quad \text{2nd order}$$

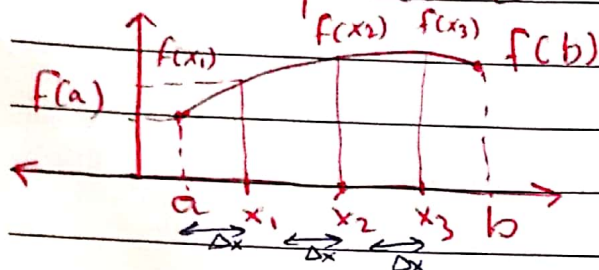


Numerical Integration

$$\int_0^1 \frac{2x^2 + x + 6}{x^3 - x^2 + 3} \cos x e^x dx$$



① Trapezoidal Rule



↓ Δx ↑ accurate ↑ more interval

$$I = \frac{f(x_1) + f(a)}{2} \Delta x + \frac{f(x_2) + f(x_1)}{2} \Delta x + \frac{f(x_3) + f(x_2)}{2} \Delta x + \frac{f(b) + f(x_3)}{2} \Delta x$$

$$= \frac{\Delta x}{2} [f(a) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(b)]$$

$$= \frac{\Delta x}{2} \left[f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b) \right]$$

$$= \frac{\Delta x}{2} \left[f(a) + 2 \sum_{i=1}^{n-1} f(a + i\Delta x) + f(x_n) \right]$$

(ex) $\int_0^1 \frac{2x^2 + x + 6}{x^3 - x^2 + 3} \cos x e^x dx$

$$\Delta x = 0.25!!$$

$$I = 0.25 \left[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1) \right]$$

$$f(0) = 2 \quad f(0.25) = 2.65 \quad f(0.5) = 3.52 \quad f(0.75) = 4.26$$

$$f(1) = 4.4$$

$$I = 0.25 \left[2 + 2(2.65 + 3.52 + 4.26) + 4.4 \right] = 3.407$$

→

② Simpson's $\frac{1}{3}$ Rule !!

[n data should be even]

$$I_{1/3} = \frac{\Delta x}{3} \left[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + f(b) \right]$$

$$\Delta x = \frac{b-a}{n} \rightarrow \text{even!!}$$

③ Simpson's $\frac{3}{8}$ rule

$$I_{3/8} = \frac{3}{8} \Delta x \left[f(a) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_4) + 3f(x_5) + 2f(x_6) + \dots + f(b) \right]$$

n $\Rightarrow$ should be divided by 3 [a, b]

$$I_{3/8} = \frac{3}{8} \left(\frac{b-a}{n} \right) \left[f(a) + 3f(a+\Delta x) + 3f(a+2\Delta x) + 2f(a+3\Delta x) + 3f(a+4\Delta x) + 3f(a+5\Delta x) + 2f(a+6\Delta x) + f(b) \right]$$

[Algorithm]

$$I_{1/3} = \frac{\Delta x}{3} \left[f_0 + 4 \sum_{i=1,3,5,\dots}^{n-1} f(x_i) + 2 \sum_{i=2,4,6,\dots}^{n-2} f(x_i) + f_n \right]$$

$$I_{3/8} = \frac{8}{3} \Delta x \left[f_0 + 3 \sum_{\substack{i=1 \\ i \neq 3,6,9}}^{n-1} f(x_i) + 2 \sum_{i=3,6,9,\dots}^{n-3} f(x_i) + f_n \right]$$

$$\text{(ex)} \int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = \textcircled{2} \rightarrow \text{Mathematically}$$

Numerically!!

Simpson's $1/3$:

$$\Delta x = \frac{b-a}{n} = \frac{\pi-0}{2} = \frac{\pi}{2}$$

$$I = \frac{\pi/2}{3} \left[f(0) + 4f(\pi/2) + f(\pi) \right] = \frac{\pi}{6} [0 + 4(1) + 0] = \frac{4}{6}\pi = 2.09$$

Let $n=3$ (wrong choice)

$\Delta x = \pi/3 \rightarrow$ wrong choice

$$I = \frac{\pi/3}{3} \left[f(0) + 4f(\pi/3) + 2f(2\pi/3) + f(\pi) \right]$$

$$= \frac{\pi}{9} \left[0 + 4\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) + 0 \right] = 1.81 \quad \times$$

Let $n=4$

$$\Delta x = \frac{b-a}{n} = \frac{\pi}{4}$$

$$I = \frac{\pi/4}{3} \left[f(0) + 4f(\pi/4) + 2f(2\pi/4) + 4f(3\pi/4) + f(\pi) \right]$$

$$= \frac{\pi}{12} \left[0 + 4 \cdot \frac{1}{\sqrt{2}} + 2(1) + 4(0.707) + 0 \right] = 2.003$$

(ex) $\int_0^{\pi} \sin x \, dx$ using Simpson's $3/8$ Rule!!

n should be divided by 3

$$n=3 \quad \Delta x = \frac{\pi-0}{3} = \frac{\pi}{3}$$

$$I_{3/8} = \frac{3}{8} \left(\frac{\pi}{3} \right) \left[f(0) + 3f(\pi/3) + 3f(2\pi/3) + f(\pi) \right]$$

$$= \frac{\pi}{8} \left[0 + \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} + 6 \right] = 2.039$$

$$n = 4$$

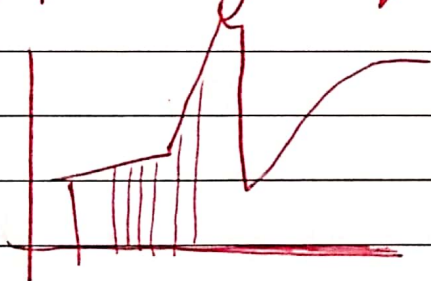
$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

$$I_{3/8} = \frac{3}{8} \left(\frac{\pi}{4} \right) \left[\sin(0) + 3 \sin(\pi/4) + 3 \sin(2\pi/4) + 2 \sin(3\pi/4) + \sin(\pi) \right] = 1.923 \quad \times$$

$$\text{let } n = 6, \quad \Delta x = \pi/6$$

$$I_{3/8} = \frac{3}{8} \left(\frac{\pi}{6} \right) \left[f(0) + 3f(\pi/6) + 3f(2\pi/6) + 2f(3\pi/6) + 3f(4\pi/6) + 3f(5\pi/6) + f(\pi) \right] = 2.009 \text{ exact}$$

Romberg integration

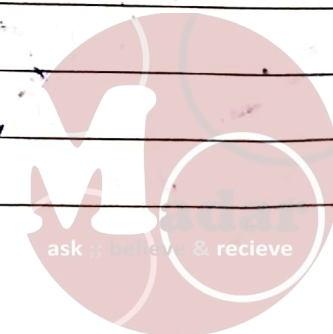


$$h_k = (b-a) 2^{-k}$$

$$k=1, h_1 = \Delta x_1 = \frac{b-a}{2}$$

$$k=2, h_2 = \Delta x_2 = \frac{b-a}{2^2} = \frac{1}{2} h_1$$

$$k=3, h_3 = \Delta x_3 = \frac{b-a}{2^3} = \frac{1}{2} h_2 = \frac{1}{4} h_1$$



$$n=1 \quad T_{11} = \frac{b-a}{2} [f(a) + f(b)]$$

$$n=2 \quad T_{21} = \frac{b-a}{2} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$T_{11} = \underbrace{I(f)}_{\text{exact}} + k_1 h^2 + o(h^4) \rightarrow \text{error} \rightarrow \text{error} = \text{overall error}$$

$$T_{21} = I(f) + k_1 \left(\frac{h}{2}\right)^2 + o(h^4)$$

$$T_{21} - T_{11} = 0 + k_1 \left[\left(\frac{h}{2}\right)^2 - h^2 \right] + o$$

$$k_1 = \frac{T_{21} - T_{11}}{h^2} \left(\frac{-4}{3} \right)$$

$$T_{21} = I(f) + k_1 \left(\frac{h}{2}\right)^2$$

$$T_{21} = I(f) + \frac{T_{21} - T_{11}}{h^2} \left(\frac{-4}{3} \right) \left(\frac{h}{2} \right)^2$$

$$I(f) = T_{21} - \frac{T_{21} - T_{11}}{h^2} \left(\frac{-4}{3} \right) \frac{h^2}{4}$$

$$I(f) = T_{21} + \frac{T_{21} - T_{11}}{3}$$

$$= T_{22}$$

$$\text{Similarly: } T_{32} = I(f) + k^2 \left(\frac{h}{2}\right)^2 + o(\text{error})$$

$$T_{j1} = \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{[2-1]^{(j-2)}} \left[f(a + (2j-1)h) \right] \right] \oplus$$

$$2 \sum_{j=1}^{[2-1]^{(j-2)}} \left[f(a + 2jh) + f(b) \right]$$

$$\textcircled{ex} \int_0^1 e^{-x^2} dx \quad !!$$

Trapezoidal rule!! $\int_a^b f(x) dx = \frac{h}{2} \left[f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right]$

where $h = b - a$

choose $n=1$, $h = \frac{b-a}{n} = \frac{1-0}{1} = 1$

$T_{11} = \frac{1}{2} [f(a) + f(b)] = 0.5 [1 + 0.367] = 0.6839$

choose $n=2$, $h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$

$f(a)$	$f(x)$	$f(b)$
1	$e^{-0.25}$	0.367

$T_{21} = \frac{h}{2} [f(a) + 2f(x) + f(b)] = 0.5 \left[1 + 2e^{-0.25} + 0.367 \right]$

$= \frac{1}{4} [1 + 2(0.7788) + 0.367] = 0.7313$

$T_{22} = T_{21} + \frac{(T_{21} - T_{11})}{3} = 0.7313 + \frac{0.7313 - 0.6839}{3}$

$= 0.747$ almost exact.

choose $n=4$, $h = \frac{1-0}{4} = \frac{1}{4}$

x	$f(x)$
0	1
0.25	$e^{-0.25}$
0.5	$e^{-0.5}$
0.75	$e^{-0.75}$
1	0.367

$T_{31} = \frac{1/4}{2} \left[f(0) + 2[f(1/4) + f(2/4) + f(3/4)] + f(1) \right]$

$= 0.7429$

π

(Ex) $\int_0^\pi \sin x dx = 2$

$T_{11} = ?$

$n=1$, $h = \frac{\pi-0}{1} = \pi$ $T_{11} = 0$

$T_{21} = ?$

$h = \frac{\pi-0}{2} = \frac{\pi}{2}$

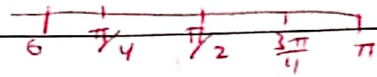
$0 \quad \pi/2 \quad \pi$

$$= \frac{h}{2} [f(a) + 2f(x) + f(b)] = \frac{\pi/2}{2} [\sin(0) + 2\sin(\frac{\pi}{2}) + \sin\pi]$$

$$= 0.785 * 2 = 1.57$$

$$T_{22} = T_{21} + \frac{(T_{21} - T_{11})}{3} = 1.57 + \left(\frac{1.57}{3}\right) = 2.09$$

$$T_{31} = ?? \quad h = \frac{\pi - 0}{4} = \frac{\pi}{4}$$



$$T_{31} = \frac{\pi/4}{2} [\sin(0) + 2\sin(\pi/4) + 2\sin(2\pi/4) + 2\sin(3\pi/4) + \sin(\pi)]$$

$$= \frac{\pi}{8} [0 + 1.41 + 2 + 1.41 + 0] = 1.89 \rightarrow \text{we have to increase from } T_{31} \rightarrow T_{33}$$

A / go, then !!

Define a, b, n

$$h_i = \Delta x = b - a$$

① Set $i=1, j=1$

$$T_{11} = \frac{h_1}{2} [f(a) + f(b)]$$

② $i = i + 1$

if $i > n \rightarrow \text{stop}$
else

$$\text{set } h_i = \frac{h_{i-1}}{2}$$

$$\textcircled{3} T_{ij} = 0.5 \left[T_{i-1,j} + h_{i-1} \sum_{k=1}^{(i-2)/2} f[a + (k-1/2)h_{i-1}] \right]$$

④ $j = j + 1$

if $j > i$ then set $j=1$ and go to step ②

else

$$\text{set } T_{i,j} = \frac{2^{2(j-1)} T_{i,j-1} - T_{i-1,j-1}}{2^{2(j-1)} - 1}$$

go to step ④ then go to the step ②

ex ① $a=0, b=\pi, n=4$

$$h_1 = \pi - 0 = \pi$$

$$i=1, j=1, T_{11} = \frac{\pi}{2} [f(0) + f(\pi)]$$

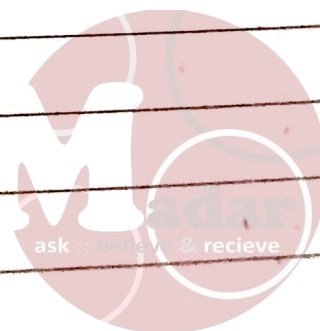
② $i = i + 1 = 2$

else, $h_2 = \frac{h_1}{2} = \frac{\pi}{2}$ to power $c^{(i-2)}$

$$T_{21} = \frac{1}{2} \left[T_{11} + h_1 \cdot \sum_{k=1}^2 [f(0) + 0.5 h_1] \right]$$

$$= \frac{1}{2} [T_{11} + h_1 (f(\pi/2))]$$

$$I_{21} = \frac{2^2 T_{21} - T_{11}}{2^2 - 1}$$



Gauss - Quadrature method

$$\int_{x=a}^{x=b} f(x) dx = \int_{t_1}^{t_2} f(t) dt = \sum_{i=1}^n C_i f(t_i)$$

$$X = \frac{(b-a)t}{2} + a + b$$

$$dx = \frac{b-a}{2} dt$$

$$\int_{x=a}^{x=b} f(x) dx = \frac{b-a}{2} \int_{-1}^1 \left(\left[\frac{b-a}{2} \right] t + a + b \right) dt$$

$$I = \frac{b-a}{2} \left[c_1 f(t_1) + c_2 f(t_2) \right] \quad \text{two points}$$

$$I = \frac{b-a}{2} \left[c_1 f(t_1) + c_2 f(t_2) + c_3 f(t_3) \right]$$

terms

c_1, c_2, c_3

t_1, t_2, t_3

2 - terms

$c_1 = c_2 = 1$

$t_1 = 0.5773$

$t_2 = -0.5773$

3 - terms

$c_1 = 0.555$

$t_1 = -0.7745$

$c_2 = 0.888$

$t_2 = 0$

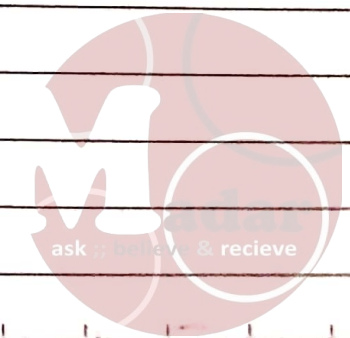
$c_3 = 0.555$

$t_3 = 0.774$

(ex): $\int_0^{\pi/2} \sin x dx = 1$

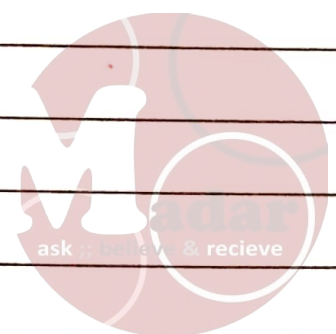
$$X = \frac{\pi/2 - 0}{2} t + \frac{\pi/2}{2} = \frac{\pi}{4} (t+1)$$

$$dx = \frac{\pi}{4} dt$$



$$\begin{aligned}
 I_2 &= \frac{\pi/2 - 0}{2} \left[1 * \sin\left(\frac{\pi}{4} (t+1)\right) + 1 * \sin\left[\frac{\pi}{4} (t_2+1)\right] \right] \\
 &= \frac{\pi}{4} \left[\sin\left(\frac{\pi}{4} * 1.5773\right) + 1 * \sin\left(\frac{\pi}{4} * 0.4227\right) \right] \\
 &= \frac{\pi}{4} \left[0.945 + 0.326 \right] = 0.998
 \end{aligned}$$

$$\begin{aligned}
 I_3 &= \frac{\pi}{4} \left[0.555 * \sin\left(\frac{\pi}{4} (1-0.7745)\right) + 0.888 * \right. \\
 &\quad \left. \sin\left(\frac{\pi}{4} (1+1)\right) + 0.555 * \sin\left(\frac{\pi}{4} (1+0.7745)\right) \right] \\
 &= 0.998
 \end{aligned}$$



* Tabulated Integrals

Ex:

$X: 0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \quad 1.2$

$F(x): 2.1 \quad 2.54 \quad 3.6 \quad 3.75 \quad 4.62 \quad 6.6 \quad 7.8$

$X: 0 \quad 0.15 \quad 0.45 \quad 0.6 \quad 0.68 \quad 0.9 \quad 1.2$

$Y: 0 \quad 0.01 \quad 0.4 \quad 0.9 \quad 2.1 \quad 15 \quad 105$

$$I_T = \frac{\Delta x}{2} [f(a) + 2f(x_1) + f(b)]$$

$$= \frac{0.2}{2} [2.1 + 2(2.54 + 3.6 + 3.75 + 4.62 + 6.6) + 7.8]$$

$$= 5.212$$

$$I_{S1/3} = \frac{\Delta x}{3} [f(a) + 2f(x_1) + 4f(x_2) + \dots + f(b)]$$

$$= \frac{0.2}{3} [2.1 + 2(2.54) + 4(3.6) + 2(3.75) + 4(4.62) + 2(6.6) + 7.8] = 4.57$$

$$I_{S3/8} = \frac{3\Delta x}{8} [f(a) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_4) + 3f(x_5) + f(b)]$$

$$= \frac{3}{8} (0.2) [2.1 + 7.8 + 3(2.54 + 3.6 + 4.62 + 6.6) + 2(3.75)]$$

$$= 5.24$$

ask :: believe & recieve

Ex1:-

X: 0 0.15 0.35 0.42 0.55 0.68

Y: 2.1 2.5 3.6 3.9 4.6 6.6

$$\Delta x_{avg} = \frac{\sum \Delta x_i}{n} \quad \left(\begin{array}{l} \text{where } \Delta x_i = x_i - x_{i-1} \\ (0.15-0) + (0.35-0.15) + (0.42-0.35) + (0.55-0.42) + (0.68-0.55) \end{array} \right)$$

data interval $\Delta x = 0.136$

$$= \left[\frac{0.68}{5} \right]$$

$$I_T = \frac{0.136}{2} \left[2.1 + 2(2.5 + 3.6 + 3.9 + 4.6) + 6.6 \right]$$
$$= 2.577$$

Example:-

$$x = 0 \quad 15 \quad 35 \quad 42 \quad 55 \quad 68$$

$$y = 2.1 \quad 2.5 \quad 3.6 \quad 3.9 \quad 4.6 \quad 6.6$$

$$\Delta x_{avg} = \frac{68-0}{5} = 13.6$$

$$I = \frac{13.6}{2} [2.1 + 2(2.5 + 3.6 + 3.9 + 4.6) + 6.6]$$

$$= \frac{13.6}{2} (37.9) = 257.7$$

$$I_{T_1} = \frac{\Delta x_1}{2} [f(a) + f(b)] = \frac{15-0}{2} [2.1 + 2.5] = 34.5$$

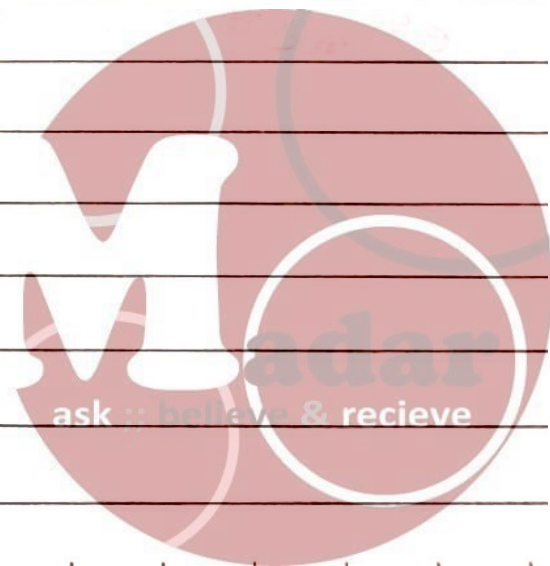
$$I_{T_2} = \frac{35-15}{2} [2.5 + 3.6] = 61$$

$$I_{T_3} = \frac{42-35}{2} [3.6 + 3.9] = 26.25$$

$$I_{T_4} = \frac{55-42}{2} [3.9 + 4.6] = 55.25$$

$$I_{T_5} = \frac{68-55}{2} [4.6 + 6.6] = 72.8$$

$$I_T = \sum I_i = 34.5 + 61 + 26.25 + 55.25 + 72.8 = 249.8$$



$$f(x) = 2.1658 + 8.6595x + 8.183x^2$$

6.8

$$\int_0^6 f(x) dx = 2.48$$

* Integral of infinite intervals

$$\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$$

$$\frac{x}{0} \quad y = \frac{1}{e^x - e^{-x}}$$

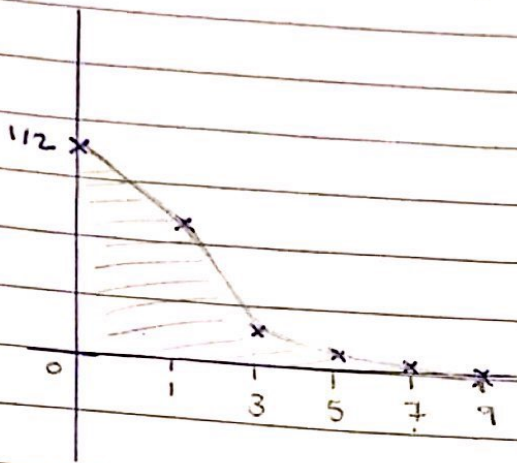
$$\frac{1}{e^1 + e^{-1}} = 0.32$$

$$\frac{3}{e^3 + e^{-3}} = 0.049$$

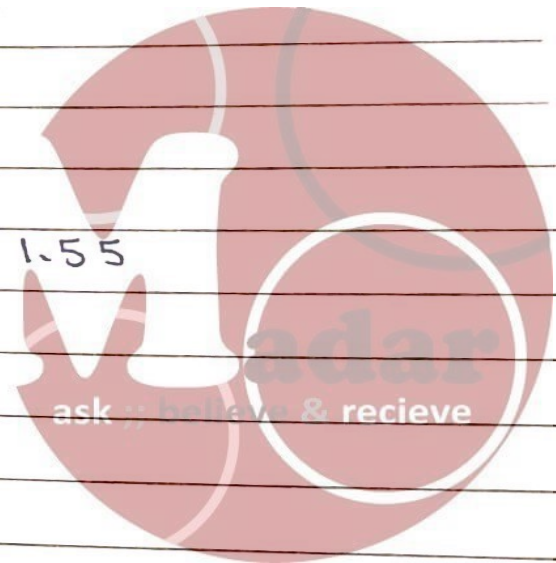
$$\frac{5}{e^5 + e^{-5}} = 8.7 \times 10^{-3}$$

$$\frac{7}{e^7 + e^{-7}} = 9.1 \times 10^{-4}$$

$$\frac{9}{e^9 + e^{-9}} = 1.23 \times 10^{-4}$$



$$\int_0^{\infty} \frac{dx}{e^x + e^{-x}} = \int_0^5 \frac{dx}{e^x + e^{-x}} + \int \frac{dx}{e^{-x}} = 1.55$$



* Gaussian Quadratic method :-

$$\int_{x=a}^{x=b} f(x) \cdot dx$$

$$= \int_{t_1}^{t_2} f(t) \cdot dt$$

$$= \sum C_i f(t_i)$$

$$X = \frac{(b-a)t + (b+a)}{2}, \quad dx = \frac{b-a}{2} \cdot dt$$

$$\int_{x=a}^{x=b} f(x) \cdot dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{(b-a)t + a+b}{2}\right) \frac{b-a}{2} \cdot dt$$

$$I = \frac{b-a}{2} (C_1 f(t_1) + C_2 f(t_2))$$

2-point

$$I = \frac{b-a}{2} (C_1 f(t_1) + C_2 f(t_2) + C_3 f(t_3))$$

3-point

*of Terms	C_1, C_2, C_3	t_1, t_2, t_3
2	$C_1 = C_2 = 1$	$t_1 = 0.5773$ $t_2 = -0.5773$
3	$C_1 = 0.555$ $C_2 = 0.888$ $C_3 = 0.555$	$t_1 = -0.7745$ $t_2 = 0$ $t_3 = 0.7745$

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⇒ Example

$$\int_0^{\pi/2} \sin x \cdot dx$$

Analytical

$$= -\cos x \Big|_0^{\pi/2}$$

$$= 1 *$$

Numr

$$x = \frac{(\frac{\pi}{2})t + \frac{\pi}{2}}{2} = \frac{\pi}{4}(t+1)$$

$$dx = \frac{\pi}{4} dt$$

$$*(2) \quad I = \frac{\pi}{4} \left(1 \cdot \sin\left(\frac{\pi}{4}(0.5773+1)\right) + \sin\left(\frac{\pi}{4}(0.5+73+1)\right) \right)$$

$$= 0.998 *$$

$$*(3) \quad I = \frac{\pi}{4} (0.097 + 0.627 + 0.5463)$$

$$= 0.998 *$$

ask ; believe & recieve

* Numerical Solution of ODE :-

$$\frac{dy}{dx} = f(x, y)$$

⇒ Example :-

$$\frac{dy}{dx} + y = x$$

OR

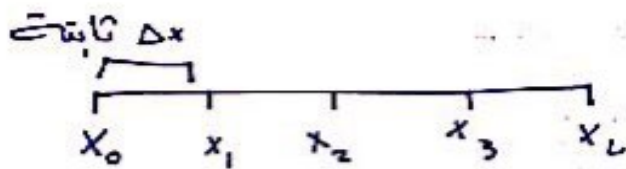
$$\frac{dy}{dx} + yx = e^x$$

- 1] Euler method
- 2] Prediction-corrector method
- 3] Runge-Kutta method
- 4] finite difference method

"Euler method"

$$\frac{dy}{dx} = f(x, y) \quad , \quad \frac{\Delta y}{\Delta x} = f(x, y)$$

when limit close To zero



$\Delta x \downarrow$ accu $\uparrow$
 $\Delta x \uparrow$ Error $\uparrow$

Condition - (a) $x = x_0$, $y = y_0$ init condition
OR (a) $x = x_L$, $y = y_L$

$$\Delta y = \Delta x (f_{x,y})$$

$$y_{i+1} - y_i = \Delta x f(x_i, y_i)$$

$$y_{i+1} = y_i + \Delta x f(x_i, y_i)$$

"Euler method"

$$\begin{array}{cccccc} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 \\ \hline x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\begin{cases} y_1 = y_0 + \Delta x f(x_0, y_0) \\ y_2 = y_1 + \Delta x f(x_1, y_1) \\ y_3 = y_2 + \Delta x f(x_2, y_2) \\ y_4 = y_3 + \Delta x f(x_3, y_3) \end{cases}$$

* Example *

$$\frac{dy}{dx} + y = x + 1$$

Solu for homogenous

$$y' + y = 0$$

$$r + 1 = 0$$

$$r = -1$$

$$y_c = c_1 e^{-x}$$

Solu for non-homogenous

$$y_p = Ax + B$$

$$y_p' = A$$

$$\Rightarrow A + Ax + B = x + 1$$

$$Ax = 1$$

$$A + B = 1$$

$$A = 1$$

$$B = 0$$

$$y_{cip} = x + c_1 e^{-x}$$

when $y(0) = 1$

$$c_1 = 1$$

$\Rightarrow$ So $y = x + e^{-x}$

This is a Analytical Solution

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* Numerical method 'solution'

$$\frac{dy}{dx} = x+1-y$$

$$0 \leq x \leq 0.1$$

$$\Delta x = \frac{b-a}{n} \text{ interval}$$

$$= \frac{0.1 - 0}{5}$$

$$= 0.02$$



Assume $n=5$

$n \uparrow$ accu $\uparrow$

$$y_1 = y_0 + \Delta x f(x_0, y_0)$$

$$y_1 = 1 + 0.02(0+1-1)$$

$$\boxed{y_1 = 1}$$

$$y_2 = y_1 + \Delta x f(x_1, y_1)$$

$$= 1 + 0.02(0.02+1-1)$$

$$\boxed{y_2 = 1.004}$$

$$y_3 = y_2 + \Delta x f(x_2, y_2)$$

$$y_3 = 1.004 + 0.002(0.04+1-1.004)$$

$$\boxed{y_3 = 1.00192}$$

$$\boxed{y_4 = 1.00268}$$

$$\boxed{y_5 = 1.00392}$$

$\Rightarrow$ Analytical Solution $\Rightarrow$ تعطي قيم x في

$$y_1 = 1.0002$$

$$y_2 = 1.00078$$

$$y_3 = 1.00176$$

$$y_4 = 1.003$$

$$y_5 = 1.0048$$

:



2] producer - euler method :- اکثر دقیق و آسان

$$y_{i+1}^{pre} = y_i + \Delta x \cdot f(x_i, y_i)$$

$$y_{i+1}^{cor} = y_i + \frac{\Delta x}{2} [f(x_i, y_i) + f(x_{i+1}, y_{i+1})]$$

→ example :-

Sol:- $\frac{dy}{dx} + y = x + 1, \quad y(0) = 1$

$$y_1^{pre} = 1$$

$$y_1^{cor} = 1 + \frac{(0.02)(0 + 1 - 1) + (0.02 + 1 - 1)}{2}$$

$$= 1.0002 \quad \text{compare with exact value} \checkmark$$

$$y_2^{pre} = 1.004 \quad \text{from Euler}$$

$$y_2^{cor} = \frac{1.002 + 0.02(0.02 + 1 - 1.002) + (0.04 + 1 - 1.0004)}{2}$$

y_1 ← باخذ آخر قیمت لا

$$= 1.00704$$

compare with exact value $\checkmark$

حائماً بستخدم آخر قيم طلت معي

ask :: believe & recieve

[3] Runge-Kutta method 5-

الطريقة الأكثر دقة

$$y_{i+1} = y_i + a_1 k_1 + a_2 k_2 + a_3 k_3 + \dots + a_n k_n$$

[To 4 almost
give
exact

$$y_{i+2} = y_i + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4]$$

$$k_1 = \Delta x f(x_i, y_i)$$

$$k_2 = \Delta x f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_1}{2}\right)$$

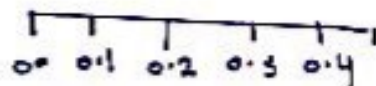
$$k_3 = \Delta x f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_2}{2}\right)$$

$$k_4 = \Delta x f(x_i + \Delta x, y_i + k_3)$$



→ example 3

$$\Delta x = 0.1$$



$$k_1 = 0.1 (0 + 1 - 1) = 0$$

$$k_2 = 0.1 (0 + \frac{0.1}{2} + 1 - 1) = 0.005$$

$$k_3 = 0.1 (0 + 0.05 + 1 - \frac{1 + 0.005}{2}) = 0.00475$$

$$k_4 = 0.1 (0 + 0.1 + 1 - 1 + 0.00475) = 0.009527$$

$$y_{\text{exact}} = x + e^{-x}$$

$$y(0.1) = e^{-0.1} + 0.1$$

$$= 1.0048$$

$$y_1 = 1 + \frac{1}{8} (0 + 2(0.005 + 0.00475))$$

$$y_1 = 1.0048$$

compare with exact

~~$$y_2 = y_1 + \frac{1}{8} (0 + 2(0.005 + 0.00475)) + 0.009527$$~~

to find y_2

$$\begin{cases} k_1 = \Delta x f(x_1, y_1) \\ \vdots \\ k_4 = \Delta x f(x_1 + \Delta x, y_1 + k_3) \end{cases}$$

$$y_2 = y_1 + \frac{1}{8} (k_1 + 2(k_2 + k_3) + k_4)$$

5.

ask ; believe & recieve

$$k_3 = \Delta x \cdot f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_2}{2}\right) \Rightarrow 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.005}{2}\right) \Rightarrow 0.00475$$

$$k_4 = 0.1 f(0 + 0.1, 1 + 0.00475) \Rightarrow 0.009527$$

$$y_1 = 1 + \frac{1}{6} [0 + 2 \cdot 0.005 + 2 \cdot 0.00475 + 0.009527] \Rightarrow 1.0048$$

[4] Finite difference methods

Exo: $\frac{d^2 y}{dx^2} - y = 0$, $y(0) = 1$, $y(1) = 2$, $0 \leq x \leq 1$

Solution: $\frac{d^2 y}{dx^2} = y_{i-1} - 2y_i + y_{i+1} \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} = y_i$

[2] when we put [1] in [2] $\Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} = y_i$

$\Rightarrow y_{i-1} - 2y_i + y_{i+1} = (\Delta x)^2 y_i$

* by choosing $\Delta x = 0.25 \Rightarrow$

Subject:

/ /

X_0	X_1	X_2	X_3	X_4	$\Delta x = 0.25$
0	0.25	0.5	0.75	1	$X_i = X_0 + i \Delta x$

$$\Rightarrow y_{i-1} - y_i [2 + (\Delta x)^2] + y_{i+1} = 0$$

$$\because \Delta x = 0.25 = \frac{1}{4} \Rightarrow (\Delta x)^2 = \frac{1}{16} \Rightarrow 2 + \Delta x^2 = 2 + \frac{1}{16} = \frac{33}{16}$$

$$\Rightarrow y_{i-1} - \frac{33}{16} y_i + y_{i+1} = 0$$

when $i=1$

$$y_0 - \frac{33}{16} y_1 + y_2 = 0 \quad \text{--- (1)}$$

Given $y_0 = 1$
 $y_4 = 2$

when $i=2$

$$y_1 - \frac{33}{16} y_2 + y_3 = 0 \quad \text{--- (2)}$$

$$\Rightarrow 1 - \frac{33}{16} y_2 + y_3 = 0$$

when $i=3$

$$y_2 - \frac{33}{16} y_3 + y_4 = 0 \quad \text{--- (3)}$$

Given $y_4 = 2$

$$y_2 - \frac{33}{16} y_3 + 2 = 0$$

$$-33y_1 + 16y_2 + 0y_3 = -16 \quad \text{--- (4)}$$

$$16y_1 - 33y_2 + 16y_3 = 0 \quad \text{--- (5)}$$

$$0y_1 + y_2 - 33y_3 = -32 \quad \text{--- (6)}$$

ask & receive

Subject:

/ /

X_0	X_1	X_2	X_3	X_4	$\Delta X = 0.25$
0	0.25	0.5	0.75	1	$X_i = X_0 + i \Delta X$

$$\Rightarrow y_{i-1} - y_i (2 + (\Delta X)^2) + y_{i+1} = 0$$

$$\Delta X = 0.25 = \frac{1}{4} \Rightarrow (\Delta X)^2 = \frac{1}{16} \Rightarrow 2 + (\Delta X)^2 = 2 + \frac{1}{16} = \frac{33}{16}$$

$$\Rightarrow y_{i-1} - \frac{33}{16} y_i + y_{i+1} = 0$$

when $i=1$

$$y_0 - \frac{33}{16} y_1 + y_2 = 0 \quad \text{--- (1)} \quad \begin{array}{l} y_0 = 1 \\ y_4 = 2 \end{array}$$

Given

when $i=2$

$$y_1 - \frac{33}{16} y_2 + y_3 = 0 \quad \text{--- (2)} \Rightarrow 1 - \frac{33}{16} y_2 + y_3 = 0$$

when $i=3$

$$y_2 - \frac{33}{16} y_3 + y_4 = 0 \quad \text{--- (3)} \Rightarrow y_2 - \frac{33}{16} y_3 + 2 = 0$$

Given

$$-33y_1 + 16y_2 + 0y_3 = -16 \quad \text{--- (1)}$$

$$16y_1 - 33y_2 + 16y_3 = 0 \quad \text{--- (2)}$$

$$0y_1 + y_2 - 33y_3 = -32 \quad \text{--- (3)}$$

ask & receive

Subject:

$$\begin{bmatrix} -33 & 16 & 0 \\ 16 & -33 & 16 \\ 0 & 16 & -33 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -16 \\ 0 \\ -32 \end{bmatrix}$$

⇒ when we solve it: $y_1 \approx 1.13$, $y_2 \approx 1.33$, $y_3 \approx 1.61$
 $x_1 \approx 0.25$, $x_2 \approx 0.5$, $x_3 \approx 0.75$

* Now when we solve the question analytically &

$$\frac{d^2y}{dx^2} - y = 0, \quad y = e^{mx}$$

$$y' = m e^{mx}$$

$$y'' = m^2 e^{mx}$$

$$\Rightarrow m^2 e^{mx} - e^{mx} = 0 \Rightarrow e^{mx}(m^2 - 1) = 0 \Rightarrow m = \pm 1$$

$$\Rightarrow y = A e^{-x} + B e^x$$

$$y(0) = 1 \Rightarrow 1 = A + B \quad \text{--- (1)}$$

$$y(1) = 2 \Rightarrow 2 = 0.367A + 2.718B \quad \text{--- (2)}$$

$$\begin{bmatrix} 1 & 1 \\ 0.367 & 2.718 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow A \approx 0.305$$

$$B \approx 0.694$$

$$y = 0.305 e^{-x} + 0.694 e^x \Rightarrow x = 0, x \approx 0.25, x \approx 0.5, x \approx 0.75$$

$$y = 1, y \approx 1.128, y \approx 1.33, y \approx 1.61$$

subject:

$$\begin{bmatrix} -33 & 16 & 0 \\ 16 & -33 & 16 \\ 0 & 16 & -33 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -16 \\ 0 \\ -32 \end{bmatrix}$$

$\Rightarrow$ when we solve it $y_1 \approx 1.13, y_2 \approx 1.33, y_3 \approx 1.61$
 $x_1 \approx 0.25, x_2 \approx 0.5, x_3 \approx 0.75$

* Now when we solve the question analytically &

$$\frac{d^2y}{dx^2} - y = 0, \quad y = e^{mx}$$

$$y' = m e^{mx}$$

$$y'' = m^2 e^{mx}$$

$$\Rightarrow m^2 e^{mx} - e^{mx} = 0 \Rightarrow e^{mx} (m^2 - 1) = 0 \Rightarrow m = \pm 1$$

$$\Rightarrow y = A e^{-x} + B e^x$$

$$y(0) = 1 \Rightarrow 1 = A + B \quad \text{--- (1)}$$

$$y(1) = 2 \Rightarrow 2 = 0.367A + 2.71B \quad \text{--- (2)}$$

$$\begin{bmatrix} 1 & 1 \\ 0.367 & 2.71 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow A \approx 0.305, B \approx 0.694$$

$$y = 0.305 e^{-x} + 0.694 e^x \Rightarrow x = 0, x = 0.25, x = 0.5, x = 0.75$$

$$y = 1, y = 1.128, y = 1.33, y = 1.61$$

ask :: believe & recieve

Subject:

/ /

* Now in general: $\frac{dy}{dx} = f(x, y) \Rightarrow \frac{y_{i+1} - y_i}{\Delta x}$

$$\frac{d^2y}{dx^2} = f(x, y, y') \Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$$

ex: $\frac{d^2y}{dx^2} + \frac{dy}{dx} = y + x$

$y_0, y_1, y_2, \dots$
 $x_0, x_1, \dots$

$$\Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} + \frac{y_{i+1} - y_i}{\Delta x} = y_i + x_i$$

$$\frac{dy}{dx} = f(x, y) \Rightarrow \frac{\Delta y}{\Delta x} = f(x, y)$$

$$\Rightarrow \Delta y = \Delta x \cdot f(x, y)$$

$$\Rightarrow y_{i+1} - y_i = \Delta x \cdot f(x, y)$$

$$\Rightarrow y_{i+1} - 2y_i + y_{i-1} + \Delta x(y_{i+1} - y_i) = \Delta x^2(y_i + x_i) \Rightarrow y_{i+1} = y_i + \Delta x f(x, y)$$

$$\Rightarrow y_{i+1} - 2y_i + y_{i-1} + \Delta x y_{i+1} - \Delta x y_i = (\Delta x)^2 y_i + x_i (\Delta x)^2$$

$$\Rightarrow (1 + \Delta x) y_{i+1} + (-2 - \Delta x - (\Delta x)^2) y_i + y_{i-1} = (\Delta x)^2 x_i$$

$$\Rightarrow \text{when } i=1 \Rightarrow y_0 + \alpha y_1 + \beta y_2 = \gamma \cdot x_1$$

$$\Rightarrow \text{when } i=2 \Rightarrow y_1 + \alpha y_2 + \beta y_3 = \gamma \cdot x_2$$

$$\Rightarrow \text{when } i=3 \Rightarrow y_2 + \alpha y_3 + \beta y_4 = \gamma \cdot x_3$$

* Both y_0, y_4 given

and we carry on

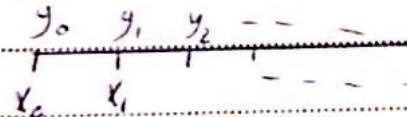
ask : ask & receive

Subject:

* Now in general let $\frac{dy}{dx} = f(x, y) \Rightarrow \frac{y_{i+1} - y_i}{\Delta x}$

$\frac{d^2y}{dx^2} = f(x, y, y') \Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$

ex: $\frac{d^2y}{dx^2} + \frac{dy}{dx} = y + x$



$\Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} + \frac{y_{i+1} - y_i}{\Delta x} = y_i + x_i$

$\frac{dy}{dx} = f(x, y) \Rightarrow \frac{\Delta y}{\Delta x} = f(x, y)$

$\Rightarrow \Delta y = \Delta x \cdot f(x, y)$

$\Rightarrow y_{i+1} - y_i = \Delta x \cdot f(x, y)$

$\Rightarrow y_{i+1} - 2y_i + y_{i-1} + \Delta x(y_{i+1} - y_i) = \Delta x^2(y_i + x_i) \Rightarrow y_{i+1} = y_i + \Delta x f(x, y)$

$\Rightarrow y_{i+1} - 2y_i + y_{i-1} + \Delta x y_{i+1} - \Delta x y_i = (\Delta x)^2 y_i + x_i (\Delta x)^2$

$\Rightarrow \underbrace{(1 + \Delta x) y_{i+1}}_{\beta} + \underbrace{(-2 - \Delta x - (\Delta x)^2) y_i}_{\alpha} + y_{i-1} = \underbrace{(\Delta x)^2 x_i}_{\gamma}$

$\Rightarrow$ when $i=1 \Rightarrow \alpha y_0 + \beta y_1 + \gamma = x_1$

$\Rightarrow$ when $i=2 \Rightarrow \alpha y_1 + \beta y_2 + \gamma = x_2$

$\Rightarrow$ when $i=3 \Rightarrow \alpha y_2 + \beta y_3 + \gamma = x_3$

* Both y_0, y_1 given ask: before & receive

and we carry on