Homogeneous Linear Systems with Constant-Coefficient.

Theorem

General Solution

If the constant matrix **A** *in the system* (1) *has a linearly independent set of n eigenvectors, then the corresponding solutions* $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}$ *in* (4) *form a basis of solutions of* (1), *and the corresponding general solution is*

(5) $$y = c_1 x^{(1)} e^{\lambda_1 t} + \cdots + c_n x^{(n)} e^{\lambda_n t}.$$

Conversion of an *n*th-Order ODE to a System

So far, the discussion has been focused on systems of first-order ODEs. The reason for this is that, beside being an important case for many physical systems, all higher order equations can be converted to a system of first order ODEs. As a general example,

An nth-order ODE

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$$

can be converted to a system of n first-order ODEs by setting

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \quad \dots, \quad y_n = y^{(n-1)}.$$

Theorem

This system is of the form

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \cdots, y_n) \end{aligned}$$

EXAMPLES

Write the following equation as a set of first order ODE's then solve using the matrix solution technique:

$$y'' + 3y' + 2y = 0$$

Write the following equation as a set of first order ODE's.

$$y''' - 5y'' - 22y' + 56y = 0.$$

How to find solutions to system of ODEs

We shall now concentrate on systems (1) with constant coefficients consisting of two ODEs

(6) $\mathbf{y}' = \mathbf{A}\mathbf{y}$; in components,

$$y_1' = a_{11}y_1 + a_{12}y_2$$
$$y_2' = a_{21}y_1 + a_{22}y_2.$$

By comparison to a single ODE of a similar form ($y' = ay$, which has a solution of the form $y = Ce^{\lambda x}$), such a system of equations has a solution of the form:

$\mathbf{y} = \mathbf{u}e^{\lambda x}$

Where u is an n X n matrix of constants. Then we obtain:

$\mathbf{y}' = \lambda \mathbf{u}e^{\lambda x} = \mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{u}e^{\lambda x}$ → $\mathbf{A}\mathbf{u}e^{\lambda x} = \lambda \mathbf{u}e^{\lambda x}$

How to find solutions to system of ODEs

Eliminating $e^{\lambda x}$ from both sides we get:

$\mathbf{Au} = \lambda \mathbf{u}$

This equation states that if we multiply matrix **A** by matrix **u** we obtain the same matrix **u** multiplied by a scalar λ.

Such a problem is called the eigenvalue problem and can be written as :

$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u} = 0$

Where (**I**) is the identity matrix and must be used to make the subtraction from (**A**) possible. The idea of solving eigenvalue problem is to find the values of λ (the eigenvalues) and the corresponding **u** (the eigenvectors) that satisfy the equation above.

How to find solutions to system of ODEs

Eigenvalues can be found by setting the determinant of ($\mathbf{A} - \lambda\mathbf{I}$) to zero i.e. $|\,\mathbf{A} - \lambda\mathbf{I}\,| = 0$ (characteristic equation) and find the values of (λ).

Here we may have three cases:

a) Distinct values (λ)

b) Repeated values of (λ)

c) Complex conjugate values of (λ)

How to find solutions to system of ODEs

a) Distinct values (λ)

After finding the eigenvalues (λ) and the corresponding eigenvectors (**u**) the solutions will be:

$\mathbf{h_1} = \mathbf{u_1}e^{\lambda 1x}$ and $\mathbf{h_2} = \mathbf{u_2}e^{\lambda 2x}$

The general solution is given by:

$\mathbf{y} = c_1\mathbf{u_1}e^{\lambda 1x} + c_2\mathbf{u_2}e^{\lambda 2x}$

How to find solutions to system of ODEs

a) Distinct values (λ)

Example 1:

Solve the following ODE's to find $y_1(x)$ and $y_2(x)$:

$y_1' = y_2$, $y_1(0) = 1$

$y_2' = y_1$, $y_2(0) = 1$

Example 2:

Solve the following ODE's to find $y_1(x)$ and $y_2(x)$:

$y_1' = -3y_1 + y_2$

$y_2' = y_1 - 3y_2$

How to find solutions to system of ODEs

a) Distinct values (λ)

Example 3:
Solve the following ODE's to find $y_1(x)$ and $y_2(x)$:
$y_1' = y_1 + y_2$
$y_2' = 4y_1 + y_2$

Example 4:
Solve the following ODE's to find $y_1(x)$ and $y_2(x)$:
$y_1' = 2y_1 + 3y_2$
$y_2' = 2y_1 + y_2$

How to find solutions to system of ODEs

b) Repeated values of (λ)

In case we have repeated Eigenvalues (i.e. $\lambda_1, = \lambda_2 = \lambda$), the first solution ($\mathbf{h_1}$) can be found as we did in the distinct eigenvalue
(i.e. $\mathbf{h_1} = \mathbf{u_1}e^{\lambda x}$). However, for the second solution to be linearly independent from the first one we use the following format (proof is available in different textbooks):

$\mathbf{h_2} = \mathbf{u_1}xe^{\lambda x} + \mathbf{P}e^{\lambda x}$

Taking the derivative of h_2 and equate it with (y') we obtain

$(\mathbf{A} - \lambda\mathbf{I})\mathbf{P} = \mathbf{u_1}$

Solving for $\mathbf{P}$ we can then find h_2. the general solution will then be:

$\mathbf{y} = c_1\mathbf{u_1}e^{\lambda x} + c_2(\mathbf{u_1}xe^{\lambda x} + \mathbf{P}e^{\lambda x})$

How to find solutions to system of ODEs

b) Repeated values of (λ)

Example 5:

Solve the following ODE's to find $y_1(x)$ and $y_2(x)$:

$y_1' = 3y_1 - 18y_2$

$y_2' = 2y_1 - 9y_2$

Example 6:

Solve the following ODE's to find $y_1(x)$ and $y_2(x)$:

$y_1' = 4y_1 + y_2$

$y_2' = -y_1 + 2y_2$

How to find solutions to system of ODEs

c) Complex conjugate values of (λ)

If eigenvalues happen to be complex conjugates ($\lambda_1 = \alpha + i\beta$) and ($\lambda_2 = \alpha - i\beta$) we can proceed as in the case of distinct eigenvalues and find the corresponding eigenvectors, which will have complex entries as well. The solutions will be:

$\mathbf{h_1} = \mathbf{u_1} e^{(\alpha + i\beta)x}$ and $\mathbf{h_2} = \mathbf{u_2} e^{(\alpha - i\beta)x}$

The general solution is given by:

$\mathbf{y} = c_1 \mathbf{u_1} e^{(\alpha + i\beta)x} + c_2 \mathbf{u_2} e^{(\alpha - i\beta)x}$

How to find solutions to system of ODEs

c) Complex conjugate values of (λ)

For example in the following set of first order ODEs:

$y_1' = 6y_1 - y_2$, $y_2' = 5y_1 + 4y_2$

We find that eigenvalue are complex conjugates ($\lambda_1 = 5 + 2i$) and ($\lambda_2 = 5 - 2i$). Eigenvector corresponding to the first eigenvalue can be found by solving:

$(1-2i)u_1 - u_2 = 0$

$5u_1 - (1+2i)u_2 = 0$ Note that the second equation is simply (1+2i) times the first one. Try it!

Solving gives

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$$

In the same way we can obtain the second eigenvector as

$$\mathbf{u_2} = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$$

How to find solutions to system of ODEs

c) Complex conjugate values of (λ)

The general solution is then given by:

$$\mathbf{y} = c_1 \begin{bmatrix} \mathbf{1} \\ \mathbf{1 - 2}\boldsymbol{i} \end{bmatrix} e^{(5 + 2i)x} + c_2 \begin{bmatrix} \mathbf{1} \\ \mathbf{1 + 2}\boldsymbol{i} \end{bmatrix} e^{(5 - 2i)x}$$

To write this solution in terms of real functions the procedure is easy.

Note that the second eigenvector is a conjugate of the first one, **so we can use only one of them to rewrite the solution in terms of real functions.**

Let us take the first eigenvalue ($\lambda_1 = 5 + 2i$) with its eigenvector

$$\begin{bmatrix} \mathbf{1} \\ \mathbf{1 - 2}\boldsymbol{i} \end{bmatrix}$$

How to find solutions to system of ODEs

c) Complex conjugate values of (λ)

The procedure is as follows:

1- write the eigenvector as a summation of two vectors:

$$\begin{bmatrix} \mathbf{1} \\ \mathbf{1-2}i \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{-2} \end{bmatrix} i$$

$\mathbf{u_1}$ $\mathbf{B_1}$ $\mathbf{B_2}$

Were $\mathbf{u_1}$ is the eigenvector, $\mathbf{B_1}$ is the real part of the eigenvector, and $\mathbf{B_2}$ is the imaginary part of the eigenvector.

2- the first solution (h_1) in terms of real functions can then be written as: $\mathbf{h_1} = [\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \cos 2x - \begin{bmatrix} \mathbf{0} \\ \mathbf{-2} \end{bmatrix} \sin 2x]e^{5x}$

3- the second solution (h_1) in terms of real functions can then be written as: $\mathbf{h_2} = [\begin{bmatrix} \mathbf{0} \\ \mathbf{-2} \end{bmatrix} \cos 2x + \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \sin 2x]e^{5x}$

How to find solutions to system of ODEs

c) Complex conjugate values of (λ)

In general, let $\lambda_1 = \alpha + \beta i$ be a complex eigenvalue of the coefficient matrix **A** in a homogeneous system of first order ODEs, and let $\mathbf{B_1}$ and $\mathbf{B_2}$ denote the real and imaginary column vectors of its eigenvector. Then the solutions to the system can be written as follows:

$$\mathbf{h_1} = [\mathbf{B_1} \cos \beta x - \mathbf{B_2} \sin \beta x]e^{\alpha x}$$
$$\mathbf{h_2} = [\mathbf{B_2} \cos \beta x + \mathbf{B_1} \sin \beta x]e^{\alpha x}$$

This gives linearly independent solutions to the system. The general solution is then written as:

$$\mathbf{y} = c_1 [\mathbf{B_1} \cos \beta x - \mathbf{B_2} \sin \beta x]e^{\alpha x} + c_2 [\mathbf{B_2} \cos \beta x + \mathbf{B_1} \sin \beta x]e^{\alpha x}$$

How to find solutions to system of ODEs

c) Complex conjugate values of (λ)

Example 7:

Solve the following ODE's to find $y_1(x)$ and $y_2(x)$ expressed in real functions:

$y_1' = 2y_1 + 8y_2$, $y_1(0) = 2$

$y_2' = -y_1 - 2y_2$, $y_2(0) = -1$

Nonhomogeneous Linear Systems of ODEs

In this section, we discuss methods for solving nonhomogeneous linear systems of ODEs

$$\mathbf{y}' = \mathbf{A}(x)\mathbf{y} + \mathbf{g}(x) \qquad \textbf{(1)}$$

where the vector $\mathbf{g}(x)$ is not identically zero. From a general solution $\mathbf{y}_h(x)$ of the homogeneous system:

$$\mathbf{y}' = \mathbf{A}(x)\mathbf{y} \qquad \textbf{(2)}$$

a **particular solution** $\mathbf{y}_p(x)$ of (1) can be obtained, and we get a general solution of (1),

$$\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p.$$

$$\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p$$

- Methods we used in finding the particular solution to solve a non-homogeneous second order ODE include the **Undetermined Coefficients** and the **Variation of Parameters**.
- Of the two methods, Variation of Parameters is the more powerful technique, which will be the focus of this section.
- As we did in finding the solution of a single non-homogeneous second order ODE, the solution of the homogeneous part is used in finding the particular solution of the non-homogeneous part of the system of ODEs.

Method of Variation of Parameters

This method can be applied to nonhomogeneous linear systems (1) [i.e. $\mathbf{y}' = \mathbf{A}(x)\mathbf{y} + \mathbf{g}(x)$]

If $\mathbf{h_1}, \mathbf{h_2}, \ldots, \mathbf{h_n}$ is a fundamental set of solutions of the homogeneous system (2) [i.e. $\mathbf{y}' = \mathbf{A}(x)\mathbf{y}$] on an interval *I*, then its general solution on the interval is the linear combination

$\mathbf{y_h} = c_1\mathbf{h_1} + c_2\mathbf{h_2} + \ldots + c_n\mathbf{h_n}$, or:

$$\boldsymbol{y_h} = c_1\begin{bmatrix} h_{11} \\ h_{21} \\ . \\ . \\ h_{n1} \end{bmatrix} + c_2\begin{bmatrix} h_{12} \\ h_{22} \\ . \\ . \\ h_{n2} \end{bmatrix} + \cdots + c_n\begin{bmatrix} h_{1n} \\ h_{2n} \\ . \\ . \\ h_{nn} \end{bmatrix} = \begin{bmatrix} c_1h_{11} + c_2h_{12} + \cdots + c_nh_{1n} \\ c_1h_{21} + c_2h_{22} + \cdots + c_nh_{2n} \\ . \\ . \\ c_1h_{n1} + c_2h_{n2} + \cdots + c_nh_{nn} \end{bmatrix}$$

Method of Variation of Parameters

The last matrix is recognized as the product of an n x n matrix with an n x 1. in other words, the general solution can be written as the product

$$\mathbf{y_h} = \boldsymbol{H}(x)\boldsymbol{C},$$

Where **C** is the n x 1 column vector of arbitrary constants (c_1, c_2, …, c_n), and **H**(x) is the n x n matrix whose columns consist of entries of the solution vectors of the system (2) $\mathbf{y}' = \mathbf{A}(x)\mathbf{y}$, i.e.

$$\boldsymbol{H}(\boldsymbol{x}) = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ h_{n1} & h_{n2} & \cdots & h_{nn} \end{bmatrix}$$

The matrix **H**(x) is called a **fundamental matrix** of the system

Method of Variation of Parameters

In the discussion that follows, we need to use two properties of a fundamental matrix:

1. a fundamental matrix $\boldsymbol{H}(x)$ is nonsingular, so it has an inverse (proof is available in many textbooks).
2. If $\boldsymbol{H}(x)$ is a fundamental matrix of a system (2) $\mathbf{y}' = \mathbf{A}(x)\mathbf{y}$, then:

$$\boldsymbol{H}'(x) = \mathbf{A}\boldsymbol{H}(x). \qquad \textbf{(3)}$$

Method of Variation of Parameters

Analogous to what was done in solving a single nonhomogeneous second order ODE, we ask whether it is possible to replace the matrix of constants $\boldsymbol{C}$ in [$\mathbf{y_h} = \boldsymbol{H}(x)\boldsymbol{C}$] by a column matrix of functions $\mathbf{U}(x) = [u_1(x), u_2(x), \ldots u_n(x)]^T$, so that:

$$y_p = \mathbf{H}(x)\mathbf{U}(x) \qquad \mathbf{(4)}$$

Is a particular solution of the nonhomogeneous system (1):

$$\mathbf{y}' = \mathbf{A}(x)\mathbf{y} + \mathbf{g}(x).$$

Substituting y_p into (1) we optain:

$\mathbf{H}(x)\mathbf{U}'(x) + \mathbf{H}'(x)\mathbf{U}(x) = \mathbf{A}\mathbf{H}(x)\mathbf{U}(x) + \mathbf{g}(x).$

If we use equation (3) to replace $\mathbf{H}'(x)$ we obtain:

$\mathbf{H}(x)\mathbf{U}'(x) + \mathbf{A}\mathbf{H}(x)\mathbf{U}(x) = \mathbf{A}\mathbf{H}(x)\mathbf{U}(x) + \mathbf{g}(x)$, and so:

$\mathbf{H}(x)\mathbf{U}'(x) = \mathbf{g}(x)$

Method of Variation of Parameters

$\mathbf{H}(x)\mathbf{U}'(x) = \mathbf{g}(x)$

It follows that $\mathbf{U}'(x)$ is given by:

$\mathbf{U}'(x) = \mathbf{H}^{-1}(x)\mathbf{g}(x)$ and so $\mathbf{U}(x) = \int \mathbf{H}^{-1}(x)\mathbf{g}(x)\, dx$

Since $y_p = \mathbf{H}(x)\mathbf{U}(x)$, we conclude that a particular solution of system (1) is given by:

$$y_p = \mathbf{H}(x) \int \mathbf{H}^{-1}(x)\mathbf{g}(x)\, dx \qquad \textbf{(5)}$$

Thus the general solution of nonhomogeneous system of linear first order ODEs [$\mathbf{y}' = \mathbf{A}(x)\mathbf{y} + \mathbf{g}(x)$] is given by:

$$y = \mathbf{H}(x)\mathbf{C} + \mathbf{H}(x) \int \mathbf{H}^{-1}(x)\mathbf{g}(x)\, dx$$

Method of Variation of Parameters

Example:

Find the general solution of the nonhomogeneous system:

$y_1' = -3y_1 + y_2 + 3x$

$y_2' = 2y_1 - 4y_2 + e^{-x}$

Example:

Find the general solution of the nonhomogeneous system:

$y_1' = 3y_1 - 5y_2 + e^{-x/2}$

$y_2' = 0.75y_1 - y_2 - e^{-x/2}$