CHAPTER 5

Section 5-1

5-1. First, $f(x,y) \geq 0$. Let R denote the range of (X,Y).

Then, $\sum_R f(x,y) = \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} = 1$

a) P(X < 2.5, Y < 3) = f(1.5,2)+f(1,1) = 1/8+1/4=3/8
b) P(X < 2.5) = f (1.5, 2) + f (1.5, 3) +f(1,1)= 1/8 + 1/4+1/4 = 5/8
c) P(Y < 3) = f (1.5, 2)+f(1,1) = 1/8+1/4=3/8
d) P(X > 1.8, Y > 4.7) = f (3, 5) = 1/8
e) E(X) = 1(1/4)+ 1.5(3/8) + 2.5(1/4) + 3(1/8) = 1.8125
E(Y) = 1(1/4)+2(1/8) + 3(1/4) + 4(1/4) + 5(1/8) = 2.875
$V(X) = E(X^2)-[E(X)]^2=[1^2(1/4)+1.5^2(3/8)+2.5^2(1/4)+3^2(1/8)]-1.8125^2=0.4961$
$V(Y) = E(Y^2)-[E(Y)]^2=[1^2(1/4)+2^2(1/8)+3^2(1/4)+4^2(1/4)+5^2(1/8)]-2.875^2=1.8594$

f) marginal distribution of *X*

x	f(x)
1	¼
1.5	3/8
2.5	¼
3	1/8

g) $f_{Y|2.5}(y) = \frac{f_{XY}(2.5,y)}{f_X(2.5)}$ and $f_X(2.5)$ = 1/4. Then,

y	$f_{Y\|2.5}(y)$
4	(1/4)/(1/4)=1

h) $f_{X|2}(x) = \frac{f_{XY}(x,2)}{f_Y(2)}$ and $f_Y(2)$ = 1/8. Then,

x	$f_{X\|2}(y)$
1.5	(1/8)/(1/8)=1

i) $E(Y|X=1.5)$ = 2(1/3)+3(2/3) =2 1/3
j) Since $f_{Y|1.5}(y) \neq f_Y(y)$, X and Y are not independent

5-2. Let R denote the range of (X,Y). Because

$\sum_R f(x,y) = c(2+3+4+3+4+5+4+5+6) = 1$, 36c = 1, and c = 1/36

a) $P(X=1, Y<3) = f_{XY}(1,1) + f_{XY}(1,2) = \frac{1}{36}(2+3) = 5/36$
b) $P(X = 1)$ is the same as part (a) = 1/4
c) $P(Y=2) = f_{XY}(1,2) + f_{XY}(2,2) + f_{XY}(3,2) = \frac{1}{36}(3+4+5) = 1/3$
d) $P(X<2, Y<2) = f_{XY}(1,1) = \frac{1}{36}(2) = 1/18$
e)

$$\begin{aligned} E(X) &= 1[f_{XY}(1,1) + f_{XY}(1,2) + f_{XY}(1,3)] + 2[f_{XY}(2,1) + f_{XY}(2,2) + f_{XY}(2,3)] \\ &+ 3[f_{XY}(3,1) + f_{XY}(3,2) + f_{XY}(3,3)] \\ &= \left(1\times\tfrac{9}{36}\right) + \left(2\times\tfrac{12}{36}\right) + \left(3\times\tfrac{15}{36}\right) = 13/6 = 2.167 \end{aligned}$$

$$V(X) = (1-\tfrac{13}{6})^2\,\tfrac{9}{36} + (2-\tfrac{13}{6})^2\,\tfrac{12}{36} + (3-\tfrac{13}{6})^2\,\tfrac{15}{36} = 0.639$$
$$E(Y) = 2.167$$
$$V(Y) = 0.639$$

f) Marginal distribution of *X*

x	$f_X(x) = f_{XY}(x,1) + f_{XY}(x,2) + f_{XY}(x,3)$
1	1/4
2	1/3
3	5/12

g) $f_{Y|X}(y) = \dfrac{f_{XY}(1,y)}{f_X(1)}$

y	$f_{Y\|X}(y)$
1	(2/36)/(1/4)=2/9
2	(3/36)/(1/4)=1/3
3	(4/36)/(1/4)=4/9

h) $f_{X|Y}(x) = \dfrac{f_{XY}(x,2)}{f_Y(2)}$ and $f_Y(2) = f_{XY}(1,2) + f_{XY}(2,2) + f_{XY}(3,2) = \frac{12}{36} = 1/3$

x	$f_{X\|Y}(x)$
1	(3/36)/(1/3)=1/4
2	(4/36)/(1/3)=1/3
3	(5/36)/(1/3)=5/12

i) $E(Y|X = 1) = 1(2/9) + 2(1/3) + 3(4/9) = 20/9$
j) Since $f_{XY}(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

5-3. $f(x,y) \geq 0$ *and* $\sum_R f(x,y) = 1$

a) $P(X < 0.5, Y < 1.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

b) $P(X < 0.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) = \frac{3}{8}$

c) $P(Y < 1.5) = f_{XY}(-1,-2) + f_{XY}(-0.5,-1) + f_{XY}(0.5,1) = \frac{7}{8}$

d) $P(X > 0.25, Y < 4.5) = f_{XY}(0.5,1) + f_{XY}(1,2) = \frac{5}{8}$

e) $E(X) = -1(\frac{1}{8}) - 0.5(\frac{1}{4}) + 0.5(\frac{1}{2}) + 1(\frac{1}{8}) = \frac{1}{8}$

$E(Y) = -2(\frac{1}{8}) - 1(\frac{1}{4}) + 1(\frac{1}{2}) + 2(\frac{1}{8}) = \frac{1}{4}$

$V(X) = (-1-1/8)^2(1/8)+(-0.5-1/8)^2(1/4)+(0.5-1/8)^2(1/2)+(1-1/8)^2(1/8)=0.4219$
$V(Y) = (-2-1/4)^2(1/8)+(-1-1/4)^2(1/4)+(1-1/4)^2(1/2)+(2-1/4)^2(1/8)=1.6875$

f) marginal distribution of *X*

x	$f_X(x)$
-1	1/8
-0.5	¼
0.5	½
1	1/8

g) $f_{Y|X}(y) = \dfrac{f_{XY}(0.5, y)}{f_X(0.5)}$

y	$f_{Y\|X}(y)$
1	1/2/(1/2)=1

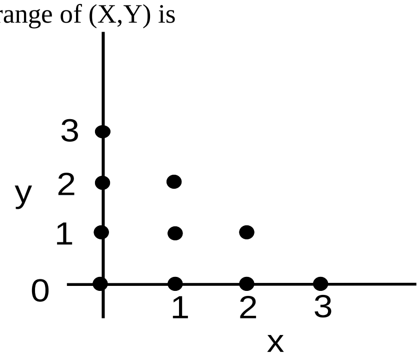
h) $f_{X|Y}(x) = \dfrac{f_{XY}(x,1)}{f_Y(1)}$

x	$f_{X\|Y}(x)$
0.5	½/(1/2)=1

i) $E(X|Y=1) = 0.5$
j) No, X and Y are not independent

5-4. Because X and Y denote the number of printers in each category,
$X \geq 0,\ Y \geq 0 \ and \ X + Y = 5$

5-5. a) The range of (X,Y) is



x,y	$f_{xy}(x,y)$
0,0	0.857375
0,1	0.1083
0,2	0.00456
0,3	0.000064
1,0	0.027075
1,1	0.00228
1,2	0.000048
2,0	0.000285
2,1	0.000012
3,0	0.000001

b)

x	$f_x(x)$
0	0.970299
1	0.029403
2	0.000297
3	0.000001

c) E(X) = 0(0.970299) + 1(0.029403) + 2(0.000297) + 3(0.000001) = 0.03
or np = 3(0.01) = 0.03

d) $f_{Y|2}(y) = \dfrac{f_{XY}(2, y)}{f_X(2)}$, $f_x(2)$ = 0.000297

y	$f_{Y\|1}(x)$
0	0.9596
1	0.0404

e) E(Y|X = 1) = 0(.920824) + 1(0.077543) + 2(0.001632) = 0.080807
g) No, X and Y are not independent because, for example, $f_Y(0) \neq f_{Y|1}(0)$.

5-6. a) The range of (X,Y) is $X \geq 0,\ Y \geq 0\ and\ X + Y \leq 4$. Here X is the number of pages with moderate graphic content and Y is the number of pages with high graphic output among a sample of 4 pages.

The following table is for sampling without replacement. Students would have to extend the hypergeometric distribution to the case of three classes (low, moderate, and high).

For example, $P(X = 1, Y = 2)$ is calculated as

$$P(X = 1, Y = 2) = \frac{\binom{60}{1}\binom{30}{1}\binom{10}{2}}{\binom{100}{4}} = \frac{60(30)(45)}{\frac{100(99)(98)(97)}{24}} = 0.02066$$

	x=0	x=1	x=2	x=3	x=4
y=4	$5.35\text{x}10^{-05}$	0	0	0	0
y=3	0.00184	0.00092	0	0	0
y=2	0.02031	0.02066	0.00499	0	0
y=1	0.08727	0.13542	0.06656	0.01035	0
y=0	0.12436	0.26181	0.19635	0.06212	0.00699

b)

	f(y)
y=4	0.0000535
y=3	0.0027600
y=2	0.0459600
y=1	0.2996000
y=0	0.6516300

c) E(Y)=

$$\sum_{0}^{4} y_i f(y_i) = 0(0.65163) + 1(0.2996) + 2(0.04596) + 3(0.00276) + 4(0.0000535) = 0.400014$$

d) $f_{Y|3}(y) = \dfrac{f_{XY}(3, y)}{f_X(3)}$, $f_x(3)$ = 0.0725

y	$f_{Y\|3}(y)$
0	0.857
1	0.143

2	0
3	0
4	0

e) E(Y|X = 3) = 0(0.857) + 1(0.143) = 0.143
f) V(Y|X = 3) = 0^2(0.857) + 1^2(0.143) - 0.143^2 = 0.123
g) No, *X* and *Y* are not independent

5-7. a) The range of (*X*,*Y*) is $X \geq 0,\ Y \geq 0\ and\ X + Y \leq 4$.

Here *X* and *Y* denote the number of defective items found with inspection device 1 and 2, respectively.

$$f(x, y) = \left[\binom{4}{x}(0.994)^x(0.006)^{4-x}\right]\left[\binom{4}{y}(0.997)^y(0.003)^{4-y}\right]$$

For *x* = 0, 1, 2, 3, 4 and *y* = 0, 1, 2, 3, 4

	x = 0	x = 1	x = 2	x = 3	x = 4
y = 0	1.05×10^{-19}	6.96×10^{-17}	1.73×10^{-14}	1.91×10^{-12}	7.91×10^{-11}
y = 1	1.4×10^{-16}	9.28×10^{-14}	2.305×10^{-11}	2.55×10^{-9}	1.054×10^{-7}
y = 2	6.96×10^{-14}	4.61×10^{-11}	1.15×10^{-8}	1.27×10^{-7}	5.24×10^{-5}
y = 3	1.54×10^{-11}	1.02×10^{-8}	2.54×10^{-6}	2.81×10^{-4}	0.0116
y = 4	1.28×10^{-9}	8.49×10^{-7}	2.11×10^{-4}	0.0233	0.965

$$f(x, y) = \left[\binom{4}{x}(0.993)^x(0.007)^{4-x}\right]\left[\binom{4}{y}(0.997)^y(0.003)^{4-y}\right]$$

b) $f(x) = \left[\binom{4}{x}(0.994)^x(0.006)^{4-x}\right]$ for $x = 1, 2, 3, 4$

	x = 0	x = 1	x = 2	x = 3	x = 4
f(x)	1.296×10^{-9}	8.59×10^{-7}	2.134×10^{-4}	0.0236	0.9762

c) Because X has a binomial distribution *E*(*X*) = *n*(*p*) = 4 × (0.994)=3.976

d) $f_{Y|2}(y) = \dfrac{f_{XY}(2, y)}{f_X(2)} = f(y)$, $f_x(2) = 2.134 \times 10^{-4}$

y	$f_{Y\|1}(y)=f(y)$
0	8.1×10^{-11}
1	1.08×10^{-7}
2	5.37×10^{-5}
3	0.0119
4	0.988

e) E(Y|X=2) = E(Y)= *n*(*p*)= 4(0.997)=3.988
f) V(Y|X=2) = V(Y)=*n*(*p*)(1-*p*)=4(0.997)(0.003)=0.0120
g) Yes, *X* and *Y* are independent.

5-8. a) $P(X = 2) = f_{XYZ}(2,1,1) + f_{XYZ}(2,1,2) + f_{XYZ}(2,2,1) + f_{XYZ}(2,2,2) = 0.5$

b) $P(X = 1, Y = 2) = f_{XYZ}(1,2,1) + f_{XYZ}(1,2,2) = 0.35$

c) $P(Z < 1.5) = f_{XYZ}(1,1,1) + f_{XYZ}(1,2,1) + f_{XYZ}(2,1,1) + f_{XYZ}(2,2,1) = 0.44$

d)
$$P(X=1\ or\ Z=2)=P(X=1)+P(Z=2)-P(X=1,Z=2)=0.5+0.56-0.3=0.76$$

e) E(X) = 1(0.5) + 2(0.5) = 1.5

f) $P(X=1|Y=1)=\dfrac{P(X=1,Y=1)}{P(Y=1)}=\dfrac{0.05+0.10}{0.21+0.2+0.1+0.05}=0.27$

g) $P(X=1,Y=1|Z=2)==\dfrac{P(X=1,Y=1,Z=2)}{P(Z=2)}=\dfrac{0.1}{0.1+0.2+0.21+0.05}=0.19$

h) $P(X=1|Y=1,Z=2)=\dfrac{P(X=1,Y=1,Z=2)}{P(Y=1,Z=2)}=\dfrac{0.10}{0.10+0.21}=0.32$

i) $f_{X|YZ}(x)=\dfrac{f_{XYZ}(x,1,2)}{f_{YZ}(1,2)}$ and $f_{YZ}(1,2)=f_{XYZ}(1,1,2)+f_{XYZ}(2,1,2)=0.31$

x	$f_{X\|YZ}(x)$
1	0.10/0.31=0.32
2	0.21/0.31=0.68

5-9. (a) $f_{XY}(x,y)=(10\%)^x(30\%)^y(60\%)^{4-x-y}$, for X+Y<=4

$f_{XY}(x,y)$	x	y
0.1296	0	0
0.0648	0	1
0.0324	0	2
0.0162	0	3
0.0081	0	4
0.0216	1	0
0.0108	1	1
0.0054	1	2
0.0027	1	3
0.0036	2	0
0.0018	2	1
0.0009	2	2
0.0006	3	0
0.0003	3	1
0.0001	4	0

(b) $f_X(x)=P(X=x)=\sum_{X+Y\le 4} f_{XY}(x,y)$.

$f_X(x)$	x
0.2511	0
0.0405	1
0.0063	2
0.0009	3
0.0001	4

(c) E(X)= $\sum xf_X(x)$ =0*0.2511+1*0.0405+2*0.0063+3*0.0009+4*0.0001= 0.0562

(d) f(y|X=3) = P(Y=y, X=3)/P(X=3)

P(Y=0, X=3) = $C^{30}_1 C^5_3 / C^{50}_4$ P(Y=1, X=3) = $C^{15}_1 C^5_3 / C^{50}_4$

P(X=3) = $C^{45}_1 C^5_3 / C^{50}_4$, from the hypergeometric distribution with N=50, n=4, k=4, x=3

Therefore

f(0|X=3) = $[C^{30}_1 C^5_3 / C^{50}_4]/[C^{45}_1 C^5_3 / C^{50}_4] = C^{30}_1 / C^{45}_1 = \frac{30}{45} = \frac{2}{3}$

f(1|X=3) = $[C^{15}_1 C^5_3 / C^{50}_4]/[C^{45}_1 C^5_3 / C^{50}_4] = \frac{C^{15}_1}{C^{45}_1} = \frac{15}{45} = \frac{1}{3}$

$f_{Y\|3}(y)$	y	x
2/3	0	3
1/3	1	3
0	2	3
0	3	3
0	4	3

(e) E(Y|X=3)=0(0.6667)+1(0.3333)=0.3333

(f) V(Y|X=3)=$(0-0.3333)^2$(0.6667)+$(1-0.3333)^2$(0.3333)=0.0741

(g) $f_X(0)$= 0.2511, $f_Y(0)$=0.1555, $f_X(0) * f_Y(0)$= 0.039046 ≠ $f_{XY}(0,0)$ = 0.1296

X and Y are not independent.

5-10. (a) P(X < 5) = 0.44 + 0.04 = 0.48

(b) E(X) = 0.43(23) + 0.44(4.2) + 0.04(11.4) + 0.05(130) + 0.04(0) = 18.694

(c) $P_{X|Y=0}(X)$ = P(X = x,Y = 0)/P(Y = 0) = 0.04/0.08 = 0.5 for x = 0 and 11.4

(d) P(X>10|Y=0) = P(X=11.4|Y=0) = 0.5

(e) E(X|Y = 0)=11.4(0.5) + 0(0.5) = 5.7

5-11. (a) $f_{XYZ}(x,y,z)$

$f_{XYZ}(x,y,z)$	Selects(X)	Updates(Y)	Inserts(Z)
0.43	23	11	12
0.44	4.2	3	1
0.04	11.4	0	0
0.05	130	120	0
0.04	0	0	0

(b) $P_{XZ|Y=0}$

$P_{XZ\|Y=0}(x,y)$	Selects(X)	updates(Y)	Inserts(Z)
4/8 = 0.5	11.4	0	0
4/8= 0.5	0	0	0

(c) P(X<6, Y<6|Z = 0) = P(X =0, Y = 0) = 0.3077

(d) E(X|Y = 0,Z = 0) = 0.5(11.4) + 0.5(0) = 5.7 where this conditional distribuition for X was determined in the previous exercise

5-12. Let X, Y, and Z denote the number of bits with high, moderate, and low distortion. Then, the joint distribution of X, Y, and Z is multinomial with n =3 and

$p_1 = 0.02,\ \ p_2 = 0.03,\ \ and\ \ p_3 = 0.95$.

a)

$$P(X=2,Y=1)=P(X=2,Y=1,Z=0)$$
$$=\frac{3!}{2!1!0!}(0.02)^2(0.03)^1 0.95^0=3.6\times10^{-5}$$

b) $P(X=0,Y=0,Z=3)=\frac{3!}{0!0!3!}(0.02)^0(0.03)^0 0.95^3=0.8574$

c) X has a binomial distribution with n = 3 and p = 0.02. Then, E(X) = 3(0.02) = 0.06 and V(X) = 3(0.02)(0.98) = 0.0588.

d) First find $P(X\mid Y=2)$

$$P(Y=2)=P(X=1,Y=2,Z=0)+P(X=0,Y=2,Z=1)$$
$$=\frac{3!}{1!2!0!}0.02(0.03)^2 0.95^0+\frac{3!}{0!2!1!}(0.02)^0(0.03)^2 0.95^1=0.0026$$
$$P(X=0\mid Y=2)=\frac{P(X=0,Y=2)}{P(Y=2)}=\left(\frac{3!}{0!2!1!}(0.02)^0(0.03)^2 0.95^1\right)\Big/0.0026=0.98654$$

$$P(X=1\mid Y=2)=\frac{P(X=1,Y=2)}{P(Y=2)}=\left(\frac{3!}{1!2!1!}(0.02)^1(0.03)^2 0.95^0\right)\Big/0.0026=0.02077$$
$$E(X\mid Y=2)=0(0.98654)+1(0.02077)=0.02077$$

$$V(X\mid Y=2)=E(X^2)-(E(X))^2=0.02077-(0.02077)^2=0.02034$$

5-13. Determine c such that $c\int_0^3\int_0^3 xydxdy=c\int_0^3 y\left.\frac{x^2}{2}\right|_0^3 dy=c(4.5\left.\frac{y^2}{2}\right|_0^3)=\frac{81}{4}c.$

Therefore, c = 4/81.

a) $P(X<2,Y<3)=\frac{4}{81}\int_0^3\int_0^2 xydxdy=\frac{4}{81}(2)\int_0^3 ydy=\frac{4}{81}(2)(\frac{9}{2})=0.4444$

b) P(X < 2.5) = P(X < 2.5, Y < 3) because the range of Y is from 0 to 3.

$$P(X<2.5,Y<3)=\frac{4}{81}\int_0^3\int_0^{2.5} xydxdy=\frac{4}{81}(3.125)\int_0^3 ydy=\frac{4}{81}(3.125)\frac{9}{2}=0.6944$$

c) $P(1<Y<2.5)=\frac{4}{81}\int_1^{2.5}\int_0^3 xydxdy=\frac{4}{81}(4.5)\int_1^{2.5} ydy=\frac{18}{81}\left.\frac{y^2}{2}\right|_1^{2.5}=0.5833$

d) $P(X>1.9,1<Y<2.5)=\frac{4}{81}\int_1^{2.5}\int_{1.9}^3 xydxdy=\frac{4}{81}(2.7)\int_1^{2.5} ydy=\frac{4}{81}(2.7)\frac{(2.5^2-1)}{2}=0.35$

e) $E(X)=\frac{4}{81}\int_0^3\int_0^3 x^2ydxdy=\frac{4}{81}\int_0^3 9ydy=\frac{4}{9}\left.\frac{y^2}{2}\right|_0^3=2$

f) $P(X<0,Y<4)=\frac{4}{81}\int_0^4\int_0^0 xydxdy=0\int_0^4 ydy=0$

g) $f_X(x)=\int_0^3 f_{XY}(x,y)dy=x\frac{4}{81}\int_0^3 ydy=\frac{4}{81}x(4.5)=\frac{2x}{9}\quad for\quad 0<x<3.$

h) $f_{Y|1.5}(y) = \frac{f_{XY}(1.5, y)}{f_X(1.5)} = \frac{\frac{4}{81} y(1.5)}{\frac{2}{9}(1.5)} = \frac{2}{9} y$ for $0 < y < 3$.

i) E(Y|X=1.5) = $\int_0^3 y\left(\frac{2}{9} y\right) dy = \frac{2}{9} \int_0^3 y^2 dy = \left.\frac{2y^3}{27}\right|_0^3 = 2$

j) $P(Y < 2 \mid X = 1.5) = f_{Y|1.5}(y) = \int_0^2 \frac{2}{9} y dy = \left.\frac{1}{9} y^2\right|_0^2 = \frac{4}{9} - 0 = \frac{4}{9}$

k) $f_{X|2}(x) = \frac{f_{XY}(x,2)}{f_Y(2)} = \frac{\frac{4}{81} x(2)}{\frac{2}{9}(2)} = \frac{2}{9} x$ for $0 < x < 3$.

5-14.

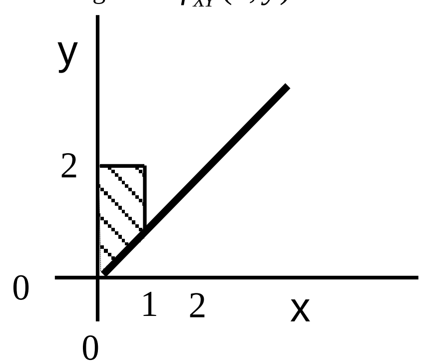
$$c\int_0^3 \int_x^{x+2} (x + y) dy dx = \int_0^3 \left. xy + \frac{y^2}{2} \right|_x^{x+2} dx$$

$$= \int_0^3 \left[x(x+2) + \frac{(x+2)^2}{2} - x^2 - \frac{x^2}{2} \right] dx$$

$$= c\int_0^3 (4x + 2) dx = \left[2x^2 + 2x\right]_0^3 = 24c$$

Therefore, c = 1/24.

a) P(X < 1, Y < 2) equals the integral of $f_{XY}(x, y)$ over the following region.

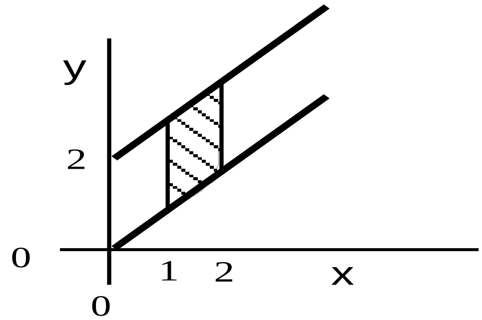


Then,

$$P(X < 1, Y < 2) = \frac{1}{24} \int_0^1 \int_x^2 (x + y) dy dx = \frac{1}{24} \int_0^1 \left. xy + \frac{y^2}{2} \right|_x^2 dx = \frac{1}{24} \int_0^3 2x + 2 - \frac{3x^2}{2} dx =$$

$$\frac{1}{24} \left[\left. x^2 + 2x - \frac{x^3}{2} \right|_0^1 \right] = 0.10417$$

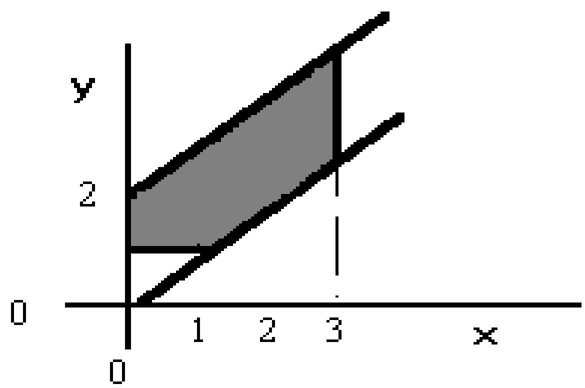
b) P(1 < X < 2) equals the integral of $f_{XY}(x, y)$ over the following region.



$$P(1<X<2)=\frac{1}{24}\int_1^2\int_x^{x+2}(x+y)dydx=\frac{1}{24}\int_1^2 xy+\frac{y^2}{2}\Big|_x^{x+2}dx$$

$$=\frac{1}{24}\int_0^3(4x+2)dx=\frac{1}{24}\left[2x^2+2x\Big|_1^2\right]=\tfrac{1}{6}.$$

c) P(Y > 1) is the integral of $f_{XY}(x,y)$ over the following region.

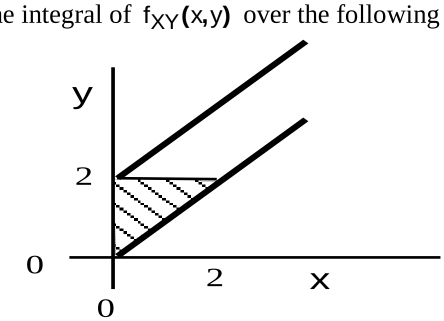


$$P(Y>1)=1-P(Y\le 1)=1-\tfrac{1}{24}\int_0^1\int_x^1(x+y)dydx=1-\frac{1}{24}\int_0^1(xy+\frac{y^2}{2})\Big|_x^1$$

$$=1-\frac{1}{24}\int_0^1 x+\frac{1}{2}-\frac{3}{2}x^2dx=1-\frac{1}{24}\left(\frac{x^2}{2}+\frac{1}{2}-\frac{1}{2}x^3\right)\Big|_0^1$$

$$=1-0.02083=0.9792$$

d) P(X < 2, Y < 2) is the integral of $f_{XY}(x,y)$ over the following region.



$$E(X)=\frac{1}{24}\int_0^3\int_x^{x+2}x(x+y)dydx=\frac{1}{24}\int_0^3 x^2y+\tfrac{xy^2}{2}\Big|_x^{x+2}dx$$
$$=\frac{1}{24}\int_0^3(4x^2+2x)dx=\frac{1}{24}\left[\frac{4x^3}{3}+x^2\Big|_0^3\right]=\frac{15}{8}$$

e)

$$E(X)=\frac{1}{24}\int_0^3\int_x^{x+2}x(x+y)dydx=\frac{1}{24}\int_0^3 x^2y+\tfrac{xy^2}{2}\Big|_x^{x+2}dx$$
$$=\frac{1}{24}\int_0^3(4x^2+2x)dx=\frac{1}{24}\left[\frac{4x^3}{3}+x^2\Big|_0^3\right]=\frac{15}{8}$$

f)

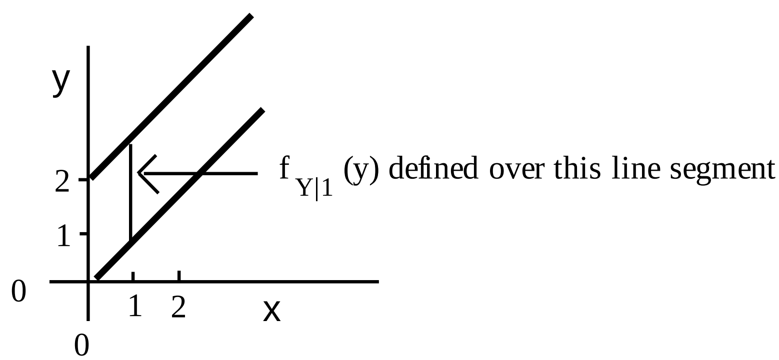
$$V(X)=\frac{1}{24}\int_0^3\int_x^{x+2}x^2(x+y)dydx-\left(\frac{15}{8}\right)^2=\frac{1}{24}\int_0^3 x^3y+\tfrac{x^2y^2}{2}\Big|_x^{x+2}dx-\left(\frac{15}{8}\right)^2$$
$$=\frac{1}{24}\int_0^3(3x^3+4x^2+4x-\frac{x^4}{4})dx-\left(\frac{15}{8}\right)^2$$
$$=\frac{1}{24}\left[\frac{3x^4}{4}+\frac{4x^3}{3}+2x^2-\frac{x^5}{20}\Big|_0^3\right]-\left(\frac{15}{8}\right)^2=\frac{31707}{320}$$

g) $f_X(x)$ is the integral of $f_{XY}(x,y)$ over the interval from x to x+2. That is,

$$f_X(x)=\frac{1}{24}\int_x^{x+2}(x+y)dy=\frac{1}{24}\left[xy+\tfrac{y^2}{2}\Big|_x^{x+2}\right]=\frac{x}{6}+\frac{1}{12}\text{ for }0<x<3.$$

h) $f_{Y|1}(y)=\frac{f_{XY}(1,y)}{f_X(1)}=\frac{\frac{1}{24}(1+y)}{\frac{1}{6}+\frac{1}{12}}=\frac{1+y}{6}$ for $1 < y < 3$.

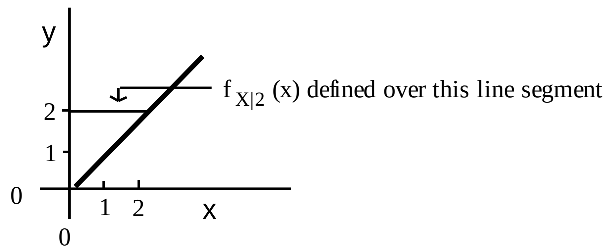
See the following graph,



i) $E(Y|X=1)=\int_1^3 y\left(\frac{1+y}{6}\right)dy=\frac{1}{6}\int_1^3(y+y^2)dy=\frac{1}{6}\left(\frac{y^2}{2}+\frac{y^3}{3}\right)\Big|_1^3=2.111$

j) $P(Y>2 \mid X=1)=\int_2^3\left(\frac{1+y}{6}\right)dy=\frac{1}{6}\int_2^3(1+y)dy=\frac{1}{6}\left(y+\frac{y^2}{2}\right)\Big|_2^3=0.5833$

k) $f_{X|2}(x)=\frac{f_{XY}(x,2)}{f_Y(2)}$. Here $f_Y(y)$ is determined by integrating over *x*. There are three regions of integration. For $0<y\le 2$ the integration is from 0 to *y*. For $2<y\le 3$ the integration is from *y*-2 to *y*. For $3<y<5$ the integration is from *y* to 3. Because the condition is *y*=2, only the first integration is needed. $f_Y(y)=\frac{1}{24}\int_0^y(x+y)dx=\frac{1}{24}\left[\frac{x^2}{2}+xy\Big|_0^y\right]=\frac{y^2}{16}$ for $0<y\le 2$.



Therefore, $f_Y(2)=1/4$ and $f_{X|2}(x)=\frac{\frac{1}{24}(x+2)}{1/4}=\frac{x+2}{6}$ for 0 < x < 3

5-15. $c\int_0^3\int_0^x xydydx=c\int_0^3 x\frac{y^2}{2}\Big|_0^x dx=c\int_0^3\frac{x^3}{2}dx\frac{x^4}{8}=\frac{81}{8}c.$ Therefore, c = 8/81

a) P(X<1,Y<2)= $\frac{8}{81}\int_0^1\int_0^x xydydx=\frac{8}{81}\int_0^1\frac{x^3}{2}dx=\frac{8}{81}\left(\frac{1}{8}\right)=\frac{1}{81}.$

b) P(1<X<2) = $\frac{8}{81}\int_1^2\int_0^x xydydx=\frac{8}{81}\int_1^2 x\frac{x^2}{2}dx=\left(\frac{8}{81}\right)\frac{x^4}{8}\Big|_1^2=\left(\frac{8}{81}\right)\frac{(2^4-1)}{8}=\frac{5}{27}.$

c)

$$P(Y>1)=\frac{8}{81}\int_1^3\int_1^x xydydx=\frac{8}{81}\int_1^3 x\left(\frac{x^2-1}{2}\right)dx=\frac{8}{81}\int_1^3\frac{x^3}{2}-\frac{x}{2}dx=\frac{8}{81}\left(\frac{x^4}{8}-\frac{x^2}{4}\right)\Big|_1^3$$

$$=\frac{8}{81}\left[\left(\frac{3^4}{8}-\frac{3^2}{4}\right)-\left(\frac{1^4}{8}-\frac{1^2}{4}\right)\right]=\frac{64}{81}=0.7901$$

d) P(X<2,Y<2) = $\frac{8}{81}\int_0^2\int_0^x xydydx=\frac{8}{81}\int_0^2\frac{x^3}{2}dx=\frac{8}{81}\left(\frac{2^4}{8}\right)=\frac{16}{81}.$

e)

$$E(X)=\frac{8}{81}\int_0^3\int_0^x x(xy)dydx=\frac{8}{81}\int_0^3\int_0^x x^2ydydx=\frac{8}{81}\int_0^3\frac{x^2}{2}x^2dx=\frac{8}{81}\int_0^3\frac{x^4}{2}dx$$
$$=\left(\frac{8}{81}\right)\left(\frac{3^5}{10}\right)=\frac{12}{5}$$

f)

$$E(Y)=\frac{8}{81}\int_0^3\int_0^x y(xy)dydx=\frac{8}{81}\int_0^3\int_0^x xy^2dydx=\frac{8}{81}\int_0^3 x\frac{x^3}{3}dx$$
$$=\frac{8}{81}\int_0^3\frac{x^4}{3}dx=\left(\frac{8}{81}\right)\left(\frac{3^5}{15}\right)=\frac{8}{5}$$

g) $f(x)=\frac{8}{81}\int_0^x xydy=\frac{4x^3}{81}\qquad 0<x<3$

h) $f_{Y|x=1}(y)=\frac{f(1,y)}{f(1)}=\frac{\frac{8}{81}(1)y}{\frac{4(1)^3}{81}}=2y\qquad 0<y<1$

i) $E(Y\mid X=1)=\int_0^1 2ydy=y^2\Big|_0^1=1$

j) $P(Y>2\mid X=1)=0$ this isn't possible since the values of y are $0<y<x$.

k) $f(y)=\frac{8}{81}\int_0^3 xydx=\frac{4y}{9}$, therefore

$$f_{X|Y=2}(x)=\frac{f(x,2)}{f(2)}=\frac{\frac{8}{81}x(2)}{\frac{4(2)}{9}}=\frac{2x}{9}\qquad 0<x<3$$

5-16. Solve for c

$$c\int_0^\infty\int_0^x e^{-3x-4y}dydx=\frac{c}{4}\int_0^\infty e^{-3x}\left(1-e^{-4x}\right)dx=\frac{c}{4}\int_0^\infty e^{-3x}-e^{-7x}dx=$$
$$\frac{c}{4}\left(\frac{1}{3}-\frac{1}{7}\right)=\frac{1}{21}c.\quad c=21$$

a) $P(X<1,Y<2)=21\int_0^1\int_0^x e^{-3x-4y}dydx=\frac{21}{4}\int_0^1 e^{-3x}(1-e^{-4x})dx=\frac{21}{4}\int_0^1 e^{-3x}-e^{-7x}dx$

$$=\frac{21}{4}\left(\frac{e^{-7x}}{7}-\frac{e^{-3x}}{3}\right)\Bigg|_0^1=0.9135$$

b) $P(1<X<2)=21\int_1^2\int_0^x e^{-3x-4y}dydx=\frac{21}{4}\int_1^2\left(e^{-3x}-e^{-7x}\right)dx$

$$= \frac{21}{4}\left(\frac{e^{-7x}}{7} - \frac{e^{-3x}}{3}\right)\Bigg|_1^2 = 0.1821$$

c) $P(Y > 3) = 21\int_3^{\infty}\int_3^{x} e^{-3x-4y}\,dy\,dx = \frac{21}{4}\int_3^{\infty} e^{-3x}(e^{-12} - e^{-4x})\,dx$

$$= \frac{21}{4}\left(\frac{e^{-7x}}{7} - \frac{e^{-12}e^{-3x}}{3}\right)\Bigg|_3^{\infty} = 7.583\times10^{-10}$$

d) $P(X < 2, Y < 2) = 21\int_0^{2}\int_0^{x} e^{-3x-4y}\,dy\,dx = \frac{21}{4}\int_0^{2} e^{-3x}(1-e^{-4x})\,dx = \frac{21}{4}\left[\left(\frac{e^{-14}}{7} - \frac{e^{-6}}{3}\right) - \left(\frac{e^{0}}{7} - \frac{e^{0}}{3}\right)\right]_0^2$

$= 0.9957$

e) E(X) = $21\int_0^{\infty}\int_0^{x} xe^{-3x-4y}\,dy\,dx = \frac{10}{21}$

f) E(Y) = $21\int_0^{\infty}\int_0^{x} ye^{-3x-4y}\,dy\,dx = \frac{29}{56}$

g) $f(x) = 21\int_0^{x} e^{-3x-4y}\,dy = \frac{21e^{-3x}}{4}(1-e^{-4x}) = \frac{21}{4}(e^{-3x} - e^{-7x})$ for $0 < x$

h) $f_{Y\backslash X=1}(y) = \frac{f_{X,Y}(1,y)}{f_X(1)} = \frac{21e^{-3-4y}}{\frac{21}{4}(e^{-3} - e^{-7})} = 4.075e^{-4y}$ $0 < y < 1$

i) $E(Y \mid X = 1) = 4.075\int_0^{1} ye^{-4y}\,dy = 0.2314$

j) $f_{X|Y=2}(x) = \frac{f_{X,Y}(x,2)}{f_Y(2)} = \frac{21e^{-3x-8}}{7e^{-14}} = 3e^{-3x+6}$ for $2 < x$,

where $f(y) = 7e^{-7y}$ for $0 < y$

5-17. $c\int_0^{\infty}\int_x^{\infty} e^{-2x}e^{-3y}\,dy\,dx = \frac{c}{3}\int_0^{\infty} e^{-2x}(e^{-3x})\,dx = \frac{c}{3}\int_0^{\infty} e^{-5x}\,dx = \frac{1}{15}c$ $c = 15$

a)

$$P(X < 1, Y < 2) = 15\int_0^{1}\int_x^{2} e^{-2x-3y}\,dy\,dx = 5\int_0^{1} e^{-2x}(e^{-3x} - e^{-6})\,dx$$

$$= 5\int_0^{1} e^{-5x}\,dx - 5e^{-6}\int_0^{1} e^{-2x}\,dx = 1 - e^{-5} + \frac{5}{2}e^{-6}(e^{-2} - 1) = 0.9879$$

b) P(1 < X < 2) = $15\int_1^{2}\int_x^{\infty} e^{-2x-3y}\,dy\,dx = 5\int_1^{2} e^{-5x}\,dy\,dx = (e^{-5} - e^{10}) = 0.0067$

c)

$$P(Y>3)=15\left(\int_0^3\int_3^\infty e^{-2x-3y}dydx+\int_3^\infty\int_x^\infty e^{-2x-3y}dydx\right)=5\int_0^3 e^{-9}e^{-2x}dx+5\int_3^\infty e^{-5x}dx$$

$$=-\frac{3}{2}e^{-15}+\frac{5}{2}e^{-9}=0.000308$$

d)

$$P(X<2,Y<2)=15\int_0^2\int_x^2 e^{-2x-3y}dydx=5\int_0^2 e^{-2x}(e^{-3x}-e^{-6})dx=$$

$$=5\int_0^2 e^{-5x}dx-5e^{-6}\int_0^2 e^{-2x}dx=\left(1-e^{-10}\right)+\frac{5}{2}e^{-6}\left(e^{-4}-1\right)=0.9939$$

e) E(X) = $15\int_0^\infty\int_x^\infty xe^{-2x-3y}dydx=5\int_0^\infty xe^{-5x}dx=\frac{1}{5^2}=0.04$

f)

$$E(Y)=15\int_0^\infty\int_x^\infty ye^{-2x-3y}dydx=\frac{-3}{2}\int_0^\infty 5ye^{-5y}dy+\frac{5}{2}\int_0^\infty 3ye^{-3y}dy$$

$$=-\frac{3}{10}+\frac{5}{6}=\frac{8}{15}$$

g) $f(x)=15\int_x^\infty e^{-2x-3y}dy=\frac{15}{3}(e^{-2z-3x})=5e^{-5x}$ for $x>0$

h) $f_X(1)=5e^{-5}$ $f_{XY}(1,y)=15e^{-2-3y}$

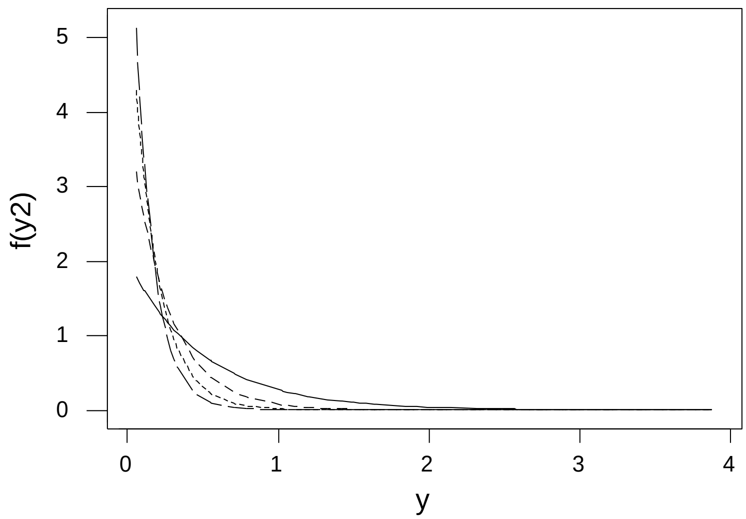
$f_{Y|X=1}(y)=\frac{15e^{-2-3y}}{5e^{-5}}=3e^{3-3y}$ for $1<y$

i) $E(Y\mid X=1)=\int_1^\infty 3ye^{3-3y}dy=-ye^{3-3y}\Big|_1^\infty+\int_1^\infty e^{3-3y}dy=4/3$

j) $\int_1^2 3e^{3-3y}dy=1-e^{-3}=0.9502$ for $0<y$, $f_Y(2)=\frac{15}{2}e^{-6}$

k) For $y>0$ $f_{X|Y=1}(y)=\frac{15e^{-2x-3}}{\frac{15}{2}e^{-3}}=2e^{-2x}$ for $0<x<1$

5-18. a) $f_{Y|X=x}(y)$, for x = 2, 4, 6, 8



b) $P(Y < 2 \mid X = 2) = \int_0^2 2e^{-2y} dy = 0.9817$

c) $E(Y \mid X = 2) = \int_0^\infty 2ye^{-2y} dy = 1/2$ (using integration by parts)

d) $E(Y \mid X = x) = \int_0^\infty xye^{-xy} dy = 1/x$ (using integration by parts)

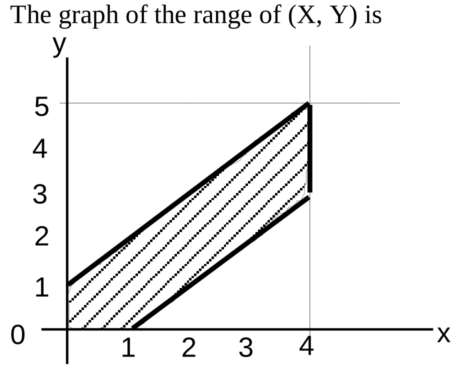
e) Use $f_X(x) = \dfrac{1}{b-a} = \dfrac{1}{10}$, $f_{Y|X}(x,y) = xe^{-xy}$, and the relationship

$$f_{Y|X}(x,y) = \frac{f_{XY}(x,y)}{f_X(x)}$$

Therefore, $xe^{-xy} = \dfrac{f_{XY}(x,y)}{1/10}$ and $f_{XY}(x,y) = \dfrac{xe^{-xy}}{10}$

f) $f_Y(y) = \int_0^{10} \dfrac{xe^{-xy}}{10} dx = \dfrac{1-10ye^{-10y} - e^{-10y}}{10y^2}$ (using integration by parts)

5-19. The graph of the range of (X, Y) is



$$\int_0^1 \int_0^{x+1} c\,dy\,dx + \int_1^4 \int_{x-1}^{x+1} c\,dy\,dx = 1$$

$$= c\int_0^1 (x+1)dx + 2c\int_1^4 dx$$

$$= \tfrac{3}{2}c + 6c = 7.5c = 1$$

Therefore, c = 1/7.5 = 2/15

a) $P(X < 0.5, Y < 1) = \int_0^{0.5}\int_0^1 \tfrac{1}{7.5}\,dy\,dx = \tfrac{1}{15}$

b) $P(X < 0.5) = \int_0^{0.5}\int_0^{x+1} \tfrac{1}{7.5}\,dy\,dx = \tfrac{1}{7.5}\int_0^{0.5}(x+1)dx = \tfrac{2}{15}\left(\tfrac{5}{8}\right) = \tfrac{1}{12}$

c)

$$E(X) = \int_0^1 \int_0^{x+1} \tfrac{x}{7.5}\,dy\,dx + \int_1^4 \int_{x-1}^{x+1} \tfrac{x}{7.5}\,dy\,dx$$

$$= \tfrac{1}{7.5}\int_0^1 (x^2 + x)dx + \tfrac{2}{7.5}\int_1^4 (x)dx = \tfrac{12}{15}\left(\tfrac{5}{6}\right) + \tfrac{2}{7.5}(7.5) = \tfrac{19}{9}$$

d)

$$E(Y) = \tfrac{1}{7.5}\int_0^1 \int_0^{x+1} y\,dy\,dx + \tfrac{1}{7.5}\int_1^4 \int_{x-1}^{x+1} y\,dy\,dx$$

$$= \tfrac{1}{7.5}\int_0^1 \tfrac{(x+1)^2}{2}dx + \tfrac{1}{7.5}\int_1^4 \tfrac{(x+1)^2-(x-1)^2}{2}dx$$

$$= \tfrac{1}{15}\int_0^1 (x^2 + 2x + 1)dx + \tfrac{1}{15}\int_1^4 4x\,dx$$

$$= \tfrac{1}{15}\left(\tfrac{7}{3}\right) + \tfrac{1}{15}(30) = \tfrac{97}{45}$$

e)

$$f(x) = \int_0^{x+1} \frac{1}{7.5}dy = \left(\frac{x+1}{7.5}\right) \quad \text{for} \quad 0 < x < 1,$$

$$f(x) = \int_{x-1}^{x+1} \frac{1}{7.5}dy = \left(\frac{x+1-(x-1)}{7.5}\right) = \frac{2}{7.5} \quad \text{for} \quad 1 < x < 4$$

f)

$$f_{Y|X=1}(y) = \frac{f_{XY}(1,y)}{f_X(1)} = \frac{1/7.5}{2/7.5} = 0.5$$

$$f_{Y|X=1}(y) = 0.5 \quad \text{for } 0 < y < 2$$

g) $E(Y \mid X = 1) = \int_0^2 \frac{y}{2}dy = \left.\frac{y^2}{4}\right|_0^2 = 1$

h) $P(Y < 0.5 \mid X = 1) = \int_0^{0.5} 0.5dy = 0.5y\Big|_0^{0.5} = 0.25$

5-20. Let X, Y, and Z denote the time until a problem on line 1, 2, and 3, respectively.

a)

$$P(X > 40, Y > 40, Z > 40) = [P(X > 40)]^3$$

because the random variables are independent with the same distribution. Now,

$P(X > 40) = \int_{40}^{\infty} \frac{1}{40} e^{-x/40} dx = -e^{-x/40}\Big|_{40}^{\infty} = e^{-1}$ and the answer is

$$\left(e^{-1}\right)^3 = e^{-3} = 0.0498$$

b) $P(30 < X < 40, 30 < Y < 40, 30 < Z < 40) = [P(30 < X < 40)]^3$ and

$$P(30 < X < 40) = -e^{-x/40}\Big|_{30}^{40} = e^{-0.75} - e^{-1} = 0.1045.$$

The answer is $0.1045^3 = 0.0011.$

c) The joint density is not needed because the process is represented by three independent exponential distributions. Therefore, the probabilities may be multiplied.

5-21. $\mu = 3.2$, $\lambda = 1/3.2$

$$P(X > 5, Y > 5) = (1/10.24)\int_5^{\infty}\int_5^{\infty} e^{-\frac{x}{3.2}-\frac{y}{3.2}} dydx = 3.2\int_5^{\infty} e^{-\frac{x}{3.2}}\left(e^{-\frac{5}{3.2}}\right)dx$$

$$= \left(e^{-\frac{5}{3.2}}\right)\left(e^{-\frac{5}{3.2}}\right) = 0.0439$$

$$P(X > 10, Y > 10) = (1/10.24)\int_{10}^{\infty}\int_{10}^{\infty} e^{-\frac{x}{3.2}-\frac{y}{3.2}} dydx = 3.2\int_{10}^{\infty} e^{-\frac{x}{3.2}}\left(e^{-\frac{10}{3.2}}\right)dx$$

$$= \left(e^{-\frac{10}{3.2}}\right)\left(e^{-\frac{10}{3.2}}\right) = 0.0019$$

b) Let *X* denote the number of orders in a 5-minute interval. Then *X* is a Poisson random variable with $\lambda = 5/3.2 = 1.5625$.

$$P(X = 1) = \frac{e^{-1.5625}(1.5625)^1}{1!} = 0.3275$$

For both systems, $P(X = 1)P(Y = 1) = 0.3275^2 = 0.1073$

c) The joint probability distribution is not necessary because the two processes are independent and we can just multiply the probabilities.

5-22. (a) X: the life time of blade and Y: the life time of bearing

$f(x) = (1/3)e^{-x/3}$ $\qquad$ $f(y) = (1/4)e^{-y/4}$

$P(X>6, Y>6)=P(X>6)P(Y>6)=e^{-6/3}e^{-6/4} = 0.0302$

(b) P(X > t, Y > t) = $e^{-t/3}e^{-t/4} = e^{-7t/12}$ = 0.95 → t = -12 ln(0.95)/7 = 0.0879 years

5-23. a) $P(X < 0.5) = \int_0^{0.5}\int_0^1\int_0^1 (10xyz)dzdydx = \int_0^{0.5}\int_0^1 (5xy)dydx = \int_0^{0.5}(2.5x)dx = 1.25x^2\Big|_0^{0.5} = 0.3125$

b) $P(X < 0.5, Y < 0.5) = \int_0^{0.5}\int_0^{0.5}\int_0^1 (10xyz)dzdydx$

$$= \int_0^{0.5}\int_0^{0.5}(5xy)dydx = \int_0^{0.5}(0.625x)dx = \frac{0.625x^2}{2}\Big|_0^{0.5} = 0.0781$$

c) P(Z < 2) = 1, because the range of Z is from 0 to 1.

d) P(X < 0.5 or Z < 2) = P(X < 0.5) + P(Z < 2) – P(X < 0.5, Z < 2). Now, P(Z < 2) =1 and P(X < 0.5, Z < 2) = P(X < 0.5). Therefore, the answer is 1.

e) $E(X) = \int_0^1\int_0^1\int_0^1 (10x^2yz)dzdydx = \int_0^1 (2.5x^2)dx = \frac{2.5x^3}{3}\Big|_0^1 = 0.833$

f) $P(X < 0.5 | Y = 0.5)$ is the integral of the conditional density $f_{X|Y}(x)$. Now,

$f_{X|0.5}(x) = \dfrac{f_{XY}(x,0.5)}{f_Y(0.5)}$ and $f_{XY}(x,0.5) = \int_0^1 (10x(0.5)z)dz = 5x0.5 = 2.5x$ for 0 < x < 1

and 0 < y < 1. Also, $f_Y(y) = \int_0^1\int_0^1 (10xyz)dzdx = 2.5y$ for 0 < y < 1; f_y (0.5) = 1.25

Therefore, $f_{X|0.5}(x) = \dfrac{2.5x}{1.25} = 2x$ for 0 < x < 1.

Then, $P(X < 0.5 | Y = 0.5) = \int_0^{0.5} 2xdx = 0.25$.

g) $P(X < 0.5, Y < 0.5 | Z = 0.8)$ is the integral of the conditional density of X and Y. Now,

$f_Z(z) = 2.5z$ for 0 < z < 1 as in part a) and

$f_{XY|Z}(x,y) = \dfrac{f_{XYZ}(x,y,z)}{f_Z(z)} = \dfrac{10xy(0.8)}{2.5(0.8)} = 4xy$ for 0 < x < 1 and 0 < y < 1.

Then, $P(X < 0.5, Y < 0.5 | Z = 0.8) = \int_0^{0.5}\int_0^{0.5}(4xy)dydx = \int_0^{0.5}(x/2)dx = \frac{1}{16} = 0.0625$

h) $f_{YZ}(y,z) = \int_0^1 (10xyz)dx = 5yz$ for 0 < y < 1 and 0 < z < 1.

Then, $f_{X|YZ}(x) = \dfrac{f_{XYZ}(x,y,z)}{f_{YZ}(y,z)} = \dfrac{10x(0.5)(0.8)}{5(0.5)(0.8)} = 2x$ for 0 < x < 1.

i) Therefore, $P(X < 0.5 | Y = 0.5, Z = 0.8) = \int_0^{0.5} 2xdx = 0.25$

5-24. $\iint_{x^2+y^2\le 4}\int_0^4 c\,dz\,dy\,dx$ = the volume of a cylinder with a base of radius 2 and a height of 4 =

$(\pi 2^2)4 = 16\pi$. Therefore, $c = \frac{1}{16\pi}$

a) $P(X^2+Y^2<2)$ equals the volume of a cylinder of radius $\sqrt{2}$ and a height of 4 (= 8π) times *c*. Therefore, the answer is $\frac{8\pi}{16\pi} = 1/2.$

b) P(Z < 2) equals half the volume of the region where $f_{XYZ}(x,y,z)$ is positive times 1/c. Therefore, the answer is 0.5.

c) $E(X) = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^4 \frac{x}{c}\,dz\,dy\,dx = c\int_{-2}^{2}\left[4xy\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\right]dx = c\int_{-2}^{2}(8x\sqrt{4-x^2})dx$.

Using substitution, $u = 4-x^2$, $du = -2x\,dx$, and

$E(X) = c\int 4\sqrt{u}\,du = \frac{-4}{c}\frac{2}{3}(4-x^2)^{\frac{3}{2}}\Big|_{-2}^{2} = 0$

d) $f_{X|1}(x) = \frac{f_{XY}(x,1)}{f_Y(1)}$ *and* $f_{XY}(x,y) = c\int_0^4 dz = \frac{4}{c} = \frac{1}{4\pi}$ *for* $x^2+y^2<4$.

Also, $f_Y(y) = c\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^4 dz\,dx = 8c\sqrt{4-y^2}$ for -2 < y < 2.

Then, $f_{X|y}(x) = \frac{4c}{8c\sqrt{4-y^2}}$ evaluated at y = 1. That is, $f_{X|1}(x) = \frac{1}{2\sqrt{3}}$ for $-\sqrt{3} < x < \sqrt{3}$

Therefore, $P(X<1 \mid Y<1) = \int_{-\sqrt{3}}^{1}\frac{1}{2\sqrt{3}}\,dx = \frac{1+\sqrt{3}}{2\sqrt{3}} = 0.7887$

e) $f_{XY|1}(x,y) = \frac{f_{XYZ}(x,y,1)}{f_Z(1)}$ *and* $f_Z(z) = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} c\,dy\,dx = \int_{-2}^{2}2c\sqrt{4-x^2}\,dx$

Because $f_Z(z)$ is a density over the range 0 < z < 4 that does not depend on Z, $f_Z(z)$ =1/4 for 0 < z < 4. Then, $f_{XY|1}(x,y) = \frac{c}{1/4} = \frac{1}{4\pi}$ for $x^2+y^2<4$.

Then, $P(X^2+Y^2<1 \mid Z=1) = \frac{area\ in\ x^2+y^2<1}{4\pi} = 1/4$

f) $f_{Z|xy}(z) = \frac{f_{XYZ}(x,y,z)}{f_{XY}(x,y)}$ and $f_{XY}(x,y) = \frac{1}{4\pi}$ *for* $x^2+y^2<4$. Therefore,

$f_{Z|xy}(z) = \frac{\frac{1}{16\pi}}{\frac{1}{4\pi}} = 1/4$ for 0 < z < 4.

5-25. Determine c such that $f(xyz) = c$ is a joint density probability over the region x > 0, y > 0 and z > 0 with x + y + z < 1

$$f(xyz) = c\int_0^1\int_0^{1-x}\int_0^{1-x-y} dzdydx = \int_0^1\int_0^{1-x} c(1-x-y)dydx = \int_0^1 \left(c(y - xy - \frac{y^2}{2})\Big|_0^{1-x} \right)dx$$

$$= \int_0^1 c\left((1-x) - x(1-x) - \frac{(1-x)^2}{2} \right)dx = \int_0^1 c\left(\frac{(1-x)^2}{2} \right)dx = c\left(\frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6} \right)\Bigg|_0^1$$

$$= c\frac{1}{6}. \quad \text{Therefore, c = 6.}$$

a) $P(X < 0.25, Y < 0.25, Z < 0.25) = 6\int_0^1\int_0^{1-x}\int_0^{1-x-y} dzdydx \Rightarrow$ The conditions $x<0.25$, $y<0.25$, $z<0.25$ and $x+y+z<1$ make a space that is a cube with a volume of 0.015625. Therefore the probability of $P(X < 0.25, Y < 0.25, Z < 0.25) = 6(0.015625) = 0.09375$

b)

$$P(X < 0.5, Y < 0.5) = \int_0^{0.5}\int_0^{0.5} 6(1-x-y)dydx = \int_0^{0.5} \left(6y - 6xy - 3y^2\right)\Big|_0^{0.5} dx$$

$$= \int_0^{0.5} \left(\frac{9}{4} - 3x \right)dx = \left(\frac{9}{4}x - \frac{3}{2}x^2 \right)\Bigg|_0^{0.5} = 3/4$$

c)

$$P(X < 0.5) = 6\int_0^{0.5}\int_0^{1-x}\int_0^{1-x-y} dzdydx = \int_0^{0.5}\int_0^{1-x} 6(1-x-y)dydx = \int_0^{0.5} 6(y - xy - \frac{y^2}{2})\Bigg|_0^{1-x}$$

$$= \int_0^{0.5} 6(\frac{x^2}{2} - x + \frac{1}{2})dx = \left(x^3 - 3x^2 + 3x\right)\Big|_0^{0.5} = 0.875$$

d)

$$E(X) = 6\int_0^1\int_0^{1-x}\int_0^{1-x-y} xdzdydx = \int_0^1\int_0^{1-x} 6x(1-x-y)dydx = \int_0^{0.5} 6x(y - xy - \frac{y^2}{2})\Bigg|_0^{1-x}$$

$$= \int_0^1 6(\frac{x^3}{2} - x^2 + \frac{x}{2})dx = \left(\frac{3x^4}{4} - 2x^3 + \frac{3x^2}{2} \right)\Bigg|_0^1 = 0.25$$

e)

$$f(x) = 6\int_0^{1-x}\int_0^{1-x-y} dzdy = \int_0^{1-x} 6(1-x-y)dy = 6\left(y - xy - \frac{y^2}{2} \right)\Bigg|_0^{1-x}$$

$$= 6(\frac{x^2}{2} - x + \frac{1}{2}) = 3(x-1)^2 \text{ for } 0 < x < 1$$

f)

$$f(x,y) = 6\int_0^{1-x-y} dz = 6(1-x-y)$$

for $x > 0,\ y > 0\ and\ x + y < 1$

g)

$$f(x \mid y = 0.5, z = 0.5) = \frac{f(x, y = 0.5, z = 0{,}5)}{f(y = 0.5, z = 0.5)} = \frac{6}{6} = 1 \text{ for } x > 0$$

h) The marginal $f_Y(y)$ is similar to $f_X(x)$ and $f_Y(y) = 3(1-y)^2$ for $0 < y < 1$.

$$f_{X|Y}(x \mid 0.5) = \frac{f(x,0.5)}{f_Y(0.5)} = \frac{6(0.5-x)}{3(0.25)} = 4(1-2x) \text{ for } x < 0.5$$

5-26. Let X denote the production yield on a day. Then,

$P(X > 635) = P(Z > \frac{635-680}{45}) = P(Z > -1) = 0.84134$.

a) Let Y denote the number of days out of five such that the yield exceeds 635. Then, by independence, Y has a binomial distribution with n = 5 and p = 0.8413. Therefore, the answer is

$P(Y = 5) = \binom{5}{5}0.8413^5(1-0.8413)^0 = 0.4215$.

b) As in part (a), the answer is

$$P(Y \geq 4) = P(Y = 4) + P(Y = 5)$$

$$= \binom{5}{4}0.8413^4(1-0.8413)^1 + 0.4215 = 0.8190$$

5-27. a) Let X denote the weight of a brick. Then,

$P(X > 2.75) = P(Z > \frac{1.2-1.5}{0.3}) = P(Z > -1) = 0.84134$.

Let Y denote the number of bricks in the sample of 20 that exceed 1.2 kg Then, by independence, Y has a binomial distribution with n = 20 and p = 0.84134. Therefore, the answer is

$P(Y = 20) = \binom{20}{20}0.84134^{20} = 0.032$.

b) Let A denote the event that the heaviest brick in the sample exceeds 1.8 kg. Then, P(A) = 1 - P(A') and A' is the event that all bricks weigh less than 1.8 kg. As in part a., P(X < 1.8) = P(Z < 1) and $P(A) = 1 - [P(Z < 1)]^{20} = 1 - 0.84135^{20} = 0.9684$.

5-28. a) Let X denote the grams of luminescent ink. Then,

$P(X < 1.14) = P(Z < \frac{1.14-1.2}{0.3}) = P(Z < -2) = 0.022750$

Let Y denote the number of bulbs in the sample of 25 that have less than 1.14 grams. Then, by independence, Y has a binomial distribution with n = 25 and p = 0.022750. Therefore, the answer is $P(Y \geq 1) = 1 - P(Y = 0) = \binom{25}{0}0.02275^0(0.97725)^{25} = 1 - 0.5625 = 0.4375$.

b)

$$P(Y \leq 5) = P(Y = 0) + P(Y = 1) + P(Y = 2) + (P(Y = 3) + P(Y = 4) + P(Y = 5)$$
$$= \binom{25}{0}0.02275^0(0.97725)^{25} + \binom{25}{1}0.02275^1(0.97725)^{24} + \binom{25}{2}0.02275^2(0.97725)^{23}$$
$$+ \binom{25}{3}0.02275^3(0.97725)^{22} + \binom{25}{4}0.02275^4(0.97725)^{21} + \binom{25}{5}0.02275^5(0.97725)^{20}.$$
$$= 0.5625 + 0.3274 + 0.09146 + 0.01632 + 0.002090 + 0.0002043 = 0.99997 \cong 1$$

c) $P(Y = 0) = \binom{25}{0}0.02275^0(0.97725)^{25} = 0.5625$

d) The lamps are normally and independently distributed. Therefore, the probabilities can be multiplied.

Section 5-2

5-29. E(X) = 2(1/8)+1(1/4)+2(1/2)+4(1/8) = 2
E(Y) = 3(1/8)+4(1/4)+5(1/2)+6(1/8)=37/8 = 4.625

$$\text{E(XY)} = [2\times 3\times(1/8)] + [1\times 4\times(1/4)] + [2\times 5\times(1/2)] + [4\times 6\times(1/8)]$$
$$= 39/4 = 9.75$$
$$\sigma_{XY} = E(XY) - E(X)E(Y) = 9.75 - (2)(4.625) = 0.5$$

V(X) $= 2^2(1/8) + 1^2(1/4) + 2^2(1/2) + 4^2(1/8) - 2^2$
$= 0.5 + 0.25 + 2 + 2 - 2^2 = 4.75 - 4 = 0.75$

V(Y) $= 3^2(1/8) + 4^2(1/4) + 5^2(1/2) + 6^2(1/8) - (37/8)^2 = 0.7344$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{0.5}{\sqrt{(0.75)(0.7344)}} = 0.6737$$

5-30. $$E(X) = -1(1/8) + (-0.5)(1/5) + 0.5(1/2) + 1(7/40) = 0.2$$
$$E(Y) = -2(1/8) + (-1)(1/5) + 1(1/2) + 2(7/40) = 0.4$$
$$\text{E(XY)} = [-1\times -2\times(1/8)] + [-0.5\times -1\times(1/5)] + [0.5\times 1\times(1/2)] + [1\times 2\times(7/40)] = 0.95$$

V(X) = 0.435
V(Y) = 1.74

$$\sigma_{XY} = 0.95 - (0.2)(0.4) = 0.87$$
$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{0.87}{\sqrt{0.435}\sqrt{1.75}} = 1$$

5-31.

$$\sum_{x=1}^{3}\sum_{y=1}^{2} c(x+y) = 21c, \qquad c = 1/21$$
$$E(X) = \frac{46}{21} \qquad E(Y) = \frac{11}{7} \qquad E(XY) = \frac{24}{7} \qquad \sigma_{xy} = \frac{24}{7} - \left(\frac{46}{21}\right)\left(\frac{11}{7}\right) = \frac{-2}{147} = -0.0136$$
$$E(X^2) = \frac{114}{21} \qquad E(Y^2) = \frac{57}{21}$$
$$V(X) = EX^2 - (EX)^2 = 0.63 \qquad V(Y) = 0.24$$
$$\rho = \frac{-0.0136}{\sqrt{0.63}\sqrt{0.24}} = -0.035$$

5-32. The marginal distribution of X is

x	f(x)
0	0.75
1	0.2
2	0.05

$$E(X) = 0(0.75) + 1(0.2) + 2(0.05) = 0.3$$

$$E(Y) = 0(0.3) + 1(0.28) + 2(0.25) + 3(0.17) = 1.29$$
$$E(X^2) = 0(0.75) + 1(0.2) + 4(0.05) = 0.4$$
$$E(Y^2) = 0(0.3) + 1(0.28) + 4(0.25) + 9(0.17) = 1.146$$

V(X) = $0.4 - 0.3^2 = 0.31$
V(Y) = $2.81 - 1.146^2 = 1.16$

$$\text{E(XY)} = [0\times 0\times(0.225)] + [0\times 1\times(0.21)] + [0\times 2\times(0.1875)] + ... + [2\times 3\times(0.0085)] = 0.387$$

$$\sigma_{XY} = 0.387 - (0.3)(1.29) = 0$$
$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = 0$$

5-33. Let X and Y denote the number of patients who improve or degrade, respectively, and let Z denote the number of patients that remain the same. If $X = 0$, then Y can equal 0,1,2,3, or 4. However, if $X = 4$ then $Y = 0$. Consequently, the range of the joint distribution of X and Y is not rectangular. Therefore, X and Y are not independent.

Var(X + Y) = Var(X) + Var(Y) + 2Cov(X,Y).
Therefore,
Cov(X,Y) = 0.5[Var(X + Y) - Var(X) - Var(Y)]

Here X and Y are binomially distributed when considered individually. Therefore,

$$f_X(x) = \frac{4!}{x!(4-x)!} 0.4^x (1-0.4)^{4-x}$$
$$f_Y(y) = \frac{4!}{y!(4-y)!} 0.1^y (1-0.1)^{4-y}$$

And
Var(X) = 4(0.4)(0.6) = 0.96
Var(Y) = 4(0.1)(0.9) = 0.36

Also, $W = X + Y$ is binomial with $n = 4$, and $p = 0.4 + 0.1 = 0.5$. Therefore,
Var($X + Y$) = 4(0.5)(0.5) = 1

Therefore, Cov(X,Y) = 0.5[1 – 0.96 – 0.36] = –0.16

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{-0.16}{\sqrt{0.96\times 0.36}} = -0.272$$

5-34.

Transaction	Frequency	Selects(X)	Updates(Y)	Inserts(Z)
New Order	43	23	11	12
Payment	44	4.2	3	1
Order Status	4	11.4	0	0
Delivery	5	130	120	0
Stock Level	4	0	0	0
Mean Value		18.694	12.05	5.6

(a) COV(X,Y) = E(XY)-E(X)E(Y) = 23*11*0.43 + 4.2*3*0.44 + 11.4*0*0.04 + 130*120*0.05 + 0*0*0.04 - 18.694*12.05=669.0713

(b) V(X)=735.9644, V(Y)=630.7875; Corr(X,Y)=cov(X,Y)/(V(X)*V(Y))$^{0.5}$ = 0.9820

(c) COV(Y,Z)=11*12*0.43+3*1*0.44+0-12.05*5.6 = -9.4
(d) V(Z)=31; Corr(X,Z)=-0.067

5-35. $\int_0^2\int_0^x cxydydx = c\int_0^2 \frac{1}{2}x^3dx = (\frac{c}{8})(2^4) = 1$, c = 1/2, $E(X)$ = 8/5, and $E(Y)$ = 16/15

$$E(XY) = \frac{1}{2}\int_0^2\int_0^x xy(xy)dydx = \frac{16}{9}$$

$$\sigma_{xy} = \frac{16}{9} - \left(\frac{8}{5}\right)\left(\frac{16}{15}\right) = 0.071$$

$$E(X^2) = \frac{8}{3} \qquad E(Y^2) = \frac{4}{3}$$

$$V(x) = 0.107, \qquad V(Y) = 0.196$$

$$\rho = \frac{0.071}{\sqrt{0.107}\sqrt{0.196}} = 0.492$$

5-36. $\int_0^1\int_0^{x+1} cdydx + \int_1^4\int_{x-1}^{x+1} cdydx = c(\frac{3}{2}) + c(8-2) = \frac{15}{2}c = 1, \qquad c = \frac{2}{15}$

$$E(X) = \frac{2}{15}\int_0^1\int_0^{x+1} xdydx + \frac{2}{15}\int_1^4\int_{x-1}^{x+1} xdydx = 2.111$$

$$E(Y) = \frac{2}{15}\int_0^1\int_0^{x+1} ydydx + \frac{2}{15}\int_1^4\int_{x-1}^{x+1} ydydx = 2.156$$

Now, $E(XY) = \frac{2}{15}\int_0^1\int_0^{x+1} xydydx + \frac{2}{15}\int_1^4\int_{x-1}^{x+1} xydydx = 5.694$

$$\sigma_{xy} = 5.694 - (2.111)(2.156) = 1.143$$

$$E(X^2) = 5.678 \qquad E(Y^2) = 6.033$$

$$V(x) = 1.222, \qquad V(Y) = 1.385$$

$$\rho = \frac{1.143}{\sqrt{1.222}\sqrt{1.385}} = 0.879$$

5-37. a) E(X) = 0 E(Y) = 0

$$E(XY) = \int_1^\infty\int_1^\infty xye^{-x-y}dxdy$$

$$= \int_1^\infty xe^{-x}dx\int_1^\infty ye^{-y}dy$$

$$= E(X)E(Y)$$

Therefore, $\sigma_{XY} = \rho_{XY} = 0$.

5-38. E(X) = 333.33, E(Y)= 833.33

$E(X^2) = 222{,}222.2$
$V(X) = 222222.2-(333.33)^2=111{,}113.31$
$E(Y^2) = 1{,}055{,}556$
$V(Y) = 361{,}117.11$

$$E(XY) = 6\times 10^{-6}\int_0^\infty\int_x^\infty xye^{-.001x-.002y}dydx = 388{,}888.9$$

$$\sigma_{xy} = 388{,}888.9-(333.33)(833.33) = 111{,}115.01$$

$$\rho = \frac{111{,}115.01}{\sqrt{111113.31}\sqrt{361117.11}} = 0.5547$$

5-39. $E(X) = -1(1/6)+1(1/6) = 0$

$$E(Y) = -1(1/3)+1(1/3) = 0$$

$$\text{E(XY)} = [-1\times 0\times(1/6)]+[-1\times 0\times(1/3)]+[1\times 0\times(1/3)]+[0\times 1\times(1/6)] = 0$$

$V(X) = 1/3$
$V(Y) = 2/3$

$$\sigma_{XY} = 0-(0)(0) = 0$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{0}{\sqrt{1/3}\sqrt{2/3}} = 0$$

The correlation is zero, but *X* and *Y* are not independent, since, for example, if y = 0, X must be –1 or 1.

5-40. If X and Y are independent, then $f_{XY}(x,y) = f_X(x)f_Y(y)$ and the range of (X, Y) is rectangular. Therefore,

$$E(XY) = \iint xyf_X(x)f_Y(y)dxdy = \int xf_X(x)dx\int yf_Y(y)dy = E(X)E(Y)$$

hence $\sigma_{XY} = 0$

5-41. Suppose the correlation between X and Y is ρ. For constants a, b, c, and d, what is the correlation between the random variables U = aX+b and V = cY+d?

Now, E(U) = a E(X) + b and E(V) = c E(Y) + d.
Also, U - E(U) = a[X - E(X)] and V - E(V) = c[Y - E(Y)]. Then,

$$\sigma_{UV} = E\{[U-E(U)][V-E(V)]\} = acE\{[X-E(X)][Y-E(Y)]\} = ac\sigma_{XY}$$

Also, $\sigma_U^2 = E[U-E(U)]^2 = a^2E[X-E(X)]^2 = a^2\sigma_X^2$ *and* $\sigma_V^2 = c^2\sigma_Y^2$. Then,

$$\rho_{UV} = \frac{ac\sigma_{XY}}{\sqrt{a^2\sigma_X^2}\sqrt{c^2\sigma_Y^2}} = \begin{cases}\rho_{XY} & \text{if a and c are of the same sign}\\ -\rho_{XY} & \text{if a and c differ in sign}\end{cases}$$

<u>Section 5-3</u>

5-42. a) board failures caused by assembly defects = $p_1 = 0.5$
board failures caused by electrical components = $p_2 = 0.4$
board failures caused by mechanical defects = $p_3 = 0.1$

$$P(X=5, Y=3, Z=2) = \frac{10!}{5!3!2!}0.5^5 0.4^3 0.1^2 = 0.0504$$

b) Because X is binomial, $P(X=8) = \binom{10}{8}0.5^8 0.5^2 = 0.0439$

c) $P(X=8|Y=1)=\dfrac{P(X=8,Y=1)}{P(Y=1)}$. Now, because x + y + z = 10,

P(X = 8, Y = 1) = P(X = 8, Y = 1, Z = 1) = $\dfrac{10!}{8!1!1!}0.5^8 0.4^1 0.1^1=0.0141$

$P(Y=1)=\binom{10}{1}0.4^1 0.6^9=0.0403$

$P(X=8|Y=1)=\dfrac{P(X=8,Y=1)}{P(Y=1)}=\dfrac{0.0141}{0.0403}=0.3499$

d) $P(X\geq 8|Y=1)=\dfrac{P(X=8,Y=1)}{P(Y=1)}+\dfrac{P(X=9,Y=1)}{P(Y=1)}$. Now, because x + y + z = 10,

P(X = 8, Y = 1) = P(X = 8, Y = 1, Z = 1) = $\dfrac{10!}{8!1!1!}0.5^8 0.4^1 0.1^1=0.0141$

P(X = 9, Y = 1) = P(X = 9, Y = 1, Z = 0) = $\dfrac{10!}{9!1!0!}0.5^9 0.4^1 0.1^0=0.0078$

$P(Y=1)=\binom{10}{1}0.4^1 0.6^9=0.0403$

$P(X\geq 8|Y=1)=\dfrac{P(X=8,Y=1)}{P(Y=1)}+\dfrac{P(X=9,Y=1)}{P(Y=1)}=\dfrac{0.0141}{0.0403}+\dfrac{0.0078}{0.0403}=0.5434$

e) $P(X=7,Y=1|Z=2)=\dfrac{P(X=7,Y=1,Z=2)}{P(Z=2)}$

P(X = 7, Y = 1, Z = 2) = $\dfrac{10!}{7!1!2!}0.5^7 0.4^1 0.1^2=0.0113$

$P(Z=2)=\binom{10}{2}0.1^2 0.9^8=0.1937$

$P(X=7,Y=1|Z=2)=\dfrac{P(X=7,Y=1,Z=2)}{P(Z=2)}=\dfrac{0.0113}{0.1937}=0.0583$

5-43. a) percentage of slabs classified as high = p_1 = 0.05
percentage of slabs classified as medium = p_2 = 0.9
percentage of slabs classified as low = p_3 = 0.05

b) X is the number of voids independently classified as high $X \geq 0$
Y is the number of voids independently classified as medium $Y \geq 0$
Z is the number of voids classified as low and $Z \geq 0$ and $X + Y + Z = 20$

c) p_1 is the percentage of slabs classified as high.

d) E(X) = np_1 = 20(0.05) = 1
V(X) = $np_1(1 - p_1)$ = 20(0.05)(0.95) = 0.95

e) $P(X=1,Y=17,Z=3)=0$ Because the point $(1,17,3)\neq 20$ is not in the range of (X,Y,Z).

f) $P(X\leq 1,Y=17,Z=3)=P(X=0,Y=17,Z=3)+P(X=1,Y=17,Z=3)$

$$=\frac{20!}{0!17!3!}0.05^0 0.9^{17} 0.05^3+0=0.02376$$

Because the point $(1,17,3)\neq 20$ is not in the range of (X, Y, Z).

g) Because X is binomial, $P(X\leq 1)=\binom{20}{0}0.05^0 0.95^{20}+\binom{20}{1}0.05^1 0.95^{19}=0.7358$

h) Because X is binomial E(Y) = np = 20(0.9) = 18

i) The probability is 0 because x + y + z > 20

j) $P(X = 2 | Y = 17) = \frac{P(X = 2, Y = 17)}{P(Y = 17)}$. Now, because x + y + z = 20,

P(X = 2, Y = 17) = P(X = 2, Y = 17, Z = 1) = $\frac{20!}{2!17!1!} 0.05^2 0.9^{17} 0.05^1 = 0.0713$

$$P(X = 2 | Y = 17) = \frac{P(X = 2, Y = 17)}{P(Y = 17)} = \frac{0.0713}{0.1901} = 0.3751$$

k) $$E(X | Y = 17) = 0\left(\frac{P(X = 0, Y = 17)}{P(Y = 17)}\right) + 1\left(\frac{P(X = 1, Y = 17)}{P(Y = 17)}\right) + 2\left(\frac{P(X = 2, Y = 17)}{P(Y = 17)}\right) + 3\left(\frac{P(X = 3, Y = 17)}{P(Y = 17)}\right)$$

$$E(X | Y = 17) = 0\left(\frac{0.02376}{0.1901}\right) + 1\left(\frac{0.07129}{0.1901}\right) + 2\left(\frac{0.0713}{0.1901}\right) + 3\left(\frac{0.02376}{0.1901}\right)$$

$= 2$

5-44. a) probability for the first landing page = $p_1 = 0.25$
probability for the second landing page = $p_2 = 0.25$
probability for the third landing page = $p_3 = 0.25$
probability for the fourth landing page = $p_4 = 0.25$

$$P(W = 6, X = 6, Y = 6, Z = 6) = \frac{24!}{6!6!6!6!} 0.25^6 0.25^6 0.25^6 0.25^6 = 0.0082$$

b) Because w+x+y+z = 24 $P(W = 6, X = 6, Y = 6) = P(W = 6, X = 6, Y = 6, Z = 6)$

$$P(W = 6, X = 6, Y = 6) = \frac{24!}{6!6!6!6!} 0.25^6 0.25^6 0.25^6 0.25^6 = 0.0082$$

c) $P(W = 7, X = 7, Y = 6 | Z = 3) = 0$ Because the point $(7,7,6,3) \neq 24$ is not in the range of (W,X,Y,Z).

d) $$P(W = 8, X = 8, Y = 5 | Z = 3) = \frac{P(W = 8, X = 8, Y = 5, Z = 3)}{P(Z = 3)}$$

P(W=8,X=8, Y=5, Z=3) = $\frac{24!}{8!8!5!3!} 0.25^8 0.25^8 0.25^5 0.25^3 = 0.0019$

$$P(Z = 3) = \binom{24}{3} 0.25^3 0.75^{21} = 0.0752$$

$$P(W = 8, X = 8, Y = 5 | Z = 3) = \frac{P(W = 8, X = 8, Y = 5, Z = 3)}{P(Z = 3)} = \frac{0.0019}{0.0752} = 0.0253$$

e) Because W is binomial,

$$P(W \le 2) = \binom{24}{0} 0.25^0 0.75^{24} + \binom{24}{1} 0.25^1 0.75^{23} + \binom{24}{2} 0.25^2 0.75^{22} = 0.0398$$

f) E(W)=np_1 = 24(0.25) = 6

g) $$P(W = 6, X = 6) = P(W = 6, X = 6, Y + Z = 12) = \frac{24!}{6!6!12!} 0.25^6 0.25^6 0.5^{12} = 0.0364$$

h) $$P(W = 6 | X = 6) = \frac{P(W = 6, X = 6)}{P(X = 6)}$$

from part g) $P(W = 6, X = 6) = 0.0364$

$$P(X = 6) = \binom{24}{6} 0.25^6 0.75^{18} = 0.1853$$

$$P(W=6\,|\,X=6)=\frac{P(W=6,X=6)}{P(X=6)}=\frac{0.0364}{0.1853}=0.1964$$

5-45. a) The probability distribution is multinomial because the result of each trial (a dropped oven) results in either a major, minor or no defect with probability 0.5, 0.4 and 0.1 respectively. Also, the trials are independent

b) Let X, Y, and Z denote the number of ovens in the sample of four with major, minor, and no defects, respectively.

$$P(X=2,Y=2,Z=0)=\frac{4!}{2!2!0!}0.5^2 0.4^2 0.1^0=0.24$$

c) $P(X=0,Y=0,Z=4)=\frac{4!}{0!0!4!}0.5^0 0.4^0 0.1^4=0.0001$

d) $f_{XY}(x,y)=\sum_R f_{XYZ}(x,y,z)$ where R is the set of values for z such that x+y+z = 4. That is, R consists of the single value z = 4-x-y and

$$f_{XY}(x,y)=\frac{4!}{x!\,y!(4-x-y)!}0.5^x 0.4^y 0.1^{4-x-y} \quad \text{for } x+y\le 4.$$

e) $E(X)=np_1=4(0.5)=2$

f) $E(Y)=np_2=4(0.4)=1.6$

g) $P(X=2\,|\,Y=2)=\frac{P(X=2,Y=2)}{P(Y=2)}=\frac{0.24}{0.3456}=0.694$

$P(Y=2)=\binom{4}{2}0.4^2 0.6^2=0.3456$ from the binomial marginal distribution of *Y*

h) Not possible, *x*+*y*+*z* = 4, the probability is zero.

i) $P(X\,|\,Y=2)=P(X=0\,|\,Y=2),P(X=1\,|\,Y=2),P(X=2\,|\,Y=2)$

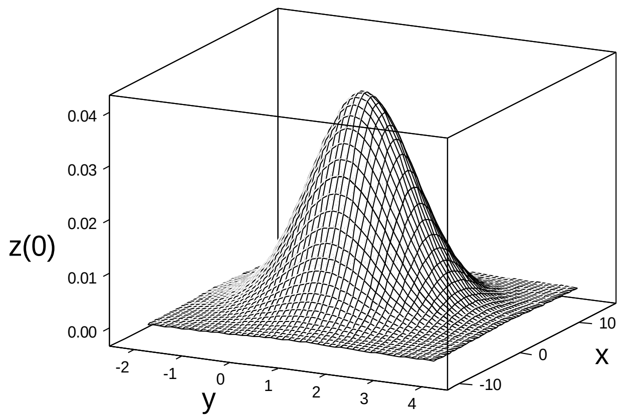
$$P(X=0\,|\,Y=2)=\frac{P(X=0,Y=2)}{P(Y=2)}=\left(\frac{4!}{0!2!2!}0.5^0 0.4^2 0.1^2\right)\Big/0.3456=0.0278$$

$$P(X=1\,|\,Y=2)=\frac{P(X=1,Y=2)}{P(Y=2)}=\left(\frac{4!}{1!2!1!}0.5^1 0.4^2 0.1^1\right)\Big/0.3456=0.2778$$

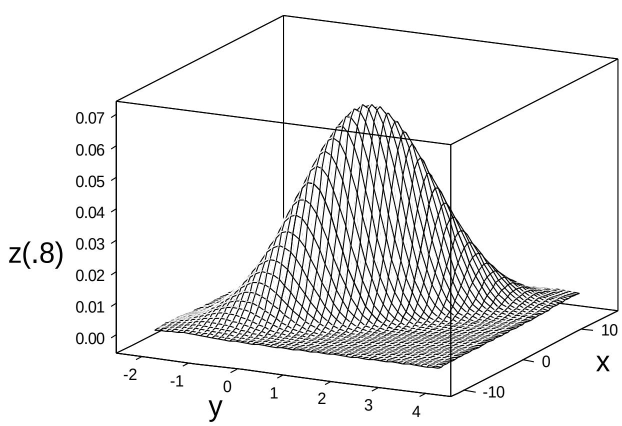
$$P(X=2\,|\,Y=2)=\frac{P(X=2,Y=2)}{P(Y=2)}=\left(\frac{4!}{2!2!0!}0.5^2 0.4^2 0.1^0\right)\Big/0.3456=0.6944$$

j) E(X|Y=2) = 0(0.0278)+1(0.2778)+2(0.6944) = 1.6666

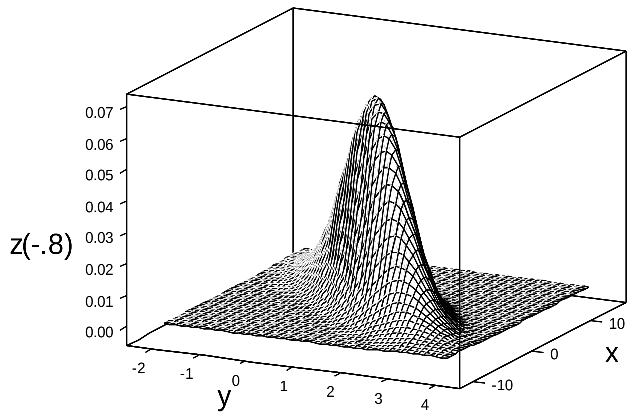
5-46. a)


0.04
0.03
0.02
z(0)
0.01
0.00
-2
-1
0
y
1
2
3
4
10
0
x
-10

b)


0.07
0.06
0.05
0.04
z(.8)
0.03
0.02
0.01
0.00
-2
-1
0
y
1
2
3
4
10
0
x
-10

c)



5-47. Because $\rho = 0$ and X and Y are normally distributed, X and Y are independent. Therefore,

(a) P(2.95 < X < 3.05) = $P(\frac{2.95-3}{0.04} < Z < \frac{3.05-3}{0.04})$ = 0.7887

(b) P(7.60 < Y < 7.80) = $P(\frac{7.6-8.0}{0.08} < Z < \frac{7.8-8.0}{0.08})$ = 0.00621

(c) P(2.95 < X < 3.05, 7.60 < Y < 7.80) = P(2.95 < X < 3.05) P(7.60 < Y < 7.80) =
$P(\frac{2.95-3}{0.04} < Z < \frac{3.05-3}{0.04})P(\frac{7.60-8.00}{0.08} < Z < \frac{7.80-8.00}{0.08}) = (0.7887)(0.00621) = 0.0049$

5-48. (a) $\rho = cov(X,Y)/\sigma_x\sigma_y = 0.6$; cov(X,Y)= 0.6*3*5=9

(b) The marginal probability distribution of X is normal with mean μ_x, σ_x.

(c) P(X < 116) =P(X-120 < -4)=P((X-120)/5 < -0.8)=P(Z < –0.8) = 0.21

(d) The conditional probability distribution of X given Y=102 is bivariate normal distribution with mean and variance

$\mu_{X|y=102}$ = 120 – 100*0.6*(5/3) +(5/3)*0.6(102) = 122

$\sigma^2_{X|y=102}$ = 25(1-0.36) =16

(e) P(X < 116|Y=102)=P(Z < (116-122)/4)=0.0668

5-49. Because $\rho = 0$ and X and Y are normally distributed, X and Y are independent. Therefore, μ_X = 0.1 mm, σ_X=0.00031 mm, μ_Y = 0.23 mm, σ_Y=0.00017 mm

Probability X is within specification limits is

$$P(0.099535 < X < 0.100465) = P\left(\frac{0.099535-0.1}{0.00031} < Z < \frac{0.100465-0.1}{0.00031}\right)$$

$$= P(-1.5 < Z < 1.5) = P(Z < 1.5) - P(Z < -1.5) = 0.8664$$

Probability that Y is within specification limits is

$$P(0.22966 < X < 0.23034) = P\left(\frac{0.22966-0.23}{0.00017} < Z < \frac{0.23034-0.23}{0.00017}\right)$$

$$= P(-2 < Z < 2) = P(Z < 2) - P(Z < -2) = 0.9545$$

Probability that a randomly selected lamp is within specification limits is (0.8664)(0.9594) = 0.8270

5-50. a) By completing the square in the numerator of the exponent of the bivariate normal PDF, the joint PDF can be written as

$$f_{Y|X=x} = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\dfrac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\left[\frac{1}{\sigma_Y^2}\left[(y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_x))\right]^2+(1-\rho^2)\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]}{2(1-\rho^2)}}}{\dfrac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{\left[\frac{x-\mu_x}{\sigma_x}\right]^2}{2}}}$$

Also, $f_x(x) = \dfrac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{\left[\frac{x-\mu_x}{\sigma_x}\right]^2}{2}}$ By definition,

$$f_{Y|X=x} = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\dfrac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\left[\frac{1}{\sigma_Y^2}\left[(y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_x))\right]^2+(1-\rho^2)\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]}{2(1-\rho^2)}}}{\dfrac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{\left[\frac{x-\mu_x}{\sigma_x}\right]^2}{2(1-\rho^2)}}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\frac{1}{\sigma_Y^2}\left[(y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_x))\right]^2}{2(1-\rho^2)}}$$

Now $f_{Y|X=x}$ is in the form of a normal distribution.

b) E(Y|X=x) = $\mu_y + \rho\dfrac{\sigma_y}{\sigma_x}(x-\mu_x)$. This answer can be seen from part a). Because the PDF is in the form of a normal distribution, then the mean can be obtained from the exponent.

c) V(Y|X=x) = $\sigma_y^2(1-\rho^2)$. This answer can be seen from part a). Because the PDF is in the form of a normal distribution, then the variance can be obtained from the exponent.

5-51.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}\right]dxdy =$$

$$\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}\right]}\right]dx\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\left[\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}\right]dy$$

and each of the last two integrals is recognized as the integral of a normal probability density function from $-\infty$ to ∞. That is, each integral equals one. Because $f_{XY}(x,y) = f(x)f(y)$, *X* and *Y* are independent.

5-52.

Let $f_{XY}(x,y)=\dfrac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^2-\frac{2\rho(X-\mu_X)(Y-\mu_X)}{\sigma_X\sigma_Y}+\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right]}{2(1-\rho^2)}}$

Completing the square in the numerator of the exponent we get:

$$\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^2-\frac{2\rho(X-\mu_X)(Y-\mu_X)}{\sigma_X\sigma_Y}+\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right]=\left[\left(\frac{Y-\mu_Y}{\sigma_Y}\right)-\rho\left(\frac{X-\mu_X}{\sigma_X}\right)\right]^2+(1-\rho^2)\left(\frac{X-\mu_X}{\sigma_X}\right)^2$$

But,

$$\left(\frac{Y-\mu_Y}{\sigma_Y}\right)-\rho\left(\frac{X-\mu_X}{\sigma_X}\right)=\frac{1}{\sigma_Y}\left[(Y-\mu_Y)-\rho\frac{\sigma_Y}{\sigma_X}(X-\mu_x)\right]=\frac{1}{\sigma_Y}\left[(Y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(X-\mu_x))\right]$$

Substituting into $f_{XY}(x,y)$, we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{XY}(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{\left[\frac{1}{\sigma_Y^2}\left[y-(\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_x))\right]^2+(1-\rho^2)\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]}{2(1-\rho^2)}}dydx$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}dx\times\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}}e^{-\left[\frac{\left(y-(\mu_y+\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x))\right)^2}{2\sigma_x^2(1-\rho^2)}\right]}dy$$

The integrand in the second integral above is in the form of a normally distributed random variable. By definition of the integral over this function, the second integral is equal to 1:

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}dx\times\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}}e^{-\left[\frac{\left(y-(\mu_y+\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x))\right)^2}{2\sigma_x^2(1-\rho^2)}\right]}dy$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}dx\times 1$$

The remaining integral is also the integral of a normally distributed random variable and therefore, it also integrates to 1, by definition. Therefore,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{XY}(x,y)=1$$

5-53.

$$f_X(x) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-0.5}{1-\rho^2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]} \right] dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{-0.5}{1-\rho^2}\frac{(x-\mu_X)^2}{\sigma_X^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-0.5}{1-\rho^2}\left\{\left[\frac{(y-\mu_Y)}{\sigma_Y} - \frac{\rho(x-\mu_X)}{\sigma_X}\right]^2 - \left[\frac{\rho(x-\mu_X)}{\sigma_X}\right]^2\right\}} dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} e^{-0.5\frac{(x-\mu_X)^2}{\sigma_X^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-0.5}{1-\rho^2}\left[\frac{(y-\mu_Y)}{\sigma_Y} - \frac{\rho(x-\mu_X)}{\sigma_X}\right]^2} dy$$

The last integral is recognized as the integral of a normal probability density with mean $\mu_Y + \frac{\sigma_Y\rho(x-\mu_X)}{\sigma_X}$ and variance $\sigma_Y{}^2(1-\rho^2)$. Therefore, the last integral equals one and the requested result is obtained.

<u>Section 5-4</u>

5-54. a) E(2X + 3Y) = 2(0) + 3(10) = 30
b) V(2X + 3Y) = 4V(X) + 9V(Y) = 101
c) 2X + 3Y is normally distributed with mean 30 and variance 101. Therefore,
$P(2X+3Y<30) = P(Z<\frac{30-30}{\sqrt{101}}) = P(Z<0) = 0.5$
d) $P(2X+3Y<40) = P(Z<\frac{40-30}{\sqrt{101}}) = P(Z<0.995) = 0.8389$

5-55. (a) E(3X+2Y) = 3*2+2*6=18
(b) V(3X+2Y) = 9*5+4*9 =81
(c) 3X+2Y ~ N(18, 81)
P(3X+2Y < 18) = P(Z < (18-18)/$81^{0.5}$)=0.5
(d) P(3X+2Y < 28) = P(Z < (28-18)/$81^{0.5}$)=P(Z < 1.1111) =0.8665

5-56. Y = 10X and E(Y) =10E(X) = 90mm.
V(Y) = 10^2V(X)=40mm^2

5-57. a) Let T denote the total thickness. Then, T = X + Y and E(T) = 6 mm,
V(T) = $0.1^2 + 0.1^2 = 0.02mm^2$, and $\sigma_T = 0.1414$ mm.
b)

$$P(T>6.3) = P\left(Z > \frac{6.3-6}{0.1414}\right) = P(Z>2.12)$$
$$= 1 - P(Z<2.12) = 1 - 0.983 = 0.0170$$

5-58. (a) X: time of wheel throwing. X~N(40,4)
Y: time of wheel firing. Y~N(60,9)
X + Y ~ N(100, 13)
P(X + Y ≤ 90) = P(Z < (90 – 100)/$13^{0.5}$) = P(Z < –2.774) = 0.0028

(b) P(X + Y > 110) = 1 – P(Z < (110 – 100)/ $13^{0.5}$) = 1 – P(Z < 2.774) = 1 – 0.9972 = 0.0028

5-59. a) *X*~N(0.1, 0.00031) and *Y*~N(0.23, 0.00017) Let T denote the total thickness.

Then, T = X + Y and E(T) = 0.33 mm,

V(T) = $0.00031^2 + 0.00017^2 = 1.25 \times 10^{-7}\, mm^2$, and $\sigma_T = 0.000354$ mm.

$P(T < 0.2340) = P\left(Z < \frac{0.2340 - 0.33}{0.000354}\right) = P(Z < -271.2) \cong 0$

b) $P(T > 0.2405) = P\left(Z > \frac{0.2405 - 0.33}{0.000354}\right) = P(Z > -253) = 1 - P(Z < 253) \cong 1$

5-60. Let D denote the width of the casing minus the width of the door. Then, D is normally distributed.

a) E(D) = 0.4 V(D) = $(0.4)^2 + (0.2)^2 = 0.2$ $\sigma_T = \sqrt{0.2} = 0.4472$

b) $P(D > 0.8) = P(Z > \frac{0.8 - 0.4}{\sqrt{0.2}}) = P(Z > 0.89) = 0.187$

c) $P(D < 0) = P(Z < \frac{0 - 0.4}{\sqrt{0.2}}) = P(Z < -0.89) = 0.187$

5-61. X = time of ACL reconstruction surgery for high-volume hospitals.
X ~ N(129,196)
$E(X_1 + X_2 + \ldots + X_{10})$ = 8* 129 =1032
$V(X_1 + X_2 + \ldots + X_{10})$ = 64*196 =8256

5-62. a) Let $\bar{X}$ denote the average fill-volume of 100 cans. $\sigma_{\bar{X}} = \sqrt{15^2/100} = 1.5$.

b) E($\bar{X}$) = 358 and $P(\bar{X} < 355) = P\left(Z < \frac{355 - 358}{1.5}\right) = P(Z < -2) = 0.023$

c) P($\bar{X}$ < 355) = 0.005 implies that $P\left(Z < \frac{355 - \mu}{1.5}\right) = 0.005$.

Then $\frac{355 - \mu}{1.5}$ = -2.58 and $\mu = 358.87$.

d) P($\bar{X}$ < 355) = 0.005 implies that $P\left(Z < \frac{355 - 358}{\sigma / \sqrt{100}}\right) = 0.005$.

Then $\frac{355 - 358}{\sigma / \sqrt{100}} = -2.58$ and $\sigma = 11.628$.

e) P($\bar{X}$ < 355) = 0.01 implies that $P\left(Z < \frac{355 - 358}{15 / \sqrt{n}}\right) = 0.01$.

Then $\frac{355 - 358}{15 / \sqrt{n}} = -2.33$ and $n = 135.72 \cong 136$.

5-63. Let $\bar{X}$ denote the average thickness of 10 wafers. Then, E($\bar{X}$) = 10 and V($\bar{X}$) = 0.1.

a) $P(8 < \bar{X} < 12) = P(\frac{8-10}{\sqrt{0.1}} < Z < \frac{12-10}{\sqrt{0.1}}) = P(-6.32 < Z < 6.32) \approx 1$.

The answer is ≈ 0

b) $P(\bar{X} > 11) = 0.05 \; and \; \sigma_{\bar{X}} = 1/\sqrt{n}$.

Therefore, $P(\bar{X} > 11) = P(Z > \frac{11-10}{1/\sqrt{n}}) = 0.05$, $\frac{11-10}{1/\sqrt{n}}$ = 1.65 and n = 2.72 which is rounded up to 3.

c) $P(\bar{X} > 11) = 0.0005 \; and \; \sigma_{\bar{X}} = \sigma/\sqrt{10}$.

Therefore, $P(\bar{X} > 11) = P(Z > \frac{11-10}{\sigma/\sqrt{10}}) = 0.0005$, $\frac{11-10}{\sigma/\sqrt{10}}$ = 3.29

$$\sigma = \sqrt{10}/3.29 = 0.9612$$

5-64. X~N(75, 225)

a) Let Y = $X_1 + X_2 + ... + X_{25}$, E(Y) = 25E(X) = 1875, V(Y) = 25^2(225) = 140625

P(Y > 1950) = $P\left(Z > \frac{1950-1875}{\sqrt{140625}}\right) = P(Z > 0.2) = 1 - P(Z < 0.2) = 1 - 0.5793 = 0.4207$

b) P(Y > x) = 0.0002 implies that $P\left(Z > \frac{x-1875}{\sqrt{140625}}\right)$ = 0.0002.

Then $\frac{x-1875}{375}$ = 3.54 and $x = 3202.5$

5-65. W: weights of parts E: measurement error.

W~ N(μ_w, σ_w^2) , E ~ N(0, σ_e^2) ,W+E ~ N(μ_w, $\sigma_w^2+\sigma_e^2$) .

W_{sp} = weights of the specification P

(a) P(W > $\mu_w + 3\sigma_w$) + P(W < $\mu_w - 3\sigma_w$) = P(Z > 3) + P(Z < -3) = 0.0027

(b) P(W+E > $\mu_w + 3\sigma_w$) + P(W+E < $\mu_w - 3\sigma_w$)

= P (Z > $3\sigma_w / (\sigma_w^2+\sigma_e^2)^{1/2}$) + P (Z < $-3\sigma_w / (\sigma_w^2+\sigma_e^2)^{1/2}$)

Because $\sigma_e^2 = 0.25\sigma_w^2$ the probability is

= P (Z > $3\sigma_w / (1.25\sigma_w^2)^{1/2}$) + P (Z < $-3\sigma_w / (1.25\sigma_w^2)^{1/2}$)

= P (Z > 2.68) + P (Z < -2.68) = 2(0.003681) = 0.0074

No.

(c) P(E + $\mu_w + 2\sigma_w > \mu_w + 3\sigma_w$) = P(E > σ_w) = P(Z > $\sigma_w/(0.25\sigma_w^2)^{1/2}$) = P(Z > 2) = 0.0228

Also, P(E + $\mu_w + 2\sigma_w < \mu_w - 3\sigma_w$) = P(E < – $5\sigma_w$) = P(Z < $-5\sigma_w/(0.25\sigma_w^2)^{1/2}$) = P(Z < -10) ≈ 0

5-66. D = A - B - C

a) E(D) = 10 - 2 - 2 = 6 mm

$$V(D) = 0.1^2 + 0.1^2 + 0.1^2 = 0.03mm^2$$

$$\sigma_D = 0.1732mm$$

b) P(D < 5.9) = P(Z < $\frac{5.9-6}{0.1732}$) = P(Z < -0.58) = 0.281.

<u>Section 5-5</u>

5-67. $f_Y(y) = \frac{1}{4}$ at y = 5, 8, 11, 14

5-68. Because $X \geq 0$, the transformation is one-to-one; that is $y = x^3$ and $x = \sqrt[3]{y}$. From equation 5-30,

$f_Y(y) = f_X(\sqrt[3]{y}) = \binom{3}{\sqrt[3]{y}} p^{\sqrt[3]{y}}(1-p)^{3-\sqrt[3]{y}}$ for y = 0, 1, 8, 27.

If p = 0.25, $f_Y(y) = \binom{3}{\sqrt[3]{y}}(0.25)^{\sqrt[3]{y}}(0.75)^{3-\sqrt[3]{y}}$ for y = 0, 1, 8, 27.

5-69. a) $f_Y(y) = f_X\left(\frac{y-10}{2}\right)\left(\frac{1}{2}\right) = \frac{y-10}{96}$ for $10 \leq y \leq 22$

b) $E(Y) = \int_{10}^{22} \frac{y^2-10y}{96}dy = \frac{1}{96}\left(\frac{y^3}{3} - \frac{10y^2}{2}\right)\Big|_{10}^{22} = 13.5$

5-70. Because y = -2 ln x, $e^{-\frac{y}{2}} = x$. Then, $f_Y(y) = f_X(e^{\frac{-y}{2}})\left|-\frac{1}{2}e^{\frac{-y}{2}}\right| = \frac{1}{2}e^{\frac{-y}{2}}$ for $0 \le e^{\frac{-y}{2}} \le 1$ or $y \ge 0$, which is an exponential distribution with λ = 1/2 (which equals a chi-square distribution with k = 2 degrees of freedom).

5-71. a) If $y = x^2$, then $x = \sqrt{y}$ for $x \ge 0$ and $y \ge 0$. Thus, $f_Y(y) = f_X(\sqrt{y})\frac{1}{2}y^{-\frac{1}{2}} = \dfrac{e^{-\sqrt{y}}}{2\sqrt{y}}$ for y > 0.

b) If $y = x^{1/3}$, then $x = y^3$ for $x \ge 0$ and $y \ge 0$. Thus, $f_Y(y) = f_X(y^3)3y = 3ye^{-y^3}$ for y > 0.

c) If y = ln x, then $x = e^y$ for $x \ge 0$. Thus, $f_Y(y) = f_X(e^y)e^y = e^y e^{-e^y} = e^{y-e^y}$ for $-\infty < y < \infty$.

5-72. a) Now, $\int_0^\infty av^2e^{-bv}dv$ must equal one. Let *u* = bv, then $1 = a\int_0^\infty (\frac{u}{b})^2 e^{-u}\frac{du}{b} = \frac{a}{b^3}\int_0^\infty u^2e^{-u}du.$

From the definition of the gamma function the last expression is $\frac{a}{b^3}\Gamma(3) = \frac{2a}{b^3}$. Therefore, $a = \frac{b^3}{2}$.

b) If $w = \frac{mv^2}{2}$, then $v = \sqrt{\frac{2w}{m}}$ for $v \ge 0,\ \ w \ge 0$.

$$f_W(w) = f_V\left(\sqrt{\tfrac{2w}{m}}\right)\frac{dv}{dw} = \frac{b^3 2w}{2m}e^{-b\sqrt{\frac{2w}{m}}}(2mw)^{-1/2}$$

$$= \frac{b^3m^{-3/2}}{\sqrt{2}}w^{1/2}e^{-b\sqrt{\frac{2w}{m}}}$$

for $w \ge 0$.

5-73. If $y = e^{x+1}$, then x = ln y-1 for $1 \le x \le 2$ and $e^2 \le y \le e^3$. Thus,

$f_Y(y) = f_X(\ln y - 1)\frac{1}{y} = \frac{1}{y}$ for $2 \le \ln y \le 3$. That is, $f_Y(y) = \frac{1}{y}$ for $e^2 \le y \le e^3$.

5-74. If y = $(x-2)^2$, then x = $2-\sqrt{y}$ for $0 \le x \le 2$ and x = $2+\sqrt{y}$ for $2 \le x \le 4$. Thus,

$$f_Y(y) = f_X(2-\sqrt{y})\,|-\tfrac{1}{2}y^{-1/2}| + f_X(2+\sqrt{y})\,|\tfrac{1}{2}y^{-1/2}|$$

$$= \frac{2-\sqrt{y}}{24\sqrt{y}} + \frac{2+\sqrt{y}}{24\sqrt{y}}$$

$$= \left(\tfrac{1}{6}\right)y^{-1/2} \quad \text{for } 0 \le y \le 4$$

Section 5-6

5-75.

a) $M_X(t) = E(e^{tX}) = \sum_{x=1}^{m} \frac{1}{m} e^{tx} = \frac{1}{m} \sum_{x=0}^{m-1} e^{t(x+1)} = \frac{e^t}{m} \sum_{x=0}^{m-1} e^{tx} = \frac{e^t(1-e^{tm})}{m(1-e^t)}$

b) $E(X) = \mu = \mu_1' = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \frac{m+1}{2}$

$$V(X) = \sigma^2 = E(X^2) - [E(X)]^2 = \mu_2' - \mu^2 = \frac{(m+1)(2m+1)}{6} - \left(\frac{m+1}{2}\right)^2 = \frac{m^2-1}{12}$$

5-76.

a) $M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda}(e^{\lambda e^t}) = e^{\lambda(e^t-1)}$

b) $E(X) = \mu = \mu_1' = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \lambda e^t \left(e^{\lambda(e^t-1)}\right)\Big|_{t=0} = \lambda$

$$V(X) = \sigma^2 = E(X^2) - [E(X)]^2 = \mu_2' - \mu^2$$

$$\mu_2' = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = \left[\lambda e^t \left(e^{\lambda(e^t-1)}\right) + \left(\lambda e^t\right)^2 \left(e^{\lambda(e^t-1)}\right)\right]_{t=0} = \lambda + \lambda^2$$

$$\sigma^2 = \mu_2' - \mu^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

5-77. $M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1} = \frac{p}{1-p} \sum_{x=1}^{\infty} (e^t(1-p))^x$

$$= \frac{p}{1-p}\left(\frac{e^t(1-p)}{1-e^t(1-p)}\right) = \frac{pe^t}{1-(1-p)e^t}$$

$$E(X) = \mu = \mu_1' = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left.\frac{pe^t\left(1-(1-p)e^t\right) - pe^t\left(-(1-p)e^t\right)}{\left(1-(1-p)e^t\right)^2}\right|_{t=0}$$

$$= \left.\frac{pe^t}{\left(1-(1-p)e^t\right)^2}\right|_{t=0} = \frac{p}{p^2} = \frac{1}{p}$$

$$V(X) = \sigma^2 = E(X^2) - [E(X)]^2 = \mu_2' - \mu^2$$

$$\mu_2' = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = \left.\frac{pe^t\left(1-(1-p)e^t\right)^2 - pe^t(2)\left(1-(1-p)e^t\right)\left(-(1-p)e^t\right)}{\left(1-(1-p)e^t\right)^4}\right|_{t=0}$$

$$= \left.\frac{pe^t\left(1+(1-p)e^t\right)}{\left(1-(1-p)e^t\right)^3}\right|_{t=0} = \frac{2-p}{p^2}$$

$$\sigma^2 = \mu_2' - \mu^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

5-78. $M_Y(t) = M_{X_1}(t).M_{X_2}(t) = \left((1-2t)^{-k_1/2}\right)\left((1-2t)^{-k_2/2}\right) = (1-2t)^{-(k_1+k_2)/2}$

As a result, Y is a chi-squared random variable with $k_1 + k_2$ degrees of freedom.

5-79

a) $M_X(t) = E(e^{tX}) = \int_{x=0}^{\infty} e^{tx}(4xe^{-2x})dx = 4\int_{x=0}^{\infty} xe^{(t-2)x}dx$

Using integration by parts, we have:

$$M_X(t) = 4\lim_{c\to\infty}\left[\frac{xe^{(t-2)x}}{t-2} - \frac{e^{(t-2)x}}{(t-2)^2}\right]_0^c = 4\lim_{c\to\infty}\left[\left(\frac{x}{t-2} - \frac{1}{(t-2)^2}\right)e^{(t-2)x}\right]_0^c = \frac{4}{(t-2)^2}$$

b) $E(X) = \mu = \mu_1' = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left.\frac{-8}{(t-2)^3}\right|_{t=0} = 1$

$$\mu_2' = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = \left.\frac{24}{(t-2)^4}\right|_{t=0} = 1.5$$

$$V(X) = \sigma^2 = E(X^2) - [E(X)]^2 = \mu_2' - \mu^2 = 1.5 - 1 = 0.5$$

5-80.

a) $M_X(t) = E(e^{tX}) = \int_{\alpha}^{\beta} e^{tx}\frac{1}{\beta-\alpha}dx = \frac{1}{\beta-\alpha}\left(\frac{e^{tx}}{t}\right)\Bigg|_{x=\alpha}^{x=\beta} = \frac{1}{\beta-\alpha}\left(\frac{e^{t\beta}}{t} - \frac{e^{t\alpha}}{t}\right) = \frac{e^{t\beta} - e^{t\alpha}}{t(\beta-\alpha)}$

b) $E(X) = \mu = \mu_1' = \left.\frac{dM_X(t)}{dt}\right|_{t=0}$

$$\frac{dM_X(t)}{dt} = \frac{(\beta e^{t\beta} - \alpha e^{t\alpha})(t(\beta-\alpha)) - (e^{t\beta} - e^{t\alpha})(\beta-\alpha)}{t^2(\beta-\alpha)^2} = \frac{t(\beta e^{t\beta} - \alpha e^{t\alpha}) - e^{t\beta} + e^{t\alpha}}{t^2(\beta-\alpha)}$$

$\frac{dM_X(t)}{dt}$ is undefined at $t = 0$ since there is t^2 in the denominator. Indeed, it has an indeterminate form of $0/0$ when it is evaluated at $t = 0$. As a result, we need to use L'Hopital's rule and differentiate the numerator and denominator.

$$E(X) = \lim_{t\to 0}\frac{dM_X(t)}{dt} = \lim_{t\to 0}\frac{t(\beta e^{t\beta} - \alpha e^{t\alpha}) - e^{t\beta} + e^{t\alpha}}{t^2(\beta-\alpha)}$$

$$= \lim_{t\to 0}\frac{(\beta e^{t\beta} - \alpha e^{t\alpha}) + t(\beta^2 e^{t\beta} - \alpha^2 e^{t\alpha}) - \beta e^{t\beta} + \alpha e^{t\alpha}}{2t(\beta-\alpha)}$$

$$= \lim_{t\to 0}\frac{\beta^2 e^{t\beta} - \alpha^2 e^{t\alpha}}{2(\beta-\alpha)} = \frac{\beta^2 - \alpha^2}{2(\beta-\alpha)} = \frac{\alpha+\beta}{2}$$

$$\mu_2' = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0}$$

$$\frac{d^2M_X(t)}{dt^2}=\frac{t^3\left(\beta^2e^{t\beta}-\alpha^2e^{t\alpha}\right)(\beta-\alpha)-2t(\beta-\alpha)\left(t\left(\beta e^{t\beta}-\alpha e^{t\alpha}\right)-e^{t\beta}+e^{t\alpha}\right)}{t^4(\beta-\alpha)^2}$$
$$=\frac{t^2\left(\beta^2e^{t\beta}-\alpha^2e^{t\alpha}\right)-2t\left(\beta e^{t\beta}-\alpha e^{t\alpha}\right)+2e^{t\beta}-2e^{t\alpha}}{t^3(\beta-\alpha)}$$

$\frac{d^2M_X(t)}{dt^2}$ has the same indefinite form of $0/0$ when it is evaluated at $t=0$. We need to use L'Hopital's rule again.

$$\mu_2'=\lim_{t\to 0}\frac{t^2\left(\beta^2e^{t\beta}-\alpha^2e^{t\alpha}\right)-2t\left(\beta e^{t\beta}-\alpha e^{t\alpha}\right)+2e^{t\beta}-2e^{t\alpha}}{t^3(\beta-\alpha)}$$
$$=\lim_{t\to 0}\frac{2t\left(\beta^2e^{t\beta}-\alpha^2e^{t\alpha}\right)+t^2\left(\beta^3e^{t\beta}-\alpha^3e^{t\alpha}\right)-2\left(\beta e^{t\beta}-\alpha e^{t\alpha}\right)-2t\left(\beta^2e^{t\beta}-\alpha^2e^{t\alpha}\right)+2\beta e^{t\beta}-2\alpha e^{t\alpha}}{3t^2(\beta-\alpha)}$$
$$=\lim_{t\to 0}\frac{t^2\left(\beta^3e^{t\beta}-\alpha^3e^{t\alpha}\right)}{3t^2(\beta-\alpha)}=\lim_{t\to 0}\frac{\beta^3e^{t\beta}-\alpha^3e^{t\alpha}}{3(\beta-\alpha)}=\frac{\beta^3-\alpha^3}{3(\beta-\alpha)}=\frac{\alpha^2+\alpha\beta+\beta^2}{3}$$

$$V(X)=\mu_2'-\mu^2=\frac{\alpha^2+\alpha\beta+\beta^2}{3}-\left(\frac{\alpha+\beta}{2}\right)^2=\frac{(\beta-\alpha)^2}{12}$$

5-81.

a) $M_X(t)=E(e^{tX})=\int_0^\infty e^{tx}\lambda e^{-\lambda x}dx=\lambda\int_0^\infty e^{(t-\lambda)x}dx$

$=\lambda\left(\frac{e^{(t-\lambda)x}}{t-\lambda}\right)\Bigg|_0^\infty$ which is finite only if $t<\lambda$.

$M_X(t)=\lambda\left(\frac{e^{(t-\lambda)x}}{t-\lambda}\right)\Bigg|_0^\infty=\lambda\left(0-\frac{1}{t-\lambda}\right)=\frac{\lambda}{\lambda-t}$ for $t<\lambda$.

b) $E(X)=\mu=\mu_1'=\frac{dM_X(t)}{dt}\Big|_{t=0}=\frac{\lambda}{(\lambda-t)^2}\Big|_{t=0}=\frac{1}{\lambda}$

$\mu_2'=\frac{d^2M_X(t)}{dt^2}\Big|_{t=0}=\frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0}=\frac{2}{\lambda^2}$

$V(X)=\mu_2'-\mu^2=\frac{2}{\lambda^2}-\left(\frac{1}{\lambda}\right)^2=\frac{1}{\lambda^2}$

5-82.

a) $M_X(t)=E(e^{tX})=\int_0^\infty e^{tx}\frac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}e^{-\lambda x}dx=\frac{\lambda^r}{\Gamma(r)}\int_0^\infty x^{r-1}e^{(t-\lambda)x}dx$

$\int_0^\infty x^{r-1}e^{(t-\lambda)x}dx$ is finite only if $t < \lambda$. Besides, we need to use integration by substitution by letting $z = (\lambda - t)x$. Note that the limits of the integration stay the same because $z \to 0$ as $x \to 0$, and $z \to \infty$ as $x \to \infty$. So, we have

$$x = \frac{z}{(\lambda - t)} \quad \text{and} \quad dx = \frac{dz}{(\lambda - t)}$$

$$M_X(t) = \frac{\lambda^r}{\Gamma(r)}\int_0^\infty x^{r-1}e^{(t-\lambda)x}dx = \frac{\lambda^r}{\Gamma(r)}\int_0^\infty \left(\frac{z}{\lambda - t}\right)^{r-1} e^{-z}\frac{dz}{\lambda - t} = \frac{\lambda^r}{\Gamma(r)(\lambda - t)^r}\int_0^\infty z^{r-1}e^{-z}dz$$

$$= \frac{\lambda^r}{\Gamma(r)(\lambda - t)^r}\Gamma(r) = \frac{\lambda^r}{(\lambda - t)^r} = \left(\frac{\lambda - t}{\lambda}\right)^{-r} = \left(1 - \frac{t}{\lambda}\right)^{-r}$$

As a result, $M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-r}$ for $t < \lambda$.

Also note that $\Gamma(r) = \int_0^\infty z^{r-1}e^{-z}dz$ for $r > 0$ by the definition of the gamma function.

b) $$E(X) = \mu = \mu_1' = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left.\left(1 - \frac{t}{\lambda}\right)^{-r}\right|_{t=0} = \left.\lambda^r(\lambda - t)^{-r}\right|_{t=0} = \left.\lambda^r r(\lambda - t)^{-r-1}\right|_{t=0} = \frac{r}{\lambda}$$

$$\mu_2' = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = \left.r(r+1)\lambda^r(\lambda - t)^{-r-2}\right|_{t=0} = \frac{r(r+1)}{\lambda^2}$$

$$V(X) = \mu_2' - \mu^2 = \frac{r(r+1)}{\lambda^2} - \left(\frac{r}{\lambda}\right)^2 = \frac{r}{\lambda^2}$$

5-83.

a) $$M_Y(t) = M_{X_1}(t).M_{X_2}(t)...M_{X_r}(t) = \frac{\lambda}{\lambda - t}\frac{\lambda}{\lambda - t}...\frac{\lambda}{\lambda - t} = \left(\frac{\lambda}{\lambda - t}\right)^r$$

b) $M_Y(t) = \left(\frac{\lambda}{\lambda - t}\right)^r = \left(1 - \frac{t}{\lambda}\right)^{-r}$ is the moment-generating function of a gamma distribution. As a result, the random variable Y has a gamma distribution with parameters r and λ.

5-84.

a) $$M_Y(t) = M_{X_1}(t)M_{X_2}(t) = \exp\left(\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right) \times \exp\left(\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right)$$

$$= \exp\left(\mu_1 t + \frac{\sigma_1^2 t^2}{2} + \mu_2 t + \frac{\sigma_2^2 t^2}{2} \right) = \exp\left((\mu_1 + \mu_2)t + (\sigma_1^2 + \sigma_2^2)\frac{t^2}{2} \right)$$

b) $M_Y(t) = \exp\left((\mu_1 + \mu_2)t + (\sigma_1^2 + \sigma_2^2)\frac{t^2}{2} \right)$ is the moment-generating function of a normal distribution with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. As a result, the random variable Y has a normal distribution with parameters $\mu_1 + \mu_2$ and $\sigma_1^2 + \sigma_2^2$.

Supplemental Exercises

5-85. The sum of $\sum_x \sum_y f(x, y) = 1$, $\left(\frac{1}{4}\right) + \left(\frac{1}{8}\right) + \left(\frac{1}{8}\right) + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right) = 1$

and $f_{XY}(x, y) \ge 0$

a) $P(X < 0.5, Y < 1) = f_{XY}(0,0) = 1/4$.

b) $P(X \le 1) = f_{XY}(0,0) + f_{XY}(0,1) + f_{XY}(1,0) + f_{XY}(1,1) = 3/4$

c) $P(Y < 1.5) = f_{XY}(0,0) + f_{XY}(0,1) + f_{XY}(1,0) + f_{XY}(1,1) = 3/4$

d) $P(X > 0.5, Y < 1.5) = f_{XY}(1,0) + f_{XY}(1,1) = 3/8$

e) E(X) = 0(3/8) + 1(3/8) + 2(1/4) = 7/8
$V(X) = 0^2(3/8) + 1^2(3/8) + 2^2(1/4) - 7/8^2 = 39/64$
E(Y) = 1(3/8) + 0(3/8) + 2(1/4) = 7/8
$V(Y) = 1^2(3/8) + 0^2(3/8) + 2^2(1/4) - 7/8^2 = 39/64$

f) $f_X(x) = \sum_y f_{XY}(x, y)$ *and* $f_X(0) = 3/8$, $f_X(1) = 3/8$, $f_X(2) = 1/4$.

g) $f_{Y|1}(y) = \frac{f_{XY}(1, y)}{f_X(1)}$ *and* $f_{Y|1}(0) = \frac{1/8}{3/8} = 1/3$, $f_{Y|1}(1) = \frac{1/4}{3/8} = 2/3$.

h) $E(Y \mid X = 1) = \sum_{x=1} y f_{Y|X=1}(y) = 0(1/3) + 1(2/3) = 2/3$

i) As is discussed in the chapter, because the range of (X, Y) is not rectangular, X and Y are not independent.

j) E(*XY*) = 1.25, E(*X*) = E(*Y*) = 0.875, V(X) = V(Y) = 0.6094
$COV(X,Y) = E(XY) - E(X)E(Y) = 1.25 - 0.875^2 = 0.4844$

$$\rho_{XY} = \frac{0.4844}{\sqrt{0.6094}\sqrt{0.6094}} = 0.7949$$

5-86. $P(X = 2, Y = 4, Z = 14) = \frac{20!}{2!4!14!} 0.05^2 0.25^4 0.70^{14} = 0.0385$

b) $P(X = 0) = 0.05^0 0.95^{20} = 0.3585$

c) $E(X) = np_1 = 20(0.05) = 1$

$V(X) = np_1(1 - p_1) = 20(0.05)(0.95) = 0.95$

d) $f_{X|Z=z}(X \mid Z = 19) \frac{f_{XZ}(x, z)}{f_Z(z)}$

$$f_{XZ}(xz) = \frac{20!}{x!z!(20-x-z)!} 0.05^x 0.25^{20-x-z} 0.7^z$$

$$f_Z(z) = \frac{20!}{z!(20-z)!} 0.3^{20-z} 0.7^z$$

$$f_{X|Z=z}(X \mid Z=19) \frac{f_{XZ}(x,z)}{f_Z(z)} = \frac{(20-z)!}{x!(20-x-z)!} \frac{0.05^x 0.25^{20-x-z}}{0.3^{20-z}} = \frac{(20-z)!}{x!(20-x-z)!}\left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{20-x-z}$$

Therefore, X is a binomial random variable with n=20-z and p=1/6. When z=19,

$f_{X|19}(0) = \frac{5}{6}$ and $f_{X|19}(1) = \frac{1}{6}$.

e) $E(X \mid Z=19) = 0\left(\frac{5}{6}\right) + 1\left(\frac{1}{6}\right) = \frac{1}{6}$

5-87. Let X, Y, and Z denote the number of bolts rated high, moderate, and low. Then, X, Y, and Z have a multinomial distribution.

a) $P(X=12, Y=6, Z=2) = \frac{20!}{12!6!2!} 0.6^{12} 0.25^6 0.15^2 = 0.0422$

b) Because X, Y, and Z are multinomial, the marginal distribution of Z is binomial with n = 20 and p = 0.15

c) E(Z) = np = 20(0.15) = 3

d) P(low>2)=1-P(low=0)-P(low=1)-P(low=2)=

$$1 - \frac{20!}{20!0!}(0.15)^0(0.85)^{20} - \frac{20!}{19!1!}(0.15)^1(0.85)^{19} - \frac{20!}{18!2!}(0.15)^2(0.85)^{18}$$

$$= 1 - 0.0388 - 0.1368 - 0.2293 = 0.5951$$

e) $f_{Z|16}(z) = \frac{f_{XZ}(16,z)}{f_X(16)}$ and $f_{XZ}(x,z) = \frac{20!}{x!z!(20-x-z)!} 0.6^x 0.25^{(20-x-z)} 0.15^z$ for

$x + z \le 20$ and $0 \le x, 0 \le z$. Then,

$$f_{Z|16}(z) = \frac{\frac{20!}{16!z!(4-z)!} 0.6^{16} 0.25^{(4-z)} 0.15^z}{\frac{20!}{16!4!} 0.6^{16} 0.4^4} = \frac{4!}{z!(4-z)!}\left(\frac{0.25}{0.4}\right)^{4-z}\left(\frac{0.15}{0.4}\right)^z$$

for $0 \le z \le 4$. That is the distribution of Z given X = 16 is binomial with n = 4 and p = 0.375

f) From part (a), E(Z) = 4 (0.375) = 1.5

g) Because the conditional distribution of Z given X = 16 does not equal the marginal distribution of Z, X and Z are not independent.

5-88. Let X, Y, and Z denote the number of calls answered in two rings or less, three or four rings, and five rings or more, respectively.

a) $P(X=8, Y=1, Z=1) = \frac{10!}{8!1!1!} 0.7^8 0.20^1 0.10^1 = 0.1038$

b) Let W denote the number of calls answered in four rings or less. Then, W is a binomial random variable with n = 10 and p = 0.90.

Therefore, P(W = 10) = $\binom{10}{10} 0.90^{10} 0.10^0 = 0.3487$.

c) E(W) = 10(0.90) = 9.

d) $f_{Z|8}(z) = \frac{f_{XZ}(8,z)}{f_X(8)}$ and $f_{XZ}(x,z) = \frac{10!}{x!z!(10-x-z)!} 0.70^x 0.2^{(10-x-z)} 0.1^z$ for

$x + z \le 10$ *and* $0 \le x, 0 \le z$. Then,

$$f_{Z|8}(z)=\frac{\frac{10!}{8!z!(2-z)!}0.70^8 0.2^{(2-z)}0.1^z}{\frac{10!}{8!2!}0.70^8 0.30^2}=\frac{2!}{z!(2-z)!}\left(\frac{0.2}{0.3}\right)^{2-z}\left(\frac{0.1}{0.3}\right)^z$$

for $0 \le z \le 2$. That is Z is binomial with n =2 and p = 0.1/0.3 = 1/3.

e) E(Z) given X = 8 is 2(1/3) = 2/3.

f) Because the conditional distribution of Z given X = 8 does not equal the marginal distribution of Z, X and Z are not independent.

5-89. $\int_0^3\int_0^2 cx^2 y\,dy\,dx=\int_0^3 cx^2 \left.\frac{y^2}{2}\right|_0^2 dx=2c\left.\frac{x^3}{3}\right|_0^3=18c$. Therefore, c = 1/18.

a) $P(X<1,Y<1)=\int_0^1\int_0^1 \frac{1}{18}x^2 y\,dy\,dx=\int_0^1 \frac{1}{18}x^2\left.\frac{y^2}{2}\right|_0^1 dx=\frac{1}{36}\left.\frac{x^3}{3}\right|_0^1=\frac{1}{108}$

b) $P(X<2.5)=\int_0^{2.5}\int_0^{2} \frac{1}{18}x^2 y\,dy\,dx=\int_0^{2.5} \frac{1}{18}x^2\left.\frac{y^2}{2}\right|_0^2 dx=\frac{1}{9}\left.\frac{x^3}{3}\right|_0^{2.5}=0.5787$

c) $P(1<Y<2)=\int_0^{3}\int_1^{2} \frac{1}{18}x^2 y\,dy\,dx=\int_0^{3} \frac{1}{18}x^2\left.\frac{y^2}{2}\right|_1^2 dx=\frac{1}{12}\left.\frac{x^3}{3}\right|_0^{3}=\frac{3}{4}$

d)

$$P(X>2,1<Y<1.5)=\int_2^3\int_1^{1.5} \frac{1}{18}x^2 y\,dy\,dx=\int_2^3 \frac{1}{18}x^2\left.\frac{y^2}{2}\right|_1^{1.5} dx=\frac{5}{144}\left.\frac{x^3}{3}\right|_2^3$$

$$=\frac{95}{432}=0.2199$$

e) $E(X)=\int_0^3\int_0^2 \frac{1}{18}x^3 y\,dy\,dx=\int_0^3 \frac{1}{18}x^3 2\,dx=\frac{1}{9}\left.\frac{x^4}{4}\right|_0^3=\frac{9}{4}$

f) $E(Y)=\int_0^3\int_0^2 \frac{1}{18}x^2 y^2\,dy\,dx=\int_0^3 \frac{1}{18}x^2 \frac{8}{3}dx=\frac{4}{27}\left.\frac{x^3}{3}\right|_0^3=\frac{4}{3}$

g) $f_X(x)=\int_0^2 \frac{1}{18}x^2 y\,dy=\frac{1}{9}x^2 \quad for\ 0<x<3$

h) $f_{Y|X}(y)=\frac{f_{XY}(1,y)}{f_X(1)}=\frac{\frac{1}{18}y}{\frac{1}{9}}=\frac{y}{2}$ for 0 < y < 2.

i) $f_{X|1}(x)=\frac{f_{XY}(x,1)}{f_Y(1)}=\frac{\frac{1}{18}x^2}{f_Y(1)}$ and $f_Y(y)=\int_0^3 \frac{1}{18}x^2 y\,dx=\frac{y}{2} \quad for\ 0<y<2$.

Therefore, $f_{X|1}(x)=\frac{\frac{1}{18}x^2}{1/2}=\frac{1}{9}x^2$ for 0 < x < 3.

5-90. The region $x^2+y^2\le 1$ and 0 < z < 4 is a cylinder of radius 1 (and base area π) and height 4. Therefore, the volume of the cylinder is 4π and $f_{XYZ}(x,y,z)=\frac{1}{4\pi}$ for $x^2+y^2\le 1$ and 0 < z < 4.

a) The region $X^2 + Y^2 \le 0.5$ is a cylinder of radius $\sqrt{0.5}$ and height 4. Therefore, $P(X^2 + Y^2 \le 0.5) = \frac{4(0.5\pi)}{4\pi} = 1/2$.

b) The region $X^2 + Y^2 \le 0.5$ and 0 < z < 1 is a cylinder of radius $\sqrt{0.5}$ and height 1. Therefore, $P(X^2 + Y^2 \le 0.5, Z < 1) = \frac{1(0.5\pi)}{4\pi} = 1/8$

c) $f_{XY|1}(x, y) = \dfrac{f_{XYZ}(x, y, 1)}{f_Z(1)}$ and $f_Z(z) = \iint\limits_{x^2+y^2\le 1} \frac{1}{4\pi} dydx = 1/4$

for 0 < z < 4. Then, $f_{XY|1}(x, y) = \dfrac{1/4\pi}{1/4} = \frac{1}{\pi}$ $\text{for } x^2 + y^2 \le 1$.

d) $f_X(x) = \int_0^4 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{4\pi} dydz = \int_0^4 \frac{1}{2\pi}\sqrt{1-x^2}\, dz = \frac{2}{\pi}\sqrt{1-x^2}$ for -1 < x < 1

e) $f_{Z|0,0}(z) = \dfrac{f_{XYZ}(0,0,z)}{f_{XY}(0,0)}$ and $f_{XY}(x, y) = \int_0^4 \frac{1}{4\pi} dz = 1/\pi$ for $x^2 + y^2 \le 1$. Then,

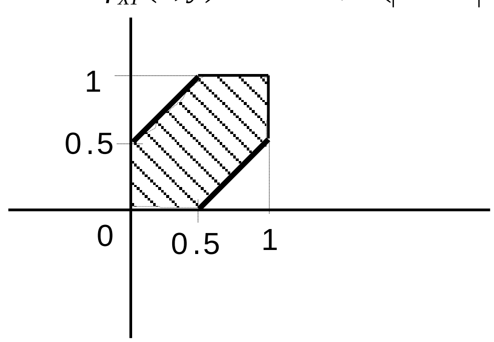
$f_{Z|0,0}(z) = \dfrac{1/4\pi}{1/\pi} = 1/4$ for 0 < z < 4 and $\mu_{Z|0,0} = 2$.

f) $f_{Z|xy}(z) = \dfrac{f_{XYZ}(x, y, z)}{f_{XY}(x, y)} = \dfrac{1/4\pi}{1/\pi} = 1/4$ for 0 < z < 4. Then, E(Z) given X = x and Y = y is

$\int_0^4 \frac{z}{4} dz = 2$.

5-91. $f_{XY}(x, y) = c$ for 0 < x < 1 and 0 < y < 1. Then, $\int_0^1 \int_0^1 c\,dxdy = 1$ and c = 1.

Because $f_{XY}(x, y)$ is constant, $P(|X - Y| < 0.5)$ is the area of the shaded region below



That is, $P(|X - Y| < 0.5)$ = 3/4.

5-92. a) Let $X_1, X_2, \ldots, X_6$ denote the lifetimes of the six components, respectively. Because of independence,

$$P(X_1 > 5000, X_2 > 5000, \ldots, X_6 > 5000) = P(X_1 > 5000)P(X_2 > 5000) \ldots P(X_6 > 5000)$$

If X is exponentially distributed with mean θ, then $\lambda = \frac{1}{\theta}$ and

$$P(X > x) = \int_x^\infty \tfrac{1}{\theta}\; e^{-t/\theta} dt = -e^{-t/\theta}\Big|_x^\infty = e^{-x/\theta}$$. Therefore, the answer is

$$e^{-1.25}e^{-1}e^{-1}e^{-0.5}e^{-0.5}e^{-0.4} = e^{-4.65} = 0.0096.$$

b) The probability that at least one component lifetime exceeds 25,000 hours is the same as 1 minus the probability that none of the component lifetimes exceed 25,000 hours. Thus,

$1 - P(X_a < 25{,}000, X_2 < 25{,}000, \ldots, X_6 < 25{,}000) = 1 - P(X_1 < 25{,}000)\ldots P(X_6 < 25{,}000)$
$= 1 - (1 - e^{-25/8})(1 - e^{-2.5})(1 - e^{-2.5})(1 - e^{-1.25})(1 - e^{-1.25})(1 - e^{-1}) = 1 - 0.2592 = 0.7408$

5-93. Let X, Y, and Z denote the number of problems that result in functional, minor, and no defects, respectively.

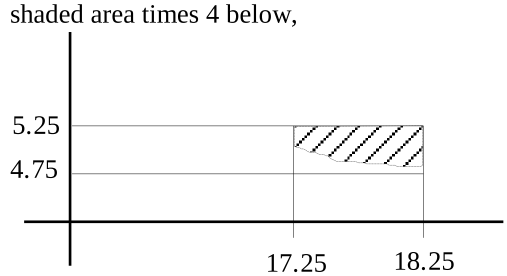
a) $P(X = 2, Y = 5) = P(X = 2, Y = 5, Z = 3) = \frac{10!}{2!5!3!} 0.1^2 0.6^5 0.3^3 = 0.053$

b) Z is binomial with n = 10 and p = 0.3.
c) E(Z) = 10(0.3) = 3.

5-94. a) Let $\bar{X}$ denote the mean weight of the 25 bricks in the sample. Then, E($\bar{X}$) = 1.5 and $\sigma_{\bar{X}} = \frac{0.1}{\sqrt{25}} = 0.02$. Then, P($\bar{X}$ < 1.48) = P(Z < $\frac{1.48-1.5}{0.02}$) = P (Z < –1) = 0.159.

b) $P(\bar{X} > x) = P(Z > \frac{x-1.5}{0.02}) = 0.99$. So, $\frac{x-1.5}{0.02}$ = –2.33 and x = 2.9534.

5-95. a) Because $\int_{17.75}^{18.25}\int_{4.75}^{5.25} c\,dy\,dx = 0.25c$, c = 4. The area of a panel is XY and P(XY > 90) is the shaded area times 4 below,



That is, $\int_{17.75}^{18.25}\int_{90/x}^{5.25} 4\,dy\,dx = 4\int_{17.75}^{18.25} 5.25 - \frac{90}{x}\,dx = 4(5.25x - 90\ln x\Big|_{17.75}^{18.25}) = 0.499$

b) The perimeter of a panel is 2X + 2Y and we want P(2X + 2Y > 46)

$$\int_{17.75}^{18.25}\int_{23-x}^{5.25} 4\,dy\,dx = 4\int_{17.75}^{18.25} 5.25 - (23 - x)dx$$

$$= 4\int_{17.75}^{18.25} (-17.75 + x)dx = 4(-17.75x + \frac{x^2}{2}\Big|_{17.75}^{18.25}) = 0.5$$

5-96. a) Let X denote the weight of a piece of candy and X~N(3, 0.3). Each package has 16 candies, then P is the total weight of the package with 16 pieces and E(P) = 16(3)=48 g and V(P) = $16^2 \times (0.3)^2$ = 23.04 g^2

b) $P(P < 48) = P(Z < \frac{48-48}{4.8}) = P(Z < 0) = 0.5$.

c) Let Y equal the total weight of the package with 17 pieces, E(Y) = 17 × (3) = 51 g and V(Y) = $17^2 \times (0.3)^2$ = 26.01 g^2

$P(Y < 1.6) = P(Z < \frac{48-51}{\sqrt{26.01}}) = P(Z < -0.59) = 0.2776$.

5-97. Let $\bar{X}$ denote the average time to locate 15 parts. Then, E($\bar{X}$) =45 and $\sigma_{\bar{X}} = \frac{30}{\sqrt{15}}$

a) $P(\bar{X} > 60) = P(Z > \frac{60-45}{30/\sqrt{15}}) = P(Z > 1.94) = 0.026$

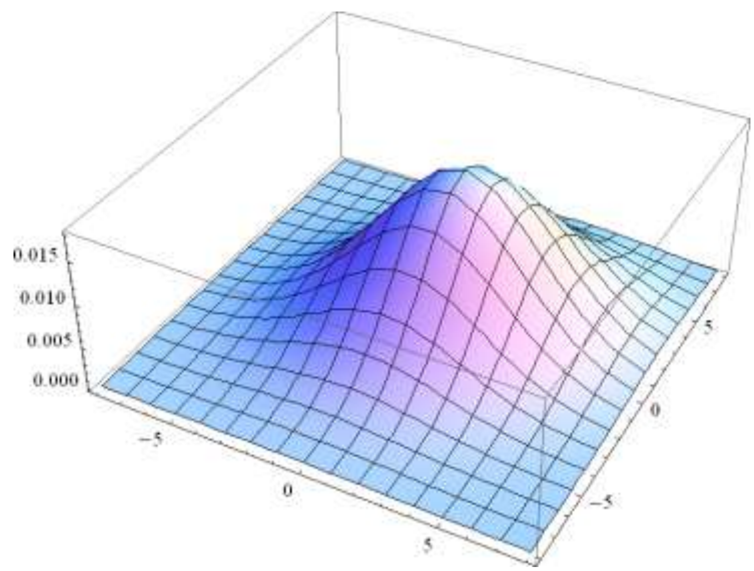
b) Let Y denote the total time to locate 15 parts. Then, Y > 600 if and only if $\bar{X}$ > 60. Therefore, the answer is the same as part a.

5-98. a) Let Y denote the weight of an assembly. Then, E(Y) = 4 + 5.5 + 10 + 8 = 27.5 and V(Y)= $0.4^2 + 0.5^2 + 0.2^2 + 0.5^2 = 0.7$.

$$P(Y > 29.5) = P(Z > \tfrac{29.5-27.5}{\sqrt{0.7}}) = P(Z > 2.39) = 0.0084$$

b) Let $\bar{X}$ denote the mean weight of 8 independent assemblies. Then, E($\bar{X}$) = 27.5 and V($\bar{X}$) = 0.7/8 = 0.0875. Also, $P(\bar{X} > 29) = P(Z > \frac{29-27.5}{\sqrt{0.0875}}) = P(Z > 5.07) = 0$.

5-99.



5-100.

$$f_{XY}(x,y) = \frac{1}{1.2\pi} e^{\left[\frac{-1}{0.72}\{(x-1)^2 - 1.6(x-1)(y-2) + (y-2)^2\}\right]}$$

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{.36}} e^{\left[\frac{-1}{2(0.36)}\{(x-1)^2 - 1.6(x-1)(y-2) + (y-2)^2\}\right]}$$

$$f_{XY}(x,y) = \frac{1}{2\pi\sqrt{1-.8^2}} e^{\left[\frac{-1}{2(1-0.8^2)}\{(x-1)^2 - 2(.8)(x-1)(y-2) + (y-2)^2\}\right]}$$

$E(X)=1,\ E(Y)=2\ V(X)=1\ V(Y)=1$ and ρ = 0.8

5-101. Let T denote the total thickness. Then, $T = X_1 + X_2$ and

a) E(T) = 0.5 + 1 = 1.5 mm
$V(T) = V(X_1) + V(X_2) + 2Cov(X_1X_2) = 0.01 + 0.04 + 2(0.014) = 0.078 mm^2$
where $Cov(XY) = \rho\sigma_X\sigma_Y = 0.7(0.1)(0.2) = 0.014$

b) $P(T<1.2)=P\left(Z<\frac{1.2-1.5}{\sqrt{0.078}}\right)=P(Z<-1.07)=0.1423$

c) Let P denote the total thickness. Then, $P = 2X_1 + 3X_2$ and
E(P) = 2(0.5) + 3(1) = 4 mm
$V(P) = 4V(X_1) + 9V(X_2) + 2(2)(3)Cov(X_1X_2)$
$= 4(0.01) + 9(0.04) + 2(2)(3)(0.014) = 0.568 mm^2$
where $Cov(XY)=\rho\sigma_X\sigma_Y=0.7(0.1)(0.2)=0.014$

5-102. Let T denote the total thickness. Then, $T = X_1 + X_2 + X_3$ and
a) E(T) = 0.5+1+1.5 =3 mm
$V(T)=V(X_1) +V(X_2) +V(X_3)+2Cov(X_1X_2)+ 2Cov(X_2X_3)+$
$2Cov(X_1X_3)=0.01+0.04+0.09+2(0.014)+2(0.03)+ 2(0.009)=0.246mm^2$
where $Cov(XY)=\rho\sigma_X\sigma_Y$

b) $P(T<1.4)=P\left(Z<\frac{1.4-3}{0.246}\right)=P(Z<-6.5)\cong 0$

5-103. Let X and Y denote the percentage returns for security one and two respectively.
If half of the total dollars is invested in each then ½X+ ½Y is the percentage return.
E(½X + ½Y) = 0.05
V(½X+ ½Y) = 1/4 V(X)+1/4V(Y)+2(1/2)(1/2)Cov(X,Y)
where $Cov(XY)=\rho\sigma_X\sigma_Y=-0.5(2)(3) = -3$
V(½X+ ½Y) = 1/4(4)+1/4(9)-1.5 = 1.75
Also, E(X) = 5 and V(X) = 4.
Therefore, the strategy that splits between the securities has a lower standard deviation of percentage return than investing $2 million in the first security.

5-104. a) The range consists of nonnegative integers with $x + y + z = 4$.
b) Because the samples are selected without replacement, the trials are not independent and the joint distribution is not multinomial.

c) $P(X=x\mid Y=2)=\frac{f_{XY}(x,2)}{f_Y(2)}$

$$P(Y=2)=\frac{\binom{7}{0}\binom{5}{2}\binom{8}{2}}{\binom{20}{4}}+\frac{\binom{7}{1}\binom{5}{2}\binom{8}{1}}{\binom{20}{4}}+\frac{\binom{7}{2}\binom{5}{2}\binom{8}{0}}{\binom{20}{4}}=0.05779+0.11558+0.04334=0.21671$$

$$P(X=0 \text{ and } Y=2)=\frac{\binom{7}{0}\binom{5}{2}\binom{8}{2}}{\binom{20}{4}}=0.05779$$

$$P(X=1 \text{ and } Y=2)=\frac{\binom{7}{1}\binom{5}{2}\binom{8}{1}}{\binom{20}{4}}=0.11558$$

$$P(X=2 \text{ and } Y=2)=\frac{\binom{7}{2}\binom{5}{2}\binom{8}{0}}{\binom{20}{4}}=0.04334$$

x	$f_{XY}(x,2)$
0	0.05779/0.21671 = 0.267
1	0.11558/0.21671 = 0.535
2	0.04334/0.21671 = 0.19999

d)
P(X=x, Y=y, Z=z) is the number of subsets of size 4 that contain x printers with graphics enhancements, y printers with extra memory, and z printers with both features divided by the number of subsets of size 4.

$$P(X=x, Y=y, Z=z) = \frac{\binom{7}{x}\binom{5}{y}\binom{8}{z}}{\binom{20}{4}} \quad \text{for } x+y+z=4.$$

$$P(X=1, Y=2, Z=1) = \frac{\binom{7}{1}\binom{5}{2}\binom{8}{1}}{\binom{20}{4}} = 0.11558$$

e) $P(X=1, Y=1) = P(X=1, Y=1, Z=2) = \frac{\binom{7}{1}\binom{5}{1}\binom{8}{2}}{\binom{20}{4}} = 0.2023$

f) The marginal distribution of X is hypergeometric with N = 20, n = 4, K = 4. Therefore, E(X) = nK/N = 4/5 and V(X) = 4(4/20)(16/20)(16/19) = 0.5389.

g) $P(X=1, Y=2 \mid Z=1) = P(X=1, Y=2, Z=1)/P(Z=1)$

$$= \left[\frac{\binom{7}{1}\binom{5}{2}\binom{8}{1}}{\binom{20}{4}}\right] \Big/ \left[\frac{\binom{8}{1}\binom{12}{3}}{\binom{20}{4}}\right] = 0.3181$$

h) $P(X=2 \mid Y=2) = P(X=2, Y=2)/P(Y=2)$

$$= \left[\frac{\binom{7}{2}\binom{5}{2}\binom{8}{0}}{\binom{20}{4}}\right] \Big/ \left[\frac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}}\right] = 0.2$$

i) Because X + Y + Z = 4, if Y = 0 and Z = 3, then X = 1. Because X must equal 1, $f_{X|YZ}(1) = 1$.

5-105. a) Let X, Y, and Z denote the risk of new competitors as no risk, moderate risk, and very high risk. Then, the joint distribution of X, Y, and Z is multinomial with n =12 and $p_1 = 0.15$, $p_2 = 0.70$, and $p_3 = 0.15$. X, Y and Z ≥ 0 and x + y + z=12

b) $P(X=1, Y=3, Z=1) = 0$, not possible since x + y + z ≠ 12

c) $P(Z \le 2) = \binom{12}{0}0.15^0 0.85^{12} + \binom{12}{1}0.15^1 0.85^{11} + \binom{12}{2}0.15^2 0.85^{10}$

$$= 0.1422 + 0.3012 + 0.2924 = 0.7358$$

d) $P(Z=2 \mid Y=1, X=10) = 0$

e) $P(X=10) = P(X=10, Y=2, Z=0) + P(X=10, Y=1, Z=1) + P(X=10, Y=0, Z=2)$

$$= \frac{12!}{10!2!0!}0.15^{10}0.70^2 0.15^0 + \frac{12!}{10!1!1!}0.15^{10}0.70^1 0.15^1 + \frac{12!}{10!0!2!}0.15^{10}0.70^0 0.15^2$$

$$= 1.86\times10^{-7} + 7.99\times10^{-8} + 8.56\times10^{-9} = 2.745\times10^{-7}$$

$$P(Z \le 1 \mid X = 10) = \frac{P(Z=0, Y=2, X=10)}{P(X=10)} + \frac{P(Z=1, Y=1, X=10)}{P(X=10)}$$
$$= \frac{12!}{10!2!0!} 0.15^{10} 0.70^{2} 0.15^{0} \Big/ 2.745 \times 10^{-7} + \frac{12!}{10!1!1!} 0.15^{10} 0.70^{1} 0.15^{1} \Big/ 2.745 \times 10^{-7}$$
$$= 0.9687$$

f) $$P(Y \le 1, Z \le 1 \mid X = 10) = \frac{P(Z=1, Y=1, X=10)}{P(X=10)}$$
$$= \frac{12!}{10!1!1!} 0.15^{10} 0.70^{1} 0.15^{1} \Big/ 6.89 \times 10^{-8}$$
$$= 0.2912$$

g) $$E(Z \mid X = 10) = (0(1.86 \times 10^{-7}) + 1(7.99 \times 10^{-8}) + 2(8.56 \times 10^{-9}) / 2.745 \times 10^{-7})$$
$$= 0.353$$

<u>Mind-Expanding Exercises</u>

5-106. By the independence,

$$P(X_1 \in A_1, X_2 \in A_2, \ldots, X_p \in A_p) = \int_{A_1} \int_{A_2} \ldots \int_{A_p} f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_p}(x_p) dx_1 dx_2 \ldots dx_p$$
$$= \left[\int_{A_1} f_{X_1}(x_1) dx_1\right]\left[\int_{A_2} f_{X_2}(x_2) dx_2\right] \ldots \left[\int_{A_p} f_{X_p}(x_p) dx_p\right]$$
$$= P(X_1 \in A_1) P(X_2 \in A_2) \ldots P(X_p \in A_p)$$

5-107. $E(Y) = c_1\mu_1 + c_2\mu_2 + \ldots + c_p\mu_p$. Also,

$$V(Y) = \int \left[c_1x_1 + c_2x_2 + \ldots + c_px_p - (c_1\mu_1 + c_2\mu_2 + \ldots + c_p\mu_p)\right]^2 f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_p}(x_p) dx_1 dx_2 \ldots dx_p$$
$$= \int \left[c_1(x_1 - \mu_1) + \ldots + c_p(x_p - \mu_p)\right]^2 f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_p}(x_p) dx_1 dx_2 \ldots dx_p$$

Now, the cross-term

$$\int c_1 c_2 (x_1 - \mu_1)(x_2 - \mu_2) f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_p}(x_p) dx_1 dx_2 \ldots dx_p$$
$$= c_1c_2\left[\int (x_1 - \mu_1) f_{X_1}(x_1) dx_1\right]\left[\int (x_2 - \mu_2) f_{X_2}(x_2) dx_2\right] = 0$$

from the definition of the mean. Therefore, each cross-term in the last integral for V(Y) is zero and

$$V(Y) = \left[\int c_1^2 (x_1 - \mu_1)^2 f_{X_1}(x_1) dx_1\right] \ldots \left[\int c_p^2 (x_p - \mu_p)^2 f_{X_p}(x_p) dx_p\right]$$
$$= c_1^2 V(X_1) + \ldots + c_p^2 V(X_p).$$

5-108. $\int_0^a \int_0^b f_{XY}(x,y) dy dx = \int_0^a \int_0^b c dy dx = cab$. Therefore, c = 1/ab. Then, $f_X(x) = \int_0^b c dy = \frac{1}{a}$ for 0 < x < a, and $f_Y(y) = \int_0^a c dx = \frac{1}{b}$ for 0 < y < b. Therefore, $f_{XY}(x,y) = f_X(x) f_Y(y)$ for all x and y and X and Y are independent.

5-109. The marginal density of X is

$$f_X(x) = \int_0^b g(x)h(u)du = g(x)\int_0^b h(u)du = kg(x) \ \ where \ k = \int_0^b h(u)du.$$ Also,

$$f_Y(y) = lh(y) \ \ where \ l = \int_0^a g(v)dv.$$ Because $f_{XY}(x,y)$ is a probability density function,

$$\int_0^a \int_0^b g(x)h(y)dydx = \left[\int_0^a g(v)dv\right]\left[\int_0^b h(u)du\right] = 1.$$ Therefore, $kl = 1$ and

$f_{XY}(x,y)=f_X(x)f_Y(y)$ for all x and y.

5-110. The probability function for X is $P(X = x) = \dfrac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$

The number of ways to select x_j items from N_j is $\binom{Nj}{xj}$.

Therefore, from the multiplication rule the total number of ways to select items to meet the conditions is $\binom{N_1}{x_1}\binom{N_2}{x_2}...\binom{N_k}{x_k}$

The total number of subsets of n items selected from N is $\binom{N}{n}$. Therefore

$$P(X_1 = x_1,...X_k = x_k) = \frac{\binom{N_1}{x_1}\binom{N_2}{x_2}...\binom{N_k}{x_k}}{\binom{N}{n}}$$