CHAPTER 7

Section 7-2

7-1. The proportion of arrivals for chest pain is 8 among 103 total arrivals. The proportion = 8/103.

7-2. The proportion is 10/80 =1/8.

7-3. $$P(2.560 \le \overline{X} \le 2.570) = P\left(\tfrac{2.560-2.565}{0.008/\sqrt{9}} \le \tfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le \tfrac{2.570-2.565}{0.008/\sqrt{9}}\right)$$
$$= P(-1.875 \le Z \le 1.875) = P(Z \le 1.875) - P(Z \le -1.875)$$
$$= 0.9696 - 0.0304 = 0.9392$$

7-4. $X_i \sim N(100, 10^2) \qquad n = 25$

$$\mu_{\overline{X}} = 100 \quad \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{25}} = 2$$
$$P[(100-1.7(2)) \le \overline{X} \le (100+1.5(2))] = P(96.6 \le \overline{X} \le 103) = P\left(\tfrac{96.6-100}{2} \le \tfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le \tfrac{103-100}{2}\right)$$
$$= P(-1.7 \le Z \le 1.5) = P(Z \le 1.5) - P(Z \le -1.7) = 0.9332 - 0.0446 = 0.8886$$

7-5. $$\mu_{\overline{X}} = 520 kN/m^2;\ \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{25}{\sqrt{6}} = 10.206$$
$$P(\overline{X} \ge 525) = P\left(\tfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \ge \tfrac{525-520}{10.206}\right)$$
$$= P(Z \ge 0.4899) = 1 - P(Z \le 0.4899)$$
$$= 1 - 0.6879 = 0.3121$$

7-6.

n = 6	n = 50
$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3.5}{\sqrt{6}} = 1.429$	$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3.5}{\sqrt{49}} = 0.495$
$\sigma_{\overline{X}}$ is reduced by 0.9339 psi	

7-7. Assuming a normal distribution,

$$\mu_{\overline{X}} = 13237; \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{345}{\sqrt{5}} = 154.289$$
$$P(17230 \le \overline{X} \le 17305) = P\left(\tfrac{17230-13237}{154.289} \le \tfrac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le \tfrac{17305-17237}{154.289}\right)$$
$$= P(-0.045 \le Z \le 0.4407) = P(Z \le 0.44) - P(Z \le -0.045)$$
$$= 0.6700 - 0.482 = 0.188$$

7-8. $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{50}{\sqrt{5}} = 22.361 psi$ = standard error of $\overline{X}$

7-9. $\sigma^2 = 36$

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

$$n = \left(\frac{\sigma}{\sigma_{\bar{X}}}\right)^2 = \left(\frac{6}{1.2}\right)^2 = 25$$

7-10. Let $Y = \bar{X} + \frac{1}{2}$

$$\mu_X = \frac{a+b}{2} = \frac{(0+1)}{2} = \tfrac{1}{2}$$

$$\mu_{\bar{X}} = \mu_X$$

$$\sigma_X^2 = \frac{(b-a)^2}{12} = \tfrac{1}{12}$$

$$\sigma_{\bar{X}}^2 = \frac{\sigma^2}{n} = \frac{\frac{1}{12}}{27} = \frac{1}{324}$$

$$\sigma_{\bar{X}} = \tfrac{1}{18}$$

$$\mu_Y = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

$$\sigma_Y^2 = \tfrac{1}{324}$$

$Y = \bar{X} + \dfrac{1}{2} \sim N(1, \tfrac{1}{324})$, approximately, using the central limit theorem.

7-11. $n = 36$

$$\mu_X = \frac{a+b}{2} = \frac{(4+1)}{2} = \frac{5}{2}$$

$$\sigma_X = \sqrt{\frac{(b-a+1)^2-1}{12}} = \sqrt{\frac{(4-1+1)^2-1}{12}} = \sqrt{\frac{15}{12}} = \sqrt{\frac{5}{4}}$$

$$\mu_{\bar{X}} = \frac{5}{2}, \sigma_{\bar{X}} = \frac{\sqrt{5/4}}{\sqrt{36}} = \frac{\sqrt{5/4}}{6}$$

$$z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Using the central limit theorem:

$$P(2.3 < \bar{X} < 2.7) = P\left(\frac{2.3-2.5}{\frac{\sqrt{5/4}}{6}} < Z < \frac{2.7-2.5}{\frac{\sqrt{5/4}}{6}}\right) = P(-1.0733 < Z < 1.0733)$$

$$= P(Z < 1.0733) - P(Z < -1.0733) = 0.7169$$

7-12.

$\mu_X = 8.2$ minutes	$n = 36$

$\sigma_X = 6$ minutes $\quad \sigma_{\bar{X}} = \dfrac{\sigma_X}{\sqrt{n}} = \dfrac{6}{\sqrt{36}} = 1$

$$\mu_{\bar{X}} = \mu_X = 8.2 \text{ mins}$$

Using the central limit theorem, $\bar{X}$ is approximately normally distributed.

a) $P(\bar{X} < 10) = P(Z < \dfrac{10-8.2}{1}) = P(Z < 1.8) = 0.9641$

b) $P(5 < \bar{X} < 10) = P(\frac{5-8.2}{1} < Z < \frac{10-8.2}{1})$

$$= P(Z < 1.8) - P(Z < -3.2) = 0.9634$$

c) $P(\bar{X} < 6) = P(Z < \dfrac{6-8.2}{1}) = P(Z < -2.2) = 0.0139$

7-13.

$n_1 = 16 \quad n_2 = 9$

$\mu_1 = 75 \quad \mu_2 = 70$

$\sigma_1 = 8 \quad \sigma_2 = 12$

$$\bar{X}_1 - \bar{X}_2 \sim N(\mu_{\bar{X}_1} - \mu_{\bar{X}_2}, \sigma^2_{\bar{X}_1} + \sigma^2_{\bar{X}_2}) \sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})$$

$$\sim N(75-70, \frac{8^2}{16} + \frac{12^2}{9}) \sim N(5,20)$$

a) $P(\bar{X}_1 - \bar{X}_2 > 5)$

$P(Z > \frac{5-5}{\sqrt{20}}) = P(Z > 0) = 0.5$

b) $P(2.5 \le \bar{X}_1 - \bar{X}_2 \le 6)$

$P(\frac{2.5-5}{\sqrt{20}} \le Z \le \frac{6-5}{\sqrt{20}}) = P(Z \le 0.2236) - P(Z \le -0.5590)$

$$= 0.5885 - 0.2881 = 0.3004$$

7-14. If $\mu_B = \mu_A$, then $\bar{X}_B - \bar{X}_A$ is approximately normal with mean 0 and variance $\frac{\sigma_B^2}{25} + \frac{\sigma_A^2}{25} = 20.48$.

Then, $P(\bar{X}_B - \bar{X}_A > 3.5) = P(Z > \frac{3.5-0}{\sqrt{20.48}}) = P(Z > 0.773) = 0.2196$

The probability that $\bar{X}_B$ exceeds $\bar{X}_A$ by 3.5 or more is not that unusual when μ_B *and* μ_A are equal. Therefore, there is not strong evidence that μ_B is greater than μ_A.

7-15. Assume approximate normal distributions.

$(\bar{X}_{high} - \bar{X}_{low}) \sim N(60-55, \frac{4^2}{16} + \frac{4^2}{16})$

$\sim N(5,2)$

$P(\bar{X}_{high} - \bar{X}_{low} \ge 2) = P(Z \ge \frac{2-5}{\sqrt{2}}) = 1 - P(Z \le -2.12) = 1 - 0.0170 = 0.983$

Section 7-3

7-16.

a) $\text{SE Mean} = \sigma_{\bar{X}} = \dfrac{s}{\sqrt{n}} = \dfrac{1.815}{\sqrt{20}} = 0.4058$

$\text{Variance} = \sigma^2 = 1.815^2 = 3.294$

b) Estimate of mean of population = sample mean = 50.184

7-17. a) $\frac{S}{\sqrt{N}} = \text{SE Mean} \rightarrow \frac{12.30}{\sqrt{N}} = 2.05 \rightarrow N = 36$

Mean $= \frac{3761.70}{36} = 104.492$, Variance $= S^2 = 12.30^2 = 151.29$

Variance $= \frac{\text{Sum of Squares}}{n-1} \rightarrow 151.29 = \frac{SS}{36-1} \rightarrow SS = 5295.15$

b) Estimate of population mean = sample mean = 104.492

7-18. a) $E(\widehat{\Theta}_1) = \mathrm{E}\left(\frac{X_1+X_2}{2}\right) = \frac{1}{2}[\mathrm{E}(X_1) + \mathrm{E}(X_2)] = \frac{1}{2}[\mu + \mu] = \mu$

Therefore, $\widehat{\Theta}_1$ is an unbiased estimator of μ

$E(\widehat{\Theta}_2) = \mathrm{E}\left(\frac{X_1+3X_2}{4}\right) = \frac{1}{4}[\mathrm{E}(X_1) + 3\mathrm{E}(X_2)] = \frac{1}{4}[\mu + 3\mu] = \mu$

Therefore $\widehat{\Theta}_2$ is an unbiased estimator of μ

b) $V(\widehat{\Theta}_1) = \mathrm{V}\left(\frac{X_1+X_2}{2}\right) = \frac{1}{4}[\mathrm{V}(X_1) + \mathrm{V}(X_2)] = \frac{1}{4}[\sigma^2 + \sigma^2] = \frac{\sigma^2}{2}$

$V(\widehat{\Theta}_2) = \mathrm{V}\left(\dfrac{X_1 + 3X_2}{4}\right) = \dfrac{1}{16}[\mathrm{V}(X_1) + 3^2\mathrm{V}(X_2)] = \dfrac{1}{16}[\sigma^2 + 9\sigma^2] = \dfrac{5\sigma^2}{8}$

7-19. $E(\widehat{\Theta}) = E\left(\sum_{i=1}^{n}(X_i - \bar{X})^2/c\right) = (n-1)\sigma^2/c$

Bias $= E(\widehat{\Theta}) - \theta = \dfrac{(n-1)\sigma^2}{c} - \sigma^2 = \sigma^2(\dfrac{n-1}{c} - 1)$

7-20. $$\mathrm{E}(\bar{X}_1) = E\left(\frac{\sum_{i=1}^{2n} X_i}{2n}\right) = \frac{1}{2n}E\left(\sum_{i=1}^{2n} X_i\right) = \frac{1}{2n}(2n\mu) = \mu$$

$$\mathrm{E}(\bar{X}_2) = E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n}(n\mu) = \mu$$

$\bar{X}_1$ and $\bar{X}_2$ are unbiased estimators of μ.

The variances are $V(\bar{X}_1) = \dfrac{\sigma^2}{2n}$ and $V(\bar{X}_2) = \dfrac{\sigma^2}{n}$; compare the MSE (variance in this case),

$$\frac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)} = \frac{\sigma^2/2n}{\sigma^2/n} = \frac{n}{2n} = \frac{1}{2}$$

Because both estimators are unbiased, one concludes that $\bar{X}_1$ is the "better" estimator with the smaller variance.

7-21. $\mathrm{E}(\hat{\Theta}_1) = \dfrac{1}{7}[E(X_1) + E(X_2) + \cdots + E(X_7)] = \dfrac{1}{7}(7E(X)) = \dfrac{1}{7}(7\mu) = \mu$

$$\text{E}\left(\hat{\Theta}_2\right) = \frac{1}{2}\left[E(2X_1) + E(X_6) + E(X_7)\right] = \frac{1}{2}[2\mu - \mu + \mu] = \mu$$

a) Both $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are unbiased estimates of μ because the expected values of these statistics are equivalent to the true mean, μ.

b)

$$V\left(\hat{\Theta}_1\right) = V\left[\frac{X_1 + X_2 + ... + X_7}{7}\right] = \frac{1}{7^2}\left(V(X_1) + V(X_2) + \cdots + V(X_7)\right)$$

$$= \frac{1}{49}(7\sigma^2) = \frac{1}{7}\sigma^2$$

$$V(\hat{\Theta}_1) = \frac{\sigma^2}{7}$$

$$V\left(\hat{\Theta}_2\right) = V\left[\frac{2X_1 - X_6 + X_4}{2}\right] = \frac{1}{2^2}\left(V(2X_1) + V(X_6) + V(X_4)\right)$$

$$= \frac{1}{4}(4V(X_1) + V(X_6) + V(X_4))$$

$$= \frac{1}{4}\left(4\sigma^2 + \sigma^2 + \sigma^2\right) = \frac{1}{4}(6\sigma^2)$$

$$V(\hat{\Theta}_2) = \frac{3\sigma^2}{2}$$

Because both estimators are unbiased, the variances can be compared to select the better estimator. Because the variance of $\hat{\Theta}_1$ is smaller than that of $\hat{\Theta}_2$, $\hat{\Theta}_1$ is the better estimator.

7-22. Because both $\hat{\Theta}_1$ and $\hat{\Theta}_2$ are unbiased, the variances of the estimators can compared to select the better estimator. Because the variance of $\hat{\theta}_2$ is smaller than that of $\hat{\theta}_1$, $\hat{\theta}_2$ is the better estimator.

Relative Efficiency = $\dfrac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)} = \dfrac{V(\bar{\Theta}_1)}{V(\hat{\Theta}_2)} = \dfrac{12}{5} = 2.4$

7-23. $E(\hat{\Theta}_1) = \theta \qquad E(\hat{\Theta}_2) = \theta/2$

$$Bias = E(\hat{\Theta}_2) - \theta$$

$$= \frac{\theta}{2} - \theta = -\frac{\theta}{2}$$

$V(\hat{\Theta}_1) = 9 \qquad V(\hat{\Theta}_2) = 5$

For unbiasedness, use $\hat{\Theta}_1$ because it is the only unbiased estimator.

As for minimum variance and efficiency we have

Relative Efficiency = $\dfrac{(V(\hat{\Theta}_1) + Bias^2)_1}{(V(\hat{\Theta}_2) + Bias^2)_2}$ where bias for θ_1 is 0.

Thus,

Relative Efficiency = $\dfrac{(9+0)}{\left(5 + \left(\dfrac{-\theta}{2}\right)^2\right)} = \dfrac{36}{\left(20 + \theta^2\right)}$

If the relative efficiency is less than or equal to 1, $\hat{\Theta}_1$ is the better estimator.

Use $\hat{\Theta}_1$, when $\frac{36}{(20+\theta^2)} \le 1$

$$36 \le (20+\theta^2)$$
$$16 \le \theta^2$$
$$\theta \le -4 \text{ or } \theta \ge 4$$

If $-4 < \theta < 4$ then use $\hat{\Theta}_2$.

For unbiasedness, use $\hat{\Theta}_1$. For efficiency, use $\hat{\Theta}_1$ when $\theta \le -4$ or $\theta \ge 4$ and use $\hat{\Theta}_2$ when $-4 < \theta < 4$.

7-24. $E(\hat{\Theta}_1) = \theta$ No bias $V(\hat{\Theta}_1) = 15 = MSE(\hat{\Theta}_1)$

$E(\hat{\Theta}_2) = \theta$ No bias $V(\hat{\Theta}_2) = 8 = MSE(\hat{\Theta}_2)$

$E(\hat{\Theta}_3) \ne \theta$ Bias $MSE(\hat{\Theta}_3) = 7$ [note that this includes (bias^2)]

To compare the three estimators, calculate the relative efficiencies:

$\frac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)} = \frac{15}{8} = 1.875$, because rel. eff. > 1 use $\hat{\Theta}_2$ as the estimator for θ

$\frac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_3)} = \frac{15}{7} = 2.14$, because rel. eff. > 1 use $\hat{\Theta}_3$ as the estimator for θ

$\frac{MSE(\hat{\Theta}_2)}{MSE(\hat{\Theta}_3)} = \frac{8}{7} = 1.14$, because rel. eff. > 1 use $\hat{\Theta}_3$ as the estimator for θ

Conclusion: $\hat{\Theta}_3$ is the most efficient estimator, but it is biased. $\hat{\Theta}_2$ is the best "unbiased" estimator.

7-25. $n_1 = 8$, $n_2 = 14$, $n_3 = 6$

Show that S^2 is unbiased.

$$E(S^2) = E\left(\frac{8S_1^2 + 14S_2^2 + 6S_3^2}{28}\right)$$
$$= \frac{1}{28}\left(E(8S_1^2) + E(14S_2^2) + E(6S_3^2)\right)$$
$$= \frac{1}{28}\left(8\sigma_1^2 + 14\sigma_2^2 + 6\sigma_3^2\right) = \frac{1}{28}\left(28\sigma^2\right) = \sigma^2$$

Therefore, S^2 is an unbiased estimator of σ^2.

7-26. Show that $\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n}$ is a biased estimator of σ^2

a)

$$E\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n}\right)$$

$$=\frac{1}{n}E\left(\sum_{i=1}^{n}(X_i-n\bar{X})^2\right)=\frac{1}{n}\left(\sum_{i=1}^{n}E(X_i^2)-nE(\bar{X}^2)\right)=\frac{1}{n}\left(\sum_{i=1}^{n}(\mu^2+\sigma^2)-n\left(\mu^2+\frac{\sigma^2}{n}\right)\right)$$

$$=\frac{1}{n}(n\mu^2+n\sigma^2-n\mu^2-\sigma^2)=\frac{1}{n}((n-1)\sigma^2)=\sigma^2-\frac{\sigma^2}{n}$$

Therefore $\frac{\sum(X_i-\bar{X})^2}{n}$ is a biased estimator of σ^2

b) Bias $=E\left[\frac{\sum(X_i^2-n\bar{X})^2}{n}\right]-\sigma^2=\sigma^2-\frac{\sigma^2}{n}-\sigma^2=-\frac{\sigma^2}{n}$

c) Bias decreases as n increases.

7-27. a) Show that $\bar{X}^2$ is a biased estimator of μ^2. Using $E(X^2)=V(X)+[E(X)]^2$

$$E(\bar{X}^2)=\frac{1}{n^2}E\left(\sum_{i=1}^{n}X_i\right)^2=\frac{1}{n^2}\left(V\left(\sum_{i=1}^{n}X_i\right)+\left[E\left(\sum_{i=1}^{n}X_i\right)\right]^2\right)$$

$$=\frac{1}{n^2}\left(n\sigma^2+\left(\sum_{i=1}^{n}\mu\right)^2\right)=\frac{1}{n^2}(n\sigma^2+(n\mu)^2)$$

$$=\frac{1}{n^2}(n\sigma^2+n^2\mu^2)E(\bar{X}^2)=\frac{\sigma^2}{n}+\mu^2$$

Therefore, $\bar{X}^2$ is a biased estimator of μ.^{2}

b) Bias = $E(\bar{X}^2)-\mu^2=\frac{\sigma^2}{n}+\mu^2-\mu^2=\frac{\sigma^2}{n}$

c) Bias decreases as *n* increases.

7-28. a) The average of the 26 observations provided can be used as an estimator of the mean pull force because we know it is unbiased. This value is 336.36 N.
b) The median of the sample can be used as an estimate of the point that divides the population into a "weak" and "strong" half. This estimate is 334.55 N.
c) Our estimate of the population variance is the sample variance or 54.16 N^2. Similarly, our estimate of the population standard deviation is the sample standard deviation or 7.36 N.
d) The estimated standard error of the mean pull force is $7.36/26^{½} = 1.44$. This value is the standard deviation, not of the pull force, but of the mean pull force of the sample.
e) No connector in the sample has a pull force measurement under 324 N.

7-29.

Descriptive Statistics

Variable	N	N*	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3	Maximum
Oxide Thickness	24	0	423.33	1.87	9.15	407.00	416.00	424.00	431.00	437.00

a) The mean oxide thickness, as estimated by Minitab from the sample, is 423.33 Angstroms.
b) The standard deviation for the population can be estimated by the sample standard deviation, or 9.15 Angstroms.
c) The standard error of the mean is 1.87 Angstroms.
d) Our estimate for the median is 424 Angstroms.

e) Seven of the measurements exceed 430 Angstroms, so our estimate of the proportion requested is 7/24 = 0.2917

7-30. a) $E(\hat{p}) = E(X/n) = \frac{1}{n}E(X) = \frac{1}{n}np = p$

b) The variance of $\hat{p}$ is $\frac{p(1-p)}{n}$ so its standard error must be $\sqrt{\frac{p(1-p)}{n}}$. To estimate this parameter we substitute our estimate of p into it.

7-31. a) $E(\overline{X}_1 - \overline{X}_2) = E(\overline{X}_1) - E(\overline{X}_2) = \mu_1 - \mu_2$

b) $s.e. = \sqrt{V(\overline{X}_1 - \overline{X}_2)} = \sqrt{V(\overline{X}_1) + V(\overline{X}_2) + 2COV(\overline{X}_1, \overline{X}_2)} = \sqrt{\frac{\sigma_1^{\ 2}}{n_1} + \frac{\sigma_2^{\ 2}}{n_2}}$

This standard error can be estimated by using the estimates for the standard deviations of populations 1 and 2.

c)

$$E(S_p^{\ 2}) = E\left(\frac{(n_1-1)\cdot S_1^{\ 2} + (n_2-1)\cdot S_2^{\ 2}}{n_1+n_2-2}\right) = \frac{1}{n_1+n_2-2}\left[(n_1-1)E(S_1^{\ 2}) + (n_2-1)\cdot E(S_2^{\ 2})\right] =$$

$$= \frac{1}{n_1+n_2-2}\left[(n_1-1)\cdot\sigma_1^{\ 2} + (n_2-1)\cdot\sigma_2^{\ 2}\right] = \frac{n_1+n_2-2}{n_1+n_2-2}\sigma^2 = \sigma^2$$

7-32. a) $E(\hat{\mu}) = E(\alpha\overline{X}_1 + (1-\alpha)\overline{X}_2) = \alpha E(\overline{X}_1) + (1-\alpha)E(\overline{X}_2) = \alpha\mu + (1-\alpha)\mu = \mu$

b)

$$s.e.(\hat{\mu}) = \sqrt{V(\alpha\overline{X}_1 + (1-\alpha)\overline{X}_2)} = \sqrt{\alpha^2 V(\overline{X}_1) + (1-\alpha)^2 V(\overline{X}_2)}$$

$$= \sqrt{\alpha^2\frac{\sigma_1^{\ 2}}{n_1} + (1-\alpha)^2\frac{\sigma_2^{\ 2}}{n_2}} = \sqrt{\alpha^2\frac{\sigma_1^{\ 2}}{n_1} + (1-\alpha)^2 a\frac{\sigma_1^{\ 2}}{n_2}}$$

$$= \sigma_1\sqrt{\frac{\alpha^2 n_2 + (1-\alpha)^2 a n_1}{n_1 n_2}}$$

c) The value of alpha that minimizes the standard error is $\alpha = \frac{an_1}{n_2 + an_1}$

d) With a = 4 and $n_1 = 1/2 n_2$, the value of α to choose is 2/3. The arbitrary value of α = 0.3 is too small and results in a larger standard error. With α = 2/3, the standard error is

$$s.e.(\hat{\mu}) = \sigma_1\sqrt{\frac{(2/3)^2 2n_1 + (1/3)^2 4n_1}{2n_1^{\ 2}}} = \frac{0.816\sigma_1}{\sqrt{n_1}}$$

If α = 0.3 the standard error is

$$s.e.(\hat{\mu}) = \sigma_1\sqrt{\frac{(0.3)^2 2n_1 + (0.7)^2 4n_1}{2n_1^{\ 2}}} = \frac{1.0344\sigma_1}{\sqrt{n_1}}$$

7-33.

a) $E(\frac{X_1}{n_1} - \frac{X_2}{n_2}) = \frac{1}{n_1}E(X_1) - \frac{1}{n_2}E(X_2) = \frac{1}{n_1}n_1 p_1 - \frac{1}{n_2}n_2 p_2 = p_1 - p_2 = E(p_1 - p_2)$

b) $\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$

c) An estimate of the standard error could be obtained substituting $\frac{X_1}{n_1}$ for p_1 and $\frac{X_2}{n_2}$ for p_2 in the equation shown in (b).

d) Our estimate of the difference in proportions is 0.02

e) The estimated standard error is 0.0386

Section 7-4

7-34. $f(x) = p(1-p)^{x-1}$

$$L(p) = \prod_{i=1}^{n} p(1-p)^{x_i - 1} = p^n (1-p)^{\sum_{i=1}^{n} x_i - n}$$

$$\ln L(p) = n \ln p + \left(\sum_{i=1}^{n} x_i - n \right) \ln(1-p)$$

$$\frac{\partial \ln L(p)}{\partial p} = \frac{n}{p} - \frac{\sum_{i=1}^{n} x_i - n}{1-p} \equiv 0$$

$$0 = \frac{(1-p)n - p\left(\sum_{i=1}^{n} x_i - n \right)}{p(1-p)} = \frac{n - np - p\sum_{i=1}^{n} x_i + pn}{p(1-p)}$$

$$0 = n - p\sum_{i=1}^{n} x_i$$

$$\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i}$$

7-35. $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $\quad L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$

$$\ln L(\lambda) = -n\lambda \ln e + \sum_{i=1}^{n} x_i \ln \lambda - \sum_{i=1}^{n} \ln x_i!$$

$$\frac{d \ln L(\lambda)}{d\lambda} = -n + \frac{1}{\lambda}\sum_{i=1}^{n} x_i \equiv 0$$

$$= -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} = 0$$

$$\sum_{i=1}^{n} x_i = n\lambda$$

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n}$$

7-36. $f(x) = (2\theta + 1)x^{\theta}$

$$L(\theta) = \prod_{i=1}^{n} (2\theta+1)x_i^{\theta} = (2\theta+1)x_1^{\theta} \times (2\theta+1)x_2^{\theta} \times \ldots = (2\theta+1)^n \prod_{i=1}^{n} x_i^{\theta}$$

$$\ln L(\theta) = n\ln(2\theta+1) + \theta \ln x_1 + \theta \ln x_2 + \ldots = n\ln(2\theta+1) + \theta \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{2n}{2\theta+1} + \sum_{i=1}^{n} \ln x_i = 0$$

$$\frac{2n}{2\theta+1} = -\sum_{i=1}^{n} \ln x_i$$

$$\hat{\theta} = \frac{n}{-\sum_{i=1}^{n} \ln x_i} - \frac{1}{2}$$

7-37. $f(x) = \lambda e^{-\lambda(x-\theta)}$ for $x \ge \theta$ $\quad L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda(x-\theta)} = \lambda^n e^{-\lambda \sum_{i=1}^{n} (x-\theta)} = \lambda^n e^{-\lambda\left(\sum_{i=1}^{n} x - n\theta\right)}$

$$\ln L(\lambda, \theta) = n\ln\lambda - \lambda \sum_{i=1}^{n} x_i + \lambda n\theta$$

$$\frac{d \ln L(\lambda,\theta)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + n\theta \equiv 0$$

$$\frac{n}{\lambda} = \sum_{i=1}^{n} x_i - n\theta$$

$$\hat{\lambda} = n \Big/ \left(\sum_{i=1}^{n} x_i - n\theta \right)$$

$$\hat{\lambda} = \frac{1}{\bar{x} - \theta}$$

The other parameter θ cannot be estimated by setting the derivative of the log likelihood with respect to θ to zero because the log likelihood is a linear function of θ. The range of the likelihood is important.

The joint density function and therefore the likelihood is zero for θ < $Min(X_1, X_2, \ldots, X_n)$. The term in the log likelihood $-n\lambda\theta$ is maximized for θ as small as possible within the range of nonzero likelihood. Therefore, the log likelihood is maximized for θ estimated with $Min(X_1, X_2, \ldots, X_n)$ so that $\hat{\theta} = x_{\min}$

b) <u>Example</u>: Consider traffic flow and let the time that has elapsed between one car passing a fixed point and the instant that the next car begins to pass that point be considered time headway. This headway can be modeled by the shifted exponential distribution.
<u>Example in Reliability</u>: Consider a process where failures are of interest. Suppose that a unit is put into operation at $x = 0$, but no failures will occur until θ time units of operation. Failures will occur only after the time θ.

7-38.

$$L(\theta) = \prod_{i=1}^{n} \frac{x_i e^{-x_i/\theta}}{\theta} \qquad \ln L(\theta) = \sum \ln(x_i) - \sum \frac{x_i}{\theta} - n\ln\theta$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{1}{\theta^2}\sum x_i - \frac{n}{\theta}$$

Setting the last equation equal to zero and solving for theta yields

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n}$$

7-39. $E(X) = \dfrac{a-0}{2} = \dfrac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$, therefore: $\hat{a} = 2\overline{X}$

The expected value of this estimate is the true parameter, so it is unbiased. This estimate is reasonable in one sense because it is unbiased. However, there are obvious problems. Consider the sample x_1=1, x_2 = 2 and x_3 =10. Now $\bar{x}$ = 4.37 and $\hat{a} = 2\bar{x} = 8.667$. This is an unreasonable estimate of a, because clearly $a \geq 10$.

7-40. a) $\int_{-1}^{1} c(1+\theta x)dx = 1 = \left(cx + c\theta\dfrac{x^2}{2}\right)\Big|_{-1}^{1} = 2c$

so that the constant c should equal 0.5

b) $E(X) = \dfrac{1}{n}\sum_{i=1}^{n} X_i = \dfrac{\theta}{3}$ $\qquad \hat{\theta} = 3 \cdot \dfrac{1}{n}\sum_{i=1}^{n} X_i$

c) $E(\hat{\theta}) = E\left(3\cdot\frac{1}{n}\sum_{i=1}^{n} X_i\right) = E(3\overline{X}) = 3E(\overline{X}) = 3\frac{\theta}{3} = \theta$

d)

$L(\theta) = \prod_{i=1}^{n} \frac{1}{2}(1+\theta X_i) \qquad \ln L(\theta) = n\ln(\frac{1}{2}) + \sum_{i=1}^{n} \ln(1+\theta X_i)$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{X_i}{(1+\theta X_i)}$$

By inspection, the value of θ that maximizes the likelihood is max (X_i)

7-41. a) $E(X^2) = 2\theta = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ so $\hat{\Theta} = \frac{1}{2n}\sum_{i=1}^{n} X_i^2$

b)

$L(\theta) = \prod_{i=1}^{n} \frac{x_i e^{-x_i^2/2\theta}}{\theta} \qquad \ln L(\theta) = \sum \ln(x_i) - \sum \frac{x_i^2}{2\theta} - n\ln\theta$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{1}{2\theta^2}\sum x_i^2 - \frac{n}{\theta}$$

Setting the last equation equal to zero, the maximum likelihood estimate is

$$\hat{\Theta} = \frac{1}{2n}\sum_{i=1}^{n} X_i^2$$

and this is the same result obtained in part (a)

c)

$$\int_0^a f(x)dx = 0.5 = 1 - e^{-a^2/2\theta}$$

$$a = \sqrt{-2\theta\ln(0.5)} = \sqrt{2\theta\ln(2)}$$

We can estimate the median (a) by substituting our estimate for θ into the equation for a.

7-42. a) $\hat{a}$ cannot be unbiased since it will always be less than a.

b) bias = $\frac{na}{n+1} - \frac{a(n+1)}{n+1} = -\frac{a}{n+1} \xrightarrow[n\to\infty]{} 0$.

c) $2\overline{X}$

d) P(Y≤y)=P(X_1, …,X_n ≤ y)=[P(X_1≤y)]^n=$\left(\frac{y}{a}\right)^n$. Thus, f(y) is as given. Thus,

bias = E(Y) – a = $\frac{an}{n+1} - a = -\frac{a}{n+1}$.

e) For any $n > 1$, $n(n+2) > 3n$ so the variance of $\hat{a}_2$ is less than that of $\hat{a}_1$. It is in this sense that the second estimator is better than the first.

7-43. a)

$$L(\beta,\delta)=\prod_{i=1}^{n}\frac{\beta}{\delta}\left(\frac{x_i}{\delta}\right)^{\beta-1}e^{-\left(\frac{x_i}{\delta}\right)^{\beta}}=e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{\beta}}\prod_{i=1}^{n}\frac{\beta}{\delta}\left(\frac{x_i}{\delta}\right)^{\beta-1}$$

$$\ln L(\beta,\delta)=\sum_{i=1}^{n}\ln\left[\tfrac{\beta}{\delta}\left(\tfrac{x_i}{\delta}\right)^{\beta-1}\right]-\sum\left(\frac{x_i}{\delta}\right)^{\beta}=n\ln\left(\tfrac{\beta}{\delta}\right)+(\beta-1)\sum\ln\left(\tfrac{x_i}{\delta}\right)-\sum\left(\tfrac{x_i}{\delta}\right)^{\beta}$$

b)

$$\frac{\partial\ln L(\beta,\delta)}{\partial\beta}=\frac{n}{\beta}+\sum\ln\left(\tfrac{x_i}{\delta}\right)-\sum\ln\left(\tfrac{x_i}{\delta}\right)\left(\tfrac{x_i}{\delta}\right)^{\beta}$$

$$\frac{\partial\ln L(\beta,\delta)}{\partial\delta}=-\frac{n}{\delta}-(\beta-1)\frac{n}{\delta}+\beta\frac{\sum x_i^{\beta}}{\delta^{\beta+1}}$$

Upon setting $\frac{\partial\ln L(\beta,\delta)}{\partial\delta}$ equal to zero, we obtain

$$\delta^{\beta}n=\sum x_i^{\beta}\quad and\quad \delta=\left[\frac{\sum x_i^{\beta}}{n}\right]^{1/\beta}$$

Upon setting $\frac{\partial\ln L(\beta,\delta)}{\partial\beta}$ equal to zero and substituting for δ, we obtain

$$\frac{n}{\beta}+\sum\ln x_i-n\ln\delta=\frac{1}{\delta^{\beta}}\sum x_i^{\beta}(\ln x_i-\ln\delta)$$

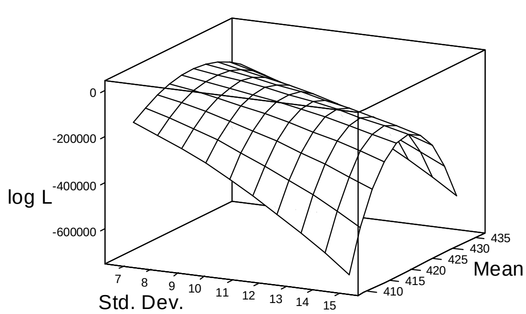
$$\frac{n}{\beta}+\sum\ln x_i-\frac{n}{\beta}\ln\left(\frac{\sum x_i^{\beta}}{n}\right)=\frac{n}{\sum x_i^{\beta}}\sum x_i^{\beta}\ln x_i-\frac{n}{\sum x_i^{\beta}}\sum x_i^{\beta}\frac{1}{\beta}\ln\left(\frac{\sum x_i^{\beta}}{n}\right)$$

and $\frac{1}{\beta}=\left[\frac{\sum x_i^{\beta}\ln x_i}{\sum x_i^{\beta}}+\frac{\sum\ln x_i}{n}\right]$

c) Numerical iteration is required.

7-44. a) Using the results from the example, we obtain that the estimate of the mean is 423.33 and the estimate of the variance is 83.7225

b)



The function has an approximate ridge and its curvature is not too pronounced. The maximum value for standard deviation is at 9.15, although it is difficult to see on the graph.

c) When n is increased to 40, the graph looks the same although the curvature is more pronounced. As n increases, it is easier to determine the maximum value for the standard deviation is on the graph.

7-45. From the example, the posterior distribution for μ is normal with mean $\dfrac{(\sigma^2/n)\mu_0 + \sigma_0^2\bar{x}}{\sigma_0^2 + \sigma^2/n}$ and variance $\dfrac{\sigma_0^2/(\sigma^2/n)}{\sigma_0^2 + \sigma^2/n}$. The Bayes estimator for μ goes to the MLE as n increases. This follows because σ^2/n goes to 0, and the estimator approaches $\dfrac{\sigma_0^2\bar{x}}{\sigma_0^2}$ (the σ_0^2's cancel). Thus, in the limit $\hat{\mu} = \bar{x}$.

7-46. a) Because $f(x\,|\,\mu) = \dfrac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ and $f(\mu) = \dfrac{1}{b-a}$ for $a \le \mu \le b$, the joint distribution is

$$f(x,\mu) = \frac{1}{(b-a)\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{ for } -\infty < x < \infty \text{ and } a \le \mu \le b.$$

Then, $f(x) = \dfrac{1}{b-a}\displaystyle\int_a^b \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}d\mu$

and this integral is recognized as a normal probability. Therefore,

$$f(x) = \frac{1}{b-a}\left[\Phi\left(\tfrac{b-x}{\sigma}\right) - \Phi\left(\tfrac{a-x}{\sigma}\right)\right]$$

where Φ(x) is the standard normal cumulative distribution function. Then

$$f(\mu\,|\,x) = \frac{f(x,\mu)}{f(x)} = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma\left[\Phi\left(\frac{b-x}{\sigma}\right) - \Phi\left(\frac{a-x}{\sigma}\right)\right]}$$

b) The Bayes estimator is

$$\tilde{\mu} = \int_a^b \frac{\mu e^{-\frac{(x-\mu)^2}{2\sigma^2}}d\mu}{\sqrt{2\pi}\sigma\left[\Phi\left(\frac{b-x}{\sigma}\right) - \Phi\left(\frac{a-x}{\sigma}\right)\right]}.$$

Let v = (x - μ). Then, dv = - dμ and

$$\tilde{\mu} = \int_{x-b}^{x-a} \frac{(x-v)e^{-\frac{v^2}{2\sigma^2}}dv}{\sqrt{2\pi}\sigma\left[\Phi\left(\frac{b-x}{\sigma}\right) - \Phi\left(\frac{a-x}{\sigma}\right)\right]} = \frac{x\left[\Phi\left(\frac{x-a}{\sigma}\right) - \Phi\left(\frac{x-b}{\sigma}\right)\right]}{\left[\Phi\left(\frac{b-x}{\sigma}\right) - \Phi\left(\frac{a-x}{\sigma}\right)\right]} - \int_{x-b}^{x-a} \frac{ve^{-\frac{v^2}{2\sigma^2}}dv}{\sqrt{2\pi}\sigma\left[\Phi\left(\frac{b-x}{\sigma}\right) - \Phi\left(\frac{a-x}{\sigma}\right)\right]}$$

Let w = $\dfrac{v^2}{2\sigma^2}$. Then, $dw = [\frac{2v}{2\sigma^2}]dv = [\frac{v}{\sigma^2}]dv$ and

$$\tilde{\mu} = x - \int_{\frac{(x-b)^2}{2\sigma^2}}^{\frac{(x-a)^2}{2\sigma^2}} \frac{\sigma e^{-w}dw}{\sqrt{2\pi}\left[\Phi\left(\frac{b-x}{\sigma}\right) - \Phi\left(\frac{a-x}{\sigma}\right)\right]} = x + \frac{\sigma}{\sqrt{2\pi}}\left[\frac{e^{-\frac{(x-a)^2}{2\sigma^2}} - e^{-\frac{(x-b)^2}{2\sigma^2}}}{\Phi\left(\frac{b-x}{\sigma}\right) - \Phi\left(\frac{a-x}{\sigma}\right)}\right]$$

7-47. a) $f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$ for x = 0, 1, 2, and $f(\lambda) = \left(\dfrac{n+1}{\lambda_0}\right)^{n+1}\dfrac{\lambda^n e^{-(n+1)\frac{\lambda}{\lambda_0}}}{\Gamma(n+1)}$ for λ > 0.

Then,

$$f(x,\lambda) = \frac{(n+1)^{n+1}\lambda^{n+x}e^{-\lambda-(n+1)\frac{\lambda}{\lambda_0}}}{\lambda_0^{n+1}\Gamma(n+1)x!}.$$

This last density is recognized to be a gamma density as a function of λ. Therefore, the posterior distribution of λ is a gamma distribution with parameters n + x + 1 and 1 + $\frac{n+1}{\lambda_0}$.

b) The mean of the posterior distribution can be obtained from the results for the gamma distribution to be

$$\frac{n+x+1}{\left[1+\frac{n+1}{\lambda_0}\right]} = \lambda_0\left(\frac{n+x+1}{n+\lambda_0+1}\right)$$

7-48. a) From the example, the Bayes estimate is $\tilde{\mu} = \frac{\frac{16}{25}(4)+1(4.85)}{\frac{16}{25}+1} = 4.518$

b.) $\hat{\mu} = \bar{x} = 4.85$ The Bayes estimate appears to underestimate the mean.

7-49. a) From the example, $\tilde{\mu} = \frac{(0.0045)(2.28)+(0.018)(2.29)}{0.0045+0.018} = 2.288$

b) $\hat{\mu} = \bar{x} = 2.29$ The Bayes estimate is very close to the MLE of the mean.

7-50. a) $f(x\,|\,\lambda) = \lambda e^{-\lambda x},\ x \ge 0$ and $f(\lambda) = 0.008e^{-0.008\lambda}$. Then,

$$f(x_1,x_2,\lambda) = \lambda^2 e^{-\lambda(x_1+x_2)}0.008e^{-0.008\lambda} = 0.008\lambda^2 e^{-\lambda(x_1+x_2+0.008)}.$$

As a function of λ, this is recognized as a gamma density with parameters 3 and $x_1 + x_2 + 0.008$.

Therefore, the posterior mean for λ is

$$\tilde{\lambda} = \frac{3}{x_1+x_2+0.008} = \frac{3}{2\bar{x}+0.008} = 0.00133.$$

b) Using the Bayes estimate for λ, P(X<1000)= $\int_0^{1000} 0.00133e^{-.00133x}dx$ = 0.736

Supplemental Exercises

7-51. $f(x_1,x_2,...,x_n) = \prod_{i=1}^{n}\lambda e^{-\lambda x_i} \quad for \quad x_1 > 0, x_2 > 0, ..., x_n > 0$

7-52. $f(x_1,x_2,x_3,x_4,x_5) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^5 \exp\left(-\sum_{i=1}^{5}\frac{(x_i-\mu)^2}{2\sigma^2}\right)$

7-53. $f(x_1,x_2,x_3,x_4,x_5) = 1 \quad for \quad 0 \le x_1 \le 1, 0 \le x_2 \le 1, 0 \le x_3 \le 1, 0 \le x_4 \le 1, 0 \le x_5 \le 1$

7-54. $\overline{X}_1 - \overline{X}_2 \sim N(100-105, \frac{2.0^2}{25} + \frac{2.5^2}{30})$

$\sim N(-5, 0.3683)$

7-55. $X \sim N(50, 289)$

$P(47 \le \overline{X} \le 53) = P\left(\frac{47-50}{17/\sqrt{25}} \le Z \le \frac{53-50}{17/\sqrt{25}}\right) = P(-0.9 \le Z \le 0.9)$

$= P(Z \le 0.9) - P(Z \le -0.9) = 0.8159 - 0.1841 = 0.6319$

Yes, because Central Limit Theorem states that with large samples ($n \ge 30$).

7-56. Assume $\overline{X}$ is approximately normally distributed.

$P(\overline{X} > 34) = 1 - P(\overline{X} \le 34) = 1 - P(Z \le \frac{34-38}{0.7/\sqrt{9}})$

$= 1 - P(Z \le -17.14) = 1 - 0 = 1$

7-57. $z = \frac{\overline{X} - \mu}{s/\sqrt{n}} = \frac{54-50}{\sqrt{3/17}} = 9.5219$

$P(Z > z) \approx 0$. The results are *very unusual*.

7-58. $P(\overline{X} \le 37) = P(Z \le -6) \approx 0$

7-59. Binomial with p equal to the proportion of defective chips and $n = 200$.

7-60. $E(a\overline{X}_1 + (1-a)\overline{X}_2 = a\mu + (1-a)\mu = \mu$

$V(\overline{X}) = V[a\overline{X}_1 + (1-a)\overline{X}_2]$

$= a^2 V(\overline{X}_1) + (1-a)^2 V(\overline{X}_2) = a^2(\frac{\sigma^2}{n_1}) + (1 - 2a + a^2)(\frac{\sigma^2}{n_2})$

$= \frac{a^2\sigma^2}{n_1} + \frac{\sigma^2}{n_2} - \frac{2a\sigma^2}{n_2} + \frac{a^2\sigma^2}{n_2} = (n_2 a^2 + n_1 - 2n_1 a + n_1 a^2)(\frac{\sigma^2}{n_1 n_2})$

$\frac{\partial V(\overline{X})}{\partial a} = (\frac{\sigma^2}{n_1 n_2})(2n_2 a - 2n_1 + 2n_1 a) \equiv 0$

$0 = 2n_2 a - 2n_1 + 2n_1 a$

$2a(n_2 + n_1) = 2n_1$

$a(n_2 + n_1) = n_1$

$a = \frac{n_1}{n_2 + n_1}$

7-61.

$$L(\theta) = \left(\frac{1}{2\theta^3}\right)^n e^{\sum_{i=1}^{n} \frac{-x_i}{\theta}} \prod_{i=1}^{n} x_i^2$$

$$\ln L(\theta) = n \ln\left(\frac{1}{2\theta^3}\right) + 2\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \frac{x_i}{\theta}$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{-3n}{\theta} + \sum_{i=1}^{n} \frac{x_i}{\theta^2}$$

Making the last equation equal to zero and solving for θ, we obtain

$\hat{\Theta} = \dfrac{\sum_{i=1}^{n} x_i}{3n}$ as the maximum likelihood estimate.

7-62.

$$L(\theta) = \theta^n \prod_{i=1}^{n} x_i^{\theta - 1}$$

$$\ln L(\theta) = n \ln \theta + (\theta - 1) \sum_{i=1}^{n} \ln(x_i)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i)$$

making the last equation equal to zero and solving for theta, we obtain the maximum likelihood estimate

$$\hat{\Theta} = \frac{-n}{\sum_{i=1}^{n} \ln(x_i)}$$

7-63.

$$L(\theta) = \frac{1}{\theta^n} \prod_{i=1}^{n} x_i^{\frac{1-\theta}{\theta}}$$

$$\ln L(\theta) = -n \ln \theta + \frac{1-\theta}{\theta} \sum_{i=1}^{n} \ln(x_i)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -\frac{n}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^{n} \ln(x_i)$$

Upon setting the last equation equal to zero and solving for the parameter of interest, we obtain the maximum likelihood estimate

$$\hat{\Theta} = -\frac{1}{n} \sum_{i=1}^{n} \ln(x_i)$$

$$E(\hat{\theta}) = E\left[-\frac{1}{n}\sum_{i=1}^{n} \ln(x_i)\right] = \frac{1}{n} E\left[-\sum_{i=1}^{n} \ln(x_i)\right] = -\frac{1}{n}\sum_{i=1}^{n} E[\ln(x_i)]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \theta = \frac{n\theta}{n} = \theta$$

$E(\ln(X_i)) = \int_0^1 (\ln x) x^{\frac{1-\theta}{\theta}} dx$ let $u = \ln x$ and $dv = x^{\frac{1-\theta}{\theta}} dx$

then, $E(\ln(X)) = -\theta \int_0^1 x^{\frac{1-\theta}{\theta}} dx = -\theta$

7-64. a) Let $E\bar{X}^2 = \theta$. Then $V(\bar{X}) = E(\bar{X}^2) - (E\bar{X})^2$. Therefore $\sigma^2/n = \theta - \mu^2$ and $\theta = \sigma^2/n + \mu^2$

Therefore, $\bar{X}^2$ is a biased estimator of the area of the square.

b) $E(\bar{X}^2 - S^2/n) = \sigma^2/n + \mu^2 - E(S^2)/n = \mu^2$

7-65. $\hat{\mu} = \bar{x} = \dfrac{23.5 = 15.6 + 17.4 + \cdots + 28.7}{10} = 21.9$

Demand for all 5000 houses is θ = 5000μ

$\hat{\theta} = 5000\,\hat{\mu} = 5000(21.9) = 109{,}500$

The proportion estimate is $\hat{p} = \frac{7}{10} = 0.7$

Mind-Expanding Exercises

7-66. $P(X_1 = 0, X_2 = 0) = \dfrac{M(M-1)}{N(N-1)}$

$P(X_1 = 0, X_2 = 1) = \dfrac{M(N-M)}{N(N-1)}$

$P(X_1 = 1, X_2 = 0) = \dfrac{(N-M)M}{N(N-1)}$

$P(X_1 = 1, X_2 = 1) = \dfrac{(N-M)(N-M-1)}{N(N-1)}$

$P(X_1 = 0) = M / N$

$P(X_1 = 1) = \dfrac{N-M}{N}$

$$P(X_2 = 0) = P(X_2 = 0 \mid X_1 = 0)P(X_1 = 0) + P(X_2 = 0 \mid X_1 = 1)P(X_1 = 1)$$
$$= \frac{M-1}{N-1} \times \frac{M}{N} + \frac{M}{N-1} \times \frac{N-M}{N} = \frac{M}{N}$$
$$P(X_2 = 1) = P(X_2 = 1 \mid X_1 = 0)P(X_1 = 0) + P(X_2 = 1 \mid X_1 = 1)P(X_1 = 1)$$
$$= \frac{N-M}{N-1} \times \frac{M}{N} + \frac{N-M-1}{N-1} \times \frac{N-M}{N} = \frac{N-M}{N}$$

Because $P(X_2 = 0 \mid X_1 = 0) = \frac{M-1}{N-1}$ is not equal to $P(X_2 = 0) = \dfrac{M}{N}$, X_1 *and* X_2 are not independent.

7-67. a)

$$c_n = \frac{\Gamma[(n-1)/2]}{\Gamma(n/2)\sqrt{2/(n-1)}}$$

b) When n = 15, c_n = 1.0180. When n = 20, c_n = 1.0132. Therefore S is a reasonably good estimator for the standard deviation even when relatively small sample sizes are used.

7-68. a) The likelihood is

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(x_i-\mu_i)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(y_i-\mu_i)^2}{2\sigma^2}}$$

The log likelihood function is

$$-2\ln(L) = \sum_{i=1}^{n} [\frac{(x_i - \mu_i)^2}{\sigma^2} + \frac{(y_i - \mu_i)^2}{\sigma^2} + 4\ln\left(\sqrt{2\pi\sigma^2}\right)]$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}[(x_i - \mu_i)^2 + (y_i - \mu_i)^2] + 4n\ln\left(\sqrt{2\pi\sigma^2}\right)$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}[x_i^2 + y_i^2 - 2\mu_i(x_i + y_i) + 2{\mu_i}^2] + 4n\ln\left(\sqrt{2\pi}\right) + 2n\ln(\sigma^2)$$

Take the derivative of each μ_i and set it to zero

$$\frac{\partial\,(-2\ln(L))}{\partial\mu_i} = \frac{-2(x_i + y_i) + 4\mu_i}{\sigma^2} = 0$$

to obtain

$$\hat{\mu}_i = \frac{x_i + y_i}{2}$$

To find the maximum likelihood estimator of σ^2, substitute the estimate for μ_i and take the derivative with respect to σ^2

$$\frac{\partial\,(-2\ln(L))}{\partial\sigma^2} = -\frac{1}{\sigma^4}\sum_{i=1}^{n}[(x_i - \hat{\mu}_i)^2 + (y_i - \hat{\mu}_i)^2] + \frac{2n}{\sigma^2}$$

$$\frac{\partial\,(-2\ln(L))}{\partial\sigma^2} = -\frac{1}{\sigma^4}\sum_{i=1}^{n}\frac{2(x_i - y_i)^2}{4} + \frac{2n}{\sigma^2}$$

$$= -\frac{\sum_{i=1}^{n}(x_i - y_i)^2}{2\sigma^4} + \frac{2n}{\sigma^2}$$

Set the derivative to zero and solve

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i - y_i)^2}{4n}$$

b)

$$E(\hat{\sigma}^2) = \frac{1}{4n}\sum_{i=1}^{n}E(y_i - x_i)^2 = \frac{1}{4n}\sum_{i=1}^{n}E(y_i^2 + x_i^2 - 2x_iy_i)$$

$$= \frac{1}{4n}\sum_{i=1}^{n}[E(y_i^2) + E(x_i^2) - E(2x_iy_i)] = \frac{1}{4n}\sum_{i=1}^{n}[\sigma^2 + \sigma^2 + 0] = \frac{\sigma^2}{2}$$

Therefore, the estimator is biased. The bias is independent of *n*.

c) An unbiased estimator of σ^2 is given by $2\hat{\sigma}^2$

7-69. $P\left(|\overline{X} - \mu| \geq \frac{c\sigma}{\sqrt{n}}\right) \leq \frac{1}{c^2}$ from Chebyshev's inequality. Then, $P\left(|\overline{X} - \mu| < \frac{c\sigma}{\sqrt{n}}\right) \geq 1 - \frac{1}{c^2}$. Given an ε, n and c can be chosen sufficiently large that the last probability is near 1 and $\frac{c\sigma}{\sqrt{n}}$ is equal to ε.

7-70. a) $P\left(X_{(n)} \le t\right) = P\left(X_i \le t \;\; for \;\; i = 1,...,n\right) = [F(t)]^n$

$P\left(X_{(1)} > t\right) = P\left(X_i > t \;\; for \;\; i = 1,...,n\right) = [1 - F(t)]^n$

Then, $P\left(X_{(1)} \le t\right) = 1 - [1 - F(t)]^n$

b)

$$f_{X_{(1)}}(t) = \frac{\partial}{\partial t} F_{X_{(1)}}(t) = n[1 - F(t)]^{n-1} f(t)$$

$$f_{X_{(n)}}(t) = \frac{\partial}{\partial t} F_{X_{(n)}}(t) = n[F(t)]^{n-1} f(t)$$

c) $P\left(X_{(1)} = 0\right) = F_{X_{(1)}}(0) = 1 - [1 - F(0)]^n = 1 - p^n$ because F(0) = 1 - p.

$P\left(X_{(n)} = 1\right) = 1 - F_{X_{(n)}}(0) = 1 - [F(0)]^n = 1 - (1 - p)^n$

d) $P(X \le t) = F(t) = \Phi\left[\frac{t-\mu}{\sigma}\right]$. From a previous exercise,

$$f_{X_{(1)}}(t) = n\left\{1 - \Phi\left[\tfrac{t-\mu}{\sigma}\right]\right\}^{n-1} \frac{1}{\sqrt{2\pi}\sigma} \; e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

$$f_{X_{(n)}}(t) = n\left\{\Phi\left[\tfrac{t-\mu}{\sigma}\right]\right\}^{n-1} \frac{1}{\sqrt{2\pi}\sigma} \; e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

e) $P(X \le t) = 1 - e^{-\lambda t}$

From a previous exercise,

$F_{X_{(1)}}(t) = 1 - e^{-n\lambda t}$	$f_{X_{(1)}}(t) = n\lambda e^{-n\lambda t}$
$F_{X_{(n)}}(t) = [1 - e^{-\lambda t}]^n$	$f_{X_{(n)}}(t) = n[1 - e^{-\lambda t}]^{n-1} \lambda e^{-\lambda t}$

7-71. $P\left(F\left(X_{(n)}\right) \le t\right) = P\left(X_{(n)} \le F^{-1}(t)\right) = t^n$ for $0 \le t \le 1$ from a previous exercise.

If $Y = F(X_{(n)})$, then $f_Y(y) = ny^{n-1}, 0 \le y \le 1$.

Then, $E(Y) = \int_0^1 ny^n dy = \dfrac{n}{n+1}$

$P\left(F\left(X_{(1)}\right) \le t\right) = P\left(X_{(1)} \le F^{-1}(t)\right) = 1 - (1-t)^n \;\; 0 \le t \le 1$ from a previous exercise.

If $Y = F(X_{(1)})$, then $f_Y(y) = n(1-t)^{n-1}, 0 \le y \le 1$.

Then, $E(Y) = \int_0^1 yn(1-y)^{n-1} dy = \dfrac{1}{n+1}$ where integration by parts is used. Therefore,

$$E[F(X_{(n)})] = \frac{n}{n+1} \;\; and \;\; E[F(X_{(1)})] = \frac{1}{n+1}$$

7-72. $E(V) = k\sum_{i=1}^{n-1} [E(X_{i+1}^2) + E(X_i^2) - 2E(X_i X_{i+1})]$

$$= k\sum_{i=1}^{n-1} (\sigma^2 + \mu^2 + \sigma^2 + \mu^2 - 2\mu^2) = k(n-1)2\sigma^2$$

Therefore, $k = \frac{1}{2(n-1)}$

7-73. a) The traditional estimate of the standard deviation, *S*, is 3.64. The mean of the sample is 13.43 so the values of $\left|X_i - \overline{X}\right|$ corresponding to the given observations are 3.43, 1.43, 5.43, 0.57, 4.57, 1.57 and 3.57. The median of these new quantities is 3.43 so the new estimate of the standard deviation is 5.08 and this value is slightly larger than the value obtained from the traditional estimator.

b) Making the first observation in the original sample equal to 50 produces the following results. The traditional estimator, *S*, is equal to 14.01. The new estimator slightly changed to 7.62.

7-74. a)

$$
\begin{aligned}
T_r = X_1 &+ \\
&X_1 + X_2 - X_1 + \\
&X_1 + X_2 - X_1 + X_3 - X_2 + \\
&\ldots + \\
&X_1 + X_2 - X_1 + X_3 - X_2 + \ldots + X_r - X_{r-1} + \\
&(n-r)(X_1 + X_2 - X_1 + X_3 - X_2 + \ldots + X_r - X_{r-1})
\end{aligned}
$$

Because X_1 is the minimum lifetime of *n* items, $E(X_1) = \dfrac{1}{n\lambda}$.

Then, $X_2 - X_1$ is the minimum lifetime of (*n*-1) items from the memoryless property of the exponential and $E(X_2 - X_1) = \dfrac{1}{(n-1)\lambda}$.

Similarly, $E(X_k - X_{k-1}) = \dfrac{1}{(n-k+1)\lambda}$. Then,

$$E(T_r) = \frac{n}{n\lambda} + \frac{n-1}{(n-1)\lambda} + \ldots + \frac{n-r+1}{(n-r+1)\lambda} = \frac{r}{\lambda} \text{ and } E\left(\frac{T_r}{r}\right) = \frac{1}{\lambda} = \mu$$

b) $V(T_r / r) = 1/(\lambda^2 r)$ is related to the variance of the Erlang distribution $V(X) = r/\lambda^2$. They are related by the value $(1/r^2)$. The censored variance is $(1/r^2)$ times the uncensored variance.