1. **(25 marks)** An extrusion die is used to produce aluminum rods. Specifications are given for the length and the diameter of the rods. For each rod, the length is classified as too short, too long, or OK, and the diameter is classified as too thin, too thick, or OK. In a population of 1000 rods, the number of rods in each class is as follows:

	Diameter		
Length	**Too thin**	**OK**	**Too thick**
Too short	**10**	**3**	**5**
OK	**38**	**900**	**4**
Too long	**2**	**25**	**13**

a) A rod is sampled at random from this population. What is the probability that it is too short?

P(too short) = (10 + 3 + 5)/1000 = 18/1000 = 0.018.

b) A rod is sampled at random from this population. What is the probability that it is either too short or too thick?

P(too short or too thick) = (10 + 3 + 5 + 4 + 13)/1000 = 35/1000 = 0.035

Another method of solution

P(too short) = 18/1000
P(too thick) = 22/1000
P(too short and too thick) = 5/1000
P(too short or too thick) = 18/1000 + 22/1000 – 5/1000 = 35/1000 = 0.035.

c) Compute the conditional probability *P*(diameter OK |length too long). Is this the same as the unconditional probability *P*(diameter OK)?

P(diameter OK) = (3 + 900 + 25)/1000 = 928/1000 = 0.928
P(diameter OK |length too long) = *P*(diameter OK and length too long)/ *P*(length too long)
= 25/(25 + 2 + 13) = 0.625.
The two probabilities are not the same i.e., the conditional probability differs from the unconditional probability.

d) A rod is sampled at random from this population; find *P*(too long) and *P*(too long |too thin). Are these probabilities different?

P(too long) = 40/1000 = 0.04
P(too long |too thin) = *P*(too long and too thin)/ *P*(too thin) = 2/50 = 0.040
The conditional and the unconditional probabilities are the same. The information that the rod is too thin does not change the probability that the rod is too long.

2. **(25 marks)** A certain radioactive mass emits alpha particles from time to time. The time between emissions, in seconds is, is random, with probability density function

$$f(x)=\begin{cases}0.1e^{-0.1x} & x>0\\ 0 & x\le 0\end{cases}$$

a) Find the expected value of time between emissions.

The expected value is defined such that

$$\int_{-\infty}^{\infty} xf(x)dx$$

For the given pdf

$$\mu=\int_0^{\infty} x(0.1e^{-0.1x})dx = 0.1\int_0^{\infty} xe^{-0.1x}dx$$

$$=-0.1e^{-0.1x}\left[\frac{x}{-0.1}-\frac{1}{(-0.1)^2}\right]\Bigg|_0^{\infty} = e^{-0.1x}\left[x+10\right]\Big|_0^{\infty}$$

$$=10\,\text{min.}$$

b) Find the median time between emissions.

The median is the solution to

$$\int_{-\infty}^{x_{50}} f(x)dx = 0.5$$

Carry out the integration for the PDF given to obtain the CDF

$$\int_0^x 0.1e^{-0.1u}du = -e^{-0.1u}\Big|_0^x = 1-e^{-0.1x}$$

Find the median which is defined as the 50th percentile i.e., the probability for values greater than the median is the same as that of values less than the median and is equal to 0.5

$$1-e^{-0.1x_{50}}=0.5$$
$$-0.1x_{50}=\ln 0.5$$
$$x_{50}=6.931\,\text{min.}$$

c) Find the 60th percentile of the times.

The probability for the 60th percentile is 0.6. Consequently,

$$1-e^{-0.1x_{60}}=0.6$$
$$-0.1x_{60}=\ln 0.4$$
$$x_{60}=9.163\,\text{min.}$$

The mean time for this distribution is higher than either the median or the 60th percentile indicating the degree of skewness of the distribution. The mean for this distribution is given as the 63rd percentile!

3. (25 marks) In oil exploration, the probability of an oil strike in the North Sea is 1 in 500 drillings. What is the probability of having exactly 3 oil-producing wells in 1000 explorations?

In this case, n = 1000 and p = 1/500 = 0.002. Therefore, the Poisson approximation to the binomial distribution is justified.

$$\lambda = np = (1000)(0.002) = 2$$

$$P(X = 3) = \frac{2^3 e^{-2}}{3!} = 0.1805.$$

Of course, you can use the binomial distribution directly to obtain the same probability

$$n = 1000$$

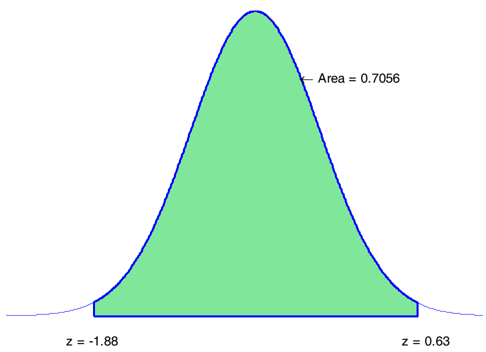
$$p = 0.002$$

$$P(X = 3) = \frac{(1000)(999)(998)(997!)}{997!3!}(0.002)^3(0.998)^{997} = 0.18063.$$

4. (25 marks) A process manufactures ball bearings whose diameters are normally distributed with mean 2.505 cm and standard deviation 0.008 cm. Specifications call for the diameter to be in the interval 2.5 ± 0.01 cm.

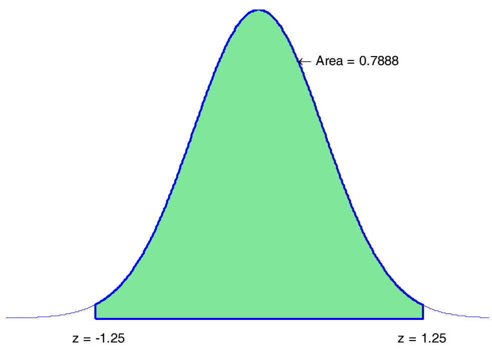
a) What proportion of the ball bearings will meet the specifications?

$$P(2.49 \le x \le 2.51) = P(z_1 \le x \le z_2) = P(z_2) - P(z_1)$$
$$z_1 = \frac{2.49 - 2.505}{0.008} = -1.88$$
$$z_2 = \frac{2.51 - 2.505}{0.008} = 0.63$$
$$P(-1.88 \le z \le 0.63) = P(z \le 0.63) - P(z \le -1.88)$$
$$= 0.735653 - 0.030054 = 0.705599.$$



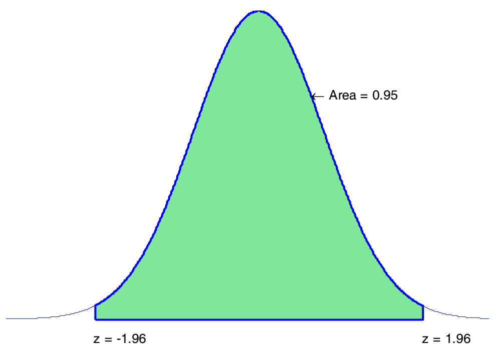
b) The process can be recalibrated so that the mean will equal 2.5 cm, the center of the specification interval. The standard deviation of the process remains 0.008 cm. What proportion of the ball bearings will meet the specifications?

$$P(2.49 \le x \le 2.51) = P(z_1 \le x \le z_2) = P(z_2) - P(z_1)$$
$$z_1 = \frac{2.49 - 2.50}{0.008} = -1.25$$
$$z_2 = \frac{2.51 - 2.50}{0.008} = 1.25$$
$$P(-1.25 \le z \le 1.25) = P(z \le 1.25) - P(z \le -1.25)$$
$$= 0.8943500 - 0.105650 = 0.7887.$$



c) Assume that the process has been calibrated so that the mean diameter is now 2.5 cm. To what value must the standard deviation be lowered so that 95% of the diameters will meet the specifications?

$$P(z_1 \le x \le z_2) = 0.95$$
$$z_1 = -1.96$$
$$z_2 = 1.96$$
$$z_2 = 1.96 = \frac{2.51 - 2.50}{\sigma} \Rightarrow \sigma = 0.0051\text{cm}.$$



Recalibration will increase the bearings within specs by about 12%. Reducing the variability of the process by reducing the standard deviation will increase the bearings within specs to 95%!