Discrete Random Variables Distributions:

Bernoulli's Trials:

Repeated independent trials, in which for each trial there are two possible outcomes: Success with probability **p** OR failure with probability **q=1-p**.

In other words, a Bernoulli's trial is an experiment which has **two complementary outcomes** with constant probabilities, and these outcomes are **Mutually Exclusive** and **Independent**

It should be noted that:

Success = Occurrence of an event while Failure = non Occurrence of the event

Binomial Distribution:

Definitions:

X : number of trials that result in a success in n Bernoulli trials

n : number of Bernoulli trials

x : number of successes

(n-x) : number of failures

p : probability of a success

The number of possible sets of outcomes [that is the number of ways of obtaining x successes in n trials] is given by:

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

The probability of obtaining each set of possible outcomes is given by:

$p^x \,.\, (1-p)^{n-x}$ since the events S and F are independent

Therefore, the probability mass function for X is:

$$f(x) = \binom{n}{x}\, p^x \,.\, (1-p)^{n-x}$$

Since the events of obtaining possible x successes are mutually exclusive, the probability of any set of x successes is the sum of their individual probabilities.

Ex. Throwing a coin 5 times. What is the probability of obtaining 3 trials with face up?

X : Trial with face up (success)

n = 5

x = 3

p = probability of a success

Example of sequences HTTHH , HHTTH , …..

Number of possible sequences:

$$\binom{5}{3} = \frac{5!}{3!\,(5-3)!} = 10$$

P(x=3) = $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3$

P(n-x=2) = $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2$

∴ the probability of a sequence = $\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$

and f(x=3) = 10 $\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$

= 10 x 0.125 x 0.25

= 0.31

Binomial expansion:

$(a+b)^n = \sum_k \binom{n}{k}\ a^k b^{n-k}$ $\quad$ *a and b are constants.*

Let a = p and b = (1-p)

$$\therefore (a+b)^n = (p+1-p)^n = 1$$

The RHS is the sum of probabilities for a binomial Random variable and it can be seen that it is equal to 1, satisfying the condition for a mass function.

Expectations of X

$$E(X) = \mu_x = np$$

$$Var(X) = \sigma_x^2 = np(1-p) = npq$$

Ex 3.18

Each sample of water has 10% chance of obtaining an organic pollutant. Samples are independent.

X: number of samples that contain the pollutant (successes)
n = number of samples =18
p= 0.1

Find P(X=2) [exactly 2]

$$f(X=2) = \binom{18}{2}\ 0.1^2 \,.\, (1-0.1)^{18-2}$$

Find P(X≥4) [at least 4 samples]

$$P(X \geq 4) = \sum_{x=4}^{18} \binom{18}{x} 0.1^x * (1-0.1)^{18-x}$$
$$=1 - P(X \leq 3)$$

$$P(X \geq 4) = 1 - \sum_{x=0}^{3} \binom{3}{x} 0.1^x * (1-0.1)^{3-x}$$

Geometric Distribution:

If X independent Bernoulli trials, each with a constant probability of success p, will be required until the event of the first success (or failure) is obtained, then the probability of obtaining this random variable i.e. probability of obtaining the first success on x^{th} trial, will be given by what is known as the geometric distribution, given by:

$$f(x) = (1-p)^{x-1} * p$$

the events of the success and the failures are independent.

$$\mu = \frac{1}{p} \qquad and \qquad \sigma^2 = \frac{q}{p^2}$$

Negative Binomial:

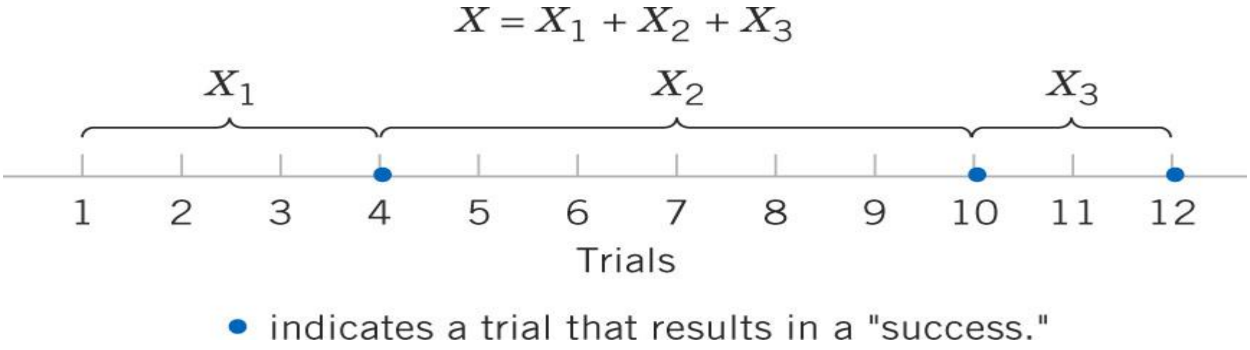
This a generalization of geometric distribution. In this case, the discrete random variable X is defined as the number of independent Bernoulli trials required to obtain r successes and has the following distribution function:

$f(x) = \binom{x-1}{r-1} (1-p)^{x-r} p^r$ x=r, r+1, r+2, ……

The range of X is r→∞ since at least r trials are required to obtain r successes.

$$\mu = \frac{r}{p} \qquad and \qquad \sigma^2 = \frac{rq}{p^2}$$

Negative binomial random variable is equivalent to the sum of geometric random variables.



Summary

Binomial: Number of successes (**x**) in **n** Bernoulli trials

$$f(x) = \binom{n}{x} p^x . (1-p)^{n-x}$$

Geometric: Number of Bernoulli trials (**x**) required until the first success is obtained

$$f(x) = (1-p)^{x-1} . p$$

Negative Binomial: Number of Bernoulli trials (**x**) required to obtain r successes.

$f(x) = \binom{x-1}{r-1} (1-p)^{x-r} p^r$ x=r, r+1, r+2, ……

Ex 3-28 Data from 250 endothermic reactions involving sodium bicarbonate

X= Final Temperature Conditions	No. of Reactions (Trials)
266 K	70
271	80
274	100

Question: Find the probability mass function of final Temperature. i.e. f(x) which is the probability of having a reaction with any final temperature X∈ [266K, 271K, or 274K].

f(x) will be in the form of a table:
First we find the individual probabilities:

P(X=266 K) = 70/250 =0.28
P(X=271 K) = 80/250 =0.32
P(X=274 K) = 100/250 =0.40

Therefore:

$$f(x) = \begin{cases} 0.28 & x = 266 \\ 0.32 & x = 271 \\ 0.40 & x = 274 \end{cases}$$

Check $\sum f(x) = 1.000$

Ex 3-144

a) Find the probability that the first reaction (X) to result in a final temperature less than 272 K is the 10^{th} reaction.
This is a geometric distribution. The success is that the reaction with a final condition of 272 is the 10^{th} reaction. Note that the final condition of 272 K does not exist, so any condition less than that will be accepted.

Therefor the probability of obtaining a final condition less than 272 is

p = probability of a success = P (Final condition < 272 K)
P (Final condition < 272 K) = P (Final condition =266) + P (Final condition =271) since they are Mutually exclusive
= 0.28 + 0.32
= 0.60

Now we can calculate the probability of having the 10th reaction resulting in a final temperature of 272 as follows:

$$f(x) = (1-p)^{x-1}.p$$

$$f(X = 10) = (1 - 0.6)^9 * 0.6$$
$$= 0.000157$$

b) What is the mean number of reactions until the first final condition is less than 272 K?

$$\mu = \frac{1}{p}$$

$$\frac{1}{0.6} = 1.67$$

c) What is the probability that the first reaction resulting in a final temperature less than 272 K occurs within 3 to fewer reactions (i.e. $X \leq 3$)?

$P(X \leq 3) = P(X=3) + P(X = 2) + P(X=1)$
$= o.4^2*0.6 + o.4^1*0.6 + o.4^0*0.6$
$= 0.936$

d) What is the mean number of reactions until two reactions result in a final temperature less than 272 K?

In this case the number of reactions is a discrete random variable having a negative binomial distribution. Therefore:

$$\mu = \frac{r}{p}$$
$$= \frac{2}{0.6}$$
$$= 3.333$$