Exponential Distribution

A random variable X is said to follow an exponential distribution if its probability distribution function is:

$$f(x) = \lambda\, e^{-\lambda x} \qquad \text{for } x \geq 0$$
$$f(x) = 0 \qquad \text{for } x < 0$$

where λ is a constant > 0.

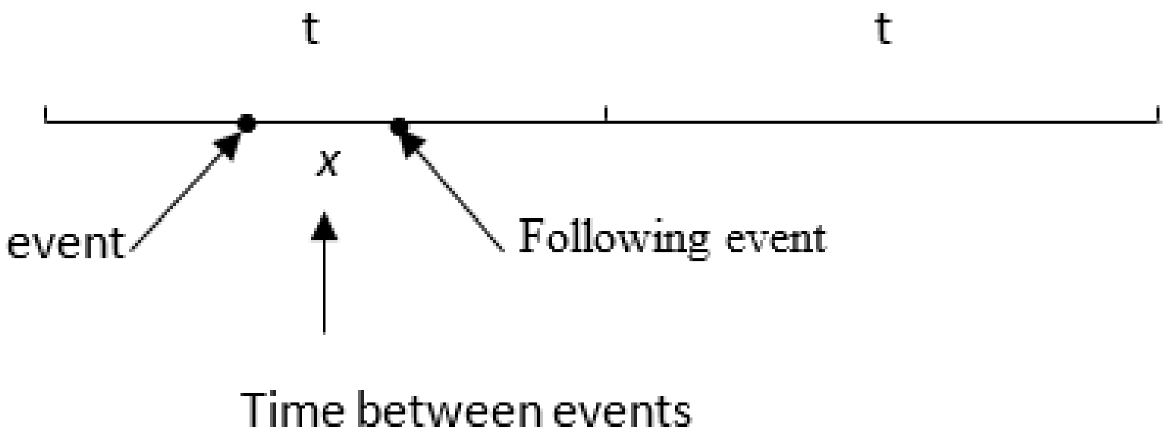
The Mean and Variance of X are:

$$\mu = \frac{1}{\lambda} \quad and \quad \sigma^2 = \frac{1}{\lambda^2}$$

The cumulative distribution function F(X) is found by integrating the PDF of X, which results in:

$$F(x_1) = \Pr[0 < X < x_1]$$
$$= \int_0^{x_1} \lambda e^{-\lambda x}\, dx$$
$$= 1 - e^{-\lambda x_1}$$

Exponential Distribution



Exponential distribution is sometimes used to model the distance between successive events (x) of a Poisson process with λ>0. This variable X is an exponential random variable with parameter λ (λ= average number of events per in the interval =at). X could represent the waiting time (distance) between any event and the next event. X at most equals t. If X>t, this means that no event took place in the interval.

$P(X \le t) = 1 - P(X>t)$

$= 1 - P[\text{no events in the interval } t]$

$= 1 - \frac{e^{-at}(at)^0}{0!} = 1 - e^{-at}$; a = average number of events per unit interval in the Poisson process. Note that in the book they use λ instead of a.

Exponential Distribution

The random variable X that equals the distance between successive events of a Poisson process with mean number of events per unit interval is an exponential random variable with parameter The probability density function of X is

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad 0 \leq x < \infty$$

Note that λ in this equation is the rate of events per unit interval.

Exponential Distribution

In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 25 log-ons per hour. What is the probability that there are no log-ons in an interval of 6 minutes?

Let X denote the time in hours from the start of the interval until the first log-on. Then, X has an exponential distribution with $\lambda = 25$ log-ons per hour. We are interested in the probability that X exceeds 6 minutes. Because λ is given in log-ons per hour, we express all time units in hours. That is, 6 minutes = 0.1 hour. The probability requested is shown as the shaded area under the probability density function in Fig. 4-23. Therefore,

$$P(X > 0.1) = \int_{0.1}^{\infty} 25e^{-25x}\,dx = e^{-25(0.1)} = 0.082$$

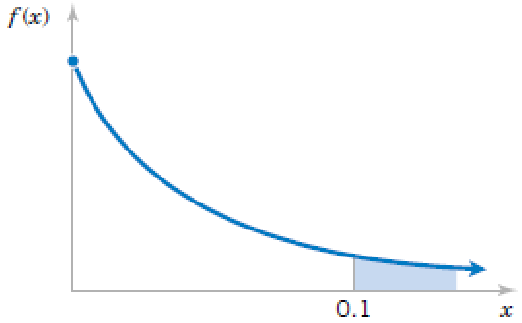


Figure 4-23 Probability for the exponential distribution in Example 4-21.

Exponential Distribution

Also, the cumulative distribution function can be used to obtain the same result as follows:

$$P(X > 0.1) = 1 - F(0.1) = e^{-25(0.1)}$$

An identical answer is obtained by expressing the mean number of log-ons as 0.417 log-ons per minute and computing the probability that the time until the next log-on exceeds 6 minutes. Try it.

What is the probability that the time until the next log-on is between 2 and 3 minutes? Upon converting all units to hours,

$$P(0.033 < X < 0.05) = \int_{0.033}^{0.05} 25e^{-25x}\,dx = -e^{-25x}\Big|_{0.033}^{0.05} = 0.152$$

An alternative solution is

$$P(0.033 < X < 0.05) = F(0.05) - F(0.033) = 0.152$$

Determine the interval of time such that the probability that no log-on occurs in the i val is 0.90. The question asks for the length of time x such that $P(X > x) = 0.90$. Now,

$$P(X > x) = e^{-25x} = 0.90$$

Take the (natural) log of both sides to obtain $-25x = \ln(0.90) = -0.1054$. Therefore,

$$x = 0.00421 \text{ hour} = 0.25 \text{ minute}$$

Exponential Distribution

Furthermore, the mean time until the next log-on is

$$\mu = 1/25 = 0.04 \text{ hour} = 2.4 \text{ minutes}$$

The standard deviation of the time until the next log-on is

$$\sigma = 1/25 \text{ hours} = 2.4 \text{ minutes}$$