

Problem 19.1

Solution:

Known quantities:

The square wave signal: $x(t) = \begin{cases} 1 & \text{for } |t| \leq \eta\tau \\ -1 & \text{for } |t| < \tau \end{cases}$, $\tau = \frac{1}{300} \text{ sec}$, $\eta = 30\%$

Find:

- The Fourier series coefficients for the square wave signal.
- Frequency spectrum of the signal for the numerical values.

Analysis:

- Fourier series coefficients:

From Fourier series theory we know that the Fourier series representation of any signal is given as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{1}{T} \int_T x(t) dt$$

$$a_n = \frac{2}{T} \int_T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_T x(t) \sin(n\omega_0 t) dt$$

where $x(t)$ is the signal, T is the period of the signal, and ω_0 is the natural frequency.

The square wave signal is an odd function, so we need to compute only the b_n Fourier coefficients. The interval of integration $0 \leq t \leq \tau$ would be convenient.

$$a_0 = \frac{1}{\tau} \left[\int_0^{\frac{\tau}{2}} dt - \int_{\frac{\tau}{2}}^{\tau} dt \right] = 0$$

$$b_n = \frac{2}{\tau} \left[\int_0^{\frac{\tau}{2}} \sin n \frac{2\pi}{\tau} t dt - \int_{\frac{\tau}{2}}^{\tau} \sin n \frac{2\pi}{\tau} t dt \right]$$

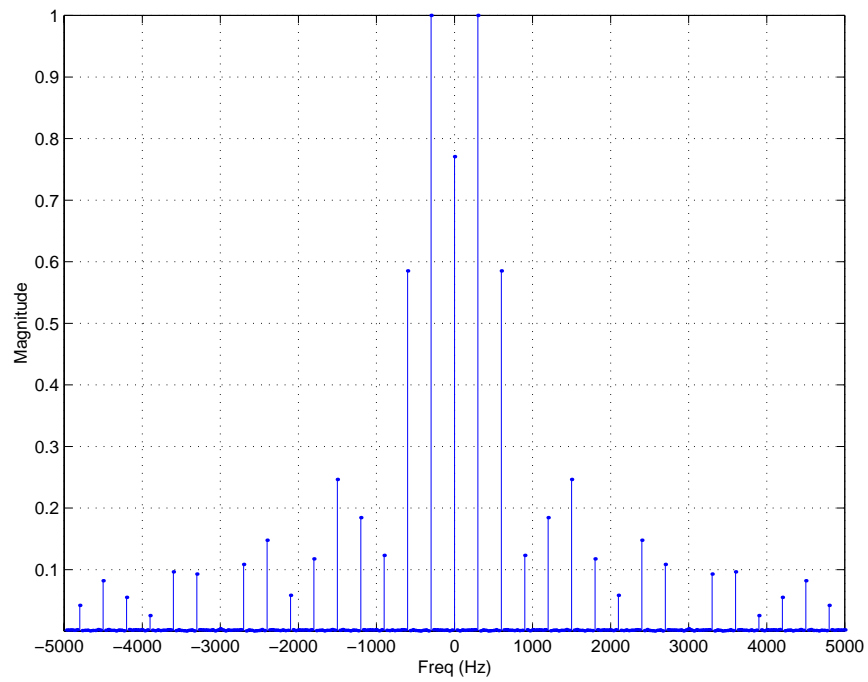
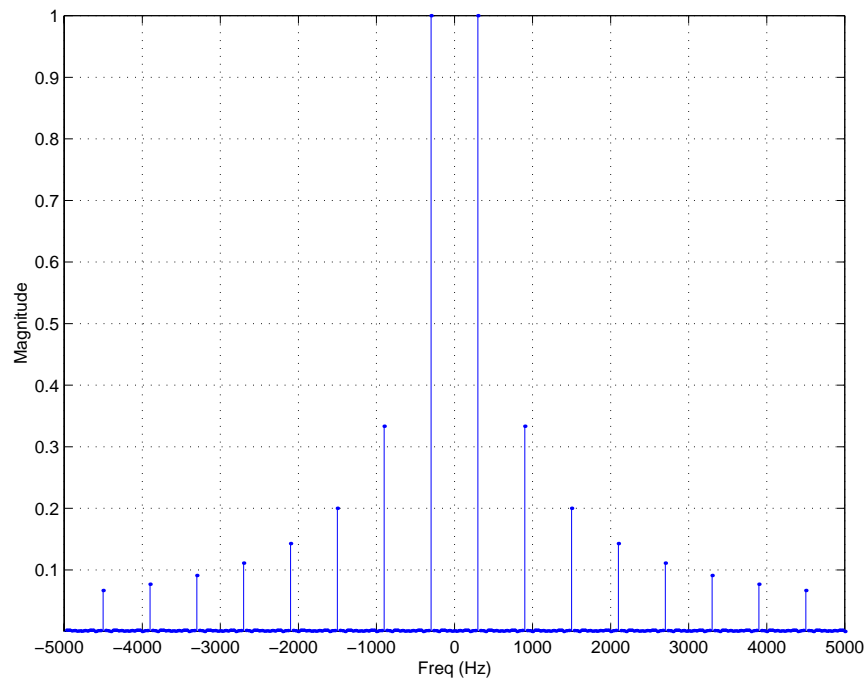
$$b_n = \frac{2}{\tau} \left[\frac{-\cos \frac{n2\pi}{\tau} t}{\frac{n2\pi}{\tau}} \Big|_0^{\tau/2} + \frac{\cos \frac{n2\pi}{\tau} t}{\frac{n2\pi}{\tau}} \Big|_{\tau/2}^{\tau} \right]$$

$$b_n = \frac{1}{n\pi} - \frac{2}{n\pi} \cos n\pi + \frac{1}{n\pi} \cos n2\pi = \frac{2}{n\pi} [1 - \cos n\pi] \quad n \neq 0$$

$$b_n = \begin{cases} \frac{4}{n\pi} & n : \text{odd} \\ 0 & n : \text{even} \end{cases}$$

$$\therefore x(t) = \frac{4}{\pi} \left[\sin \frac{2\pi t}{\tau} + \frac{1}{3} \sin \frac{3\pi t}{\tau} + \dots \right]$$

- b. Frequency spectrum of the signal for $\eta = 50\%$ is shown in Figure 19.xx and for $\eta = 30\%$ is shown in Figure 19.xx



Problem 19.2

Solution:

Known quantities:

Functional form of a full-wave rectified sinusoidal signal of time period T sec, $x(t) = |\sin(\omega_o t)|$, and natural

frequency $\omega_o = 200\pi \frac{\text{rad}}{\text{s}}$.

Find:

- The Fourier series coefficients.
- Frequency spectrum of the signal.

Analysis:

The rectified sine wave signal is an even function. Hence, we need to compute only the a_n coefficients of the Fourier series.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} |\sin(\omega_o t)| d(\omega_o t) = \frac{2}{\pi} \int_0^{\pi} \sin(\omega_o t) d(\omega_o t)$$

$$a_0 = \frac{4}{\pi}$$

$$a_1 = \frac{2}{\pi} \int_0^{2\pi} |\sin(\omega_o t)| \cos(\omega_o t) d(\omega_o t) = \frac{4}{\pi} \int_0^{\pi} \sin(\omega_o t) \cos(\omega_o t) d(\omega_o t) = \frac{4}{\pi} \int_0^{\pi} \frac{1}{2} \sin(2\omega_o t) d(\omega_o t)$$

$$a_1 = 0$$

$$a_2 = \frac{2}{\pi} \int_0^{2\pi} |\sin(\omega_o t)| \cos(2\omega_o t) d(\omega_o t) = \frac{4}{\pi} \int_0^{\pi} \sin(\omega_o t) \cos(2\omega_o t) d(\omega_o t)$$

$$= \frac{4}{\pi} \int_0^{\pi} \frac{1}{2} [\sin(3\omega_o t) - \sin(\omega_o t)] d(\omega_o t)$$

$$a_2 = \frac{4}{\pi} \frac{2}{3}$$

$$a_3 = 0$$

$$a_4 = -\frac{4}{\pi} \frac{2}{15}$$

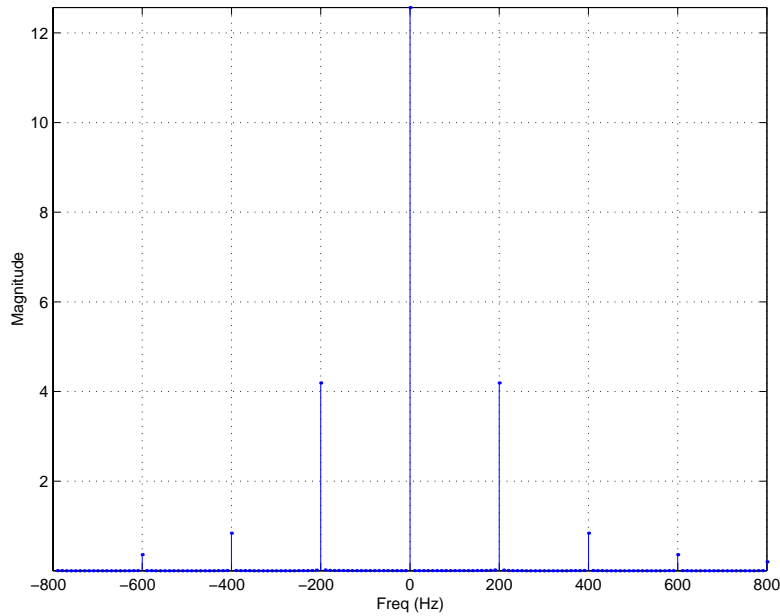
$$a_n = \begin{cases} \frac{4}{\pi} \frac{2}{(n-1)(n+1)} & n : \text{even} \\ 0 & n : \text{odd} \end{cases}$$

$$b_n = 0$$

$$x(t) = \frac{4}{\pi} \left[1 + \frac{2}{3} \cos(2\omega_o t) - \frac{2}{15} \cos(4\omega_o t) + \frac{2}{65} \cos(6\omega_o t) \cdots \right]$$

- The Frequency spectrum for the full wave rectified sinusoid of frequency $\omega_o = 200\pi \frac{\text{rad}}{\text{s}}$ is shown in

Figure 19.xx.



Problem 19.3

Solution:

Known quantities:

Functional form of a full-wave rectified cosine wave of time period T sec, $x(t) = |\sin(\omega_o t)|$, and natural

frequency $\omega_o = 150\pi \frac{\text{rad}}{\text{s}}$.

Find:

- The Fourier series coefficients.
- Frequency spectrum of the signal.

Analysis:

The rectified cosine wave signal is same as the rectified sine wave with a phase shift of $\frac{\pi}{2}$ rad. The Fourier series

coefficients are to be found over a period of π rad. If we consider the period from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ rad, the analysis is

same as a full wave rectified sinusoid. Hence, the Fourier series coefficients are the same as computed in Problem 19.2.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} |\cos(\omega_o t)| d(\omega_o t) = \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(\omega_o t) d(\omega_o t)$$

$$a_0 = \frac{4}{\pi}$$

$$a_1 = \frac{2}{\pi} \int_0^{2\pi} |\cos(\omega_o t)| \cos(\omega_o t) d(\omega_o t) = \frac{4}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(\omega_o t) \cos(\omega_o t) d(\omega_o t) = \frac{4}{\pi} \int_0^{\pi} \frac{1}{2} \sin(\omega_o t) d(\omega_o t)$$

$$a_1 = 0$$

$$a_2 = \frac{2}{\pi} \int_0^{2\pi} |\cos(\omega_o t)| \cos(2\omega_o t) d(\omega_o t) = \frac{4}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(\omega_o t) \cos(2\omega_o t) d(\omega_o t)$$

$$= \frac{4}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2} [\sin(3\omega_o t) - \sin(\omega_o t)] d(\omega_o t)$$

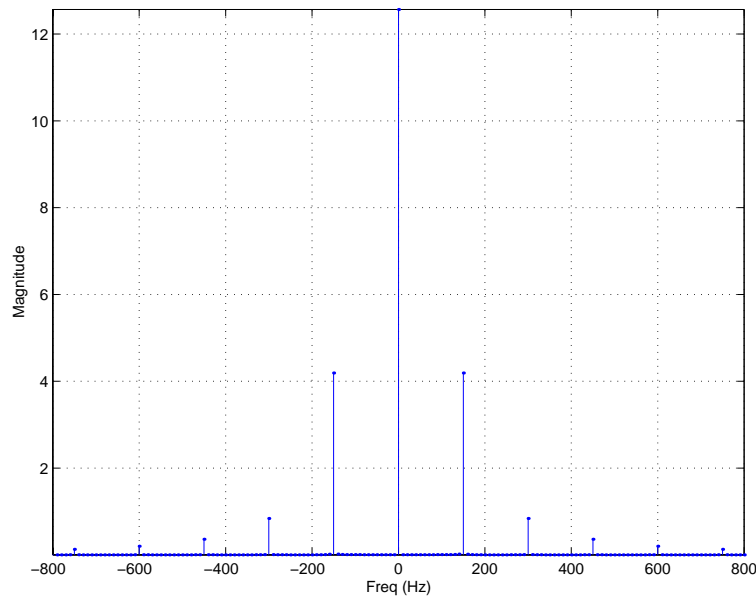
$$a_2 = \frac{4}{\pi} \frac{2}{3}$$

$$a_n = \begin{cases} \frac{4}{\pi} \frac{2}{(n-1)(n+1)} & n : \text{even} \\ 0 & n : \text{odd} \end{cases}$$

$$b_n = 0$$

$$x(t) = \frac{4}{\pi} \left[1 + \frac{2}{3} \cos(2\omega_o t) - \frac{2}{15} \cos(4\omega_o t) + \frac{2}{65} \cos(6\omega_o t) \cdots \right]$$

- b. The Frequency spectrum for the full wave rectified cosine wave of frequency $\omega_o = 150\pi \frac{\text{rad}}{\text{s}}$ is shown in Figure 19.xx.



Problem 19.4

Solution:

Known quantities:

Functional form of a cosine burst as shown in Fig. 19.xx and mathematically defined as:

$$x(t) = \cos\left(\frac{\pi}{T}t\right)$$

Find:

Fourier series coefficients for the cosine burst.

Analysis:

The Fourier series coefficients can be calculated as follows:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_T x(t) dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{\pi}{T}t\right) dt = \frac{1}{T} \left[\frac{T}{\pi} \sin\left(\frac{\pi}{T}t\right) \right]_{-\frac{T}{2}}^{\frac{T}{2}} \\ a_0 &= \frac{2}{\pi} \\ a_n &= \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{\pi}{T}t\right) \cos\left(\frac{n2\pi}{T}t\right) dt \\ &= \frac{2}{T} \frac{1}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\cos\left(\frac{(2n+1)\pi}{T}t\right) + \cos\left(\frac{(2n-1)\pi}{T}t\right) \right] dt \\ &= \frac{1}{T} \left[\frac{T}{(2n+1)} \sin\left(\frac{(2n+1)\pi}{T}t\right) + \frac{T}{(2n-1)} \sin\left(\frac{(2n-1)\pi}{T}t\right) \right]_{-\frac{T}{2}}^{\frac{T}{2}} \\ a_n &= \frac{1}{\pi} \left[\frac{1}{(2n+1)} \sin\left((2n+1)\frac{\pi}{2}\right) + \frac{1}{(2n-1)} \sin\left((2n-1)\frac{\pi}{2}\right) \right] \\ a_n &= (-1)^{n+1} \frac{2}{(2n+1)(2n-1)\pi} \end{aligned}$$

Since the cosine burst is an even signal the b_n coefficients are all zero. Hence $b_n = 0$.

The cosine burst signal can be written as:

$$x(t) = \frac{2}{\pi} + \frac{2}{3\pi} \cos\left(\frac{2\pi}{T}t\right) - \frac{2}{15\pi} \cos\left(\frac{4\pi}{T}t\right) + \dots$$

Problem 19.5

Solution:

Known quantities:

Functional form of a triangular pulse signal as shown in Fig. 19.xx and mathematically defined as:

$$x(t) = A \left[1 - \frac{|t|}{T} \right] (u(t+T) - u(t-T))$$

Find:

- Fourier transform of the function.
- Plot the frequency spectrum of the triangular pulse of period, $T = 0.01$ sec and amplitude, $A = 0.5$.

Analysis:

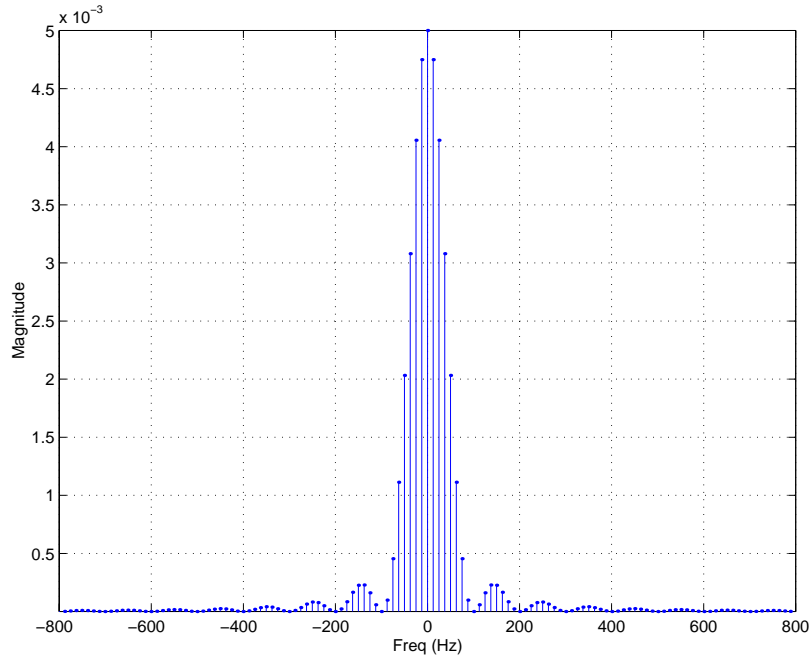
The mathematical equation for the triangular pulse can be split into a function defined over different periods as follows:

$$x(t) = \begin{cases} A \left[1 + \frac{t}{T} \right] & \text{for } -T \leq t < 0 \\ A \left[1 - \frac{t}{T} \right] & \text{for } 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases}$$

The Fourier transform is defined as: $X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) dt$ and can be computed for the triangular pulse as follows:

$$\begin{aligned} X(f) &= \int_{-T}^T A \left[1 - \frac{|t|}{T} \right] \exp(-j2\pi ft) dt \\ &= \int_{-T}^0 A \left[1 + \frac{t}{T} \right] \exp(-j2\pi ft) dt + \int_0^T A \left[1 - \frac{t}{T} \right] \exp(-j2\pi ft) dt \\ &= A \left\{ \frac{j \exp(j\pi fT)}{2\pi^2 f^2 T} \left[\frac{\exp(-j\pi fT) - \exp(j\pi fT)}{2j} \right] + \frac{j \exp(-j\pi fT)}{2\pi^2 f^2 T} \left[\frac{\exp(j\pi fT) - \exp(-j\pi fT)}{2j} \right] \right\} \\ &= A \left\{ \frac{j \exp(j\pi fT)}{2\pi^2 f^2 T} [-\sin(\pi fT)] + \frac{j \exp(-j\pi fT)}{2\pi^2 f^2 T} [\sin(\pi fT)] \right\} \\ &= \frac{A}{\pi^2 f^2 T} \sin(\pi fT) \left[\frac{-\exp(-j\pi fT) + \exp(j\pi fT)}{2j} \right] \\ &= \frac{AT}{(\pi fT)^2} \sin(\pi fT) \sin(\pi fT) \\ X(f) &= AT \text{sinc}^2(fT) \quad \text{where } \text{sinc}(fT) = \frac{\sin(\pi fT)}{\pi fT} \end{aligned}$$

- The Frequency spectrum for the signal as computed from Matlab is shown in Figure 19.xx.



Problem 19.6

Solution:

Known quantities:

Functional form of a exponential pulse signal as shown in Fig. 19.xx and mathematically defined as:

$$x(t) = \begin{cases} (\exp)^{-at}, & \text{for } t > 0 \\ 0, & \text{for } t = 0 \\ -(\exp)^{at}, & \text{for } t < 0 \end{cases}$$

Find:

Fourier transform for the exponential pulse.

Analysis:

We can formulate a compact notation for the pulse signal by using “signum function” which equals +1 for positive time and -1 for negative time. This function is defined as:

$$\text{sgn}(t) = \begin{cases} 1, & \text{for } t > 0 \\ 0, & \text{for } t = 0 \\ -1, & \text{for } t < 0 \end{cases}$$

The signal $x(t)$ can be written as:

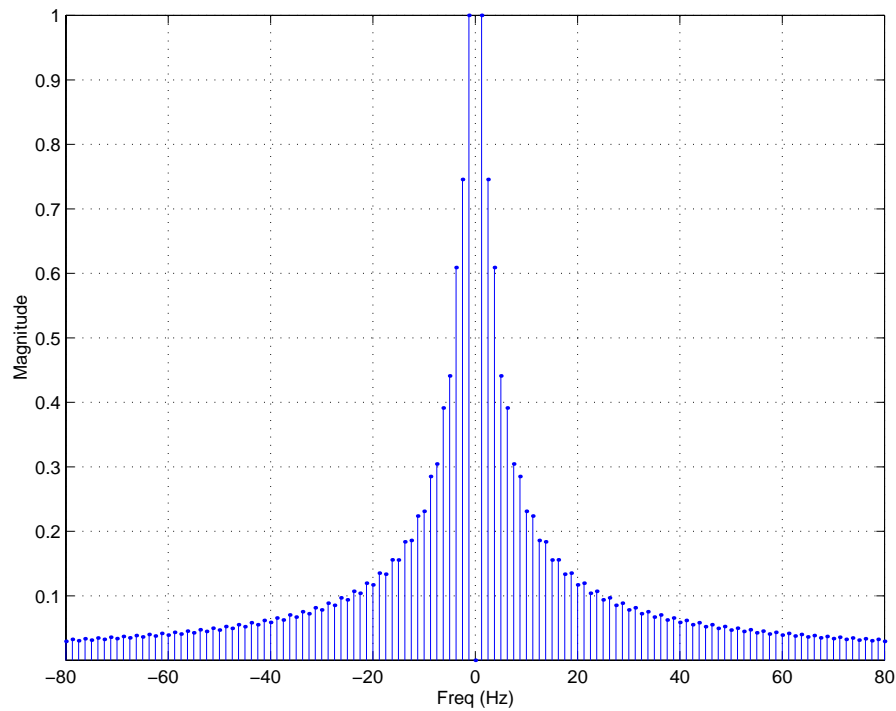
$$x(t) = \exp(-a |t|) \text{sgn}(t)$$

The Fourier transform is now calculated as follows:

$$X(f) = \int_{-\infty}^{\infty} \exp(-a |t|) \text{sgn}(t) \exp(-j2\pi ft) dt$$

$$\begin{aligned}
 &= \int_{-\infty}^0 -\exp((a - j2\pi f)t)dt + \int_0^{\infty} \exp(-(a + j2\pi f)t)dt \\
 &= -\frac{1}{a - j2\pi f} + \frac{1}{a + j2\pi f} \\
 X(f) &= \frac{-j4\pi f}{a^2 + 4\pi^2 f^2}
 \end{aligned}$$

b. The Frequency spectrum for the signal is shown in Figure 19.xx.



Problem 19.7

Solution:

Known quantities:

$$x(t) = \exp(-at) \cos(2\pi f_c t) u(t)$$

Functional form of a damped sinusoid signal as shown in Fig. 19.xx and mathematically defined as:

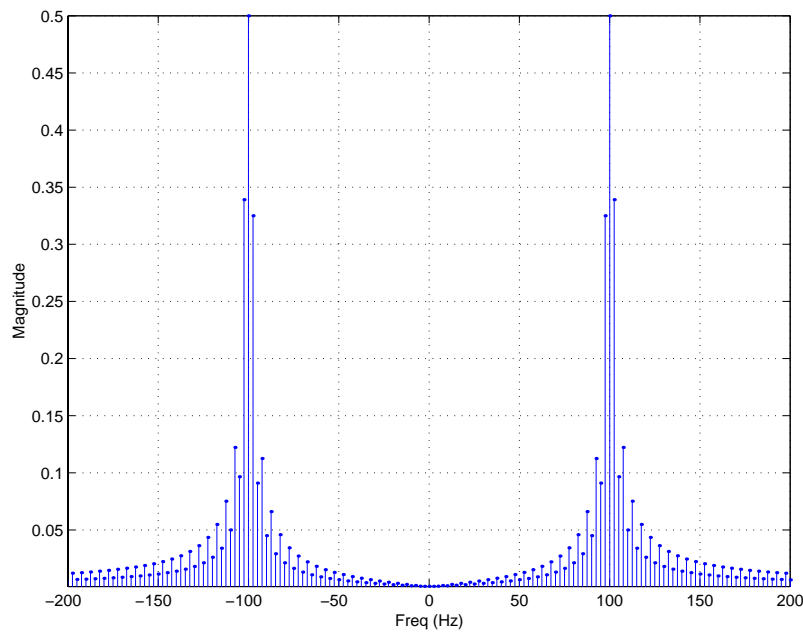
Find:

Fourier transform for the damped sinusoid.

Analysis:

$$\begin{aligned}
 X(f) &= \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) dt \\
 &= \int_{-\infty}^{\infty} \exp(-at) \cos(2\pi f_c t) u(t) \exp(-j2\pi ft) dt \\
 &= \int_0^{\infty} \exp(-at) \frac{1}{2} [\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)] \exp(-j2\pi ft) dt \\
 &= \frac{1}{2} \int_0^{\infty} \{\exp(-[a + j2\pi(f - f_c)]t) + \exp(-[a + j2\pi(f + f_c)]t)\} dt \\
 X(f) &= \frac{1}{2} \left[\frac{1}{a + j2\pi(f - f_c)} + \frac{1}{a + j2\pi(f + f_c)} \right]
 \end{aligned}$$

b. The Frequency spectrum from Matlab is shown in Figure 19.xx.



Problem 19.8

Solution:

Known quantities:

Functional form of an ideal sampling function of frequency $\frac{1}{T_0}$ Hz as shown in Figure 19.xx and having

mathematical equation: $\delta_{T_0} = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$

Find:

- The Fourier transform for the periodic signal.
- Frequency spectrum of the signal for $T_0 = 0.01 \text{ sec}$.

Analysis:

In a limiting sense, Fourier transforms can be defined for periodic signals. Therefore, it is reasonable to represent that a periodic signal can be represented in terms of a Fourier transform, provided that this transform is permitted to include delta functions. An ideal sampling function consists of an infinite sequence of uniformly spaced delta functions. We observe that the generating function for the ideal sampling function is simply a delta function $\delta(t)$. The periodic signal can be represented in terms of the complex exponential Fourier series:

$$\delta_{T_0} = \sum_{m=-\infty}^{\infty} c_n \exp(j2\pi n f_0 t)$$

where c_n is the complex Fourier series coefficients defined as:

$$\begin{aligned} c_n &= \frac{1}{T_0} \int_{-\infty}^{\infty} \delta(t) \exp(-j2\pi n f_0 t) dt \\ &= f_0 G(nf_0) \end{aligned}$$

where $G(nf_0)$ is the Fourier transform of $\delta(t)$ evaluated at the frequency nf_0 . For the delta function:

$$G(nf_0) = 1 \quad \text{for all } n$$

Therefore, using the relation for Fourier transform pair for a periodic signal $g_{T_0}(t)$ with a generating function $g(t)$ and period T_0 :

$$\sum_{m=-\infty}^{\infty} g(t - mT_0) \Leftrightarrow f_0 \sum_{n=-\infty}^{\infty} G(nf_0) \delta(f - nf_0)$$

We get the Fourier transform pair for the ideal sampling function as:

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) \Leftrightarrow f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

We can see that the Fourier transform of a periodic train of delta functions, spaced T_0 seconds apart, consists of another set of delta functions weighted by a factor $f_0 = \frac{1}{T_0}$ and regularly spaced f_0 Hz apart along the frequency axis.

Problem 19.9**Solution:****Known quantities:**

Functional form of the modulating signal $m(t)$, the carrier signal $c(t)$, and the modulation index μ .

$$c(t) = A_c \cos(2\pi f_c t)$$

$$m(t) = A_m \cos(2\pi f_m t)$$

Find:

The average power delivered to a 1-ohm resistor.

Analysis:

The AM signal is given by:

$$s(t) = A_c [1 + \mu \cos(2\pi f_m t)] \cos(2\pi f_c t)$$

Expressing the product of two cosines as the sum of sinusoidal waves, we get:

$$s(t) = A_c \cos(2\pi f_c t) + \frac{1}{2} \mu A_c \cos[2\pi(f_c + f_m)t] + \frac{1}{2} \mu A_c \cos[2\pi(f_c - f_m)t]$$

The Fourier transform of $s(t)$ is therefore:

$$\begin{aligned} S(f) &= \frac{1}{2} A_c [\delta(f - f_c) + \delta(f + f_c)] \\ &+ \frac{1}{4} \mu A_c [\delta(f - f_c - f_m) + \delta(f + f_c + f_m)] \\ &+ \frac{1}{4} \mu A_c [\delta(f - f_c + f_m) + \delta(f + f_c - f_m)] \end{aligned}$$

Thus the spectrum of an AM wave, for sinusoidal modulation, consists of delta functions at $\pm f_c, f_c \pm f_m, -f_c \pm f_m$ as seen from its Fourier transform.

In practice, the AM wave $s(t)$ is a voltage or current wave. In either case, the average power delivered to a 1-ohm resistor by $s(t)$ is comprised of three components:

$$\text{Average power} = \frac{1}{T} \int_T x^2(t) dt$$

Using Parseval's energy relation we can find average power in frequency domain as:

$$\text{Average power} = |X(f)|^2 \quad \text{at } f = 0$$

Hence the carrier frequency, upper side-frequency and lower side-frequency power is:

$$\begin{aligned} \text{Carrier Power} &= \left| \frac{1}{2} A_c [\delta(-f_c) + \delta(f_c)] \right|^2 \\ &= \frac{1}{2} A_c^2 \end{aligned}$$

$$\begin{aligned} \text{Upper side - frequency power} &= \left| \frac{1}{4} A_c [\delta(-f_c - f_m) + \delta(f_c + f_m)] \right|^2 \\ &= \frac{1}{8} \mu^2 A_c^2 \end{aligned}$$

$$\begin{aligned} \text{Lower side - frequency power} &= \left| \frac{1}{4} A_c [\delta(-f_c + f_m) + \delta(f_c - f_m)] \right|^2 \\ &= \frac{1}{8} \mu^2 A_c^2 \end{aligned}$$

For a load resistor R different from 1-ohm, which is usually the case in practice, the expression for carrier power, upper side-frequency power, and lower side-frequency power are merely scaled by the factor $1/R$ or R , depending on whether the modulated wave $s(t)$ is a voltage or current, respectively.

Problem 19.10

Solution:

Known quantities:

Carrier signal frequency, $f_c = 0.82$ MHz, upper side-band frequency components at frequencies

$f_{s1} = 0.825$ MHz, $f_{s2} = 0.83$ MHz, $f_{s3} = 0.84$ MHz, their amplitudes and the modulation index $\mu = 1$.

Find:

- Modulating signal equation.
- Plot spectrum of the modulating signal.
- Plot the spectrum of the AM signal including the lower side-band.

Analysis:

- We know from the theory for AM that the upper side-band frequency has frequency components at frequencies:

$f_{sn} = f_c + f_{mn}$ where n is the number of frequency components in the modulating signal

and, their amplitudes in the AM signal are $\frac{1}{2}$ times the original amplitude of the modulating signal for a

modulation index $\mu = 1$. Hence, we can find the modulating signal components to be:

$$m_1(t) = 0.8 \sin(2\pi 5000t)$$

$$m_2(t) = 0.4 \sin(2\pi 10000t)$$

$$m_3(t) = 0.5 \sin(2\pi 20000t)$$

Hence the modulating signal is:

$$m(t) = 0.8 \sin(2\pi 5000t) + 0.4 \sin(2\pi 10000t) + 0.5 \sin(2\pi 20000t)$$

- The spectrum for the modulating signal is shown in Figure 19.xx
- The spectrum for the AM modulated signal with the lower side-band is shown in Figure 19.xx

Problem 19.11

Known quantities:

AM frequency spectrum from 525 kHz to 1.7 MHz, bandwidth for each channel is 10 kHz

Find:

- Number of channels that can be transmitted in the given frequency range
- The maximum modulating frequency that can be transmitted without overlap.

Analysis:

Assume: No guardband between channels.

- The frequency range allocated for AM broadcast is

$$f_R = 1700 - 525 = 1175 \text{ kHz}$$

This range is partitioned to allow 10 kHz of separation between each channel; therefore, the total number of channels, N is

$$N = \frac{1175}{10} \approx 118 \text{ channels}$$

- The carriers of two separate channels are separated by 10 kHz. If we let the maximum frequency of the message signal increase, the outer edges of both sidebands move away from the carrier frequency and into each other, thereby increasing the bandwidth of each AM channel. The maximum allowable message frequency will occur at the midpoint of the spacing between the carriers. Hence, the maximum message frequency is half the frequency spacing between the carriers.

$$f_{m(\max)} = 5 \text{ kHz}$$

