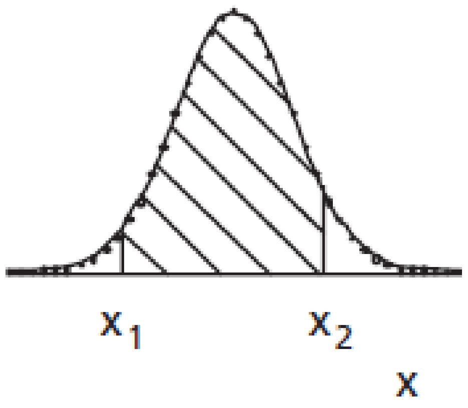
Normal Distribution

Standard Normal

The probability that a variable, *X*, *is between* x_1 *and* x_2 *according to the normal* distribution is given by:

$$\Pr[x_1 < X < x_2] = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$



The cumulative probability function is given by:

$$\Pr[-\infty < X < x] = F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

The use of these equations is dependent on the values of the parameters **μ** and **σ**

Normal Distribution

The calculations can be made easier and generalised by making a the following change of variable:

$$\text{Let} \quad z = \frac{x - \mu}{\sigma}$$

It can be seen that the variable Z is dimensionless. It is the ratio between (***x*- *μ***) and **σ.**

The parameters **μ** and **σ** are constants for any distribution.

$dz = \frac{1}{\sigma} dx \quad and\ dx = \sigma dz$ and:

$$\Pr[x_1 < X < x_2] = \int_{z_1}^{z_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sigma\, dz$$

$$= \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

where

$$z_1 = \frac{x_1 - \mu}{\sigma} \quad \text{and } z_2 = \frac{x_2 - \mu}{\sigma}$$

Normal Distribution

The process of making the change of variable is called normalization of the normal random variable X, and the new variable Z is also normal random variable and it is called the standard normal random variable.

If X is a normal variable with E(X) = **μ** and V(X) = **σ,** the random variable

$$Z = \frac{X - \mu}{\mu}$$

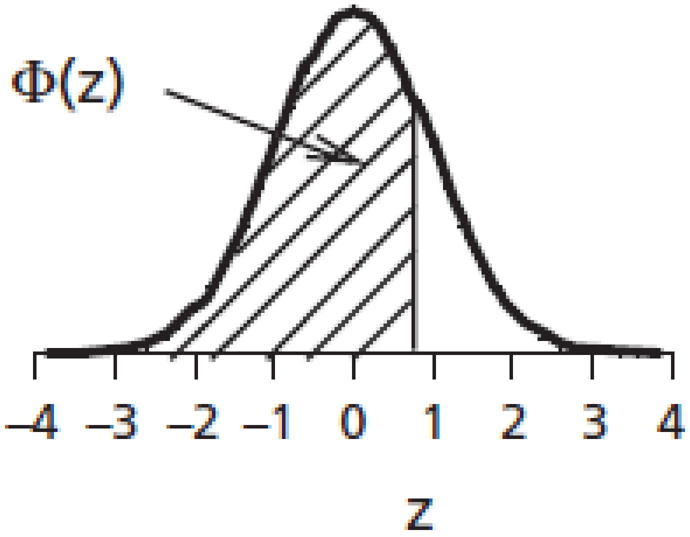
Is a normal variable with E(Z) = 0 and V(Z) = 1. That is Z is a standard normal variable.

Normal Distribution

We define a cumulative distribution function, Φ(*z), as a function of z,* as follows:

$$\Phi(z_1) = \Pr[-\infty < Z < z_1] = \Pr[Z < z_1]$$
$$= \int_{-\infty}^{z_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

Which corresponds to the area under the curve:

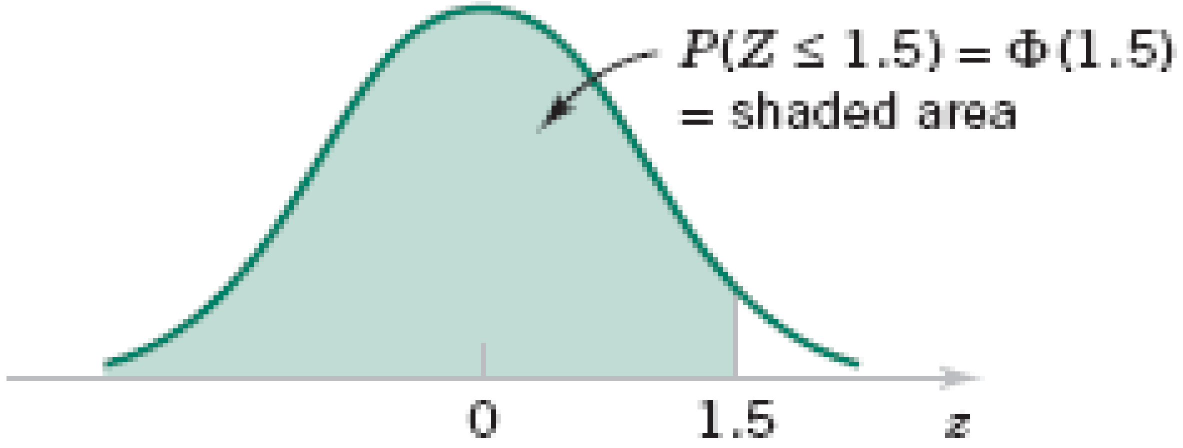


Instead of calculating ϕ(Z) can be found from tables as will be shown later.

Normal Distribution

Example (from Book)

> Assume Z is a standard normal random variable. Appendix Table II provides probabilities of the form $P(Z \leq z)$. The use of Table II to find $P(Z \leq 1.5)$ is illustrated in Fig. 4-13. Read down the z column to the row that equals 1.5. The probability is read from the adjacent column, labeled 0.00, to be 0.93319.
>
> The column headings refer to the hundredth's digit of the value of z in $P(Z \leq z)$. For example, $P(Z \leq 1.53)$ is found by reading down the z column to the row 1.5 and then selecting the probability from the column labeled 0.03 to be 0.93699.



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
⋮		⋮		
1.5	0.93319	0.93448	0.93574	0.93699

Figure 4-13 Standard normal probability density function.

Table II Cumulative Standard Normal Distribution

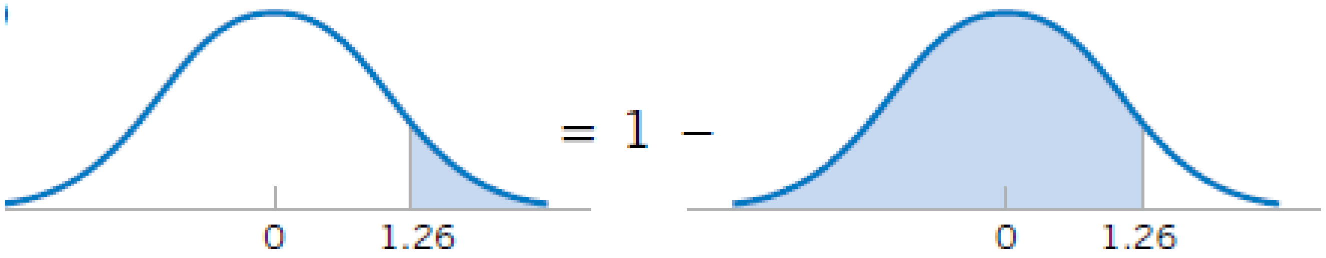
z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108
−3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159
−3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233
−3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337
−3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483
−3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687
−3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968
−3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350
−2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866

Table II Cumulative Standard Normal Distribution (*continued*)

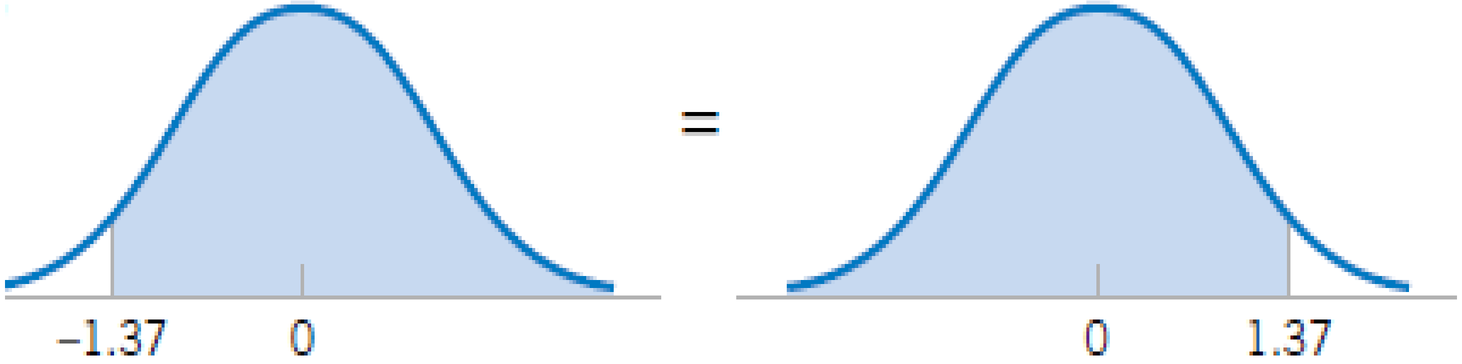
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083

Normal Distribution

$$P(Z > 1.26) = 1 - P(Z \leq 1.26) = 1 - 0.89616 = 0.10384$$



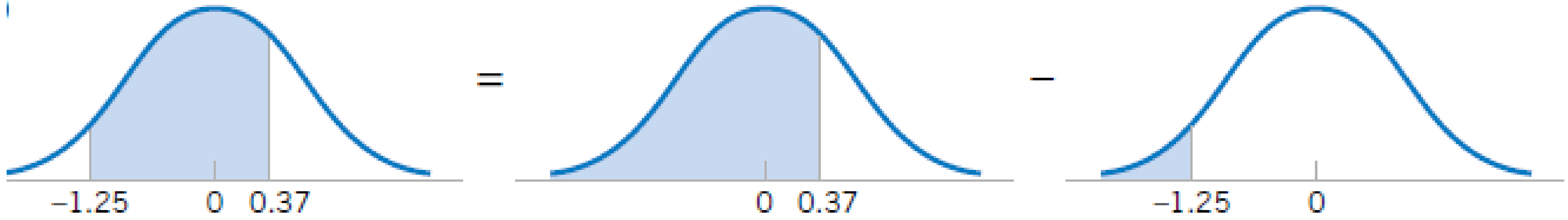
$$P(Z > -1.37) = P(Z < 1.37) = 0.91465$$



Normal Distribution

$$P(-1.25 < Z < 0.37).$$

This probability can be found from the difference of two areas,



$$P(Z < 0.37) - P(Z < -1.25).$$

$$P(Z < 0.37) = 0.64431 \quad \text{and} \quad P(Z < -1.25) = 0.10565$$

$$P(-1.25 < Z < 0.37) = 0.64431 - 0.10565 = 0.53866$$

Normal Distribution

Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 $(\text{milliamperes})^2$. What is the probability that a measurement will exceed 13 milliamperes?

Let X denote the current in milliamperes. The requested probability can be represented as $P(X > 13)$. Let $Z = (X - 10)/2$. The relationship between the several values of X and the transformed values of Z are shown in Fig. 4-15. We note that $X > 13$ corresponds to $Z > 1.5$. Therefore, from Appendix Table II,

$$P(X > 13) = P(Z > 1.5) = 1 - P(Z \leq 1.5) = 1 - 0.93319 = 0.06681$$

Rather than using Fig. 4-15, the probability can be found from the inequality $X > 13$. That is,

$$P(X > 13) = P\left(\frac{(X - 10)}{2} > \frac{(13 - 10)}{2}\right) = P(Z > 1.5) = 0.06681$$

Normal Distribution

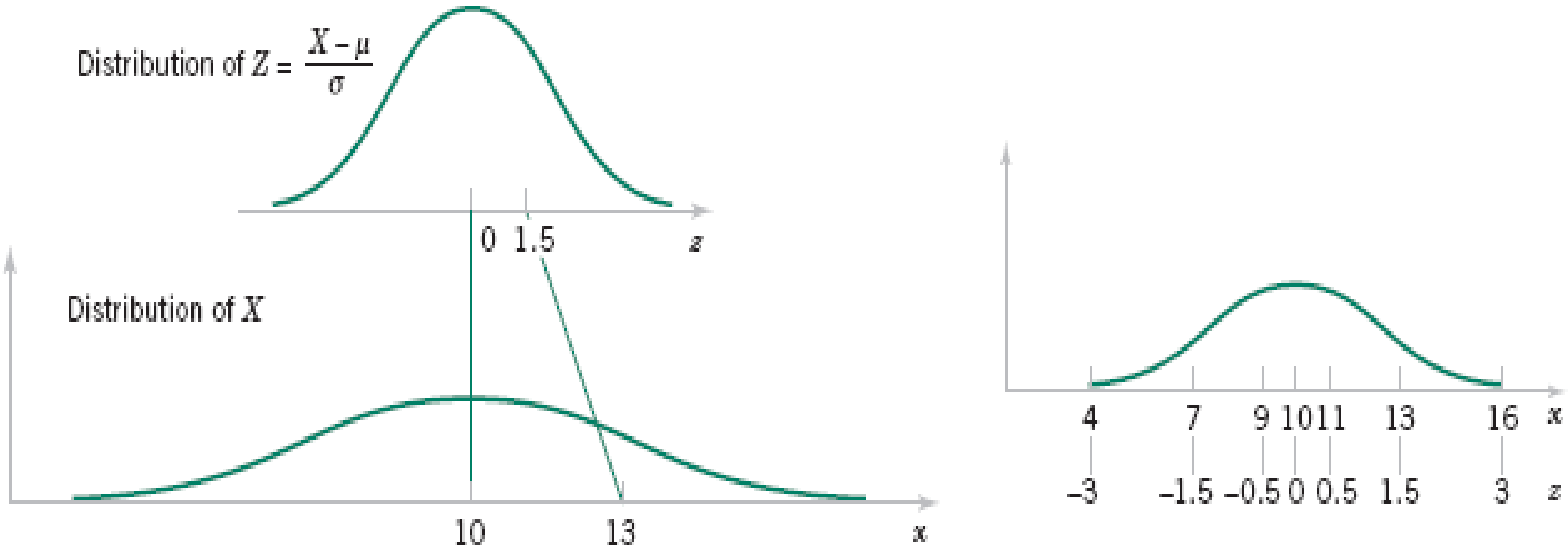


Figure 4-15 Standardizing a normal random variable.

How to standardize to calculate probabilities

Suppose X is a normal random variable with mean μ and variance σ^2. Then,

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = P(Z \leq z) \qquad (4\text{-}11)$$

where Z is a **standard normal random variable**, and $z = \frac{(x-\mu)}{\sigma}$ is the z-value obtained by **standardizing** X.

The probability is obtained by entering Appendix Table II with $z = (x - \mu)/\sigma$.

Continuing the previous example, what is the probability that a current measurement is between 9 and 11 milliamperes? From Fig. 4-15, or by proceeding algebraically, we have

$$\begin{aligned} P(9 < X < 11) &= P((9-10)/2 < (X-10)/2 < (11-10)/2) \\ &= P(-0.5 < Z < 0.5) = P(Z < 0.5) - P(Z < -0.5) \\ &= 0.69146 - 0.30854 = 0.38292 \end{aligned}$$

Determine the value for which the probability that a current measurement is below this value is 0.98. The requested value is shown graphically in Fig. 4-16. We need the value of x such that $P(X < x) = 0.98$. By standardizing, this probability expression can be written as

$$\begin{aligned} P(X < x) &= P((X - 10)/2 < (x - 10)/2) \\ &= P(Z < (x - 10)/2) \\ &= 0.98 \end{aligned}$$

Appendix Table II is used to find the z-value such that $P(Z < z) = 0.98$. The nearest probability from Table II results in

$$P(Z < 2.05) = 0.97982$$

Therefore, $(x - 10)/2 = 2.05$, and the standardizing transformation is used in reverse to solve for x. The result is

$$x = 2(2.05) + 10 = 14.1 \text{ milliamperes}$$

Normal Distribution

Example 4-14 (continued)

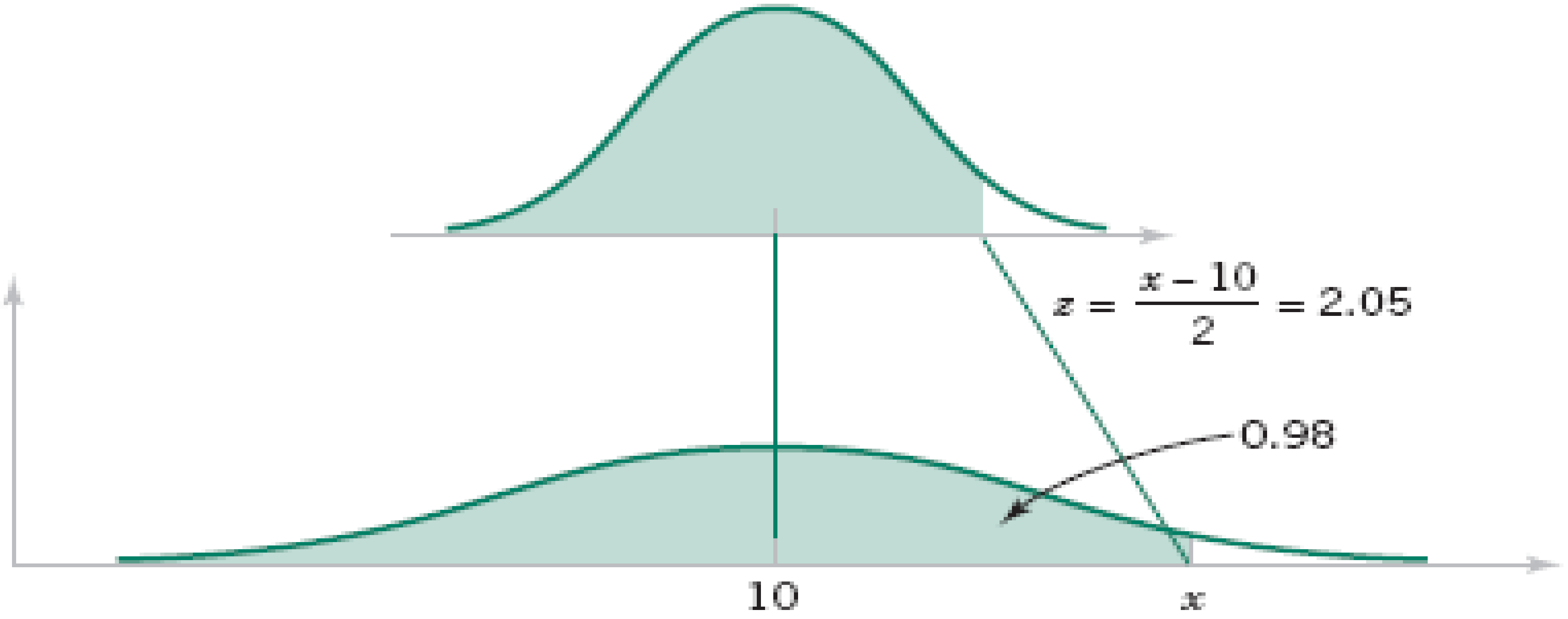


Figure 4-16 Determining the value of *x* to meet a specified probability.

Normal Approximation to the Binomial and Poisson Distributions

Under certain conditions, the normal distribution can be used to approximate the binomial distribution and the Poisson distribution.

Approximation to Binomial

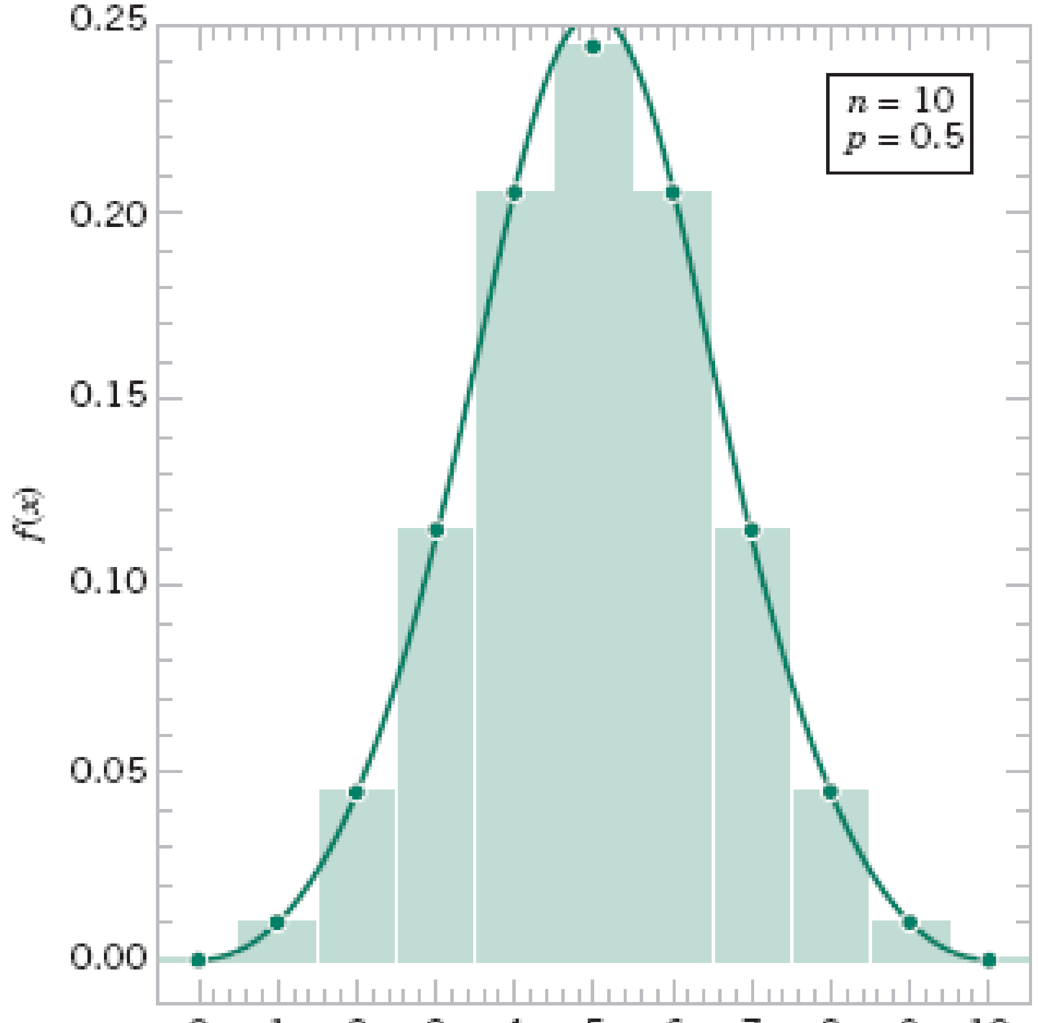


Figure 4-19 Normal approximation to the binomial.

Normal Distribution

Example 4-17

In a digital communication channel, assume that the number of bits received in error can be modeled by a binomial random variable, and assume that the probability that a bit is received in error is 1×10^{-5}. If 16 million bits are transmitted, what is the probability that more than 150 errors occur?

Let the random variable X denote the number of errors. Then X is a binomial random variable and

$$P(X > 150) = 1 - P(x \leq 150) = 1 - \sum_{x=0}^{150} \binom{16{,}000{,}000}{x} (10^{-5})^x (1 - 10^{-5})^{16{,}000{,}000-x}$$

Clearly, the probability is difficult to calculate. The normal distribution can be used to provide an appropriate approximation in this case.

Normal Distribution

Normal Approximation to the Binomial Distribution

If X is a binomial random variable, with parameters n and p

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \qquad (4\text{-}12)$$

is approximately a standard normal random variable. To approximate a binomial probability with a normal distribution a **continuity correction** is applied as follows

$$P(X \le x) = P(X \le x + 0.5) \cong P\left(Z \le \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

and

$$P(x \le X) = P(x - 0.5 \le X) \cong P\left(\frac{x - 0.5 - np}{\sqrt{np(1-p)}} \le Z\right)$$

The approximation is good for $np > 5$ and $n(1 - p) > 5$.

Normal Distribution

The digital communication problem in the previous example is solved as follows:

$$P(X > 150) = P\left(\frac{X - 160}{\sqrt{160(1 - 10^{-5})}} > \frac{150 - 160}{\sqrt{160(1 - 10^{-5})}}\right)$$

$$= P(Z > -0.79) = P(Z < 0.79) = 0.785$$

Because $np = (16 \times 10^6)(1 \times 10^{-5}) = 160$ and $n(1 - p)$ is much larger, the approximation is expected to work well in this case.

Conditions for approximating binomial probabilities:

binomial distribution	$\approx$ $np > 5$ $n(1-p) > 5$	normal distribution

Normal Approximation to the Poisson Distribution

If X is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}} \tag{4-13}$$

is approximately a standard normal random variable. The approximation is good for

$$\lambda > 5$$

The same continuity correction for binomial distribution can be applied to this approximation.

Normal Approximation to the Poisson Distribution

Assume that the number of asbestos particles in a squared meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a squared meter of dust is analyzed, what is the probability that less than 950 particles are found?

This probability can be expressed exactly as

$$P(X \leq 950) = \sum_{x=0}^{950} \frac{e^{-1000} x^{1000}}{x!}$$

The computational difficulty is clear. The probability can be approximated as

$$P(X \leq x) = P\left(Z \leq \frac{950 - 1000}{\sqrt{1000}}\right) = P(Z \leq -1.58) = 0.057$$